중력파 파형 & 모델

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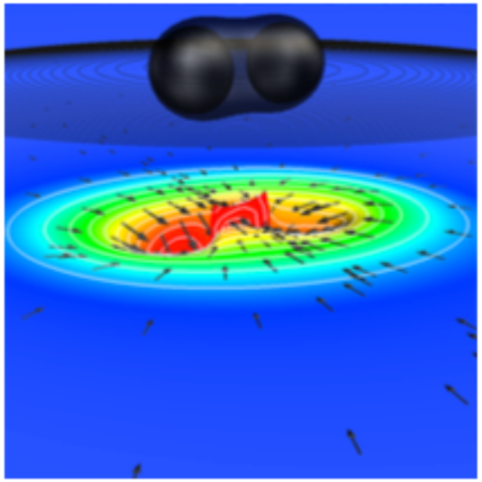
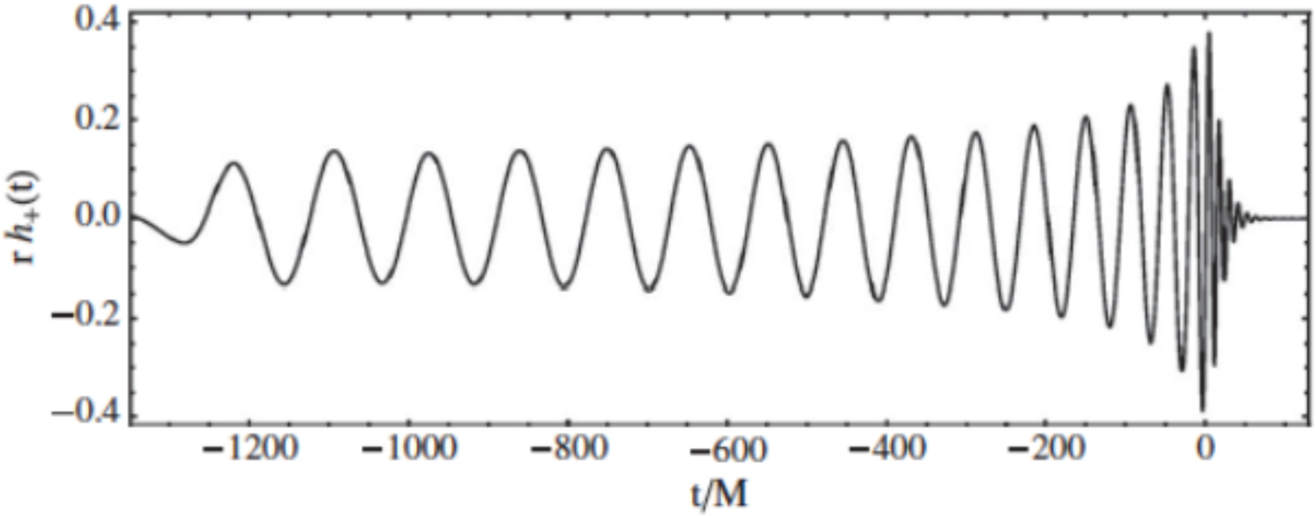




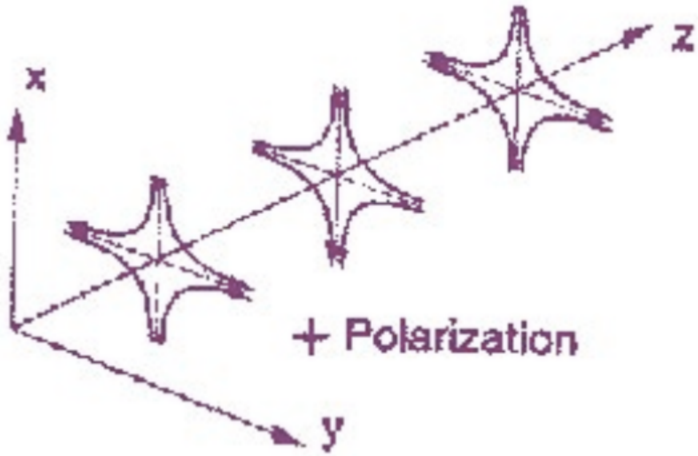
Outline

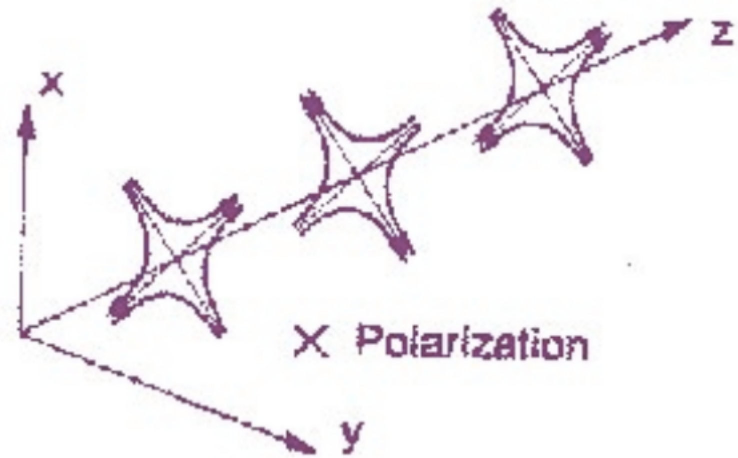
- Binary Black Hole (BBH) waveforms
  - nonspinning
  - precessing (amplitude modulation)
  - PN phase evolution
  - waveform models

