중성자별의 내부 구조 및 상태방정식

기본 원리

이창환 / 부산대학교 물리학과

Contents

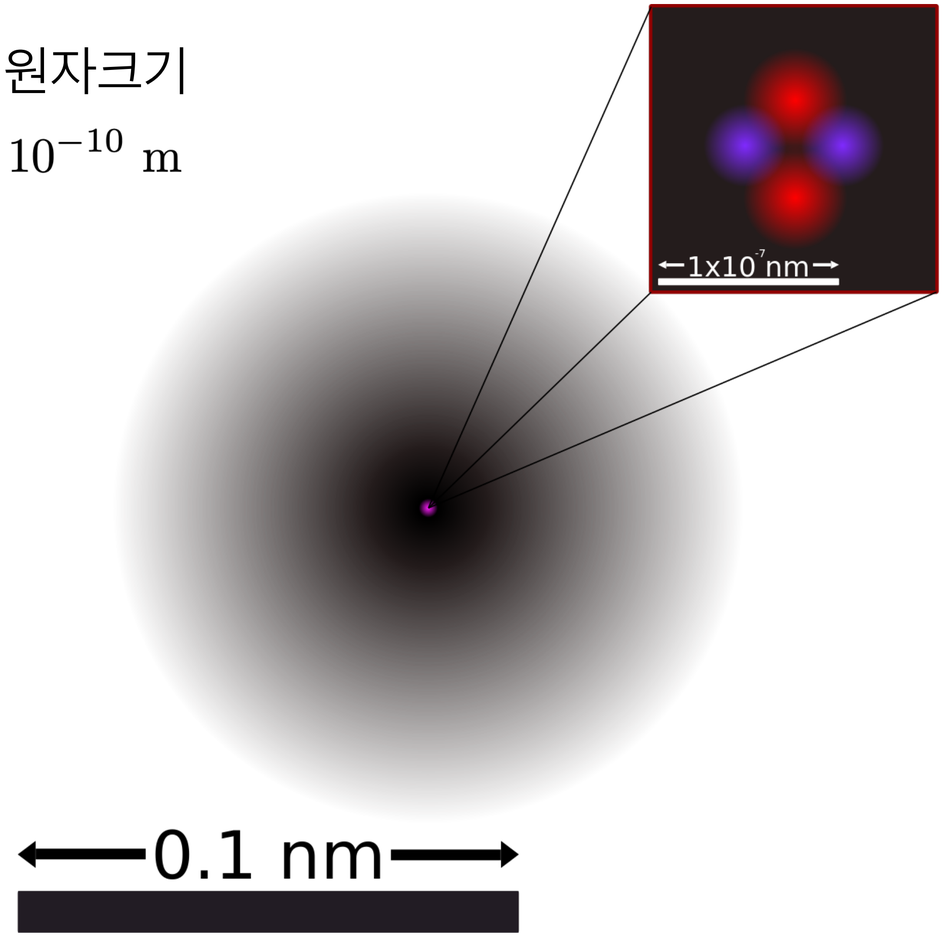
중성자 (Chadwick, 1932)

원자크기

10^{-10} m

양성자/중성자 크기

10^{-15} m



원자핵 밀도 $\approx 2 \times 10^{17} \mathrm{kg/m^3}$

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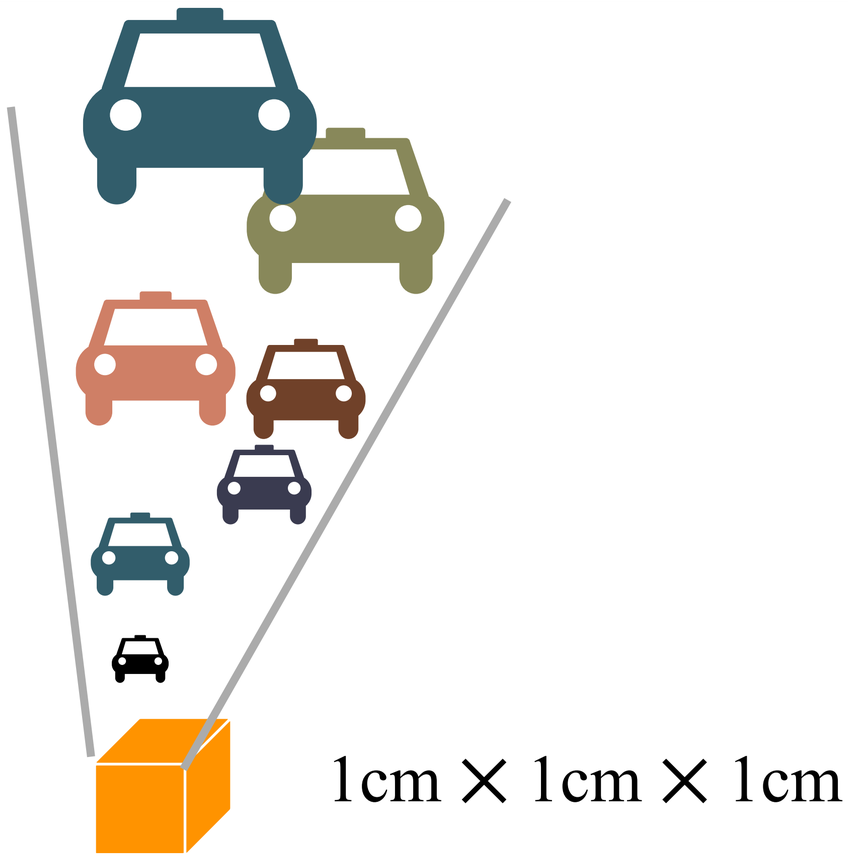
원자핵 밀도 $\approx 2 \times 10^{17} \mathrm{kg/m^3}$



원자핵밀도 ~ 2톤 자동차 10^8(1억)대를 각설탕 크기로 압축한 밀도

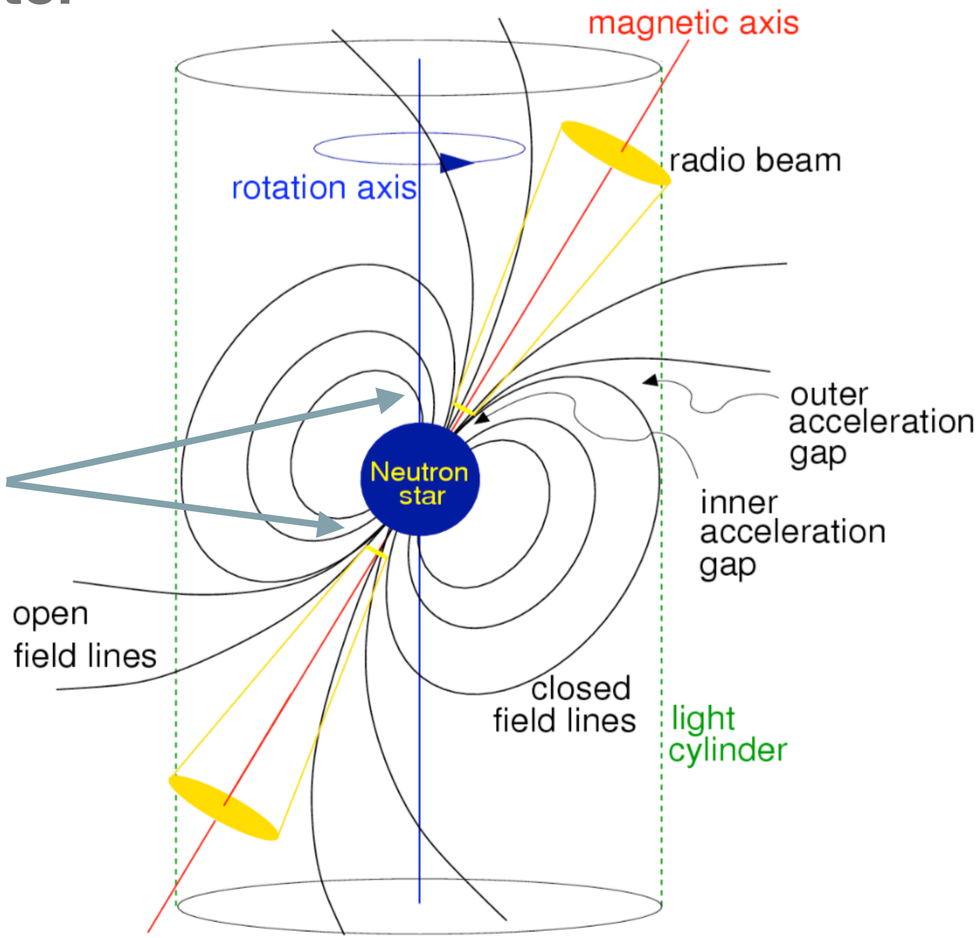
Why Neutron Star (중성자별) ?

태양반경 : 696,340 km

Ultimate testing place for physics of dense matter

$$M = 1.5 \sim 2.0 M_\odot$$
$$R = 10 \sim 15 \text{ km}$$
$$A \sim 10^{57} \text{ nucleons}$$

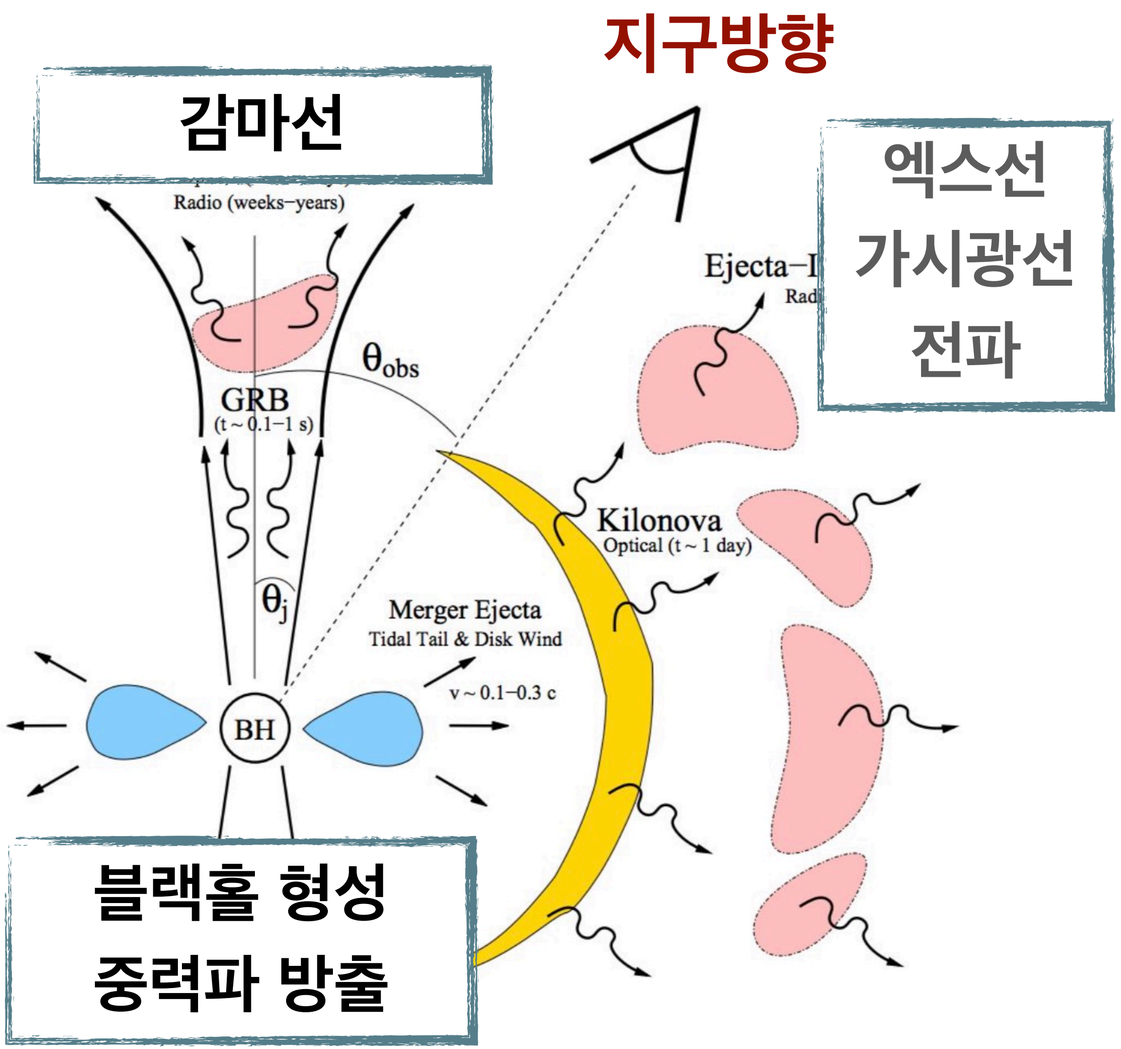
e^+e^- pair creation

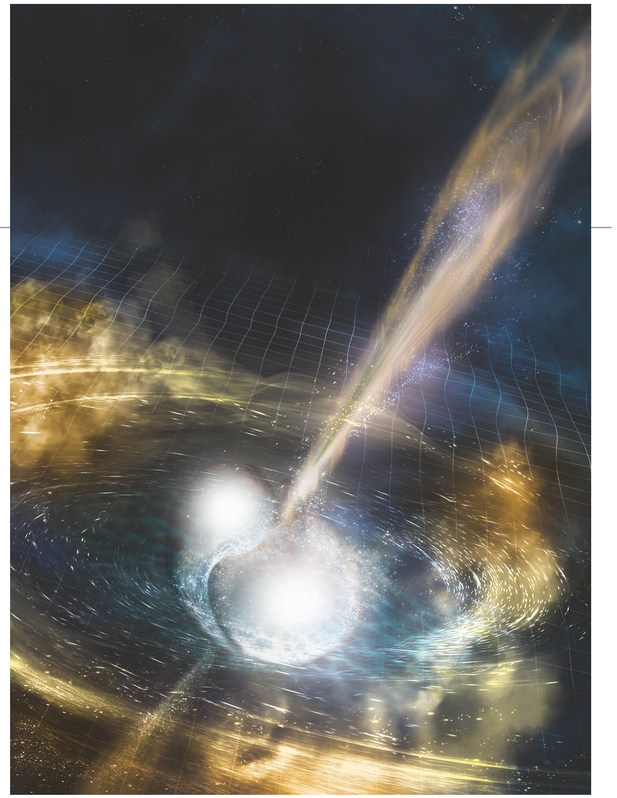


$$\rho_{\text{center}} \approx \text{several} \times \rho_0$$

$$n_0 \approx 0.16 \text{ fm}^{-3}$$
$$\approx 1.6 \times 10^{44} \text{m}^{-3}$$
$$\rho_0 \approx 2.04 \times 10^{17} \text{kg} \cdot \text{m}^{-3}$$

중성자별 충돌 모형도


지구방향
감마선
Radio (weeks–years)
θobs
GRB
(t ~ 0.1–1 s)
θj
Ejecta–I
Rad
엑스선
가시광선
전파
Kilonova
Optical (t ~ 1 day)
Merger Ejecta
Tidal Tail & Disk Wind
v ~ 0.1–0.3 c
BH
블랙홀 형성
중력파 방출



Nuclear matter is not an ideal gas

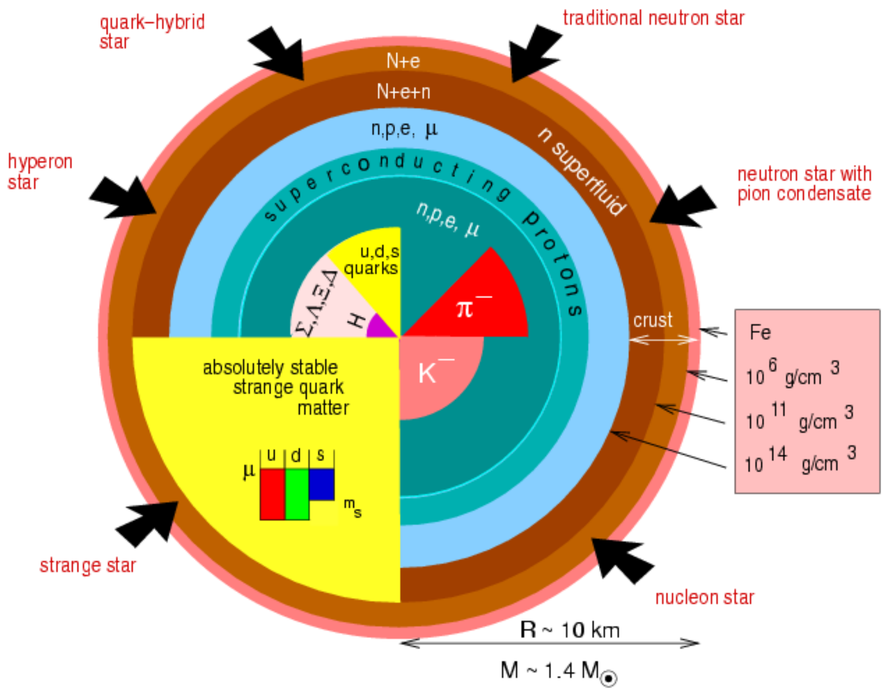
White Dwarf / Condensed Matter:

- 전자 겹침(축퇴)
- 전자기 상호작용

Neutron Star / Nuclei:

