Non-radial oscillations of compact stars

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Outline

1. Overview of Compact Stars
2. Structure
3. Gravitational waves & its impact on EoS and Structure of Compact Stars
4. Oscillation of Compact Stars & its connection with GWs
5. Non-radial oscillation properties of Compact Star with Cowling approximation
   - Neutron stars
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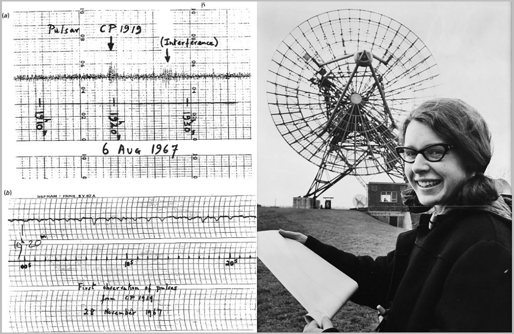
6. Non-radial oscillation of proto-neutron stars with General Relativistic Treatment

Compact Stars and Pulsar Observations

Compact stars are remnants of massive stars after thermo-nuclear runaway. They are mostly observed as pulsars with particular and very accurate rotational period and emission of electromagnetic and gravitational waves.

4000+ pulsars detected to date



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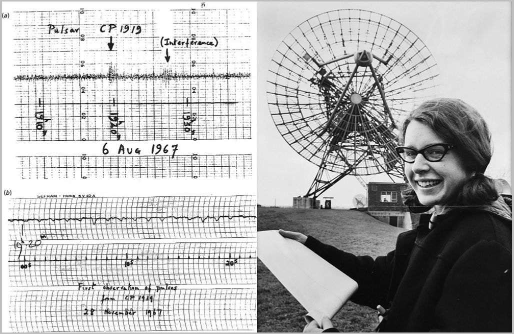


Present Day Astrophysical Constraints

Maximum mass (PSR J0740+6620) : $M = 2.08 \pm 0.07\ M_\odot$

Corresponding radius (NICER) : $R = 13.7^{+2.6}_{-1.5}$ km and $R = 12.39^{+1.30}_{-0.98}$ km

Radius of $1.4 M_\odot$ **compact star (GW170817) :** $8.9 \leq R_{1.4} \leq 13.2$ km

Corresponding deformability : $70 \leq \Lambda_{1.4} \leq 580$

Mass of PSR J0030+0451 (NICER) : M=$1.34^{+0.15}_{-0.16}\ M_\odot$; M=$1.44^{+0.15}_{-0.14}\ M_\odot$

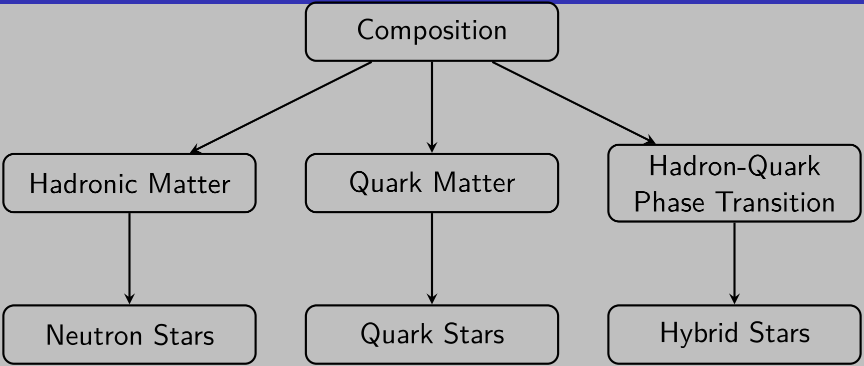
Corresponding radius (NICER) : R=$12.71^{+1.14}_{-1.19}$ km; R=$13.02^{+1.24}_{-1.06}$ km

Characteristics of Compact Stars

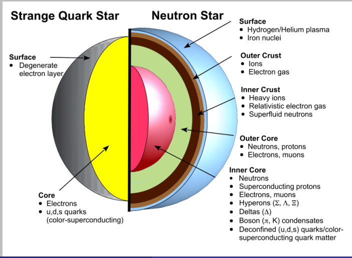
Density : $\rho \approx (5-10)\rho_0$
Compactness : M/R $\approx$ 0.2 and
Temperature : a few Kelvin $\approx$ 0 MeV (negligible)

Experiments till date have examined nuclear matter at saturation density $\rho_0 = 0.16$ fm^{-3} and up to $\approx$4.5ρ_0. So, the composition and equation of state (EoS) of compact stars are still inconclusive. Therefore, we rely on the theoretical modeling of compact star matter to obtain the structural properties of compact stars.
$\longrightarrow$ a lot of uncertainties pertaining to the EoS $\longrightarrow$ partially constrained by the astrophysical observations

Composition and EoS of Compact Stars

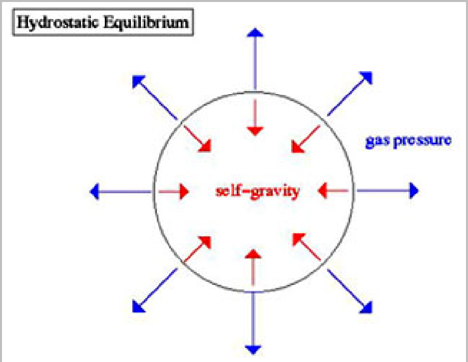


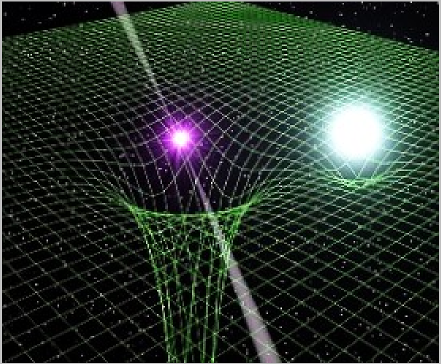
Based on the composition, we obtain the EoS of the compact stars as the energy density ($\varepsilon(\rho)$) and pressure ($P(\rho)$) as a function of baryon number density (ρ).



Structural Properties of Compact Star

Under Static Conditions





Structural Properties of Compact Star

The metric for a static spherically symmetric star is given as

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Here, Φ(- gravitational potential) and λ are the metric functions.
From this metric, the Tolman-Oppenheimer-Volkoff (TOV) equations are obtained as

$$\begin{aligned}
\frac{dP(r)}{dr} &= -\Big(\varepsilon(r) + P(r)\Big)\frac{d\Phi(r)}{dr}, \\
\frac{d\Phi(r)}{dr} &= \frac{M(r) + 4\pi r^3 P(r)}{r\Big(r - 2M(r)\Big)}, \\
\frac{dM(r)}{dr} &= 4\pi r^2 \varepsilon(r).
\end{aligned}$$

where, the mass function $M(r) = \frac{r}{2}\Bigg(1 - e^{-2\lambda(r)}\Bigg)$ satisfies the last equation.

Numerical Solution

Boundary conditions :

1. At the center : $P(0) = P_c$ & $\varepsilon(0) = \varepsilon_c$,

$$r = 0 \ (\sim \delta R) \quad \& \quad M(0) = 0 \ (\sim \frac{4}{3}\pi\delta R^3\varepsilon_c)$$

where, ε_c and P_c are the central energy density and central pressure.

2. At the surface : $P(R) = 0$,

$$r = R \quad \& \quad e^{\Phi(R)} = 1 - \frac{2M}{R}$$

By solving the TOV, we obtain the mass (M) and radius (R) of the compact star

Structural Properties of Compact Star

Tidal deformability : quadrupole deformation of a compact star in a binary system due to the tidal field created by its companion star. For an isolated star, it is calculated in the limit where source of the external tidal field is very far away.