GW polarizations

$$h_+ = -\frac{2G\mu}{c^2 R} x(1 + \cos^2\Theta)\cos 2\psi$$

$$h_\times = -\frac{2G\mu}{c^2 R} x(2\cos\Theta)\sin 2\psi$$





GW polarizations: nonspinning

$$h_+ = -\frac{2G\mu}{c^2 R} x (1 + \cos^2\Theta) \cos 2\psi$$

$$h_\times = -\frac{2G\mu}{c^2 R} x (2\cos\Theta) \sin 2\psi$$

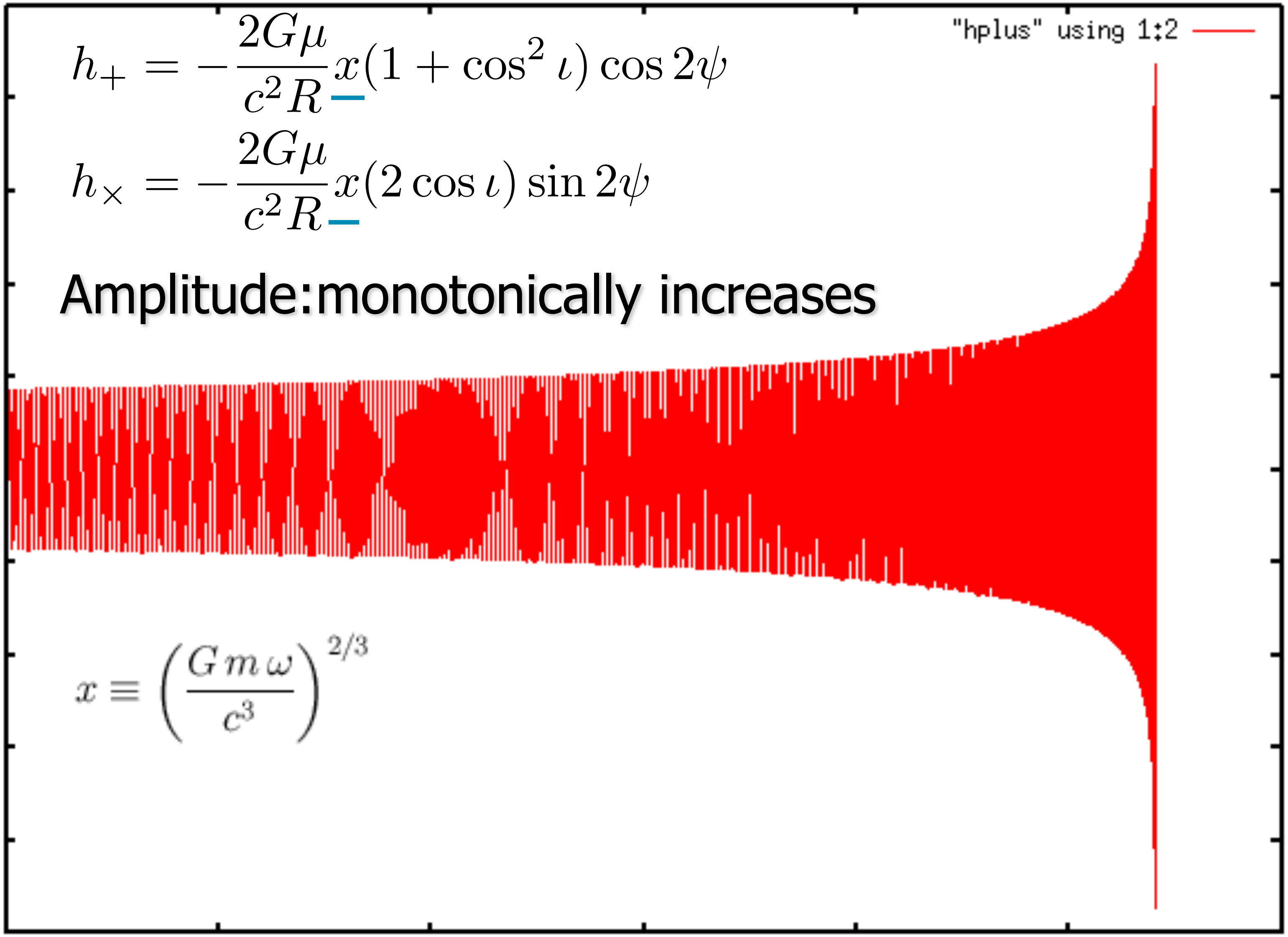
Orbital phase

Inclination

PN parameter

$$x \equiv \left(\frac{G\, m\, \omega}{c^3}\right)^{2/3}$$