- 양성자/중성자 (강입자) 겹침
- 강상호 작용

F. Weber 2005



별의 운명

Pressure

$$n = N/V$$

number density

Radiation pressure of photon & neutrino (빛과 중성미자)

$$P_{\rm rad} = \frac{1}{3} a T^4$$

Thermal (kinetic) pressure of ideal gas (입자들의 운동)

$$P_{\rm kin} = nkT \qquad (PV = NkT)$$

Quantum degeneracy pressure of ideal gas (양자 겹침 압력)

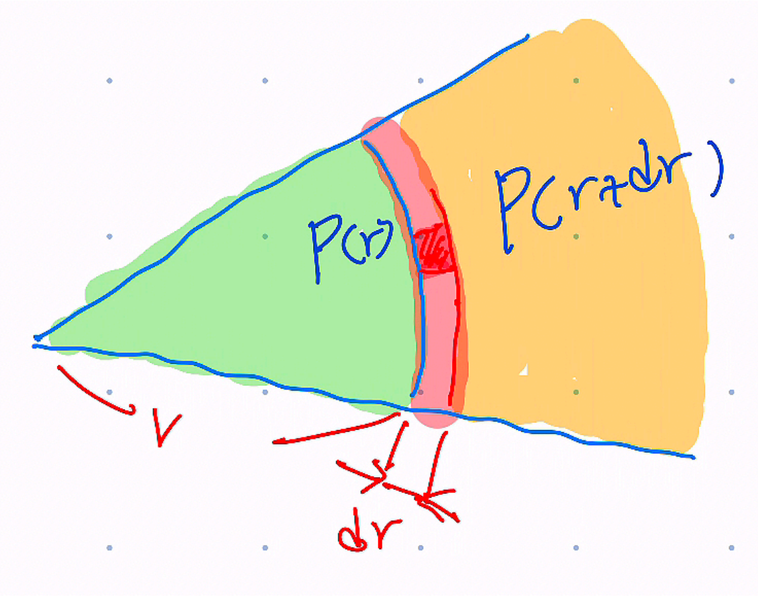
$$P_{\rm deg} = ? \qquad \text{(Pauli exclusion)}$$

Pressure from strong interactions (강상호작용에 의한 압력)

$$P_{\rm nuclear\ matter} = ?$$

압력과 중력의 균형

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$



밖으로 미는 힘의 크기

$$\propto -\frac{P(r+dr) - P(r)}{dr} \propto -\frac{dP}{dr}$$

안으로 당기는 중력 크기

$$\propto \frac{GM\rho}{r^2}$$

$$-\frac{dP}{dr} = \frac{GM\rho}{r^2}$$

Virial theorem

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = \int_0^R \left(-\frac{GM}{r^2}\rho\right) 4\pi r^3 dr$$

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = \cancel{P \cdot 4\pi r^2 \Big|_0^R} - \int_0^R 3P \cdot 4\pi r^2 dr$$

$$E_{\text{therm}} = \int_0^R \left(\frac{3}{2}nkT\right) 4\pi r^2 dr = \frac{1}{2}\int_0^R 3P \cdot 4\pi r^2 dr$$

$$\int_0^R \left(-\frac{GM}{r}\right) 4\pi r^2 \rho dr$$

$$-2E_{\text{therm}} = E_G$$

$$E_{\text{therm}} = -\frac{1}{2}E_G = \frac{1}{2}|E_G|$$

$$E_{\text{tot}} = E_G + E_{\text{therm}} = \frac{1}{2}E_G = -\frac{1}{2}|E_G| < 0$$

Virial theorem

$$E_{\rm therm} = -\frac{1}{2}E_G = \frac{1}{2}|E_G|$$

$$E_{\rm tot} = E_G + E_{\rm therm} = \frac{1}{2}E_G = -\frac{1}{2}|E_G| < 0$$

- Star forms by slow gravitational contraction
- Total energy becomes more negative
  - T & density increase as star loses energy

Fate of stars

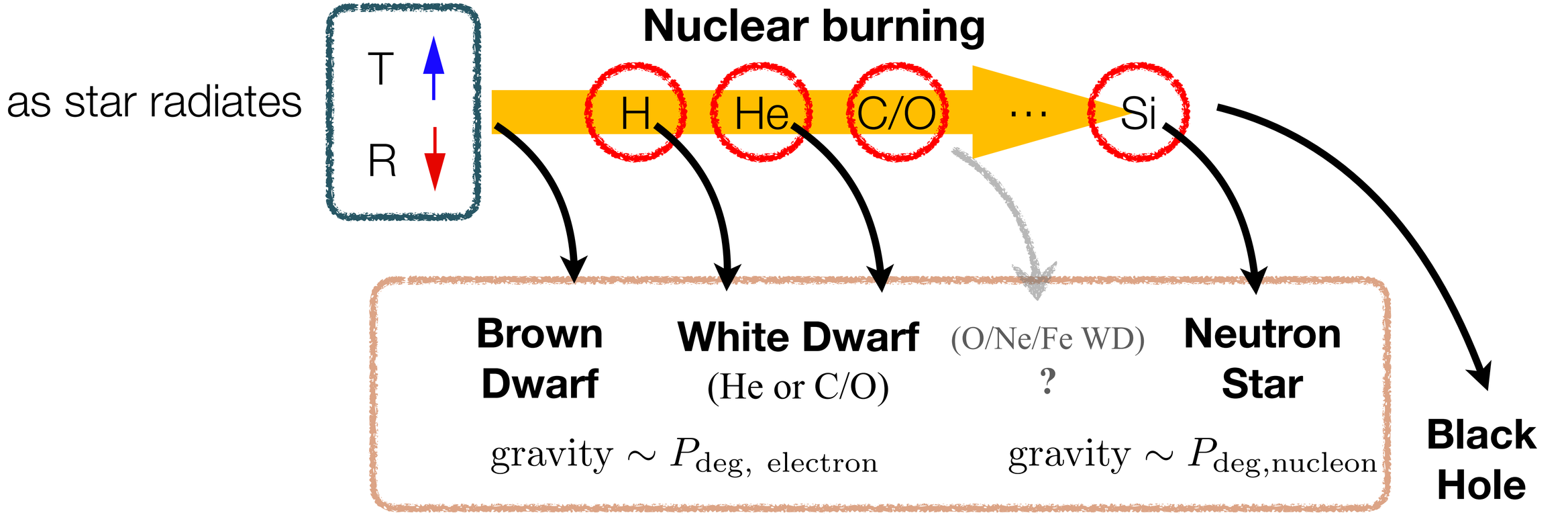
T_c

- start nuclear burning
- thermal pressure dominates

$\rho_{\rm quantum}$

- reach quantum state
- degeneracy pressure dominates
- contraction stops and cools down

Evolution of star

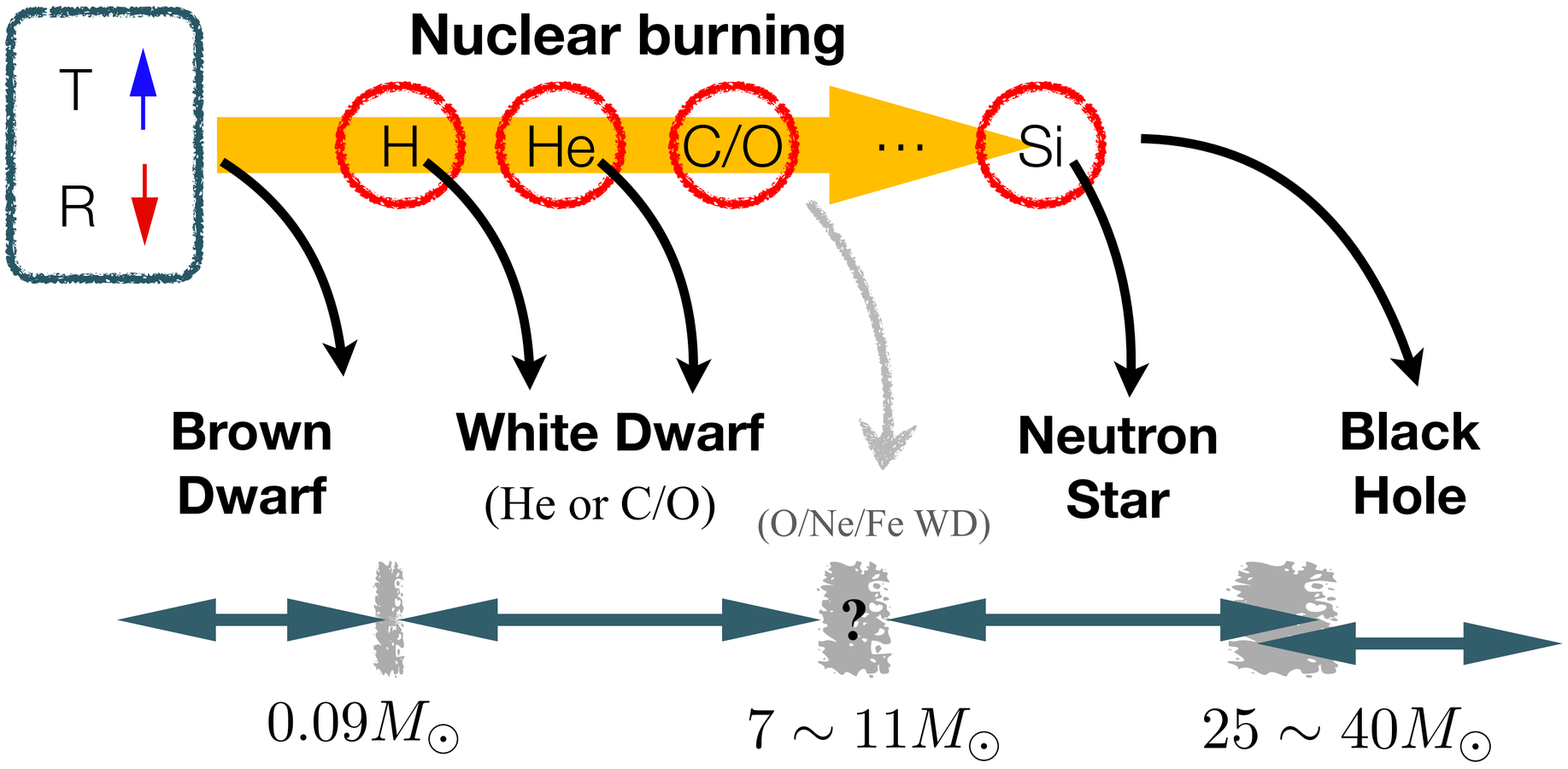


gravity $\gg P_{\mathrm{deg,nucleon}}$

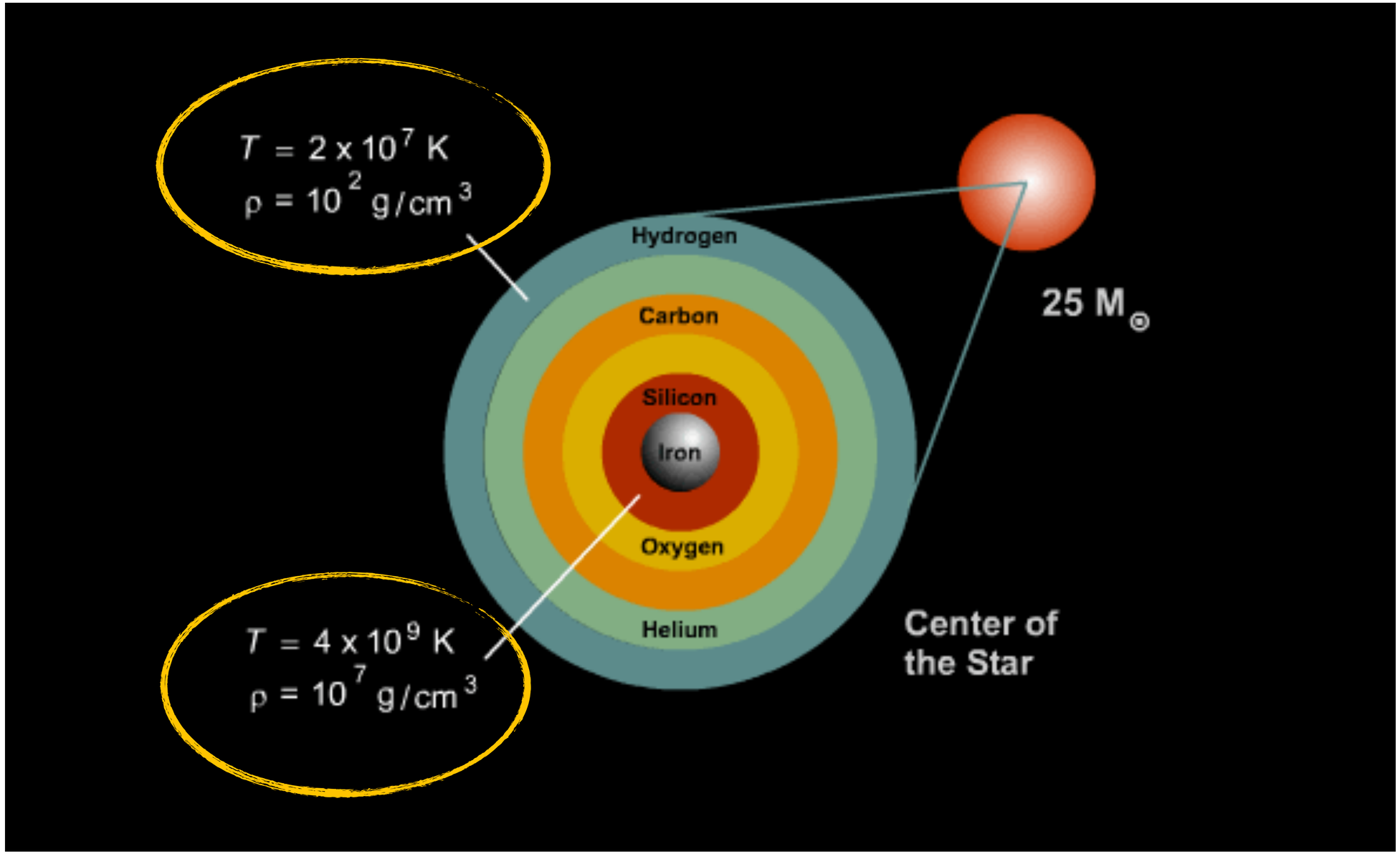
gravity ~ P_{quantum}?

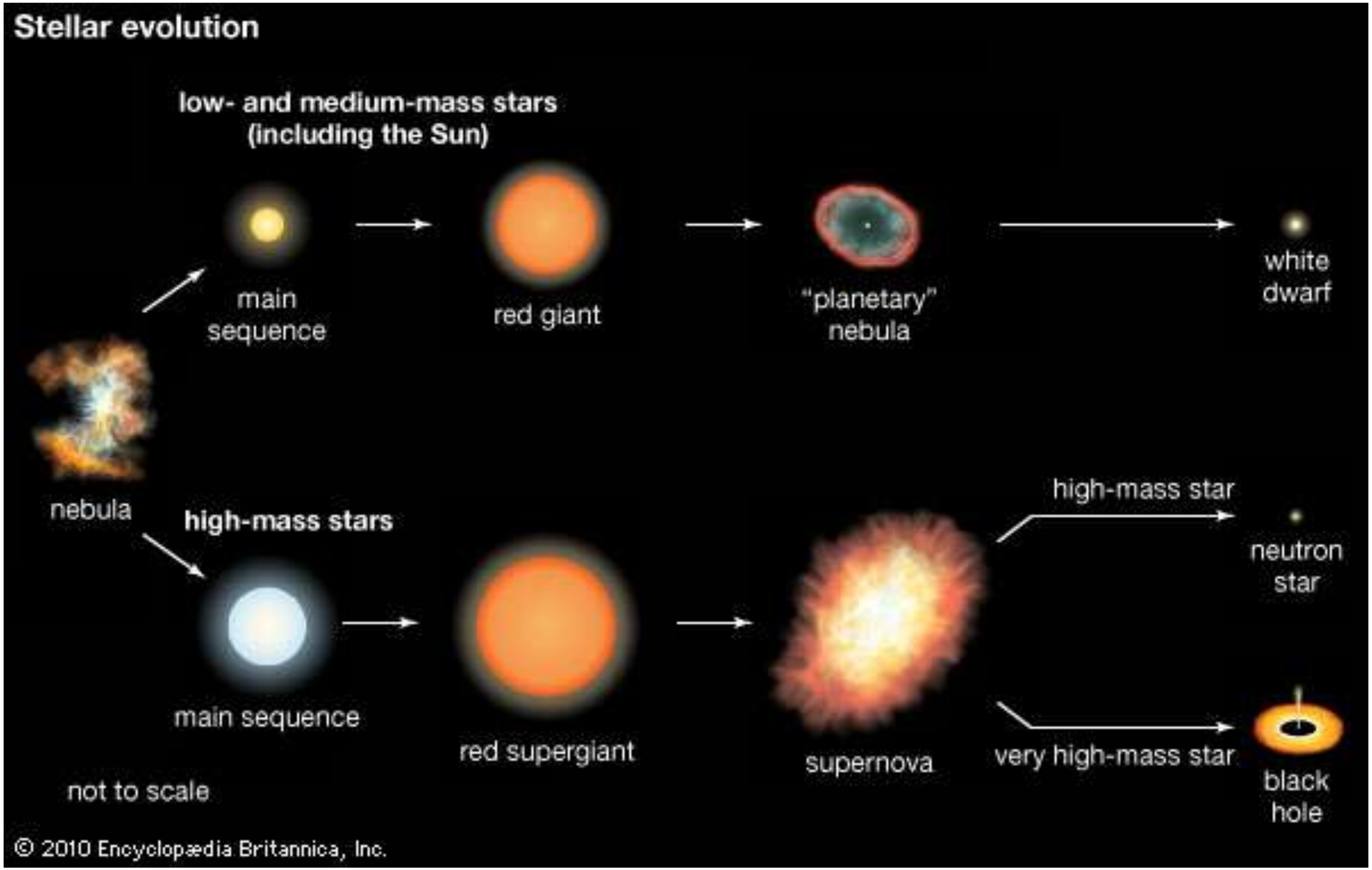
as star radiates R ~ fixed (degeneracy pressure)
T ↓