Perturbation metric

$$
\begin{aligned}
ds^2 = &- e^{2\Phi(r)}(1 + H_0 Y^l_m e^{i\omega t})dt^2 + e^{2\lambda(r)}(1 - H_0 Y^l_m e^{i\omega t})dr^2 \\
&+ r^2(1 - K Y^l_m e^{i\omega t})(d\theta^2 + \sin^2\theta d\phi^2).
\end{aligned}
$$

where, $K'(r) = H'_0(r) + 2H'_0(r)\Phi'(r)$; primes denote derivatives with respect to r.
$l \rightarrow$ harmonic order=2, and $m \rightarrow$ azimuthal order=0.

Structural Properties of Compact Star

The perturbation functions $H_0(r)$, $\beta(r) = dH_0/dr$ obey

$$\frac{dH_0}{dr} = \beta$$

$$\frac{d\beta}{dr} = 2\eta H_0\Bigg[-2\pi\Bigg\{5\varepsilon(r) + 9P(r) + f\Bigg(\varepsilon(r) + P(r)\Bigg)\Bigg\} + \frac{3}{r^2} + \eta\left(\frac{M(r)}{r^2} + 4\pi rP(r)\right)^2\Bigg] + \frac{2\beta}{r}\eta\left\{-1 + \frac{M(r)}{r} + 2\pi r^2\Bigg(\varepsilon(r) - P(r)\Bigg)\right\}$$

where, $f = \frac{d\varepsilon}{dP}$ and $\eta = \left(1 - 2\frac{M(r)}{r}\right)$

These equations must be solved simultaneously with the TOV equations.

Structural Properties of Compact Star

Numerical Solution

Boundary conditions :

At the center ($r \sim \delta R$), the functions behave as

$$H_0(r) = a_0 r^2 \text{ and } \beta(r) = 2a_0 r$$

where, a_0 is an arbitrary constant.

We define $y = \frac{R\beta(R)}{H_0(R)}$

The tidal Love number is defined as

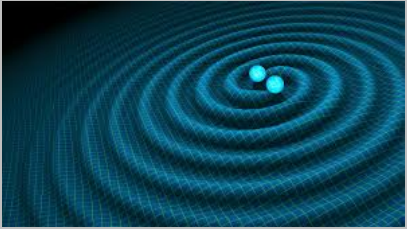
$$k_2 = k_2(y, C) \text{ where, the compactness C} = \text{M/R}$$

The dimensionless tidal deformability is given as

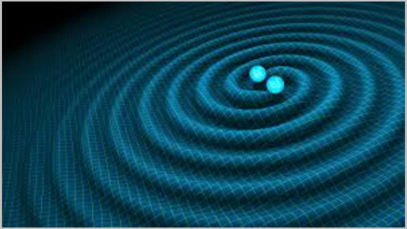
$$\Lambda = \frac{2}{3} k_2 R^5$$

Gravitational waves from compact star mergers

Gravitational waves (GWs) from compact star mergers are ripples in spacetime generated by collision of two compact stars.



Gravitational waves from compact star mergers



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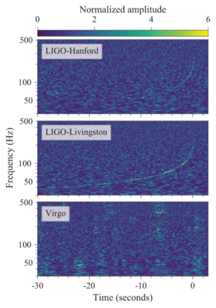
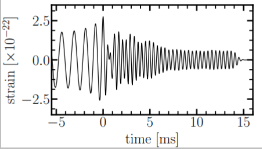




GW170817 was the first-ever direct detection of a BNS merger by LIGO-Virgo collaboration in 2017.

Two component NSs, each of mass $M_{NS} \sim 1.4 M_\odot \longrightarrow$ merged into a black hole of mass $M_{BH} \sim 2.74 M_\odot$

B. P. Abbott et al., PRL 119, 161101 (2017); N. Sarin and P. D. Lasky, Gen Relativ Gravit 53, 59 (2021)

Impact on EoS

The GW170817 observation provided stringent limits as $R_{1.4} = 11.9 \pm 1.4$ km and $\Lambda_{1.4} = 190^{+390}_{-190} \longrightarrow$ constraint on EoS upto a certain density.

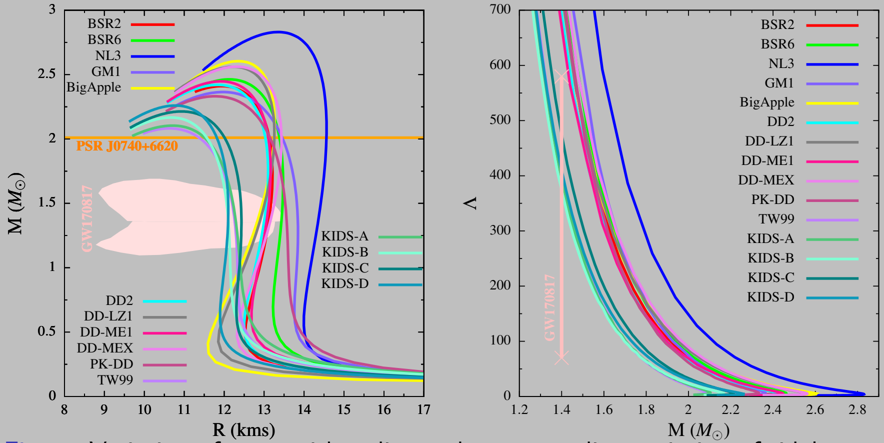
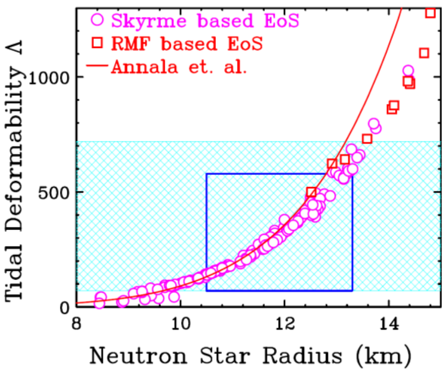
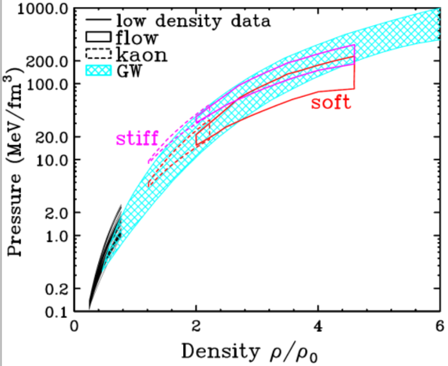


Figure: Variation of mass with radius and corresponding variation of tidal deformability with mass of NSs with β-equilibrated matter

Impact on EoS





C. Y. Tsang et al., arXiv:1807.06571 [nucl-ex]

Oscillation of compact stars in connection with GWs

- The manifestation of the oscillations is expected through GWs or electromagnetic radiation.
- Compact star oscillations, triggered by mergers or internal mechanisms $\longrightarrow$ cause non-radial or radial movements of matter $\longrightarrow$ emit specific, detectable GWs.
- The next generation GW detectors like Advanced LIGO (aLIGO), A+, Cosmic Explorer (CE1), and Einstein Telescope (ET) are likely to measure the frequencies of several oscillation modes of compact stars through GW detection.
- Gravitational asteroseismology : measurement of oscillation frequencies is directly linked to the structural properties (mass, radius, and tidal deformability) of compact stars $\longrightarrow$ constraints on EoS

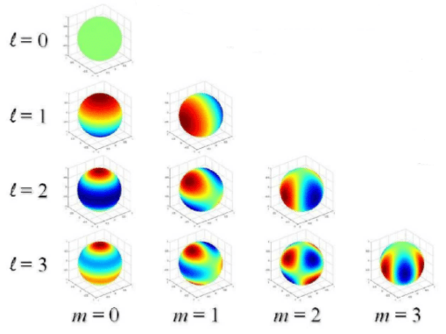
Classification of oscillation modes based on restoring force

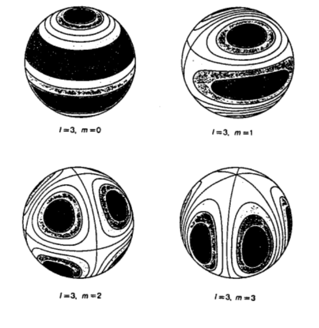
The GW oscillation spectra consist of different modes depending upon the different driving restoring force that acts on a displaced mass element.