Waveform: nonspinning



$$h_+ = -\frac{2G\mu}{c^2 R} x (1 + \cos^2 \iota) \cos 2\psi$$

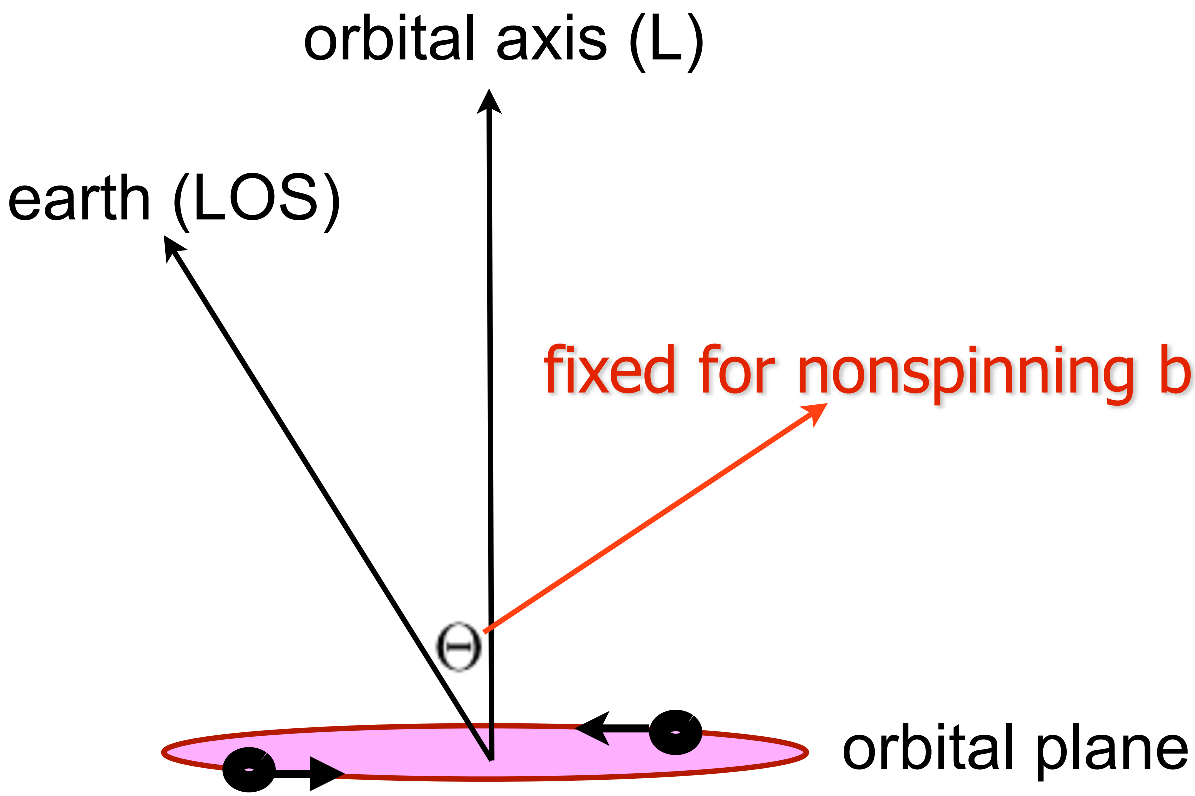
$$h_\times = -\frac{2G\mu}{c^2 R} x (2 \cos \iota) \sin 2\psi$$

$$x \equiv \left(\frac{G m \omega}{c^3} \right)^{2/3}$$

Inclination (Θ)

$$h_+ = -\frac{2G\mu}{c^2 R} x (1 + \cos^2 \underline{\Theta}) \cos 2\psi$$

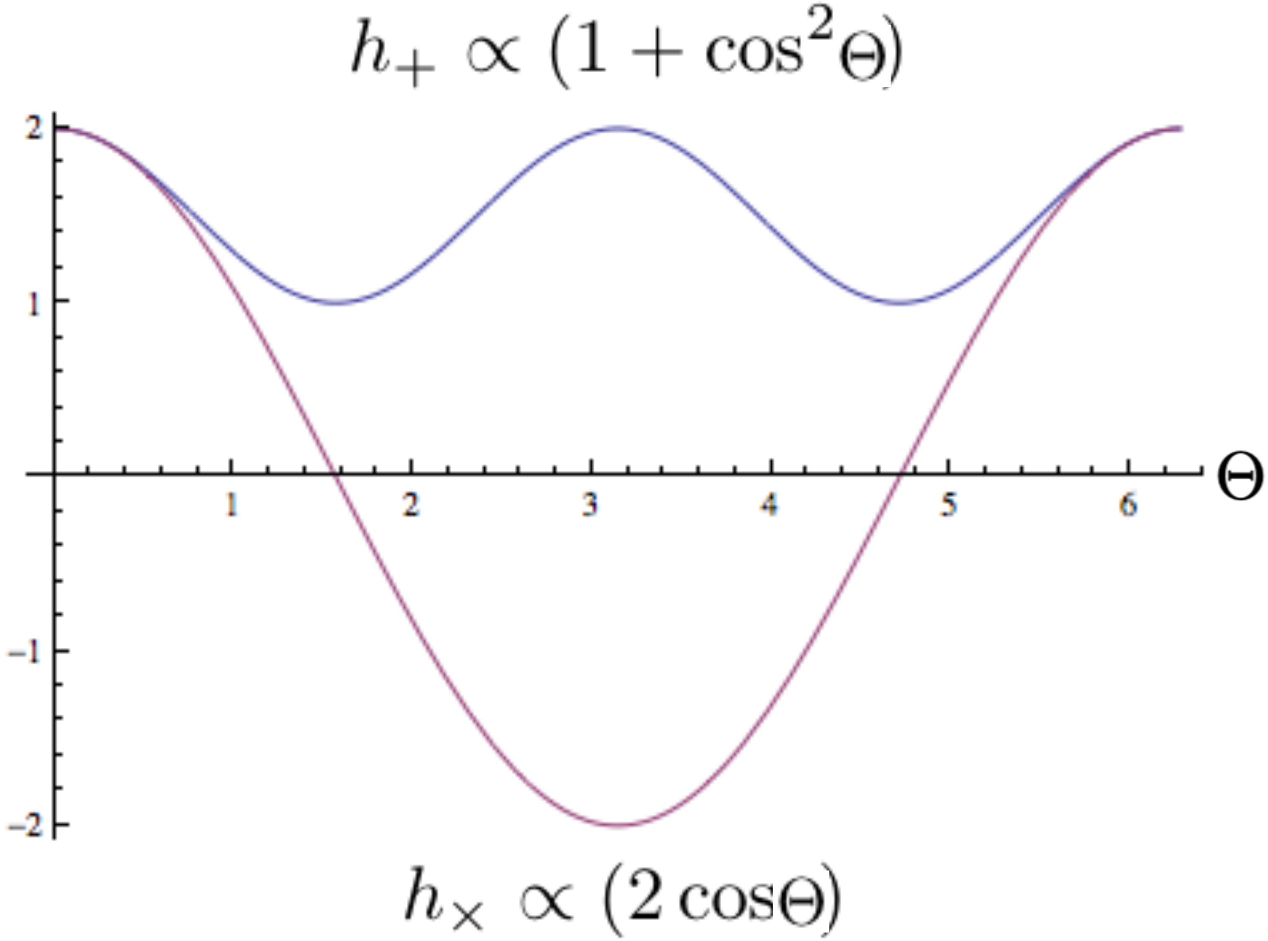
$$h_\times = -\frac{2G\mu}{c^2 R} x (2 \cos \underline{\Theta}) \sin 2\psi$$