Initial mass of progenitor star (with solar metallicity)



ZAMS (zero age main sequence) mass


$T = 2 \times 10^7$ K
$\rho = 10^2$ g/cm^3
Hydrogen
Carbon
Silicon
Iron
Oxygen
Helium
25 $M_\odot$
Center of the Star
$T = 4 \times 10^9$ K
$\rho = 10^7$ g/cm^3


Stellar evolution
low- and medium-mass stars
(including the Sun)
main
sequence
red giant
"planetary"
nebula
white
dwarf
nebula
high-mass stars
main sequence
red supergiant
supernova
high-mass star
neutron
star
very high-mass star
black
hole
not to scale
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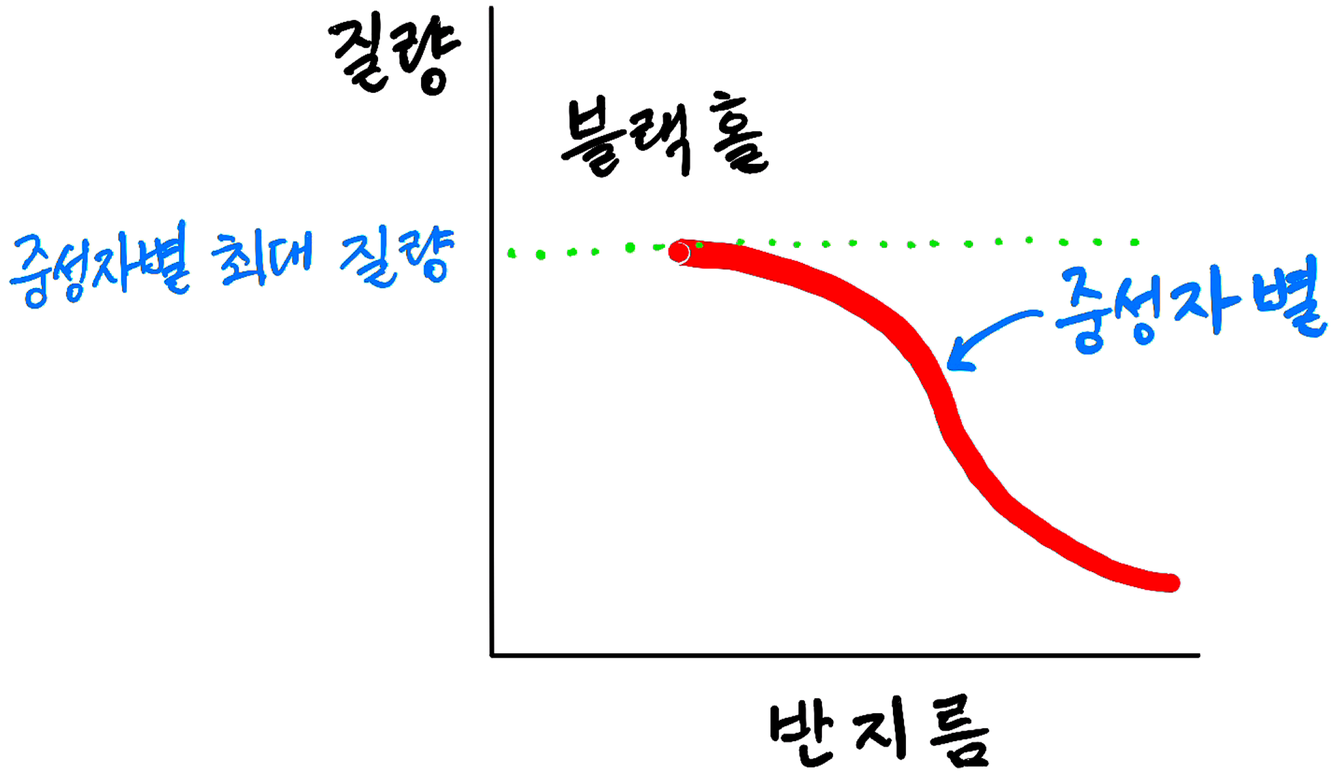
백색왜성의 구조

전자 겹침 압력

중성자별 최대 질량을 넘으서면 블랙홀 외에는 대안이 없음 !!

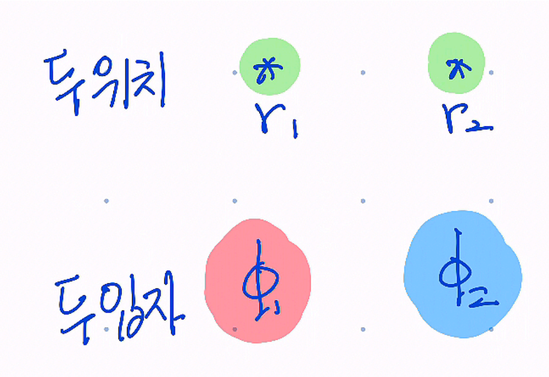


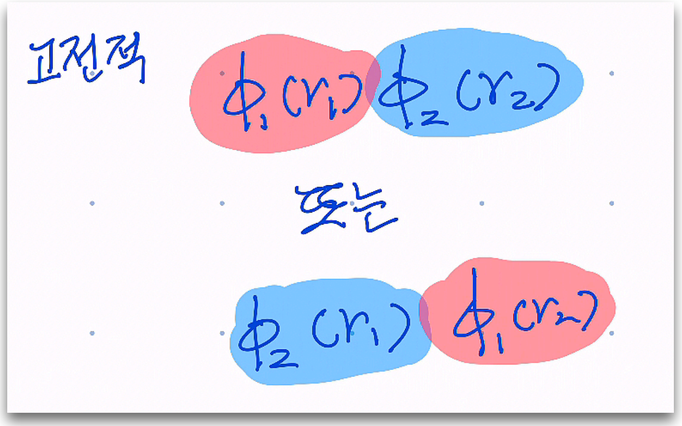
중성자별은 질량이 증가할 수록 크기는 작아진다 (백색왜성도 같은 성질을 가짐)

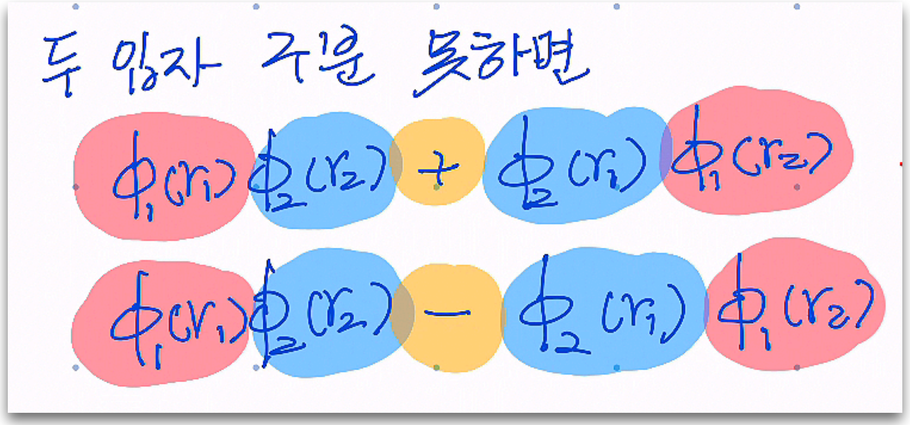
(양자) 겹침 압력 (축퇴압력)

- 두 입자는 구분할 수 없다
- 전자, 양성자, 중성자는

  - 같은 장소에 같은 성질을 가진 두 입자가 들어갈 수 없다 (파울리 배타원리)







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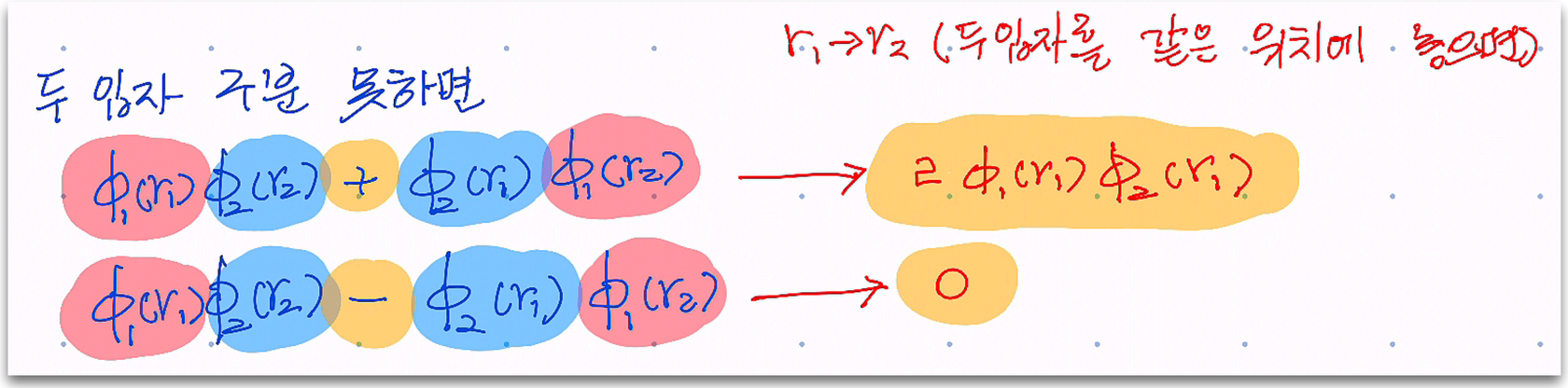
  - 같은 장소에 같은 성질을 가진 두 입자가 들어갈 수 없다 (파울리 배타원리)
- 같은 장소에 두 개가 못 들어가면 서로 밀어내는 것과 같다

  - 양자 통계에 의한 척력의 출현



백색왜성의 특성 (보통의 백색왜성)

$$\rho_{\rm WD} \sim 10^6 \ {\rm g/cm^3}$$

$$T_{\rm WD} \sim 10^7 \ {\rm K}$$

Quantum degeneracy pressure dominates : *thermal pressure negligible*

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$P_{\rm deg} \approx 3 \times 10^{22} {\rm dyne/cm^2}$$ 겹침 압력

$$P_{\rm th} \approx 2 \times 10^{20} {\rm dyne/cm^2}$$ 열압력

Thermodynamics

zero temperature limit

without nuclear burning

Hydrostatic equilibrium	$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$
Mass continuity	$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$
Radiative energy transport	$\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 4acT(r)^3}$
Energy conservation	$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$

$$P = P(\rho, T, \text{composition})$$

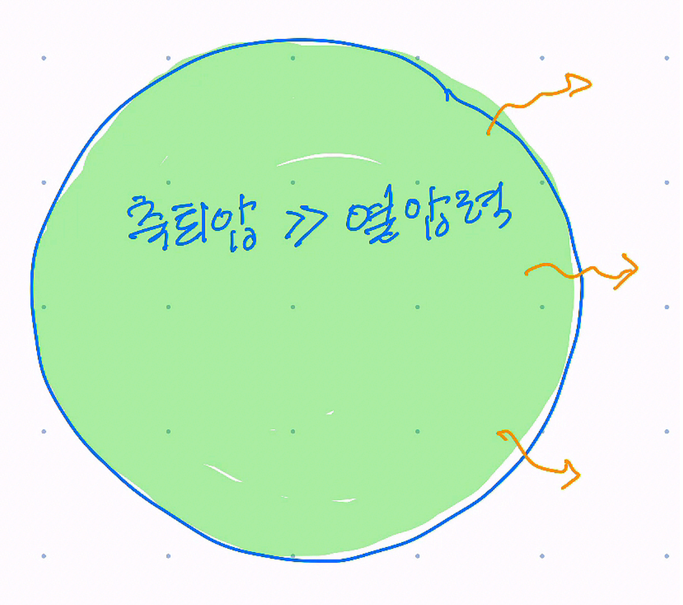
$$\kappa = \kappa(\rho, T, \text{composition})$$

$$\epsilon = \epsilon(\rho, T, \text{composition})$$

(양자) 겹침 압력이 지배하는 별의 탄생 (겹침 압력 / 축퇴압력)

- 양자 겹침 압력이 열압력보다 매우 큼.
- 별이 에너지(빛)를 방출하여도, 별은 더이상 압축 안됨
- 연료가 소진되면 핵반응 멈춤
- 별의 크기는 거의 변하지 않고, 빛을 방출하면서 식어감
- (양자)겹침 압력이 지배하는 별의 탄생 (백색왜성, 중성자별)
- 외부에서 추가로 물질이 유입되지 않는 한,

  온도가 식으면서 죽어감 (방출하는 빛의 양이 줄어듬)



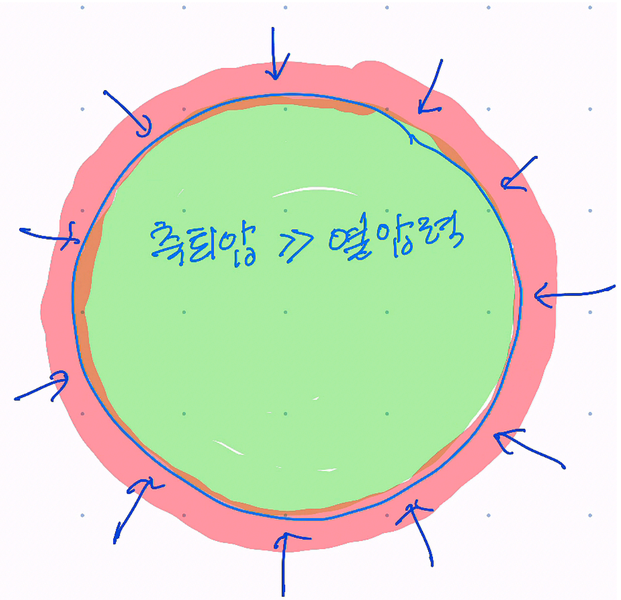
(양자)겹침 압력이 지배하는 별의 특성

- 별의 질량이 증가하면, 중력에 의해 별의 중심부로 당기는 힘이 증가함
- 이 때, 압력에 의해 밖으로 미는 힘도 증가는 하지만,

  중력에 의한 힘이 겹침 압력에 의한 힘보다 (상대적으로) 더 빨리 증가함
- 질량을 더하면 더할 수록 반경이 줄어듬
- 별 내부 입자들의 운동 속도가 빛의 속도에 가까워지면

  (입자들의 속도가 빛의 속도 이상으로 커지지는 않으므로)

  특수상대론의 적용을 받아 겹침 압력보다 중력-압력이 점 점 더 커짐
- 별은 더 이상 중력을 버티지 못하고 붕괴 됨.