- **Fundamental (f-mode)** : restoring force - pressure. No radial node.
- **Pressure (p-mode)** : restoring force - pressure. One radial node (p_1).
- **Gravity (g-mode)** : restoring force - gravity. No radial node. Arises due to fluctuations in composition/density/temperature.
- **Rotational (r-mode)** : restoring force - Coriolis force. Only in rotating stars.
- **Torsional (t-mode)** : restoring force - torsional/shear at the crust.
- **Interfacial (i-mode)** : restoring force - buoyancy. Appears at the boundaries of different layers due to discontinuities of density and shear modulus at the crust-core boundary.
- **Spacetime (w-mode)** : restoring force - curvature of spacetime. Focuses on the oscillation of only spacetime metric.

Types of oscillation based on angular and azimuthal modes

Non-radial - radial + angular components; Radial - angular components absent





Zonal modes ($m = 0$) divide the star into oscillating areas with nodal lines following the meridians only.
Sectoral modes ($m = l$) divide the star into oscillating areas with nodal lines that are parallel to the circles of latitude only.
Tesseral modes ($m \neq l$) have nodal lines along both meridians and circles of latitude

https://www.accimt.ac.lk & "Oscillations in Neutron Stars" by Gudrun Kristine Høye

Non-radial oscillation properties in static conditions with Cowling approximation

Cowling approximation neglects the spacetime metric perturbation.
Considers only fluid perturbations.

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

The perturbtion of the fluid inside the star is quantified by the Lagrangian displacement vector $\boldsymbol{\xi}\left(\xi^r, \xi^\theta, \xi^\phi\right)$

$$\xi^r = r^{l-1} e^{-\lambda} W(r) Y^l_m e^{i\omega t},$$
$$\xi^\theta = -r^{l-2} V(r) \partial_\theta Y^l_m e^{i\omega t}$$
$$\xi^\phi = -r^l (r\sin\theta)^{-2} V(r) \partial_\phi Y^l_m e^{i\omega t}$$

where, $V(r)$ and $W(r)$ are fluid perturbation functions.

Non-radial oscillation properties in static conditions with Cowling approximation

We take harmonic order $l = 2$ and azimuthal order $m = 0$ and solve two coupled differential equations

$$
\begin{aligned}
\frac{dW(r)}{dr} &= \frac{d\varepsilon(r)}{dP(r)}\left[\omega^2 r^2 e^{\lambda(r)-2\Phi(r)} V(r) + \frac{d\Phi(r)}{dr} W(r)\right] \\
&- l(l+1) e^{\lambda(r)} V(r) \\
\frac{dV(r)}{dr} &= 2\frac{d\Phi(r)}{dr} V(r) - e^{\lambda(r)} \frac{W(r)}{r^2}
\end{aligned}
$$

where, the coefficients of the eigenfunctions $V(r)$ and $W(r)$ are already obtained from solutions of TOV equations.

Numerical Solution with Cowling approximation

Boundary conditions :

1. At the center ($r = 0$) of the star, $W(r)$ and $V(r)$ behave asymptotically as

$$W(r) = Ar^{l+1} \text{ and } \; V(r) = -Ar^{l}/l;$$

where, A is an arbitrary constant.

2. At the surface ($r = R$),

$$\omega^2 R^2 e^{\lambda(R) - 2\Phi(R)} V(R) + \left.\frac{d\Phi(r)}{dr}\right|_{r=R} W(R) = 0.$$

The above equations are integrated from the center to the surface of the star by assuming an initial guess value of ω^2. After each integration the value of ω^2 is improved using Ridders' method until the above equation is satisfied.

Models Considered

1. Non-relativistic Korea-IBS-Daegu-SKKU (KIDS) model

For the cold β-equilibrated neutron star matter (n,p,e,μ), where ρ is the matter density, the energy per particle in homogeneous nuclear matter can be expanded as

$$\mathcal{E}(\rho, \delta) = \mathcal{T}(\rho, \delta) + \sum_{i=0}^{2} \alpha_i \rho^{1+i/3} + \delta^2 \sum_{i=0}^{3} \beta_i \rho^{1+i/3}.$$

$\delta = (\rho_n - \rho_p)/\rho$ is the neutron-proton asymmetry, so the terms corresponding to α_i and β_i describe the strong forces in symmetric and asymmetric nuclear matter, respectively.

2. Relativistic mean field (RMF) model

The Lagrangian density is given as

$$\begin{aligned}\mathcal{L}_{RMF} &= \bar{\psi}[\gamma_\mu(i\partial^\mu - g_\omega\omega^\mu - g_\rho\vec{\rho_\mu}\cdot\vec{\tau}) - (M + g_\sigma\sigma)]\psi \\ &+ \frac{1}{2}\partial_\mu\sigma\partial^\mu - \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{3}g_2\sigma^3 - \frac{c}{4}g_3\sigma^4 \\ &- \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu + \frac{1}{4}c_3(\omega_\mu\omega^\mu)^2 - \frac{1}{4}\vec{R}_{\mu\nu}\cdot\vec{R}^{\mu\nu} + \frac{1}{2}m_\rho^2\vec{\rho_\mu}\cdot\vec{\rho^\mu}.\end{aligned}$$

baryons: n, p

mesons: σ, vector ω and iso-vector ρ

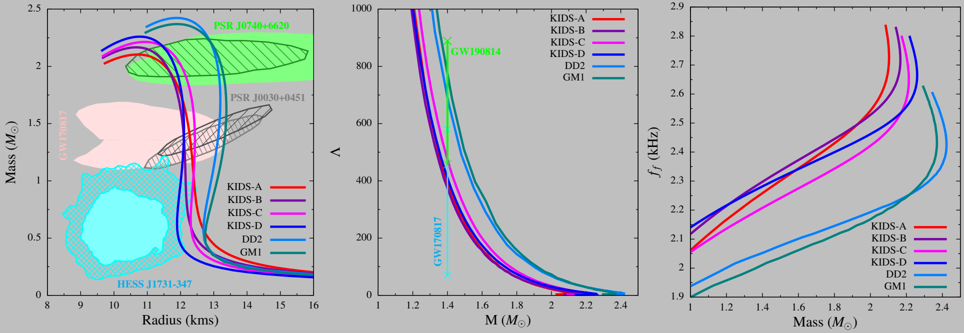
We consider GM1 and DD2 models with respective baryon-meson couplings.

Non-radial oscillation properties of neutron star

Table: The nuclear saturation properties like incompressibility (K_0), symmetry energy (J), slope (L), and the curvature parameter (K_{sym}) of the nuclear symmetry energy for the KIDS-A, B, C, D, DD2 and GM1 models.