Dependence on inclination

$$h_+ = -\frac{2G\mu}{c^2 R} x \underline{(1 + \cos^2\Theta)} \cos 2\psi$$

$$h_\times = -\frac{2G\mu}{c^2 R} x \underline{(2\cos\Theta)} \sin 2\psi$$

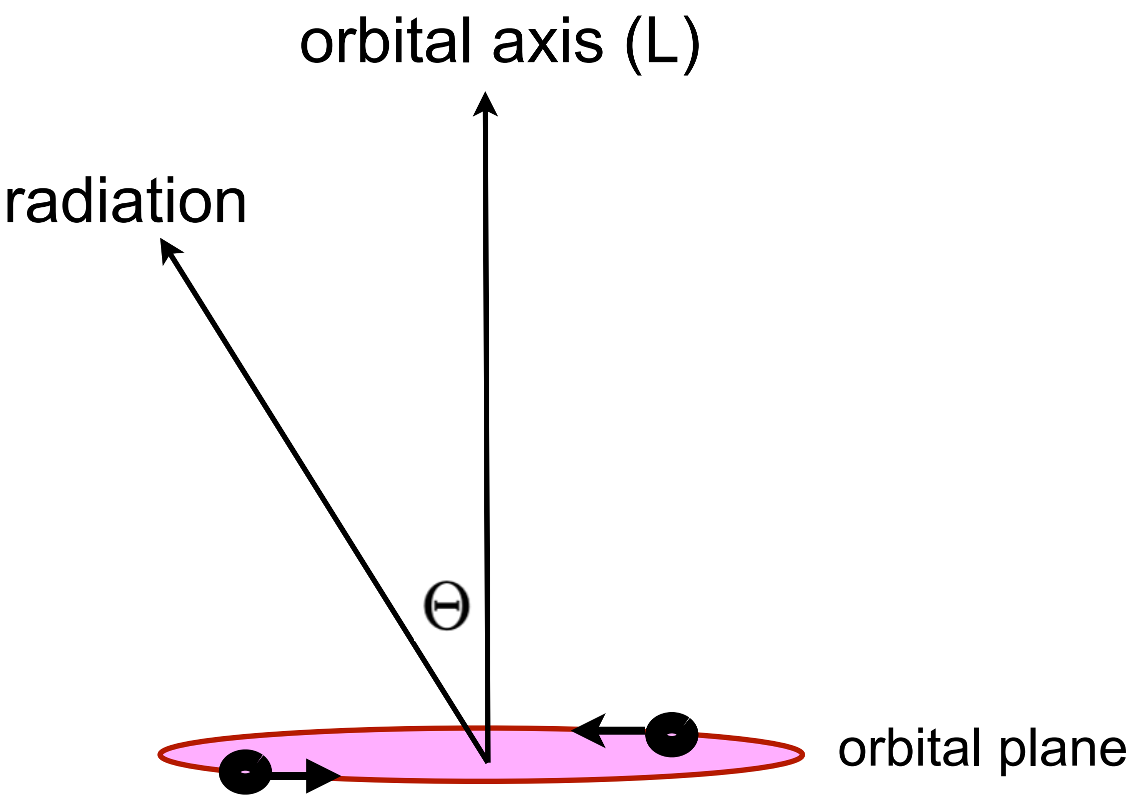


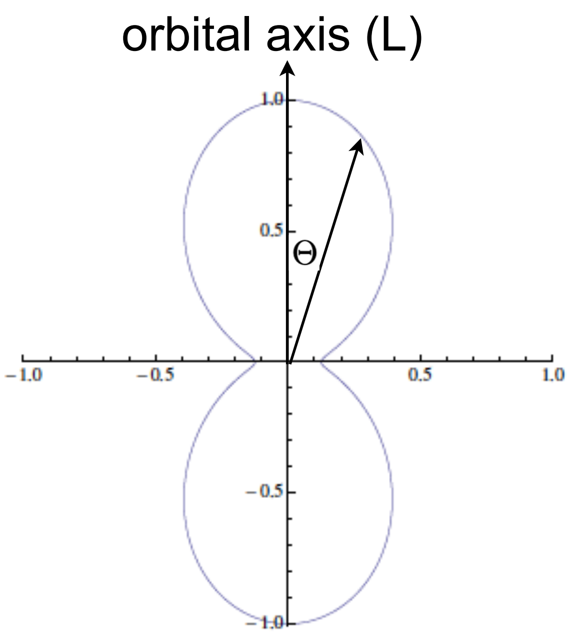
Radiation power (source frame)