중력 증가
축퇴압 증가
힘겨루기

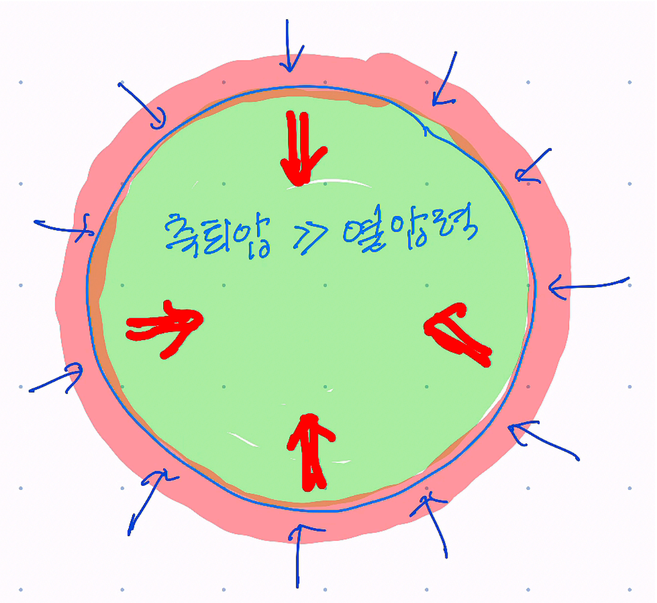
(양자)겹침 압력이 지배하는 별의 특성

- 백색왜성

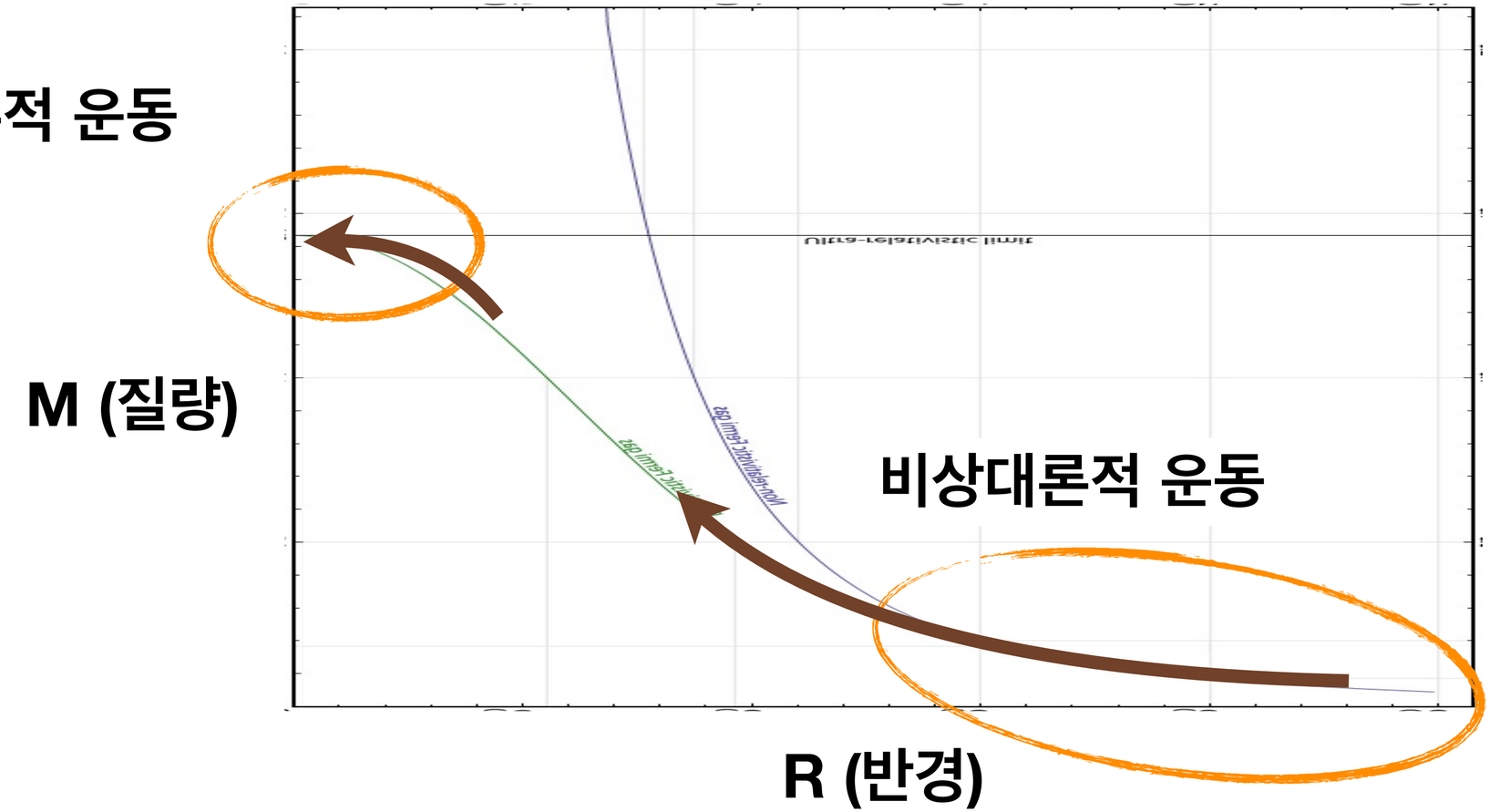
  - 백색왜성은 헬륨, 탄소, 산소 등으로 구성됨.
  - 한계 질량에 다다른 백색왜성이 질량을 더하면, 겹침 압력이 중력을 버티지 못하고 별이 붕괴
  - 별이 수축하면서 온도가 증가하여 다음 핵반응이 진행되어 갑자기 많은 열이 발생
  - 핵반응이 연쇄적으로 진행되어 초신성 폭발이 일어남

- 중성자별

  - 한계 질량에 다다른 중성자별에 질량을 더하면
  - 겹침 압력이 중력을 버티지 못하고 별이 붕괴
  - 별이 수축하면서 (다음 핵반응이 진행되지 못하므로) 블랙홀이 형성됨



백색왜성 (전자들의 겹침 압력이 결정적)



Workout Problem

[Problem 1] The ideal Fermi gas equation of states of white dwarfs or neutron stars, in which quantum degeneracy pressure dominates, can be represented by polytropic form;

$$P_{\rm deg} = K_\Gamma \rho^\Gamma = K_n \rho^{(n+1)/n}$$

where $K_\Gamma(K_n)$ and Γ are constants, ρ is density and n is called the polytropic index.

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

---

a) For the comparison, consider an adiabatic expansion of an ideal monoatomic gas for which thermal (kinetic) pressure, $P_{\rm kin} = (\rho/m)kT$, dominates and degeneracy pressure is negligible. Show that

$$TV^{\gamma-1} = \text{constant}, \qquad P_{\rm kin}V^\gamma = \text{constant}, \qquad P_{\rm kin} \propto \rho^\gamma$$

where $\gamma = c_P/c_V = 5/3$ is the ratio of specific heats (c_P: specific heat at constant pressure, c_V: specific heat at constant volume).

Polytropic Structure

n : polytropic index

ideal Fermi gas $\quad p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$

cf)

Thermal pressure for an adiabatic process

$$pV^{\gamma} = \text{constant}$$

$$\gamma = \frac{c_p}{c_V}$$ ratio of specific heats

$$E = \frac{3}{2}\frac{N}{V}kT$$

$$P_{\text{kin}} = \frac{N}{V}kT$$

$$\gamma = \frac{5}{3}$$ mono-atomic gas

Workout Problem

$$TV^{\gamma-1} = \text{constant}, \qquad P_{\text{kin}}V^{\gamma} = \text{constant}, \qquad P_{\text{kin}} \propto \rho^{\gamma}$$

$$E = \frac{3}{2}nkT$$

$$P_{\text{kin}} = nkT$$

$$\Delta E_{th} = W + Q = -p\Delta V + Q$$

단열과정 $\Delta E_{th} + p\Delta V = 0 \quad (Q=0)$

E_{th} 대입 $\Downarrow$ $\Delta E_{th} = n C_V \Delta T$

$$n C_V dT + p\, dV = 0$$

$pV = nRT$

dT 대입 $\Downarrow$ $p dV + dp\, V = nR\, dT$

$$\frac{C_V}{R}(p dV + dp V) + p dV = 0$$

$\Downarrow$ $\frac{C_V+R}{R} p dV + C_V V dp = 0$

$$\frac{dP}{P} + \frac{C_V+R}{C_V}\frac{dV}{V} = 0$$

$$\frac{dP}{P} + \frac{C_V+R}{C_V}\frac{dV}{V} = 0$$

$\Downarrow$ $\gamma \equiv \frac{C_V+R}{C_V} = \frac{C_P}{C_V}$

$$\frac{dP}{P} + \gamma\frac{dV}{V} = 0$$

$\Downarrow$ $\ln P + \gamma \ln V = C$

$\ln P + \ln V^{\gamma} = \ln PV^{\gamma} =$ 상수

$$PV^{\gamma} = \text{상수}$$

$$\gamma = \frac{5}{3} \qquad \text{mono-atomic gas}$$

Workout Problem

Ideal Degenerate Electron Gas

$$\Delta x \Delta p_x > h \qquad d^3p dV > h^3$$

$$\rho_{\text{quantum}} \approx \frac{m_p}{(\lambda_e/2)^3} = \frac{8m_p(3m_e kT)^{3/2}}{h^3}$$

$$\rho_{\text{quantum}}(\text{sun, center}) \approx 640 \text{ g/cm}^3$$

$$\rho_{\text{quantum}}(T = 10^8 \text{ K}) \approx 11,000 \text{ g/cm}^3$$

$$n_e = \int_0^{p_f} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_f^3 = Zn_+ = Z\frac{\rho}{Am_p}$$

n_e, n_+ number density

white dwarfs

$$\rho_{\text{WD}} \sim 10^6 \text{ g/cm}^3$$

Workout Problem

Degeneracy Pressure / Polytropic Structure

n : polytropic index

ideal Fermi gas $\quad p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$

Non-relativistic

$$\gamma = \frac{5}{3} \qquad n = \frac{3}{2} \qquad P_e = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e m_p^{5/3}} \left(\frac{Z}{A}\right)^{5/3} \rho^{5/3}$$

Ultra-relativistic

$$\gamma = \frac{4}{3} \qquad n = 3 \qquad P_e = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4 m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}$$

$$P_e \propto \int_0^{\infty} n_e(p) p v dp \propto \int_0^{p_F} \frac{p^4}{\sqrt{p^2 + m^2}} dp$$

p : momentum transfer

v : number of particles

$p = \gamma m v$

b) For ideal Fermi gas, in the zero temperature limit, show that the number density of gas (n_g) and Fermi momentum (p_F) are related by

$$n_g = g \times \frac{2\pi}{3h^3} p_F^3$$

where g is the degeneracy.

Consider simple case ($T=0$)

isotropic distribution

$$d^3p \to 4\pi p^2 dp$$

Fermi momentum

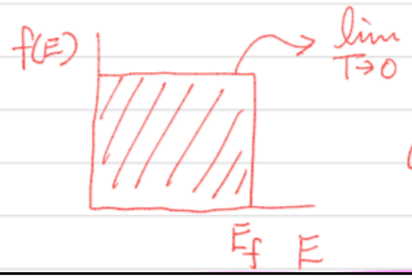
$$dN(p)dp = 2 \times \frac{d^3p\,dV}{h^3} = \begin{cases} 2 \times \dfrac{4\pi p^2 dp\, dV}{h^3} & \text{if } |\vec{p}| \le p_f \\ 0 & \text{if } |\vec{p}| > p_f \end{cases}$$

$$n_e(p)dp = \frac{8\pi p^2 dp}{h^3}$$

$$n_e = \int_0^{p_F} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_F^3$$

Normal gas. $P(T=0)=0$.

Quantum gas $P(T=0) \neq 0$ from $\Delta x^3 \Delta p^3 > h^3$



All states with $E \le E_F$ are occupied (degenerate)

Workout Problem

Non-relativistic

c) In the non-relativistic limit, show that

$$P_{\text{deg}} = g \times \frac{4\pi}{3h^3} \times \frac{1}{5m} \times p_F^5$$

and $\Gamma = 5/3$ and $n = 3/2$. Why is Γ the same as γ obtained in a) despite the difference in their physical origin?

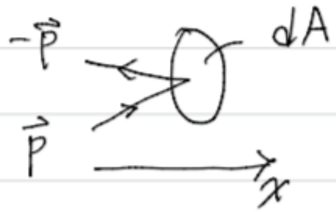
Ultra-relativistic

d) In the full relativistic limit, show that

$$P_{\text{deg}} = g \times \frac{4\pi}{3h^3} \times \frac{c}{4} \times p_F^4$$

and $\Gamma = 4/3$ and $n = 3$.

Degeneracy Pressure



$$\Delta p_x = 2p_x$$

$$v_x = \frac{dx}{dt}$$

change of momentum

# of particles

$$\frac{dF_x}{dA} = \frac{2p_x}{dA\,dt} = \frac{2p_x}{dA\,(dx/v_x)} = \frac{2p_x v_x}{dV}$$

The force per unit area due to each collision.

$$P = \int_0^\infty \frac{dN(p)}{2}\,\frac{2p_x v_x}{dV}\,dp$$

only half of electrons are moving $+x$ at any given time.

$$p_x v_x = \frac{1}{3} p v.$$

$$\begin{cases} v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{3 v_x^2} \\ p = \sqrt{3 p_x^2} \\ vp = 3\, p_x v_x \end{cases}$$

$$P_e = \frac{1}{3}\int_0^\infty n_e(p)\, p\, v\, dp$$

$$P_e = \frac{8\pi}{3h^3}\int_0^{p_F} v\, p^3\, dp$$

Workout Problem

Degeneracy Pressure

$$P_e = \frac{8\pi}{3h^3}\int_0^{P_F} v\, p^3 dp$$

$$p = \gamma m v \quad , \quad E = \gamma m c^2$$

$$v = \frac{p}{m\gamma} = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}}$$

$$P_e = \frac{8\pi}{3h^3}\int_0^{P_F} \frac{p^4c^2}{\sqrt{p^2c^2 + m_e^2c^4}}\, dp$$

$$n_e = \frac{X\rho}{m_H} + \frac{(1-X)\rho}{2m_H} = \frac{\rho}{2m_H}(1+X)$$

$$n_e = \frac{\rho}{\mu_e m_H} \quad ; \quad \mu_e = \frac{2}{1+X}$$ (mean molecular weight of electron)

from

$$n_e = \frac{8\pi}{3h^3} P_F^3$$

$$P_F = \left(\frac{3h^3\rho}{8\pi\mu_e m_H}\right)^{1/3}$$

Workout Problem

non-relativistic

c) In the non-relativistic limit, show that

$$P_{\text{deg}} = g \times \frac{4\pi}{3h^3} \times \frac{1}{5m} \times p_F^5$$

and $\Gamma = 5/3$ and $n = 3/2$. Why is Γ the same as γ obtained in a) despite the difference in their physical origin?

$$P_e = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}}\, dp$$

i) non-relativistic $\quad \sqrt{p^2c^2 + m_e^2c^4} \approx m_e c^2$

$$P = \frac{8\pi}{15\, h^3 m_e} p_F^5$$

$$P = K_1 \rho^{5/3}$$

$$K_1 = \frac{3^{2/3}}{20\, \pi^{2/3}} \frac{h^2}{m_e m_H^{5/3} \mu_e^{5/3}} = \frac{1.00 \times 10^7}{\mu_e^{5/3}}$$

Workout Problem

ultra-relativistic

d) In the full relativistic limit, show that

$$P_{\text{deg}} = g \times \frac{4\pi}{3h^3} \times \frac{c}{4} \times p_F^4$$

and $\Gamma = 4/3$ and $n = 3$.

$$P_e = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}}\, dp$$

ii) fully relativistic $\quad \sqrt{p^2c^2 + m_e^2c^4} \approx pc$

$$P = \frac{2\pi c}{3h^3} p_F^4$$

$$P = K_2 \rho^{4/3}$$

$$K_2 = \frac{3^{1/3}}{8\pi^{1/3}} \frac{hc}{m_H^{4/3} \mu_e^{4/3}} = \frac{1.24 \times 10^{10}}{\mu_e^{4/3}}$$

Workout Problem

e) For typical white dwarfs ($\rho \sim 10^6$ g cm^{-3}) and neutron stars ($\rho \sim 10^{14}$ g cm^{-3}), compare the magnitudes of degeneracy pressure ($P_{\rm deg}$), kinetic pressure of the ideal gas ($P_{\rm kin}$) and radiation pressure at $kT = 1$ MeV, 1 keV and 1 eV.

$$P_{\rm total} = P_{\rm deg} + \frac{\rho}{m}kT + \frac{1}{3}aT^4$$

where $k = 1.4 \times 10^{-16}$ erg K^{-1} $= 8.6 \times 10^{-5}$ eV K^{-1}.

$$\left(20^\circ\mathrm{C} = 293K \approx \frac{1}{40}\mathrm{eV}\right)$$

Without nuclear reaction, star will contract to the quantum limit.

de Broglie wavelength

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} \approx \frac{h}{\sqrt{3mkT}}$$

$$E \sim \frac{3}{2}kT$$

* electron will reach quantum domain first

Check.

If we assume inter-particle separation $\sim \frac{\lambda}{2}$

→ quantum domain

$$\rho_{\rm quantum} \approx \frac{m_p}{(\lambda/2)^3} = \frac{8m_p(3m_e kT)^{3/2}}{h^3}$$

With $T = 15 \times 10^6$ K

$$\rho_{\rm quantum} \approx 640 \text{ g/cm}^3$$

$$\rho_{\rm center,\,sun} \sim 150 \text{ g/cm}^3$$

↑ classical.