	KIDS-A	KIDS-B	KIDS-C	KIDS-D	DD2	GM1
K_0	230	240	250	260	242.7	300.5
J	33	32	31	30	31.7	32.5
L	66	58	58	47	55	94
K_{sym}	-139.5	-162.1	-91.5	-134.5	93	18

Non-radial oscillation properties of neutron star

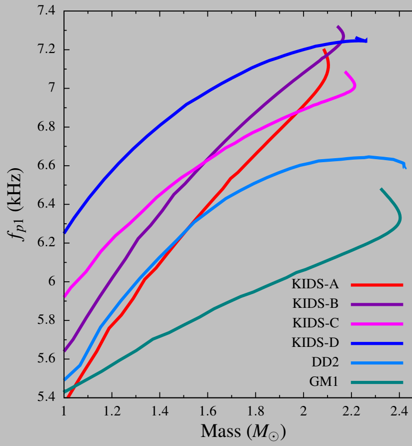


Corelation: $R_{1.4}$ - $\Lambda_{1.4}$ - $f_{f\,1.4}$

A. Guha, D. Sen, and C. H. Hyun, Eur.Phys.J.C 85 (2025) 4, 442

Non-radial oscillation properties of neutron star

KIDS-A : L=66 MeV
KIDS-B : L=58 MeV
KIDS-C : L=58 MeV
KIDS-D : L=47 MeV
DD2 : L=55 MeV
GM1 : L=94 MeV



Corelation: L - $f_{p_{1.4}}$

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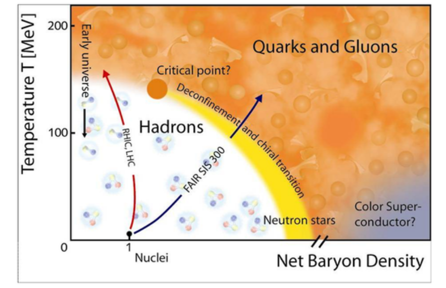
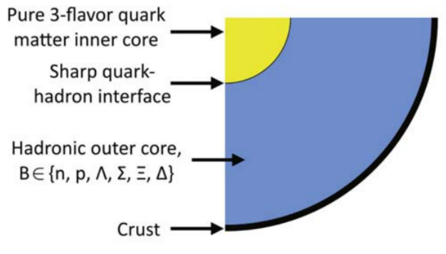
Non-radial oscillation properties of hybrid star





Quark model: Vector MIT bag (v-Bag)

The Lagrangian for the v-Bag model is given by

$$\mathcal{L}_{vBag} = \sum_{f=u,d,s} \left[\bar{\psi}_f \left\{\gamma^\mu (i\partial_\mu - g_{qqV} V_\mu) - m_f\right\} \psi_f - B\right] \Theta(\bar{\psi}_f \psi_f) + \frac{1}{2} m_V^2 V_\mu V^\mu - \frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \overline{\psi}_e (i\gamma_\mu \partial^\mu - m_e)\psi_e,$$

V_μ denotes the vector field, g_{qqV} is the quark-vector meson coupling constant, B is the bag constant. For the vector coupling constant, we assume $g_{uuV} = g_{ddV}$, $X_V = g_{ssV}/g_{uuV} = 0.4$ and $G_V = (g_{uuV}/m_V)^2$.

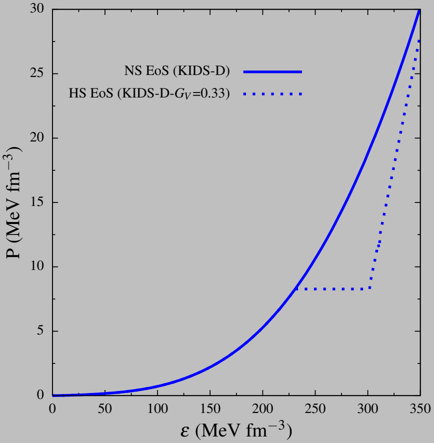
Phase Transition with Maxwell construction

Local charge neutrality:

$q_H(\mu_B, \mu_e) = 0$, $q_Q(\mu_B, \mu_e) = 0$

Maxwell criteria:

$P_H(\mu_B, \mu_e) = P_Q(\mu_B, \mu_e)$,
$\mu_B^H = \mu_B^Q$



Jump in density $\longrightarrow$ Emergence of g-mode

Non-radial oscillation properties of hybrid star

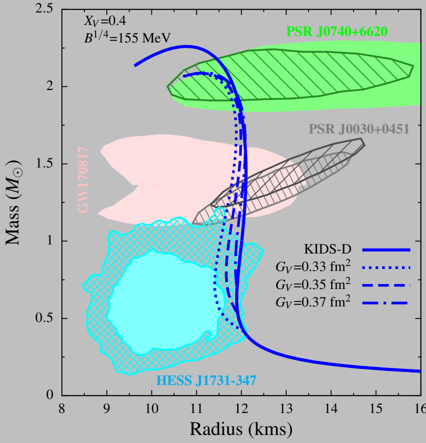
For HSs, with distinct discontinuity in density, the g mode oscillation frequency is also excited.
To account for the density jump for HSs, the following junction conditions for W and V are taken into account :

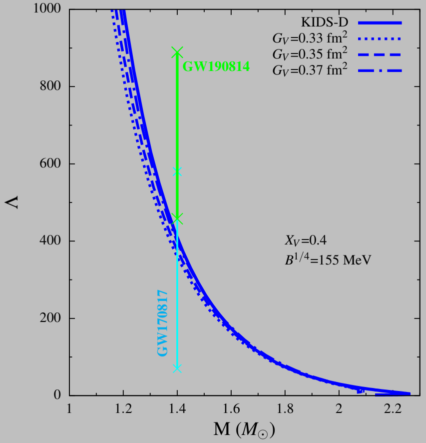
$$W_+ = W_-,$$

$$V_+ = \frac{e^{2\Phi}}{\omega^2 R_g^2}\left\{\frac{\varepsilon_- + P}{\varepsilon_+ + P}\left[\omega^2 R_g^2 e^{-2\Phi} V_- + e^{-\lambda}\frac{d\Phi}{dr} W_-\right] - e^{-\lambda}\frac{d\Phi}{dr} W_+\right\}.$$

Here R_g indicates the position of the discontinuity corresponding to the jump in density. W_-, V_-, and ε_- are the values of W, V, and ε at $r = R_g - \Delta r$ (quark phase) while W_+, V_+, and ε_+ are the ones at $r = R_g + \Delta r$ (hadronic phase), respectively.

Non-radial oscillation properties of hybrid star





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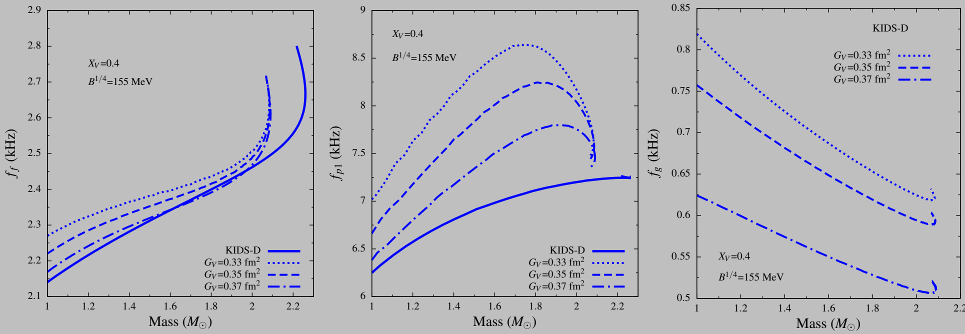
Non-radial oscillation properties of hybrid star



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- Corelation: $R_{1.4}$ - $\Lambda_{1.4}$ - $f_{f\,1.4}$ $\longrightarrow$ independent of composition
- p_1-mode may distinguish between NSs and HSs
- g-mode may provide an idea of the strength of quark repulsion in HSs