$$h_+ = -\frac{2G\mu}{c^2 R} x(1+\cos^2\Theta)\cos 2\psi$$

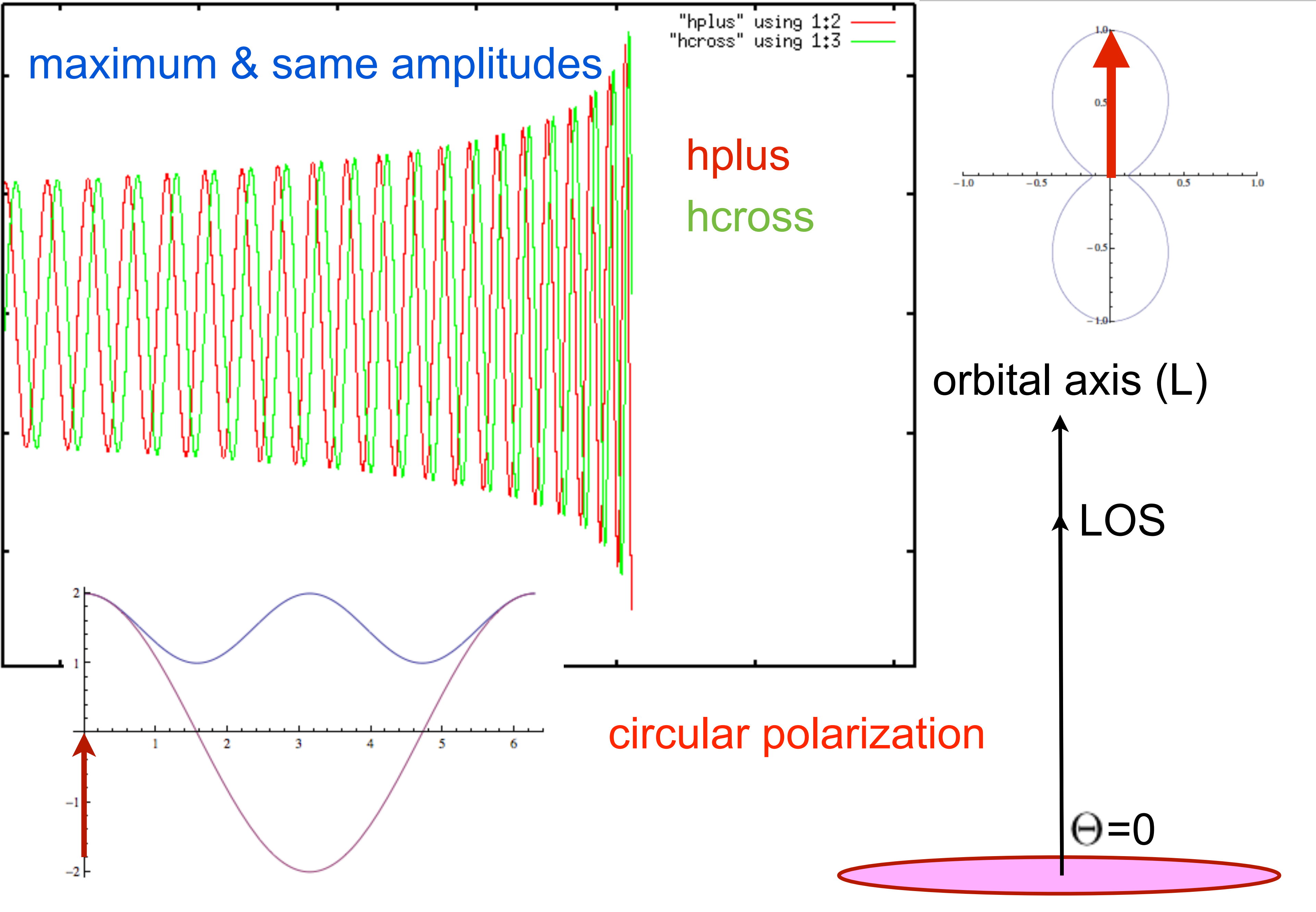
$$h_\times = -\frac{2G\mu}{c^2 R} x(2\cos\Theta)\sin 2\psi$$

$$\left(\frac{dP}{d\Omega}\right) = \frac{2G\mu^2 a^4 \omega^6}{\pi c^5} g(\Theta),$$