Lane-Emden Equation

$$n = 1/(\gamma - 1)$$
$$\gamma = (n+1)/n$$

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$
$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

$$P = K\rho^\gamma$$
$$\rho = \rho_c \theta^n$$
$$r = a\xi$$

$$a = \sqrt{\frac{(n+1)K_n \rho_c^{(1-n)/n}}{4\pi G}}$$

$$\boxed{\frac{1}{\xi}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n}$$

$$\rho = \rho_c \theta^n$$

center $\theta(0) = 1, \quad \theta'(0) = 0$

surface $\theta(\xi_1) = 0 \longrightarrow P = \rho = 0$

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \right|$$

$$R = \sqrt{\frac{(n+1)K_n \rho_c^{(1-n)/n}}{4\pi G}}\, \xi_1$$

Constants of Lane-Emden equation

$$r = a\xi$$

n	ξ_1	$-\xi_1^2\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800
5.0	∞	1.73205	∞

Stellar Structure by S. Chandrasekhar

Polytropic Structure

n : polytropic index

ideal gas $\quad p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \right|$$

Ultra-relativistic

$$\gamma = \frac{4}{3} \qquad n = 3 \qquad P_e = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}$$

Chandrasekhar limit

$$\mu_e = A/Z \approx 2 \text{ for He, C, O, ..}$$

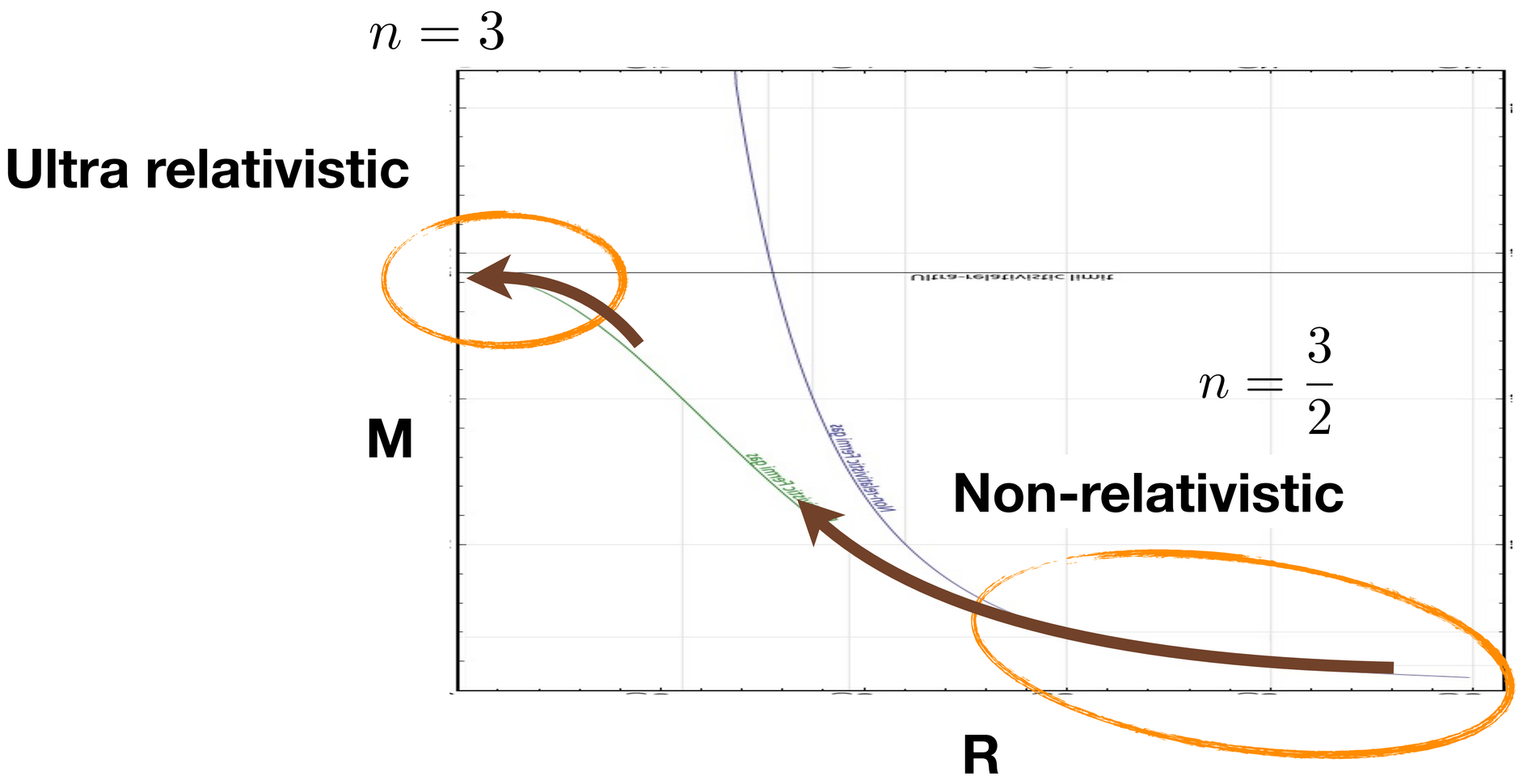
Ultra-relativistic $\gamma = \frac{4}{3}$ $n = 3$

$$M = 1.457 \left(\frac{2}{\mu_e}\right)^2 M_\odot$$

$$R = 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}}\right)^{-1/3} \left(\frac{2}{\mu_e}\right)^{2/3} \text{km}$$

Mass is independent of central density

White Dwarfs - electron degeneracy pressure



White Dwarfs

$\mu_e = A/Z \approx 2$ for He, C, O, ..

Non-relativistic $\gamma = \frac{5}{3} \quad n = \frac{3}{2}$

$$R = 1.122 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}}\right)^{-1/6} \left(\frac{2}{\mu_e}\right)^{5/6} \text{km}$$

$$M = 0.7011 \left(\frac{R}{10^4 \text{ km}}\right)^{-3} \left(\frac{2}{\mu_e}\right)^{5} M_\odot$$

Ultra-relativistic $\gamma = \frac{4}{3} \quad n = 3$

$$R = 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}}\right)^{-1/3} \left(\frac{2}{\mu_e}\right)^{2/3} \text{km}$$

$$M = 1.457 \left(\frac{2}{\mu_e}\right)^{2} M_\odot$$

independent of central density

Workout Problem

[Problem 2] At zero temperature limit, when the compact star is in a hydrostatic equilibrium with spherical symmetry, compact star equation of state can be obtained by solving TOV (Tolman-Oppenheimer-Volkoff) equation

$$\frac{dP(r)}{dr} = -\frac{GM(r)\epsilon(r)}{c^2r^2}\left(1+\frac{P(r)}{\epsilon(r)}\right)\left(1+\frac{4\pi r^3P(r)}{M(r)c^2}\right)\left(1-\frac{2GM(r)}{c^2r}\right)^{-1}$$

where $P(r)$ is the pressure, $\epsilon(r)$ is the energy density and $M(r)$ is the enclosed gravitational mass $M_G(r)$ for a given radius r. The gravitational and baryon masses of the star are defined by

$$M_G(r) = \int_0^R 4\pi r^2 \frac{\epsilon(r)}{c^2}$$

$$M_A(r) = m_A \int_0^R dr 4\pi r^2 n(r)\left(1-\frac{2GM(r)}{c^2r}\right)^{-1/2}$$

where m_A is baryon mass and $n(r)$ is the baryon number density.

Workout Problem

a) In the Newtonian limit ($P \ll \epsilon$ and $GM/c^2 \ll r$), with polytropic EOS, show that TOV equation can be reduced to Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n$$

with $\rho = \rho_c\theta^n$, $r = a\xi$, and

$$a = \sqrt{\frac{(n+1)K_n\rho_c^{(1-n)/n}}{4\pi G}},$$

where ρ_c is the central density of a star.

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(-GM_r\right) = -4\pi G\rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Workout Problem

b) The Lane-Emden equation can be solved with boundary conditions at the center;

$$\theta(0) = 1, \quad \theta'(0) = 0.$$

For $n < 5$ (or $\Gamma > 6/5$), the solution decreases monotonically and have a zero at a finite value $\xi = \xi_1$: $\theta(\xi_1) = 0$ (see Table 1 for numerical values). This point corresponds to the surface of the star, where $P = \rho = 0$. Show that the mass of the star is given as

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \right|.$$

Table 1: The constants of the Lane-Emden functions.

n	ξ_1	$-\xi_1^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
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4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800
5.0	∞	1.73205	∞

Workout Problem

c) In the full relativistic and Newtonian limit, show that the mass become independent of radius. This implies that there exist maximum mass (Chandrasekhar mass) for the compact stars (for which quantum degeneracy pressure dominates). What is the value of the radius-independent mass?

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \right|$$

n	ξ_1	$-\xi_1^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
3.0	6.89685	2.01824	54.1825

$$M_{ch} = \frac{\sqrt{6}}{32\pi}\left(\frac{\hbar c}{G}\right)^{3/2}\left(\frac{2}{\mu_e}\right)^2 \frac{\xi_1^2 |\theta'(\xi_1)|}{m_H^2}$$

$$\xi_1^2 |\theta'(\xi_1)| = 2.018$$

$$M_{ch} = 1.46 \left(\frac{2}{\mu_e}\right)^2 M_\odot$$

Workout Problem

d) In the Newtonian limit, show that the radius becomes independent of mass when $n = 1$. Note that $n = 1$ is possible only when the system is far from an ideal Fermi gas; i.e., interactions are non-negligible. What is the value of the mass-independent radius?

---

with $\rho = \rho_c \theta^n$, $r = a\xi$, and

$$a = \sqrt{\frac{(n+1)K_n \rho_c^{(1-n)/n}}{4\pi G}}$$

Table 1: The constants of the Lane-Emden functions.

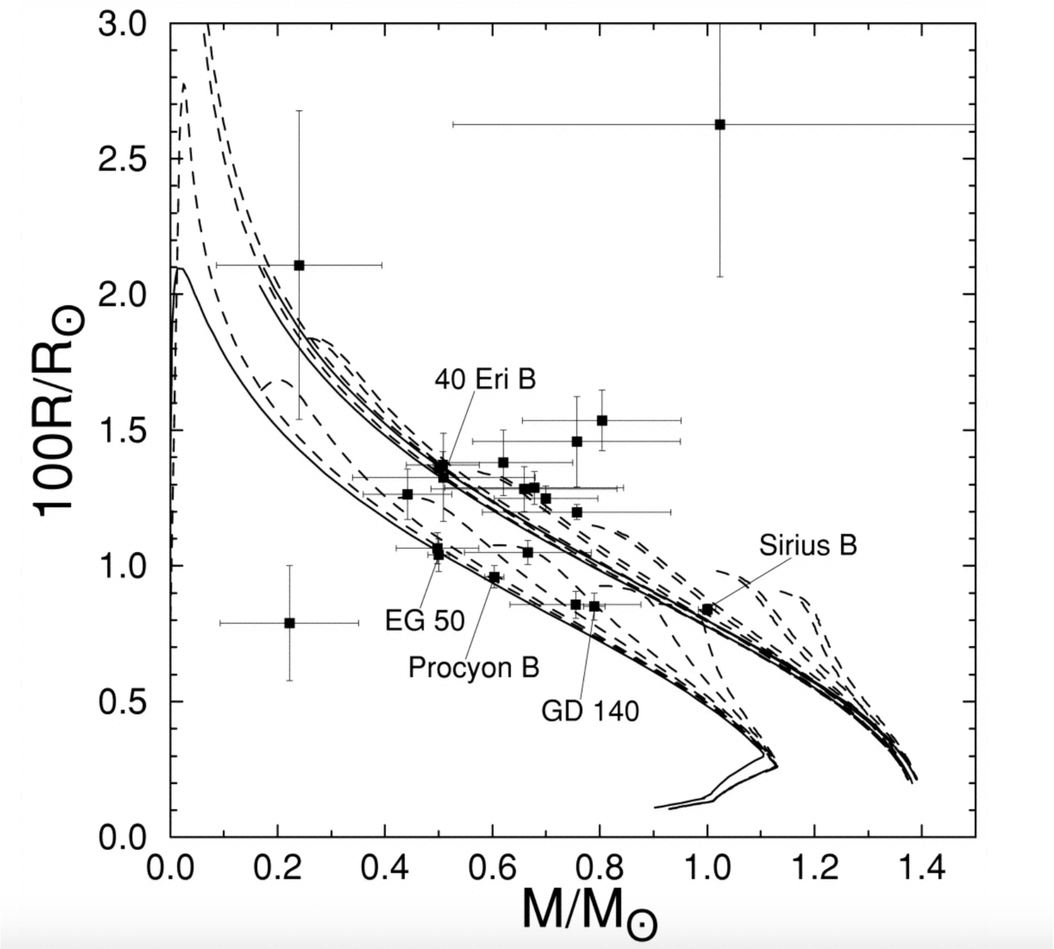
n	ξ_1	$-\xi_1^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987

But, we don't know K_1 unless we know the interactions !

White Dwarfs

ApJ 530, 949 (2000)

반경

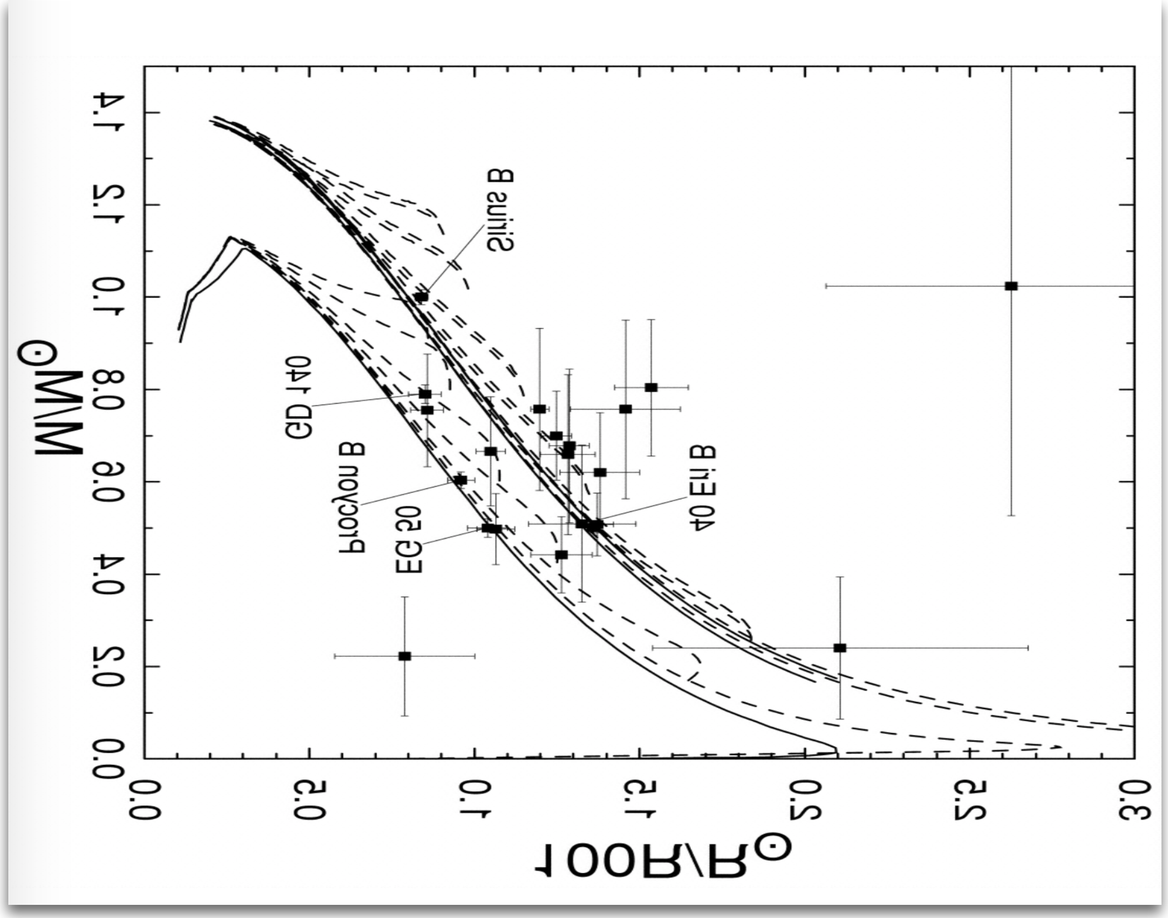


질량

White Dwarfs

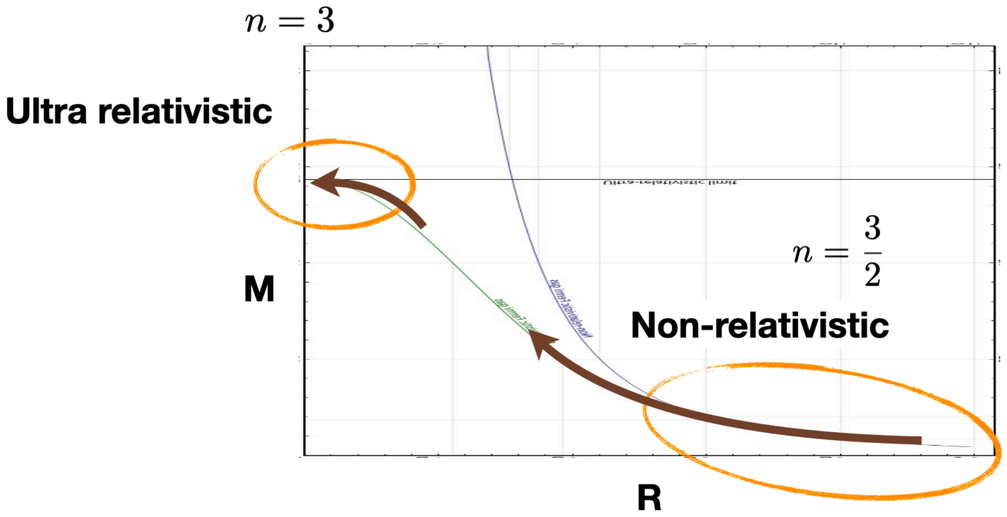
ApJ 530, 949 (2000)

질량



반경

White Dwarfs - electron degeneracy pressure



중성자별의 구조

중성자 겹침 압력

Neutron Star Formation from Fe Core Collapse

When Fe core reaches Chandrasekhar Limit

(n : neutron)