Non-radial oscillation properties of hybrid star

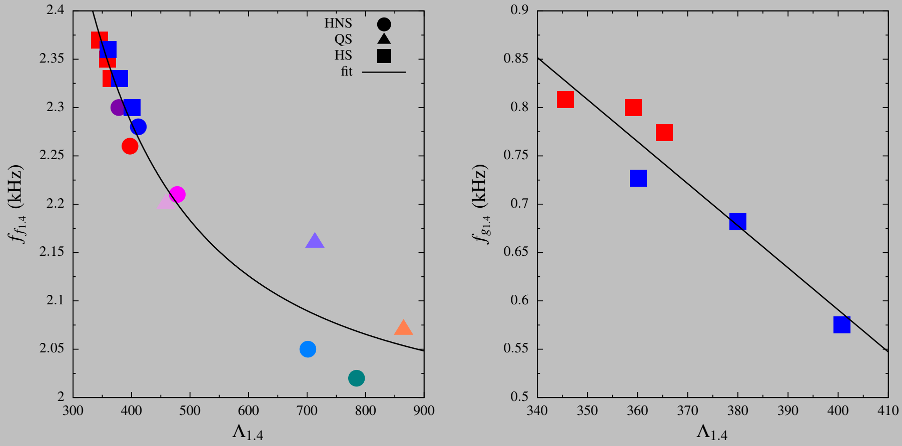

Figure: Variation of f and g mode frequencies with respect to tidal deformability of 1.4 $M_\odot$ of hybrid stars.

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Non-radial oscillation properties of hybrid star

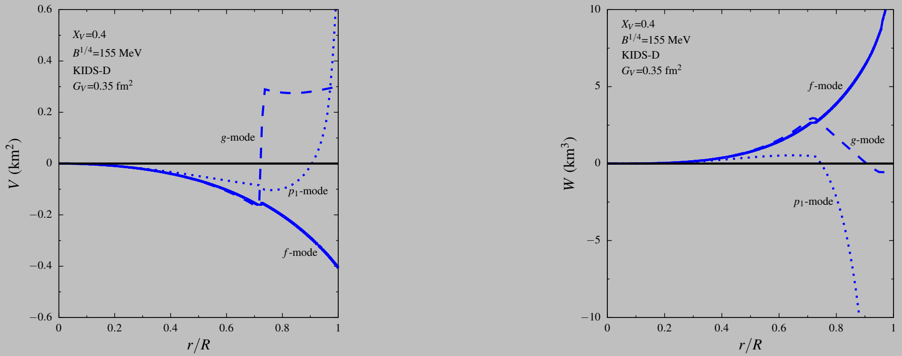

Figure: Radial variation of the eigenfunctions W and V for 1.4 $M_\odot$ hybrid stars with KIDS-D model and G_V=0.35 fm^2.

- Number of nodes is visibly manifested → f-mode, g-mode - 0 and p_1-mode - 1
- g-mode shows glitch around r/R=0.7 → phase transition

Conclusions

- We compute the fundamental (f) and the first pressure (p_1) mode frequencies (f_f and f_{p1}, respectively) for neutron star (NS) and hybrid star (HS) with Cowling approximation.
- For HSs, we also study the gravity (g) mode frequency (f_g).
- The f mode frequency ($f_{f_{1.4}}$) of the canonical 1.4 $M_\odot$ compact star (both NS and HS) is highly correlated with the canonical radius ($R_{1.4}$) and tidal deformability ($\Lambda_{1.4}$).
- $f_{p_{1.4}}$ of NS is well correlated with slope parameter of the symmetry energy (L).
- $f_{g_{1.4}}$ is linearly correlated with $\Lambda_{1.4}$.
- Should g modes be detected, they could not only support the existence of HSs, but f_g could be useful to understand the strength of quark repulsion in HSs.

Non-radial oscillation with General Relativistic Treatment

General relativistic treatment: fluid perturbation + metric perturbation
Perturbation of Regge-Wheeler-metric

$$ds^2 = -e^{2\Phi}(1 + r^l H_0 Y^l_m e^{i\omega t})dt^2 - 2i\omega r^{l+1} H_1 Y^l_m e^{i\omega t} dtdr$$
$$+ e^{2\lambda}(1 - r^l H_0 Y^l_m e^{i\omega t})dr^2 + r^2(1 - r^l K Y^l_m e^{i\omega t})(d\theta^2 + \sin^2\theta d\phi^2).$$

where, $H_0(r)$, $H_1(r)$, and $K(r)$ are the metric perturbation functions; complex oscillation frequency $\omega = 2\pi f + i/\tau$ (τ - damping time)

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where, $H_0(r)$, $H_1(r)$, and $K(r)$ are the metric perturbation functions;
complex oscillation frequency $\omega = 2\pi f + i/\tau$ (τ - damping time)
The perturbtion of the fluid inside the star is quantified by the Lagrangian displacement vector $\boldsymbol{\xi}$ $\left(\xi^r, \xi^\theta, \xi^\phi\right)$

$$\xi^r = r^{l-1} e^{-\lambda} W(r) Y^l_m e^{i\omega t},$$
$$\xi^\theta = -r^{l-2} V(r) \partial_\theta Y^l_m e^{i\omega t},$$
$$\xi^\phi = -r^l (r\sin\theta)^{-2} V(r) \partial_\phi Y^l_m e^{i\omega t}$$

where, $H_0(r)$, $H_1(r)$, and $K(r)$ are the metric perturbation functions.
$n \rightarrow$ radial order, $l \rightarrow$ harmonic order, and $m \rightarrow$ azimuthal order.

Solution to obtain the non-radial oscillation frequency

Additional fluid perturbation function $X(r)$, related to the Lagrangian pressure variations

$$\Delta P = -r^l e^{-\Phi} X Y^l_m e^{i\omega t}$$

6 perturbation functions - $H_0(r)$, $H_1(r)$, $K(r)$, $W(r)$, $V(r)$, and $X(r)$
But the perturbations of a spherical star can have 4 degrees of freedom.

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But the perturbations of a spherical star can have 4 degrees of freedom.
$\longrightarrow$ four independent variables $H_1(r)$, $W(r)$, $K(r)$, and $X(r)$ while $H_0(r)$ and $V(r)$ are converted as functionals of independent variables as

$$H_0 = \left\{8\pi r^3 e^{-\Phi} X - \left[\frac{1}{2} l(l+1)\left(M + 4\pi r^3 P\right) - \omega^2 r^3 e^{-2(\lambda+\Phi)}\right] H_1 + \left[\frac{1}{2}(l+2)(l-1)r - \omega^2 r^3 e^{-2\Phi} - \frac{e^{2\lambda}}{r}\left(M + 4\pi r^3 P\right)\left(3M - r + 4\pi r^3 P\right)\right] K\right\} \times \left\{3M + \frac{1}{2}(l+2)(l-1)r + 4\pi r^3 P\right\}^{-1} \text{and}$$

$$V = \left\{\frac{X}{\varepsilon + P} + \frac{1}{r}\frac{dP}{dr} e^{\Phi-\lambda}\frac{W}{\varepsilon + P} - \frac{1}{2} e^{\Phi} H_0\right\} \times \frac{e^{\Phi}}{\omega^2}$$