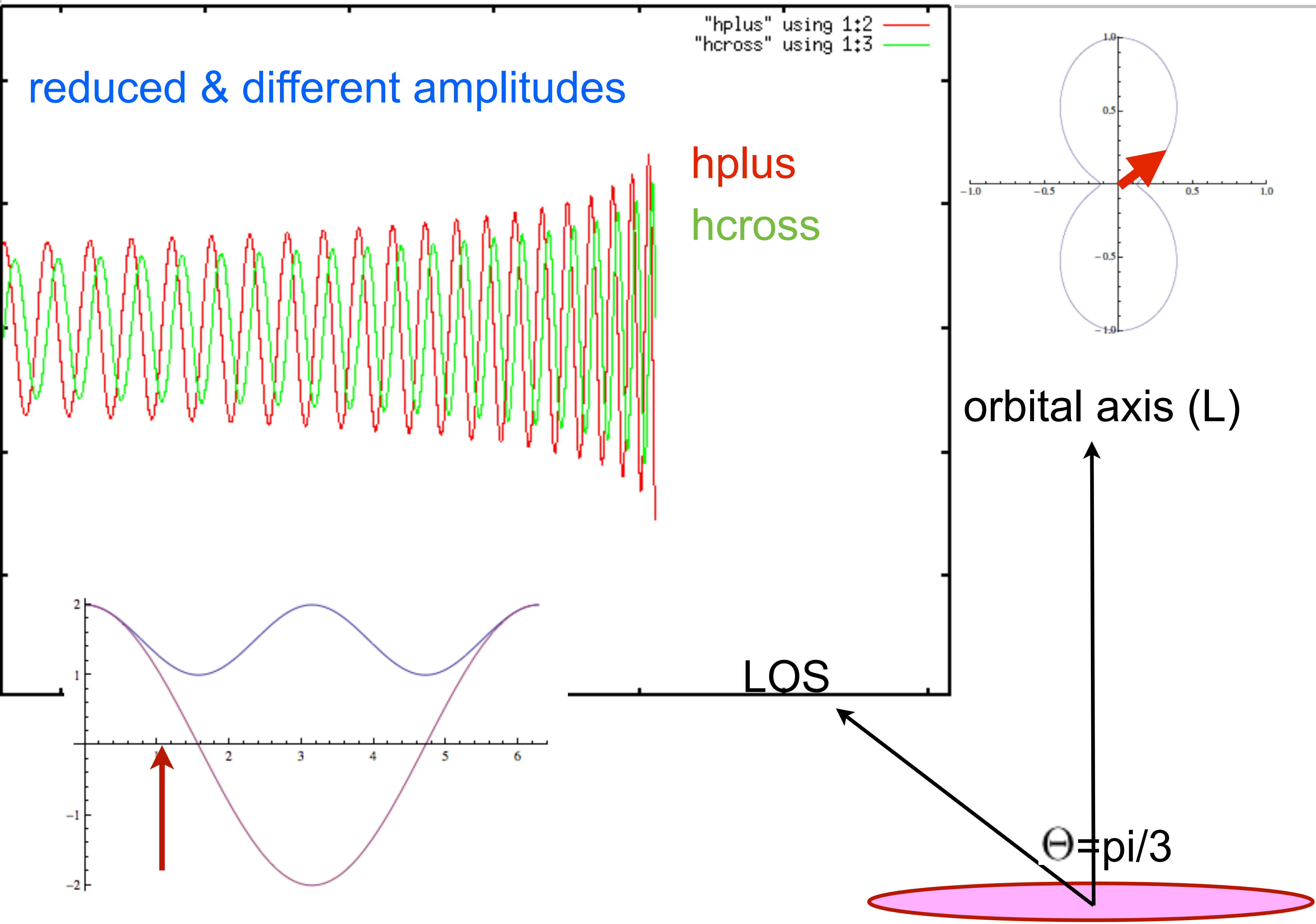




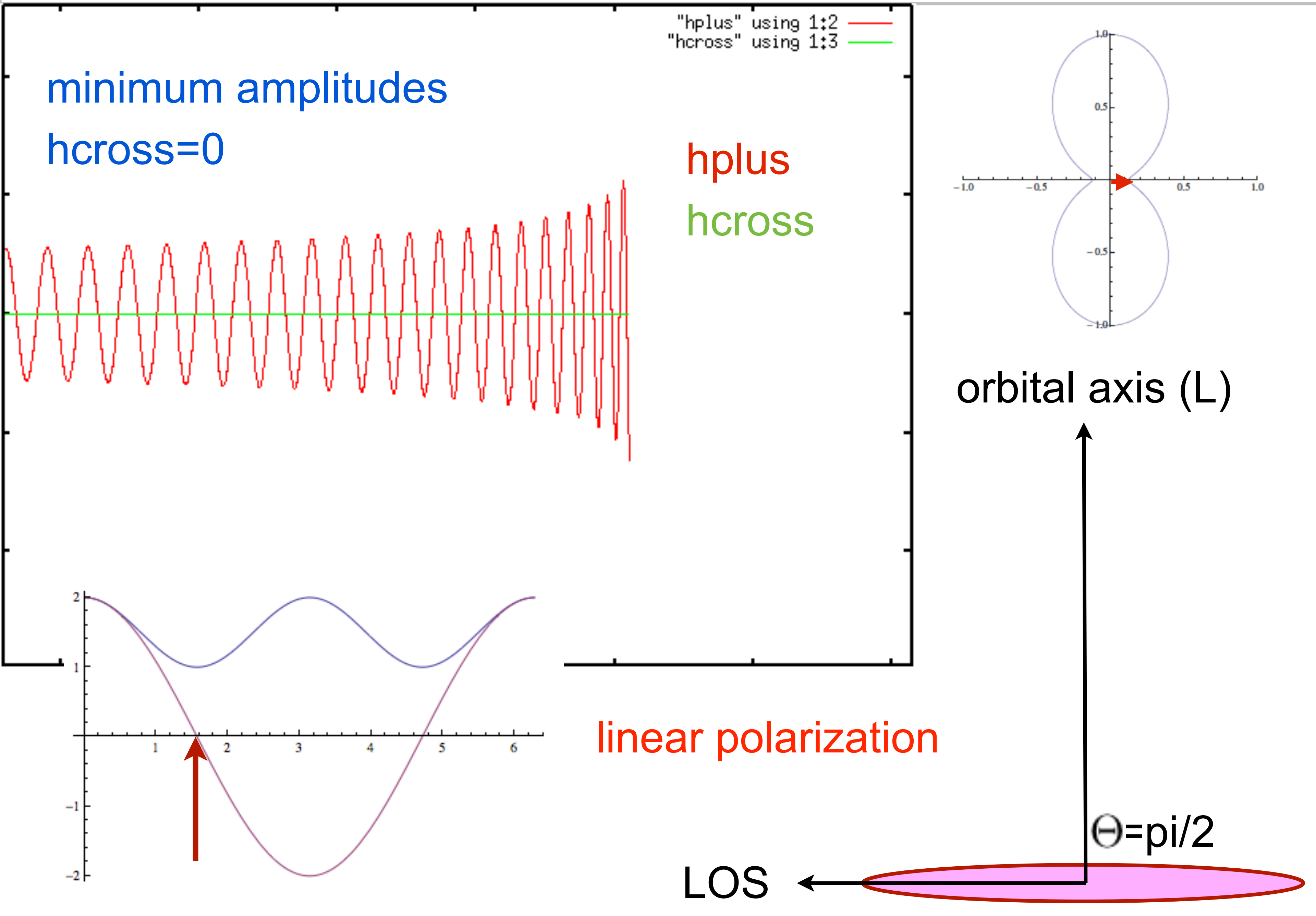
Non spinning GW polarizations (theta=0)



Non spinning polarizations (theta=pi/3)



Non spinning polarizations (theta=pi/2)