Nuclear Photodisintegration

$$\gamma + {}^{56}\mathrm{Fe} \rightarrow 13\ {}^{4}\mathrm{He} + 4\ \mathrm{n}\ (124\ \mathrm{MeV\ is\ consumed})$$

$$\gamma + {}^{4}\mathrm{He} \rightarrow 2\ \mathrm{p} + 2\ \mathrm{n}\ (28.3\ \mathrm{MeV\ is\ consumed})$$

$$E_{\mathrm{consumed}}(10^{57}\mathrm{p}) \sim 1.4 \times 10^{52}\ \mathrm{erg} \sim 10^{11}\mathrm{year\ with}\ L_{\odot}$$

Neutralisation

$$\mathrm{e}^{-} + \mathrm{p} \rightarrow \mathrm{n} + \nu_{\mathrm{e}}$$

$$\mathrm{e}^{-} + {}^{56}\mathrm{Fe} \rightarrow {}^{56}\mathrm{Mn} + \nu_{\mathrm{e}}$$

$$\mathrm{e}^{-} + {}^{\mathrm{Mn}}\mathrm{Fe} \rightarrow {}^{56}\mathrm{Cr} + \nu_{\mathrm{e}}$$

core collapse → **NS + Supernova**

$$R_{\mathrm{Fe}} \sim 1500\mathrm{km}$$

$$R_{\mathrm{NS}} \sim \mathcal{O}(10\mathrm{km})$$

$$\tau_{\mathrm{fall}} \sim \mathrm{a\ few\ seconds}$$

General Relativity

hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi P r^3}{Mc^2}\right)\left(1 - \frac{2GM}{rc^2}\right)^{-1}$$

$$\frac{dM}{dr} = 4\pi r^2 \left(\frac{\epsilon}{c^2}\right)$$

include all energy sources

physics of dense nuclear matter (strong interaction)

Without general relativity

n : polytropic index

ideal gas $$p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$$

nonrelativistic

$$\gamma = \frac{5}{3} \qquad n = \frac{3}{2} \qquad P_N = \left(\frac{3}{\pi}\right)^{2/3} \left(\frac{h^2}{20 m_N}\right) \left(\frac{\rho}{m_N}\right)^{5/3}$$

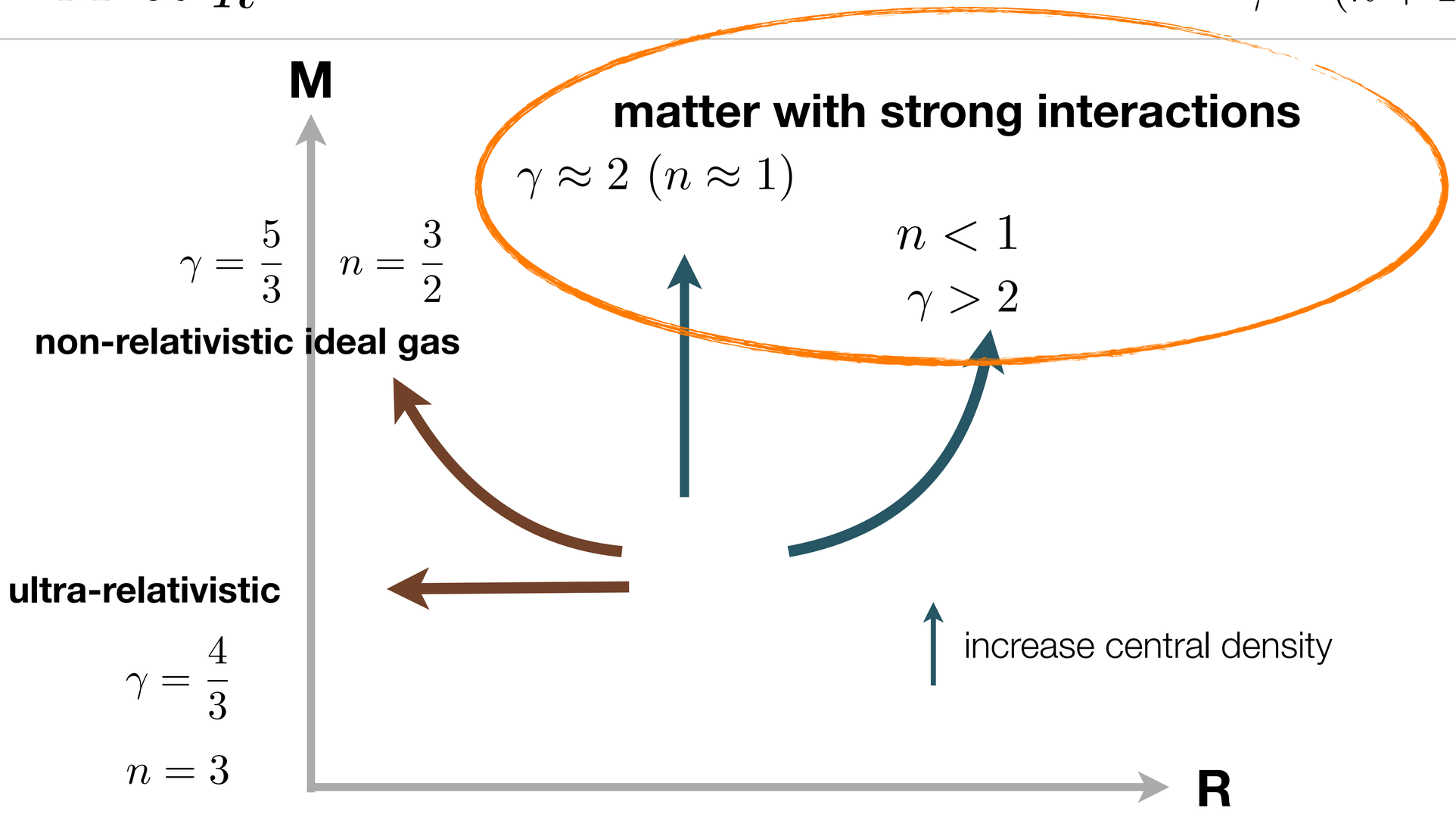
ultrarelativistic

$$\gamma = \frac{4}{3} \qquad n = 3 \qquad P_N = \left(\frac{3}{8\pi}\right)^{1/3} \left(\frac{hc}{4}\right) \left(\frac{\rho}{m_N}\right)^{4/3}$$

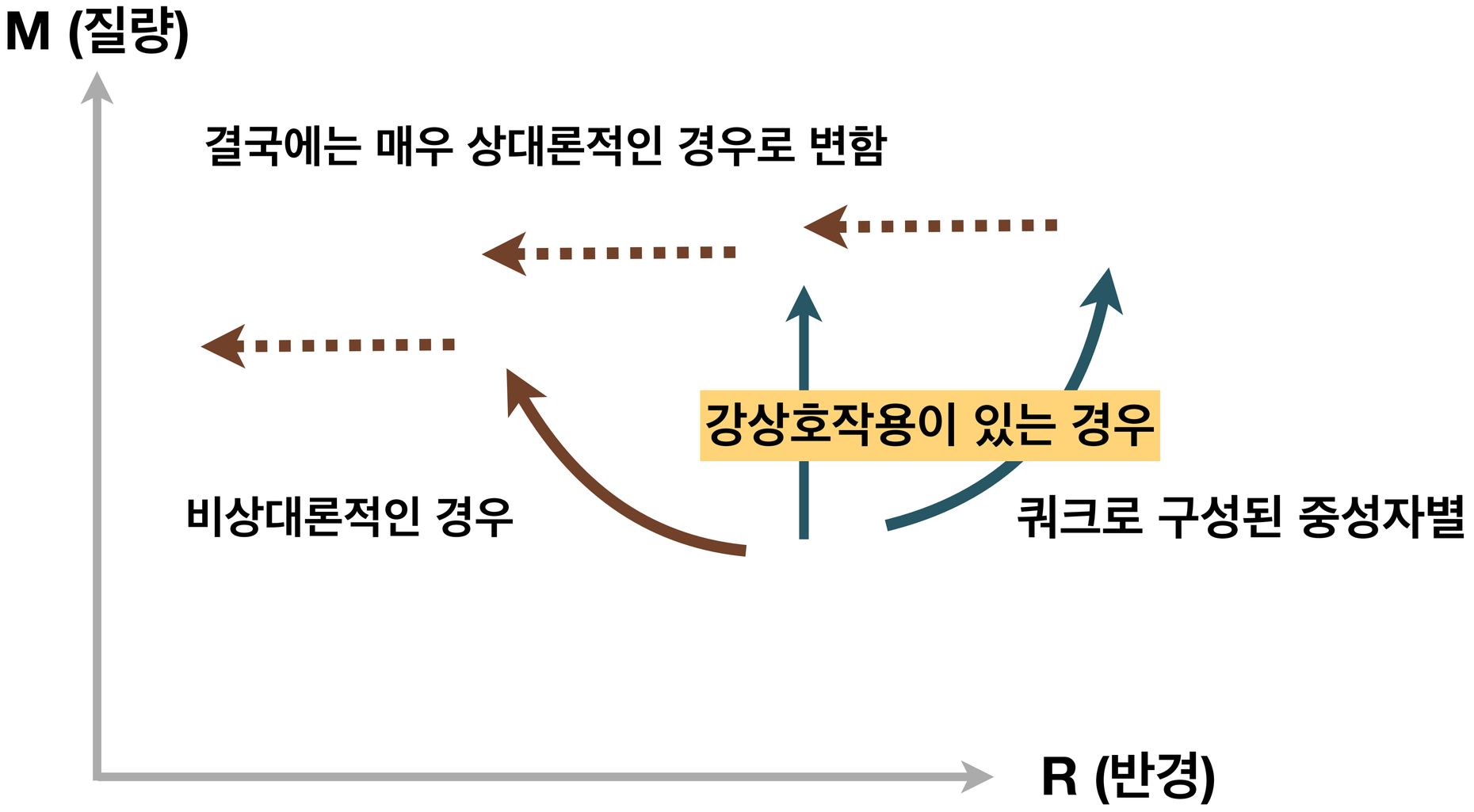
Without general relativity

$$M \propto R^{(\gamma-2)/(3\gamma-4)}$$

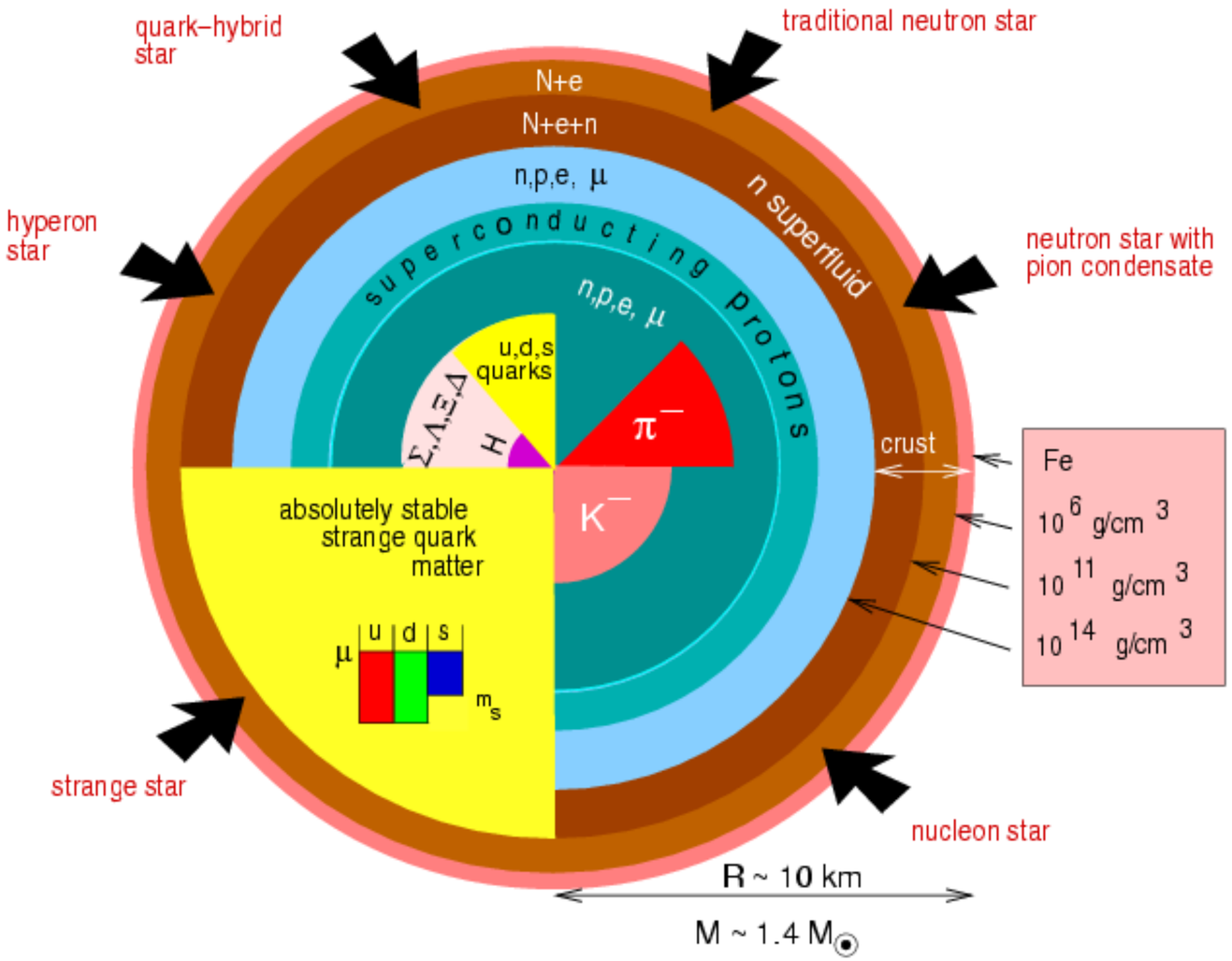
$$n = 1/(\gamma - 1)$$
$$\gamma = (n+1)/n$$



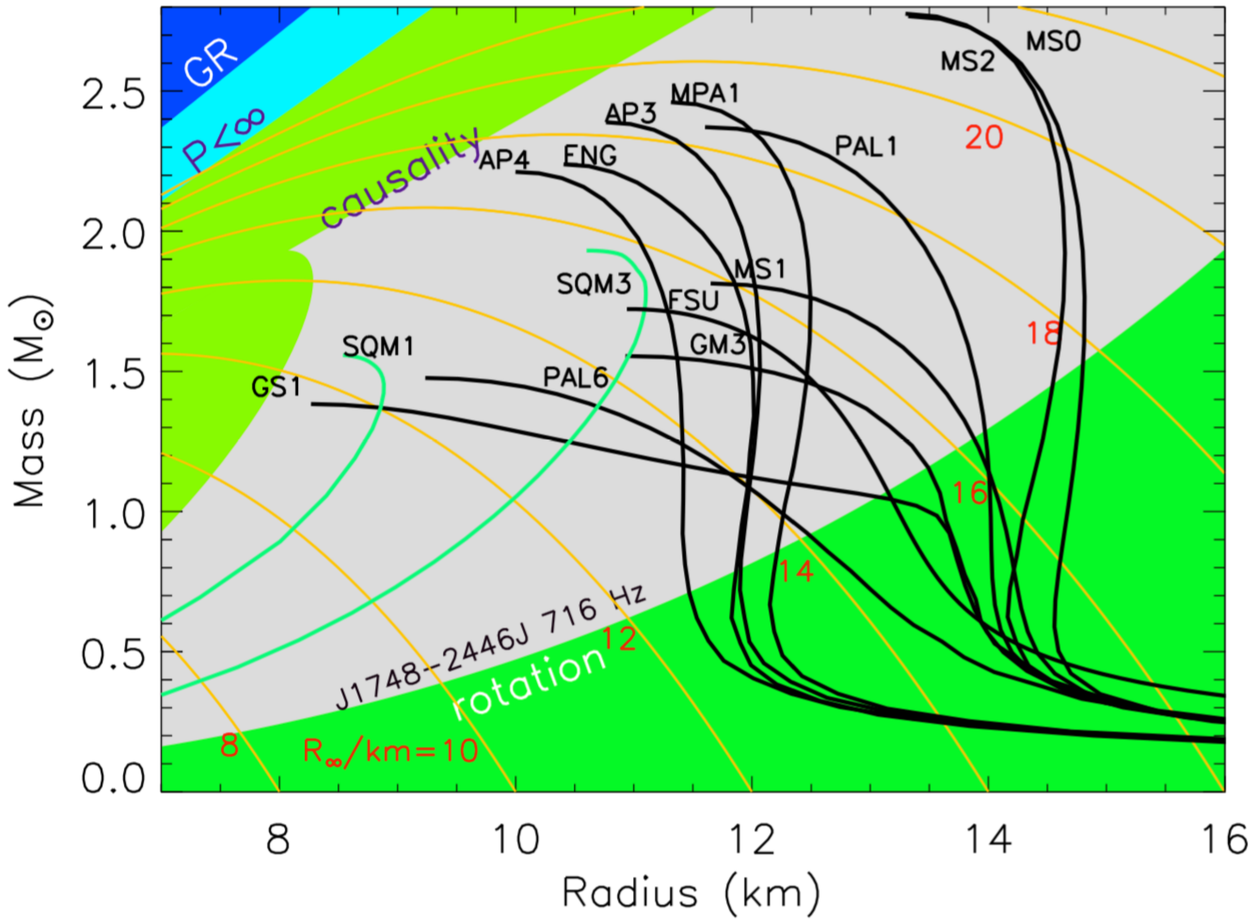
화살표방향 (중심부 질량 증가)



중성자별 내부구조는 아직도 잘 모름 !