Solution to obtain the non-radial oscillation frequency

The homogeneous linear differential equations for the independent perturbation functions

$$r\frac{dH_1}{dr} = -\left[l + 1 + \frac{2M}{r}e^{2\lambda} + 4\pi r^2 e^{2\lambda}(P - \varepsilon)\right] H_1 + e^{2\lambda}\left[H_0 + K\right],$$

$$r\frac{dK}{dr} = H_0 + \frac{1}{2}l(l+1)H_1 - \left[(l+1) - r\frac{d\Phi}{dr}\right] K - 8\pi(\varepsilon + P)e^{\lambda}W,$$

$$r\frac{dW}{dr} = -(l+1)W + r^2 e^{\lambda}\left[(\gamma P)^{-1}e^{-\Phi}X - \frac{l(l+1)}{r^2}V + \frac{1}{2}H_0 + K\right],$$

$$r\frac{dX}{dr} = -lX + (\varepsilon + P)e^{\Phi}\left\{\frac{1}{2}(1 - r\frac{d\Phi}{dr})H_0 + \frac{1}{2}\left[r^2\omega^2 e^{-2\Phi} + \frac{1}{2}l(l+1)\right]H_1\right.$$

$$\left. + \frac{1}{2}(3r\frac{d\Phi}{dr} - 1)K - \frac{l(l+1)}{r}\frac{d\Phi}{dr}V - \left[4\pi(\varepsilon + P)e^{\lambda} + \omega^2 e^{\lambda - 2\Phi} - r^2\frac{d}{dr}\left(\frac{e^{-\lambda}}{r^2}\frac{d\Phi}{dr}\right)\right]W\right\}$$

where, adiabatic index $\gamma = \frac{(\varepsilon+P)}{P}\frac{dP}{d\varepsilon}$.

Solution to obtain the non-radial oscillation frequency

System of perturbation functions as $Y(r) = \{H_1, K, W, X\}$. The system is singular at $r = 0$, where the perturbation functions are expanded in the power of r

$$Y(r) = Y(0) + \frac{1}{2} Y_{,rr}(0) r^2 + \mathcal{O}(r^4)$$

At $r = 0$, the lowest order terms satisfy the relations

$$H_0(0) = K(0), \quad H_1(0) = \frac{1}{l(l+1)} \left[2lK(0) + 16\pi(\varepsilon_c + P_c) W(0) \right] \text{ and}$$

$$X(0) = (\varepsilon_c + P_c) e^{\Phi_0} \left\{ \left[\frac{4\pi}{3}(\varepsilon_c + 3P_c) - \omega^2 e^{-2\Phi_0} l^{-1} \right] \times W(0) + \frac{1}{2} K(0) \right\}$$

For higher order terms: J.L. Lu and W.M. Suen, Chin. Phys. B 20, 040401 (2011)
4 degrees of freedom → 4 independent solutions of the perturbation function equations

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Boundary conditions :
1. At $r = 0$, $W(0) = 1$, $K(0) = \pm(\varepsilon_c + P_c) \longrightarrow$ 2 independent solutions of the perturbation function equations
2. At $r = R$, $X(R) = 0 \longrightarrow$ 3 independent solutions of the perturbation function equations

Solution to obtain the non-radial oscillation frequency

To avoid singularity :
Integration of differential equations for $Y = \{H_1, K, W, X\} \longrightarrow$ Forward integration and Backward integration
For a given value of ω (real),
Forward integration : r_0 (point very close to $r = 0$) to $r_c (= R/2)$
Backward integration : R_G (point very close to $r = R$) to $r_c (= R/2)$
$\longrightarrow$ 2 independent forward solutions and 3 independent backward solutions

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To satisfy the boundary conditions, the linear combination of the two independent forward solutions and the linear combination of the three independent backward solutions are matched at r_c.

$$f_1 Y_1(r_c) + f_2 Y_2(r_c) = b_1 Y_3(r_c) + b_2 Y_4(r_c) + b_3 Y_5(r_c)$$

coefficients f_1, f_2, b_1, b_2, b_3 are solved taking one of them equal to one.
$\longrightarrow$ most general solutions for H_1, K, W, X obtained.

Solution to obtain the non-radial oscillation frequency

But the most general solutions for H_1, K, W, X is valid only inside the star and GW detectors will measure the oscillation frequency far away from the star where the value of ω must be different.

Outside the star the fluid perturbation functions vanish $\longrightarrow W = X = 0$.

Solution to obtain the non-radial oscillation frequency

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Outside the star the fluid perturbation functions vanish $\longrightarrow W = X = 0$.

Redefining the surviving perturbation functions, we obtain a system described by the Zerilli equation

$$\frac{d^2 Z}{dr^{*2}} + \left[\omega^2 - v(r^*)\right] Z = 0$$

where, the effective potential is

$$v(r^*) = \frac{2(1 - 2M/r)}{r^3(nr + 3M)^2}\left[n^2(n+1)r^3 + 3n^2 M r^2 + 9nM^2 r + 9M^3\right]$$

r^* is the tortoise coordinate and given in terms of r as

$$r^* = r + 2M \log\left(\frac{r}{2M} - 1\right)$$

Solution to obtain the non-radial oscillation frequency

In terms of $H_0(r)$ and $K(r)$ outside the star, we have

$$Z(r^*) = \frac{k(r)K(r) - a(r)H_0(r) - b(r)K(r)}{k(r)g(r) - h(r)} \quad \text{and}$$

$$\frac{dZ(r^*)}{dr^*} = \frac{h(r)K(r) - a(r)g(r)H_0(r) - b(r)g(r)K(r)}{h(r) - k(r)g(r)}$$

where, the radial functions are defined as

$$a(r) = -(nr + 3M)/\left[\omega^2 r^2 - (n+1)M/r\right],$$

$$b(r) = \frac{\left[nr(r-2M) - \omega^2 r^4 + M(r-3M)\right]}{(r-2M)\left[\omega^2 r^2 - (n+1)M/r\right]},$$

$$g(r) = \frac{\left[n(n+1)r^2 + 3nMr + 6M^2\right]}{r^2(nr+3M)},$$

$$h(r) = \frac{\left[-nr^2 + 3nMr + 3M^2\right]}{(r-2M)(nr+3M)},$$

$$k(r) = -r^2/(r-2M)$$

The starting inputs of $H_0(r)$ and $K(r)$ to be given from the solutions inside the star.

Solution to obtain the non-radial oscillation frequency

Two independent solutions of the Zerilli equation : incoming $Z_+(r^*)$ and outgoing $Z_-(r^*)$ waves. General solution :

$$Z(r^*) = A(\omega)Z_-(r^*) + B(\omega)Z_+(r^*)$$

At very large r, the expansion of Z_- and Z_+ can be represented as

$$Z_-(r^*) = e^{-i\omega r^*}\sum_{j=0}^{\infty}\beta_j r^{-j}, \quad Z_+(r^*) = e^{i\omega r^*}\sum_{j=0}^{\infty}\bar{\beta}_j r^{-j}$$

where, $\bar{\beta}_j$ is the complex conjugate of β_j.
At large radius ($r \leq 50\omega^{-1}$), up to second order

$$Z_- = e^{-i\omega r^*}\left[\beta_0 + \frac{\beta_1}{r} + \frac{\beta_2}{r^2} + \mathcal{O}(r^3)\right],$$
$$\frac{dZ_-}{\mathrm{d}r^*} = -i\omega e^{-i\omega r^*}\left[\beta_0 + \frac{\beta_1}{r} + \frac{\beta_2 - i\beta_1(1-2m/r)/\omega}{r^2}\right]$$

giving,

$$\beta_1 = \frac{-i(n+1)\beta_0}{\omega}, \qquad \beta_2 = \frac{[-n(n+1) + im\omega(3/2 + 3/n)]\beta_0}{2\omega^2}$$

Solution to obtain the non-radial oscillation frequency

For the real values of ω, $A(\omega) = B^*(\omega)$
At $r = 50\omega^{-1}$, we obtain the numerical values of Z and $\frac{dZ}{dr^*}$ by solving Zerilli equation $\rightarrow B(\omega)$
$B(\omega)$ is in general complex for each ω. So, we interpolate $B(\omega)$ as an analytic function of ω in power series ($\mathcal{O}(2)$) along the real axis.

$$B(\omega) \approx \gamma_0 + \gamma_1\omega + \gamma_2\omega^2 \,, \gamma_i \rightarrow \text{unknown}$$

For three guess values of real ω, we obtain the values of γ_i.
In the absence of incoming wave,

$$\gamma_0 + \gamma_1\omega + \gamma_2\omega^2 = 0 \,, \gamma_i \rightarrow \text{known}$$

Once we obtain the values of γ_i, we get the final solution for the complex eigen frequency $\omega = 2\pi f + i/\tau$, where τ is the damping time and f is the frequency of the GW. $\rightarrow$ Convergence with multiple set of guess values.