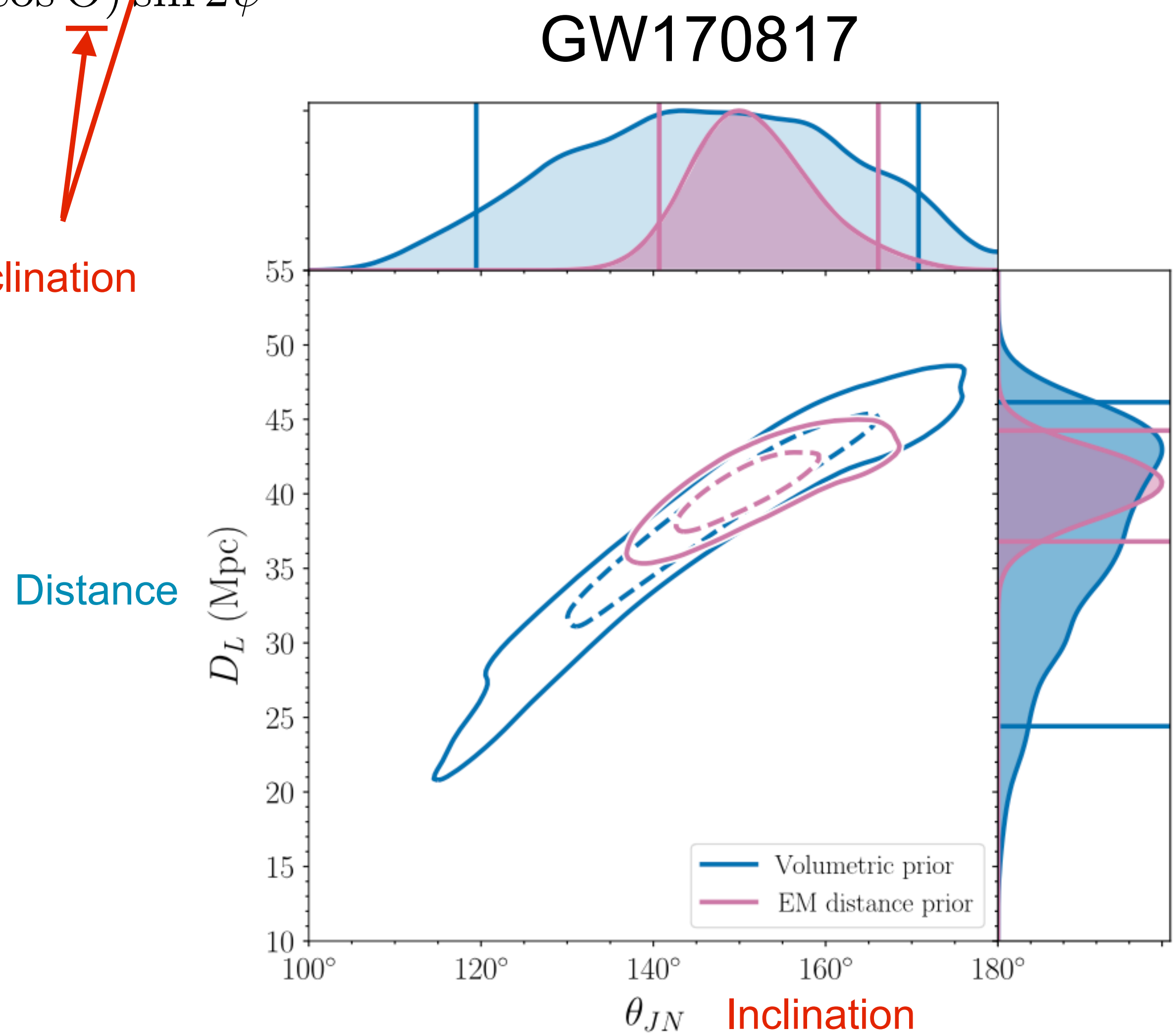
distance-inclination degeneracy

$$h_+ = -\frac{2G\mu}{c^2 R} x(1+\cos^2\Theta)\cos 2\psi$$

$$h_\times = -\frac{2G\mu}{c^2 R} x(2\cos\Theta)\sin 2\psi$$

Inclination

Distance



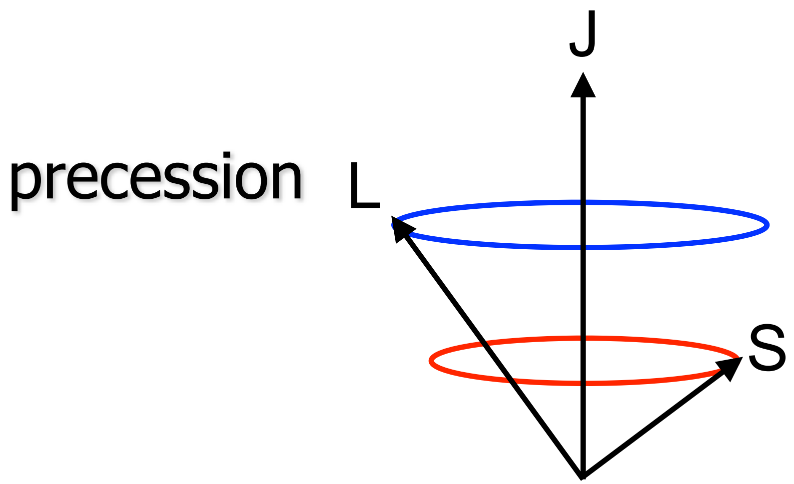
Precession of a spinning binary

Spin-Orbit coupling, Spin-Spin coupling

$$\dot{\mathbf{S}}_1 = \frac{(M\omega)^2}{2M}\left\{\eta\,(M\omega)^{-1/3}\left(4+3\frac{m_2}{m_1}\right)\hat{\mathbf{L}}_N + \frac{1}{M^2}\left[\mathbf{S}_2 - 3(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)\hat{\mathbf{L}}_N\right]\right\}\times\mathbf{S}_1$$

$$\dot{\mathbf{S}}_2 = \frac{(M\omega)^2}{2M}\left\{\eta\,(M\omega)^{-1/3}\left(4+3\frac{m_1}{m_2}\right)\hat{\mathbf{L}}_N + \frac{1}{M^2}\left[\mathbf{S}_1 - 3(\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)\hat{\mathbf{L}}_N\right]\right\}\times\mathbf{S}_2$$

$$\dot{\hat{\mathbf{L}}}_N = -\frac{(M\omega)^{1/3}}{\eta M^2}\dot{\mathbf{S}} = \frac{\omega^2}{2M}\left\{\left[\left(4+3\frac{m_2}{m_1}\right)\mathbf{S}_1 + \left(4+3\frac{m_1}{m_2}\right)\mathbf{S}_2\right]\times\hat{\mathbf{L}}_N - \frac{3\,\omega^{1/3}}{\eta\,M^{5/3}}\left[(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_1 + (\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_2\right]\times\hat{\mathbf{L}}_N\right\}$$