다양한 중성자별 구조 상태방정식



with general relativity & strong interactions !

Q) Which EOS ?

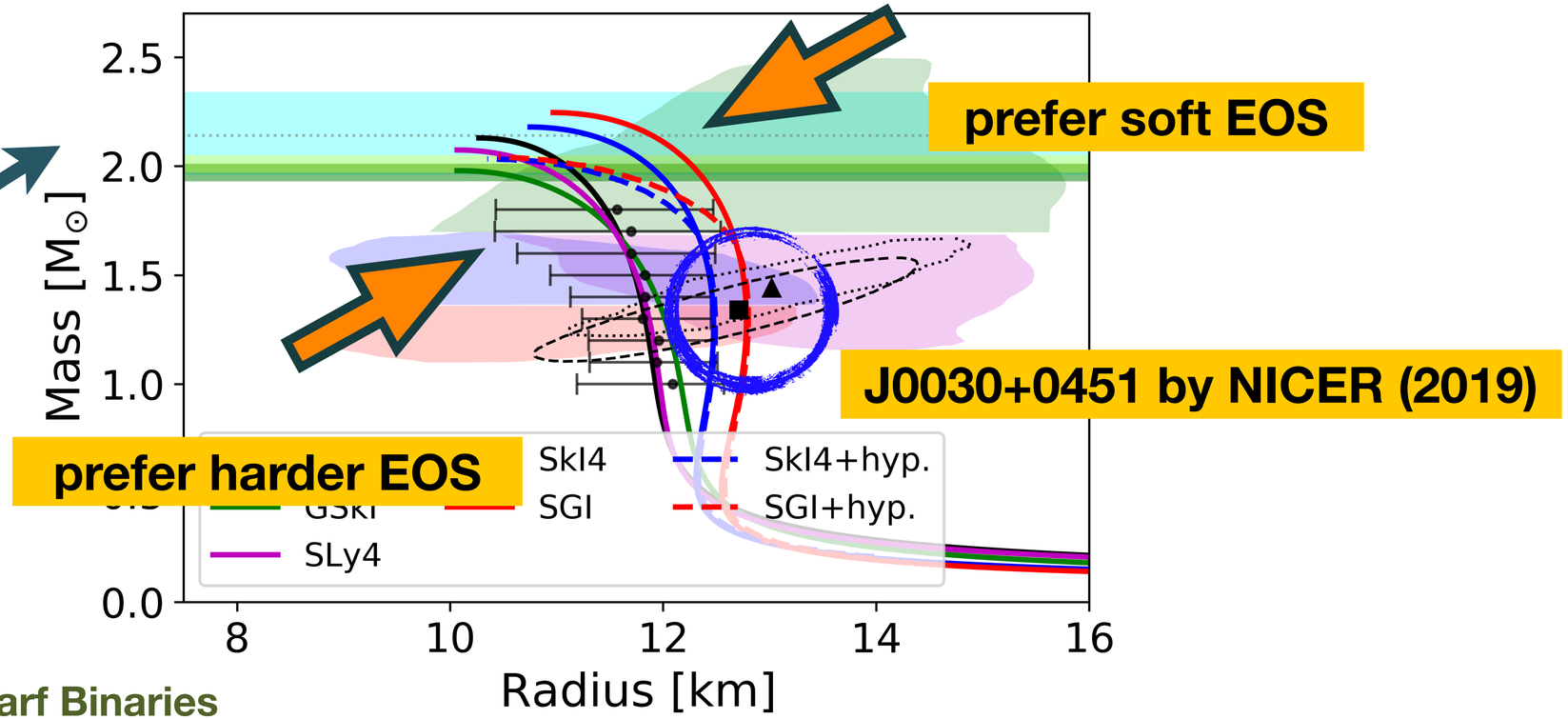
H. T. Cromartie, *et al.* Nature Astronomy (2019).

MSP J0740+6620

$2.14^{+0.20}_{-0.18}\,M_\odot$

Tidal deformability of NS from GW

Strangeness in NS



Neutron Star-White Dwarf Binaries

1.97 solar mass NS : Nature 467 (2010) 1081

2.01 solar mass NS : Science 340 (2013) 6131

IJMPE (2020) Kim, Lee, Kim, Kwak, Lim, Hyun

블랙홀 내부 ?

What's inside black hole ?

quantum gravity

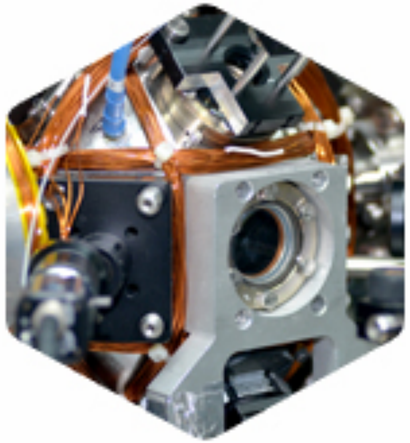
$$\text{gravity} \sim P_{\text{quantum}}?$$

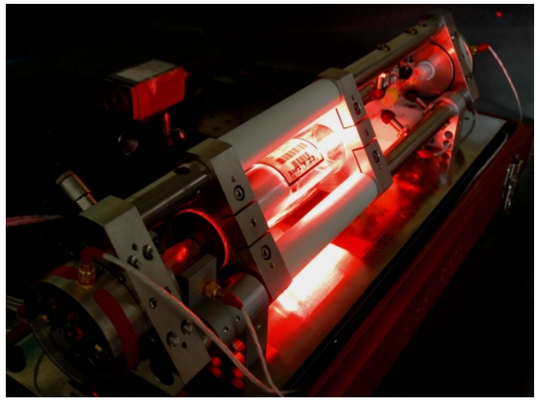
$$\frac{GMm}{l} \sim \frac{G}{l}\frac{E}{c^2}m \sim \frac{G}{lc^2}\frac{\hbar c}{l}m = \frac{G\hbar}{l^2 c}m \sim mc^2$$

$$M \sim \frac{E}{c^2} \qquad E \sim \frac{hc}{\lambda} \sim \frac{hc}{2\pi l} = \frac{\hbar c}{l} \qquad l \sim \sqrt{\frac{G\hbar}{c^3}}$$

$$l_{\text{Planck}} \sim 10^{-33}\ \text{cm}$$

$$t_{\text{Planck}} = \frac{l_{\text{Planck}}}{c} \sim 10^{-43}\ \text{s}$$

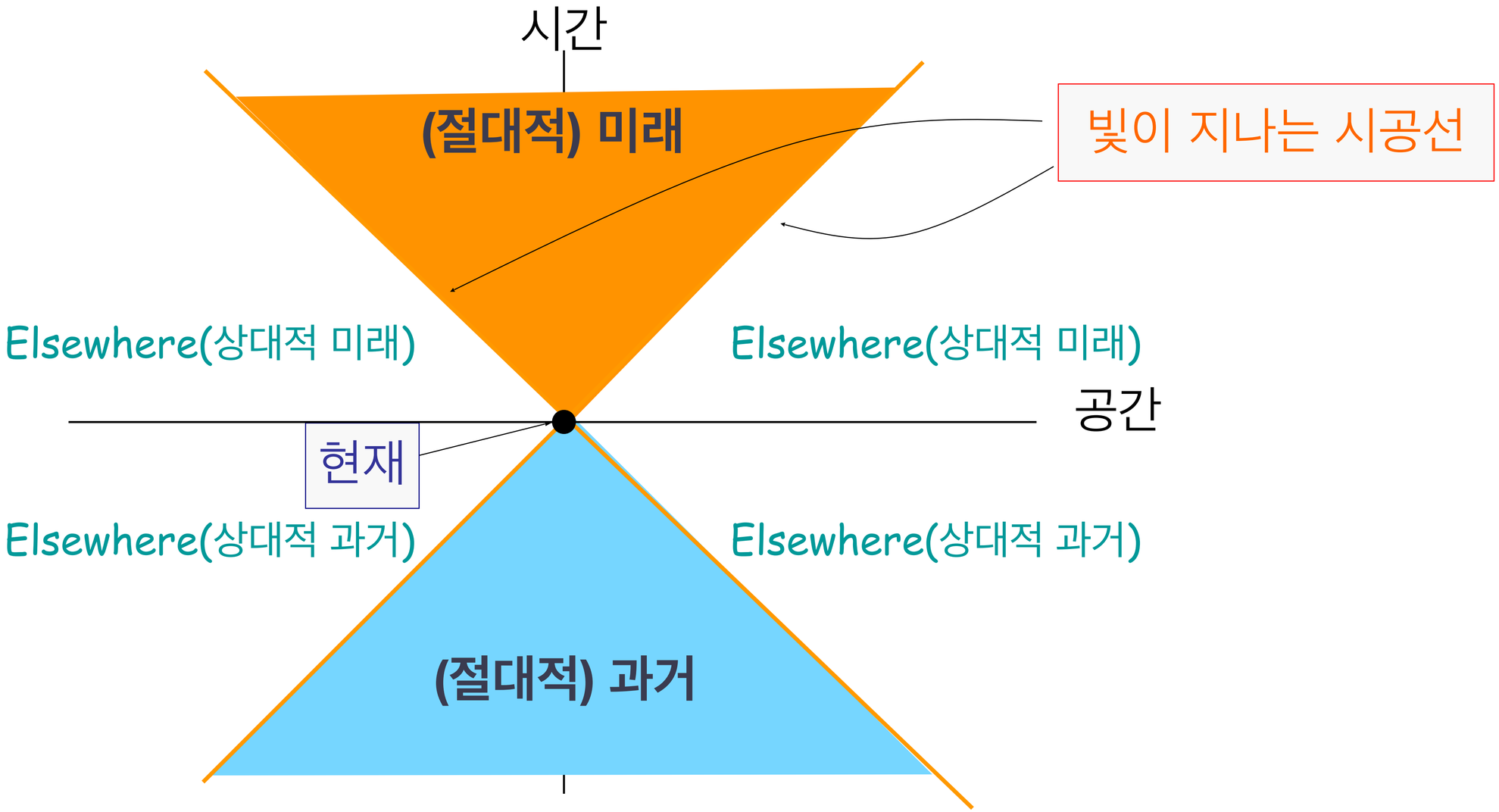




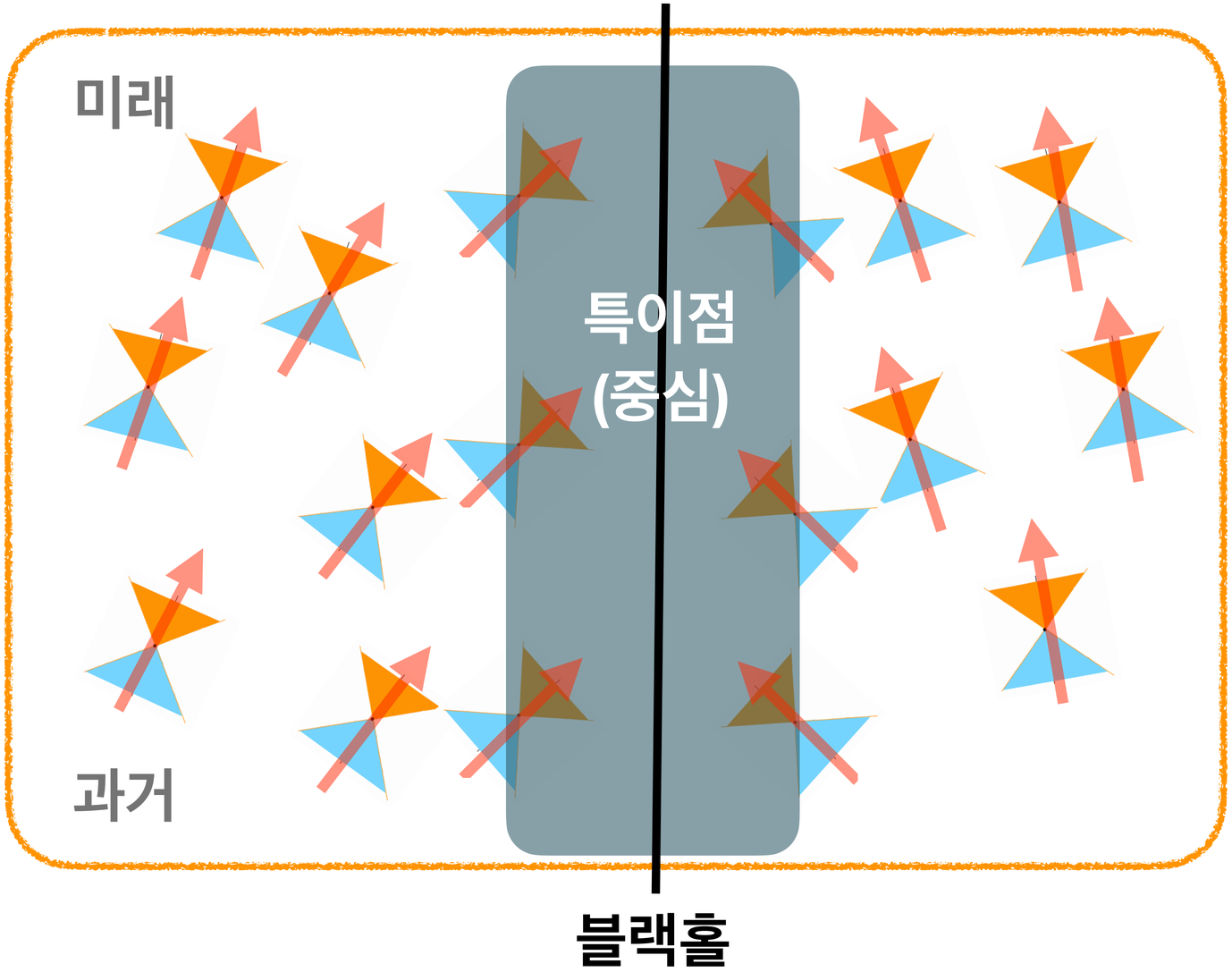
빛으로 이해하는 우주 / 시공간 자체의 변화

시간(세슘-133원자)과 길이의 기준
시간과 3차원 공간, 그리고 물질은 광속불변의 법칙에 의해 얽혀 있음

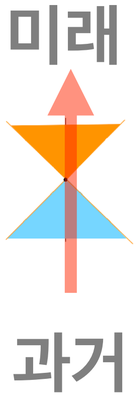
아인슈타인 - 과거, 현재, 미래, Elsewhere



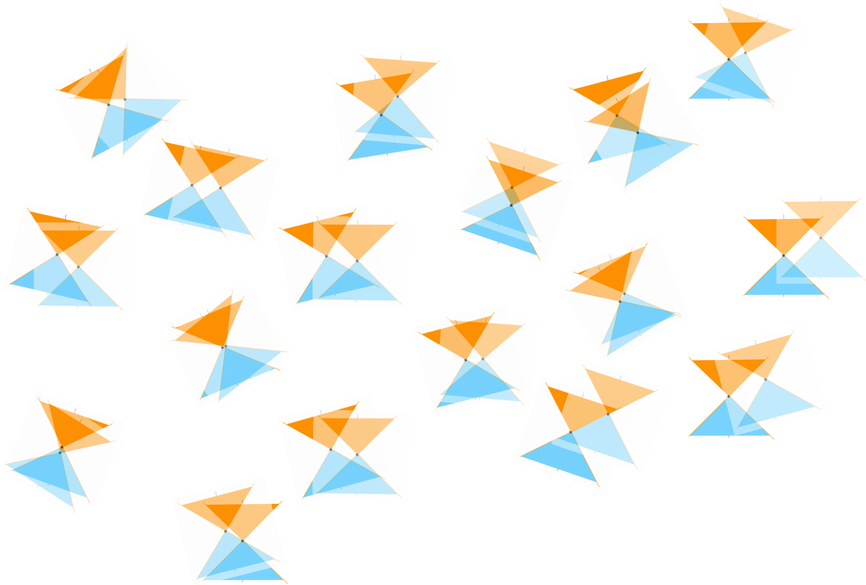
시공간의 구조 / 블랙홀

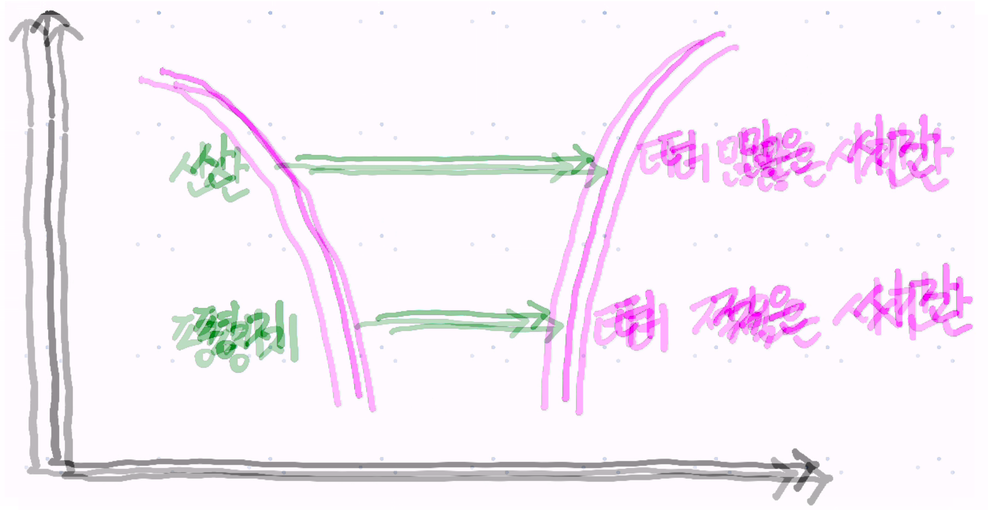


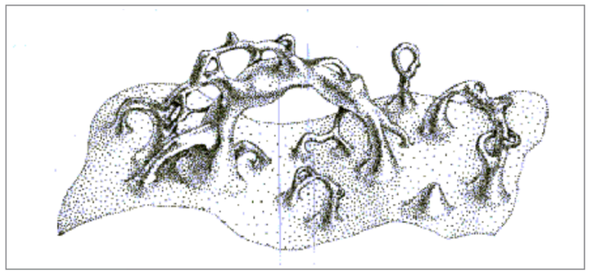
- 블랙홀 안에서 미래는 특이점에 갇혀 버린다.
- 시간도 끝이 있다.
- 그럼, 시간의 시작은 ?