Non-radial oscillation properties of proto-neutron star

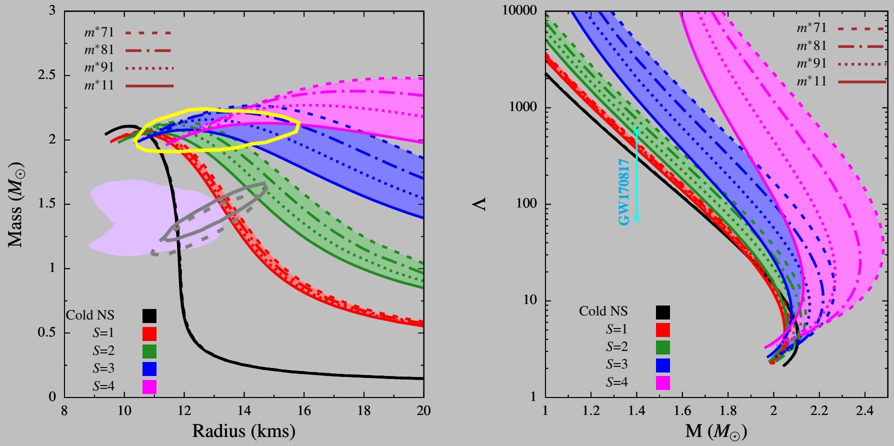
Evolved massive star ($\gtrsim 8M_\odot$) $\xrightarrow[\text{collapse}]{\text{gravitational}}$ Supernova explosion $\longrightarrow$ PNS $\xrightarrow[\nu\text{ emission}]{\text{cooling}}$ NS

PNS structure is largely affected by the composition of the star, temperature, entropy, and the number of trapped neutrinos.

Composition Considered : homogeneous β-stable nuclear matter (n, p, γ, e, e^+, ν_e and $\bar{\nu}_e$)
Model Considered : KIDS0
with fixed isovector effective mass $\mu_V^*(= m_V^*/m_N)$=1 and different values of isoscalar effective mass $\mu_S^*(= m_S^*/m_N)$ of nucleons
$\longrightarrow$ m^*71 - (μ_S^*=0.7, μ_V^*=1)

Different values of entropy per nucleon (S) indicate different stages of PNS evolution.
$\longrightarrow$ $S = 4$ - supernova stage

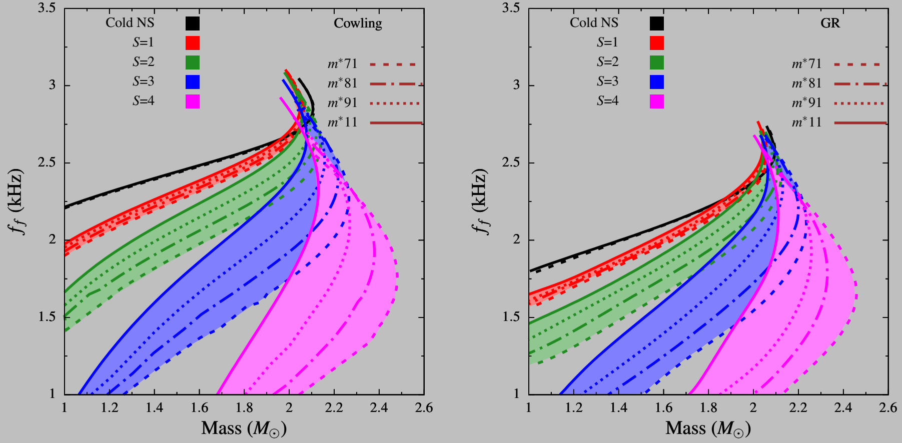
Non-radial oscillation properties of proto-neutron star



- Radius is greatly affected by thermal effects
- $M - R$ and $M - \Lambda$ variation bands widen high values of S $\longrightarrow$ uncertainty of EoS due to μ_S^* in earlier stages evolution.

Non-radial oscillation properties of proto-neutron star

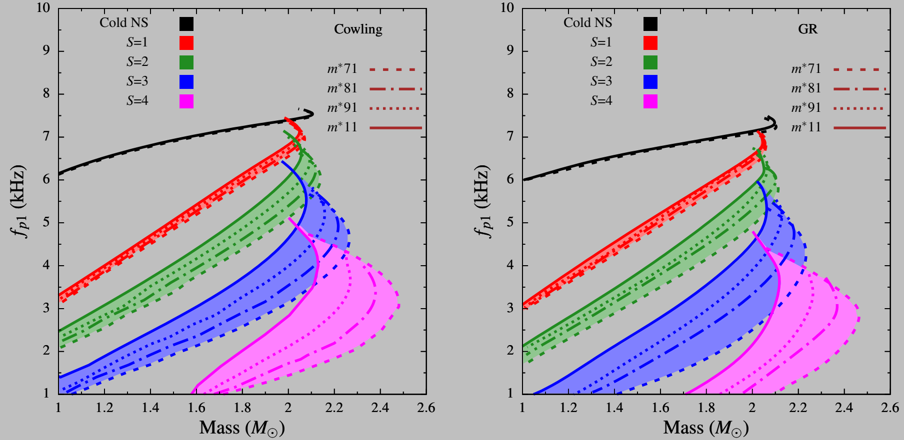
f-mode oscillation



- GR treatment estimates the f_f smaller than the Cowling approximation
- $f_{f\,PNS} < f_{f\,NS}$ for low and intermediate mass

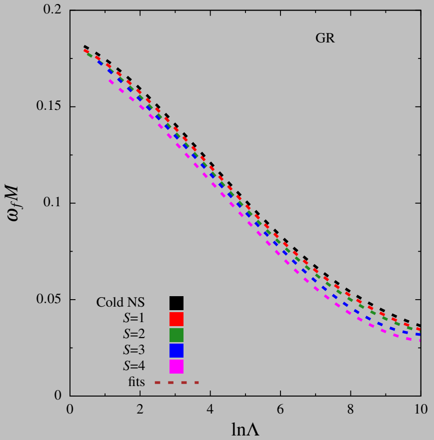
Non-radial oscillation properties of proto-neutron star

p-mode oscillation



- f_{p_1} is less affected by the treatment chosen; difference becomes slightly significant at early stages of evolution
- $f_{p\,PNS} < f_{p\,NS}$ even for high mass $\longrightarrow$ detection of f_f and f_{p_1} of PNSs facilitated $\longrightarrow$ indicate the evolutionary stage

Non-radial oscillation properties of proto-neutron star



- mass-scaled angular frequency ($\omega_f M$) is correlated to tidal deformability (Λ) as non-linear fits that shift upwards in the $\omega_f M - \Lambda$ plane as the star evolves

Prospects of detection of f-mode frequency

The amplitude of the GW strain

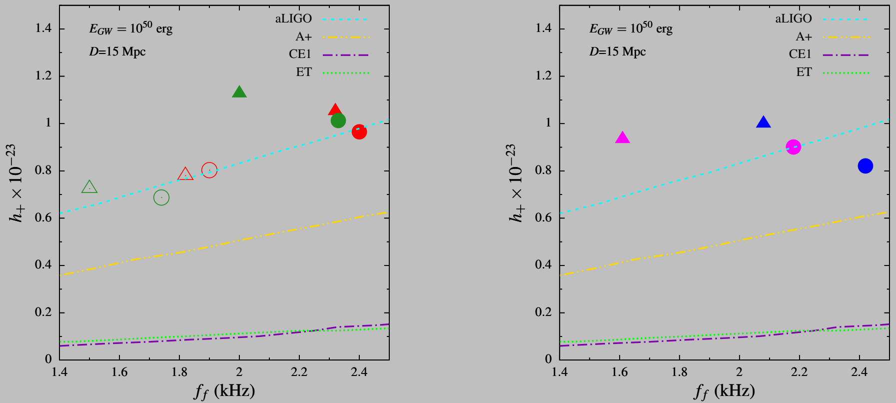
$$h_+ = \frac{\sqrt{30 E_{GW}/\tau}}{2\pi D f}$$

where, D=15 Mpc (star in the Virgo cluster) is the distance of the detector to the source.

Total released energy in case of a core-collapse supernova $\sim 10^{53}$ erg and the kinetic energy of mass ejecta is $\sim 10^{51}$ erg.

Assuming that $\approx$ 10% of the total available energy is efficiently converted into f-mode GW radiation $\longrightarrow E_{GW} = 10^{50}$ erg

Non-radial oscillation properties of proto-neutron star



The empty points correspond to $1.4M_\odot$ and filled points corresponds to $2M_\odot$. The red and green colors indicate $S = 1$ and $S = 2$, respectively while circles represent $\mu_S^* = 1.0$ and triangles represent $\mu_S^* = 0.7$. The blue and magenta colors indicate $S = 3$ and $S = 4$, respectively while circles represent $\mu_S^* = 1.0$ and triangles represent $\mu_S^* = 0.7$.

- If supernova explosion happens in the Virgo cluster, f_f could be detected by the upcoming detectors

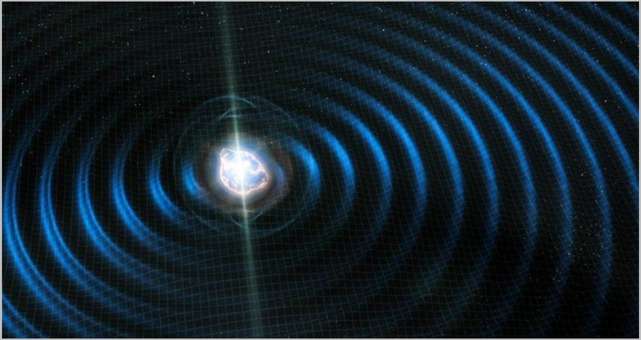
Conclusions

- At each stage of evolution of a proto-neutron star (PNS), the fundamental f-mode and first pressure p_1-mode frequencies (f_f and f_{p_1}) are computed under isentropic conditions by varying the effective isoscalar effective mass (μ_S^*) of nucleons.
- f_f and f_{p_1} are calculated adopting both the formalisms of complete general relativity (GR) and Cowling approximation.
- GR treatment estimates the f_f smaller than the Cowling approximation by about $15-20\,\%$, specially at later stages of evolution of the star. f_{p_1} is less affected by the treatment chosen
- Thermal effects reduce the values of f_f and f_{p_1} of PNSs compared to those of cold NSs $\longrightarrow$ detection of f_f and f_{p_1} of PNSs facilitated.
- For high-mass PNSs, this reduction is more pronounced for f_{p_1} than f_f $\longrightarrow$ detection of f_{p_1} by upcoming GW detectors, could potentially indicate the evolutionary stage of a star from a supernova to NS via the PNS stage
- f_f and f_{p_1} are also strongly dependent on μ_S^*, specially at early stages $\longrightarrow$ Simultaneous measurement of f_f and f_{p_1} can reduce the uncertainty of EoS due to μ_S^*
- mass-scaled angular frequency ($\omega_f M$) is correlated to tidal deformability (Λ)

Simultaneous detection of tidal waves and non-radial modes of oscillation from BNS mergers to enrich our understanding

BEST WISHES TO THE GW DETECTION GROUPS WORLDWIDE 🙂

GW from isolated compact stars



Continuous gravitational waves from isolated compact stars are long-lasting ripples in spacetime.

They can be generated by several mechanisms like elastic deformations (irregularities) of the crust, distortions caused by high magnetic field, r-modes fluid oscillations due to rapid rotation, and free precession (wobbling)

Non-radial oscillation properties of neutron star

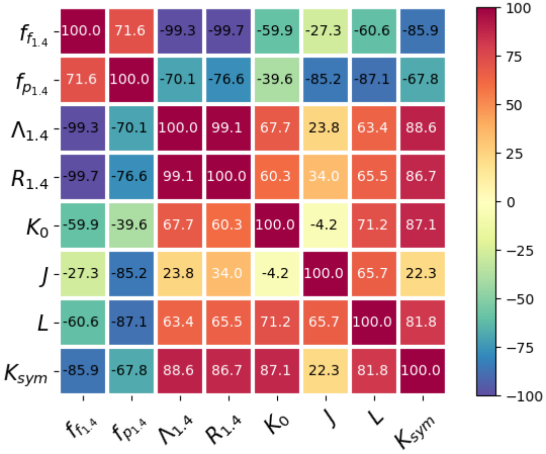


Figure: Correlation (in percentage) between the nuclear matter parameters, structural, and oscillation properties of 1.4 $M_\odot$ neutron star.

A. Guha, D. Sen, and C. H. Hyun, Eur.Phys.J.C 85 (2025) 4, 442

Oscillation properties of hybrid star

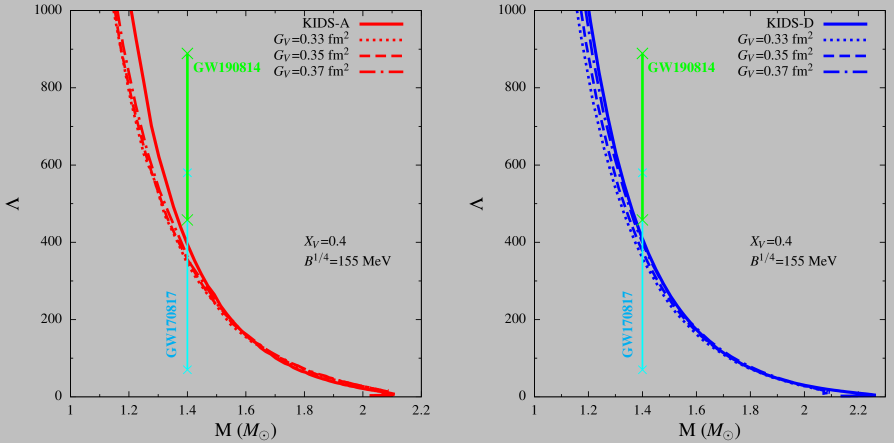


Figure: Variation of tidal deformability with mass of hybrid stars

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Non-radial oscillation under static conditions

Perturbation in oscillation spectra can be decomposed into spherical harmonics with even and odd parity components.

- Odd parity perturbations are important only for rotating stars and manifested as r-mode of oscillations.
- For non-rotating stars, odd parity perturbations are trivial zero mode.
- We consider only even parity perturbations