시간의 양자중첩







10^{-33} cm (10^{-43}초)

(비교: 양성자 10^{-13} cm)

3차원 양자진공 요동에 의해
시간의 흐름과 길이가 무작위로 변함

Gödel & Einstein

괴델이 준 충격

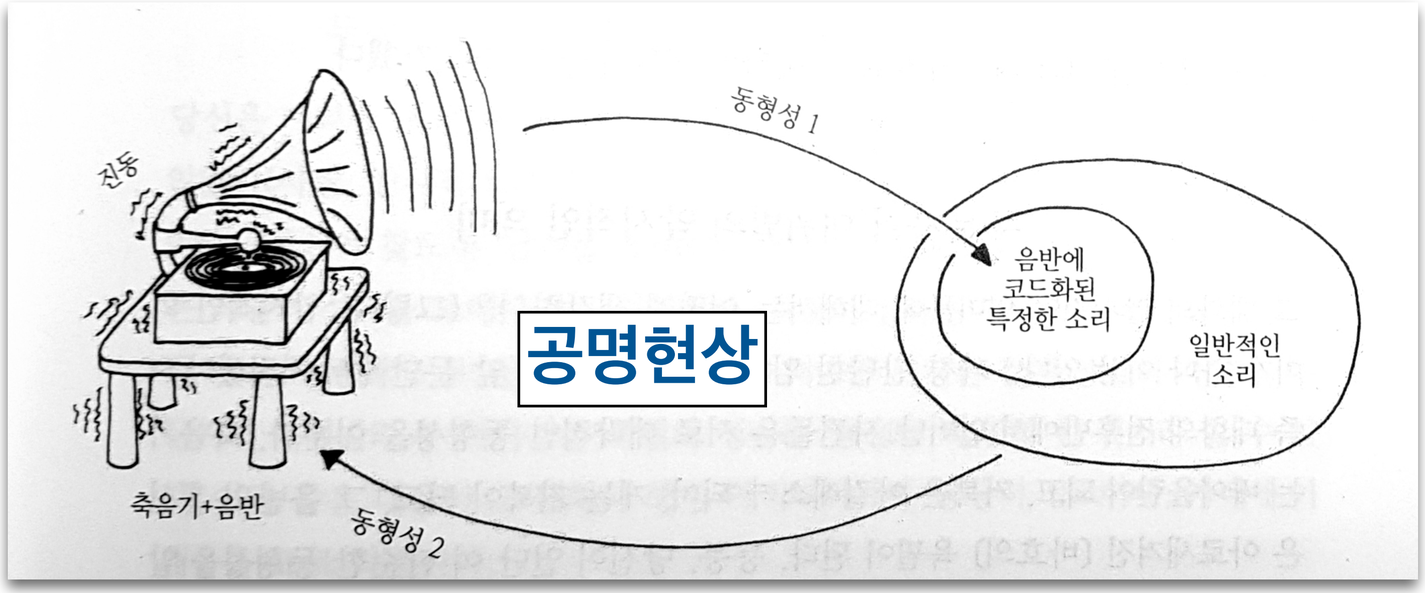
Kurt Gödel 1906-1978

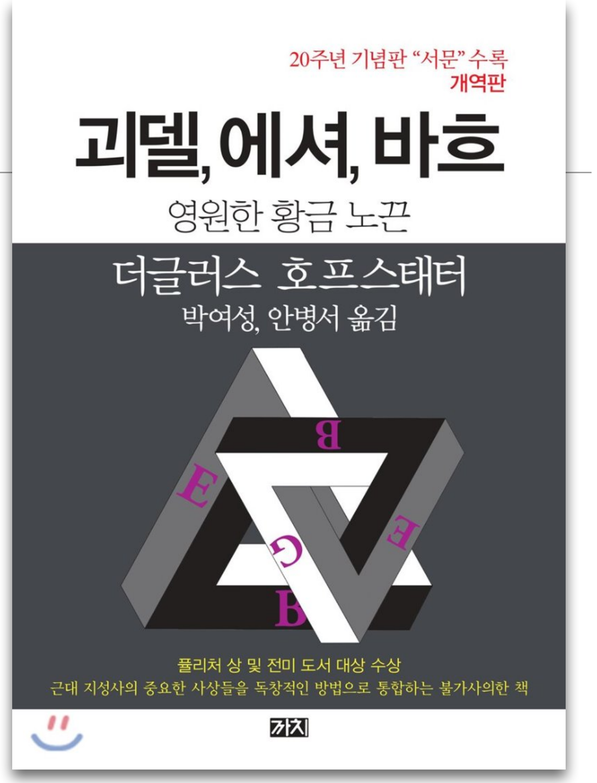
괴델이 준 충격 - Now 시간의 물리학 / 리쳐드 뮬러

- 아인슈타인은 물리학이 '불완전'하다는 개념, 물리학은 실재에 대한 완전한 기술이 아니며 과거는 미래를 완벽하게 결정하지 않는다는 개념을 견딜 수 없었다
- **괴델의 정리**
  - **20세기 가장 위대한 수학적 성취**
  - 모든 수학적 체계 안에는 증명될 수 없는 진리가 포함되어 있다

괴델의 정리 / 축음기

모든 수학적 체계 안에는 증명될 수 없는 진리가 포함되어 있다



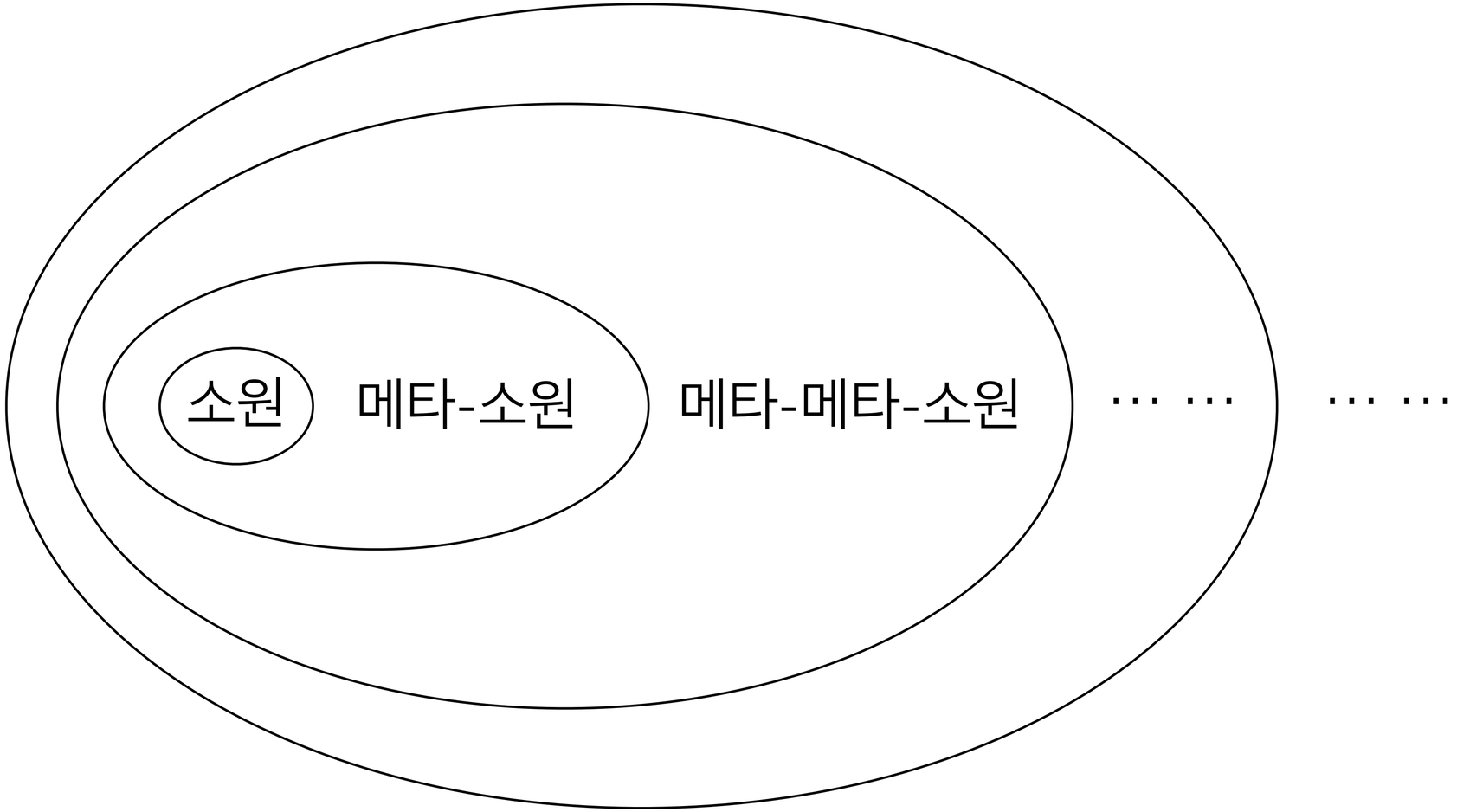


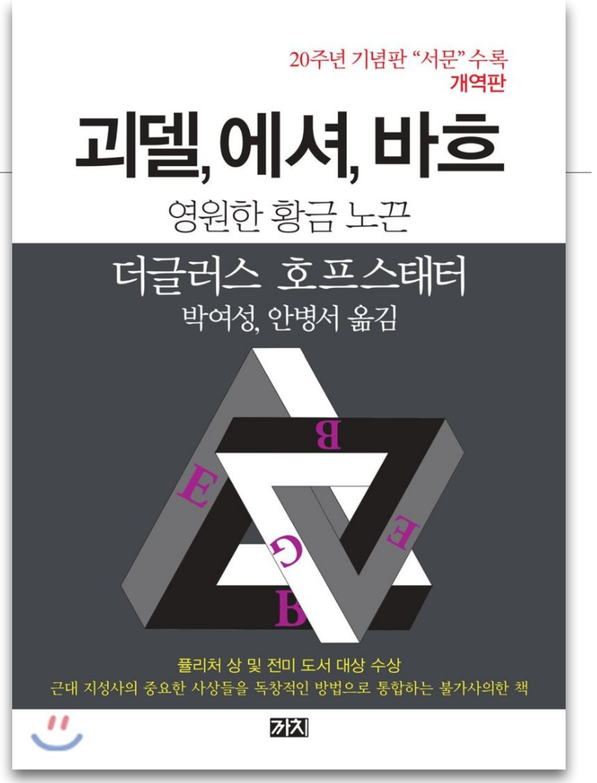
그 어떤 축음기로도 소리를 재생할 수 없는 음반이 존재한다.

왜냐하면 그 소리가 … 축음기의 자기파괴를 유발할 것이기 때문이다.

소원의 역설

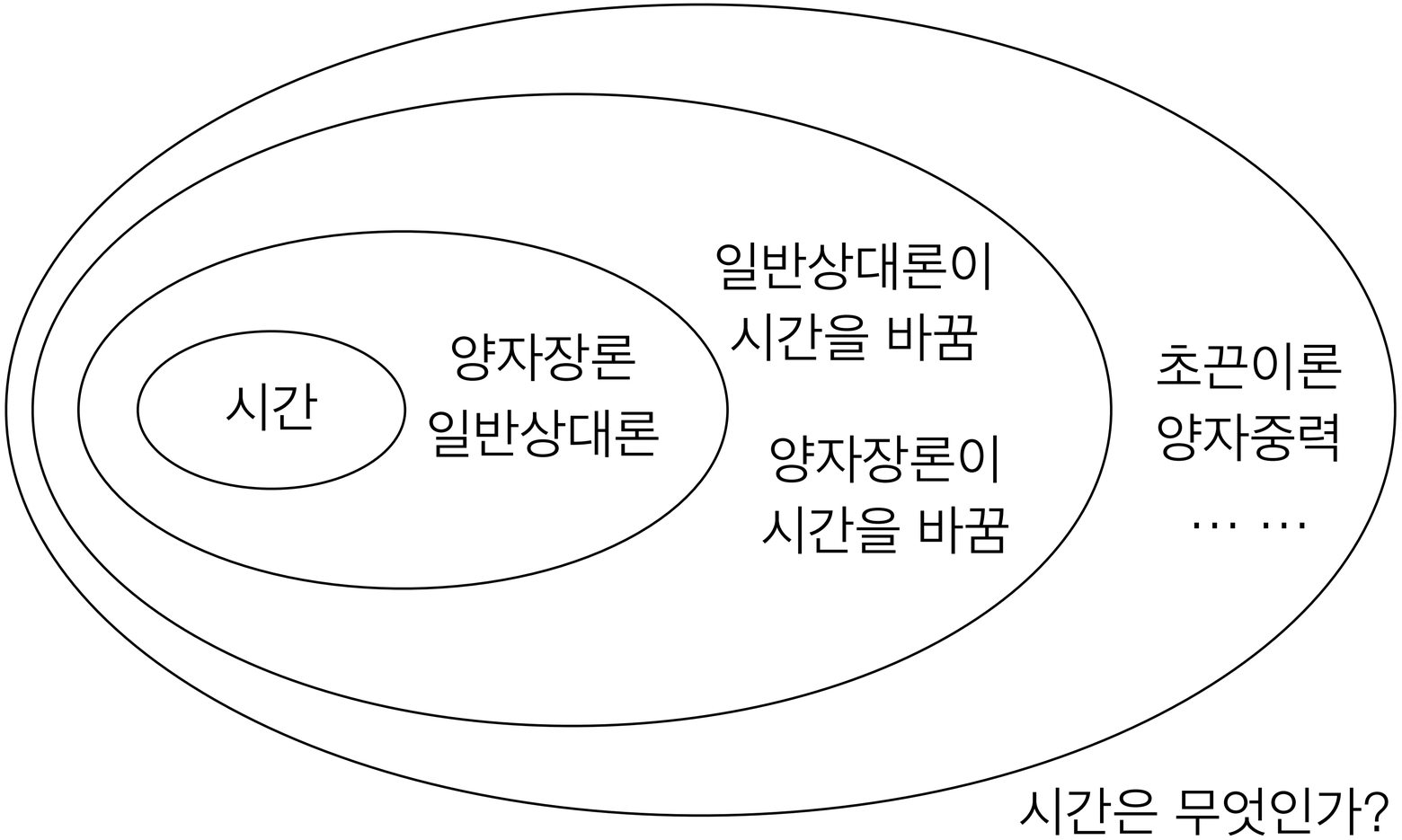
난 내 소원이 이루어지지 않기를 바라네 !



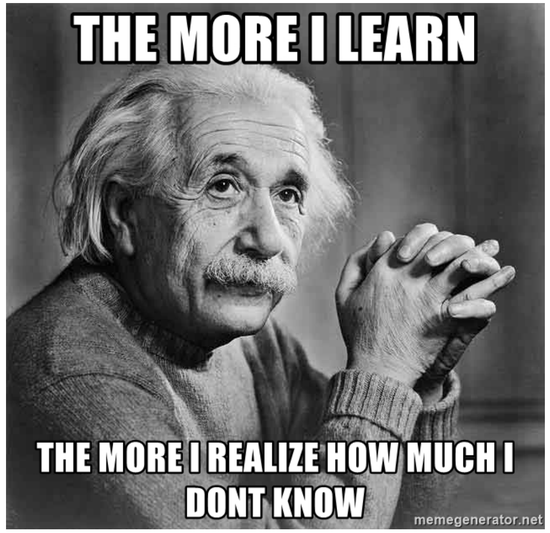


시간의 역설

난 이론이 시간을 설명하기를 바라네 !



더닝-크루거 효과 / Dunning-Kruger Effect



2000년 이그 노벨상 Ig Nobel Prize (심리학)

지식의 섬

휠러 교수 / **블랙홀** 이름 유행시키고 **웜홀** 이름을 지음

“우리는 무지의 바다에 둘러 쌓인 섬에 살고 있다.
지식의 섬이 커질 수록, 무지의 해안선도 증가한다.”