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$$\dot{\mathbf{S}}_2 = \frac{(M\omega)^2}{2M}\left\{\eta\,(M\omega)^{-1/3}\left(4+3\frac{m_1}{m_2}\right)\hat{\mathbf{L}}_N + \frac{1}{M^2}\left[\mathbf{S}_1 - 3(\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)\hat{\mathbf{L}}_N\right]\right\}\times\mathbf{S}_2$$

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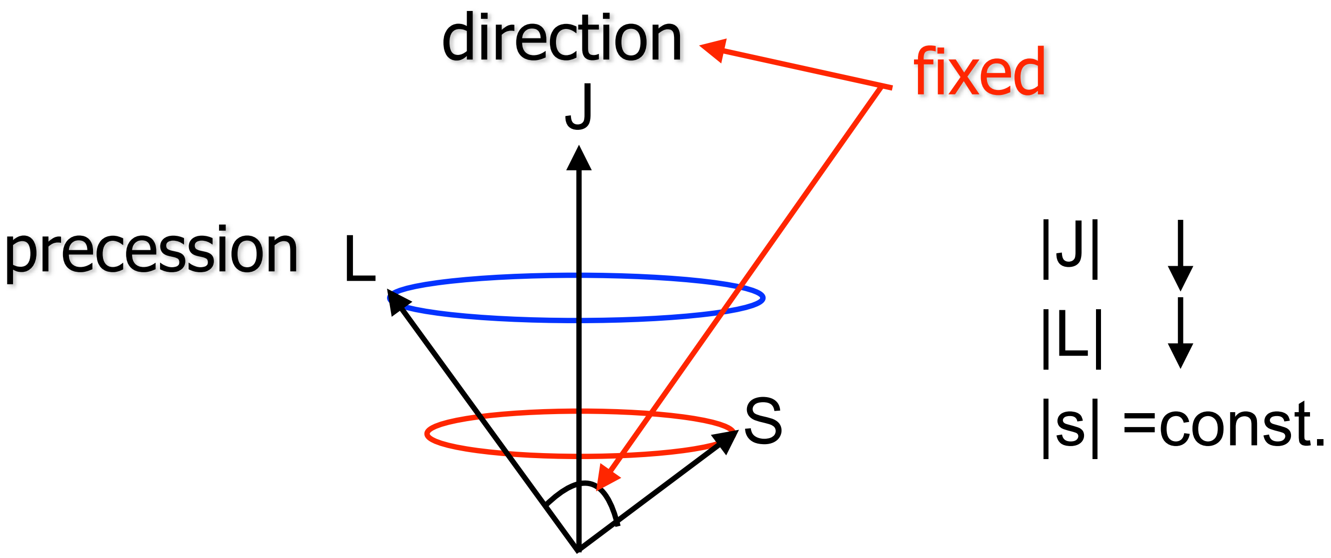
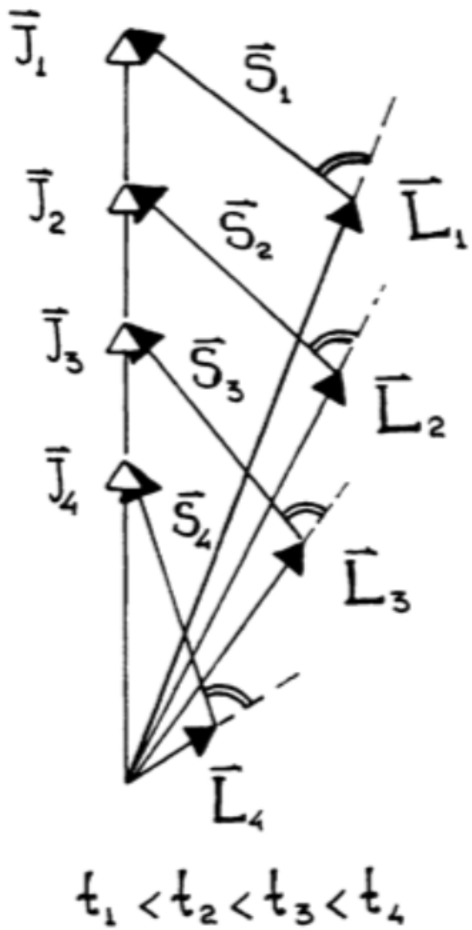
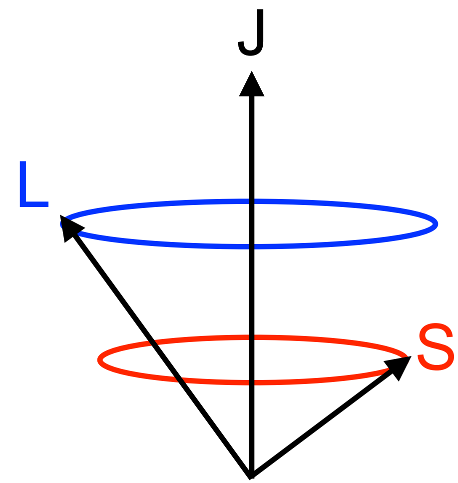
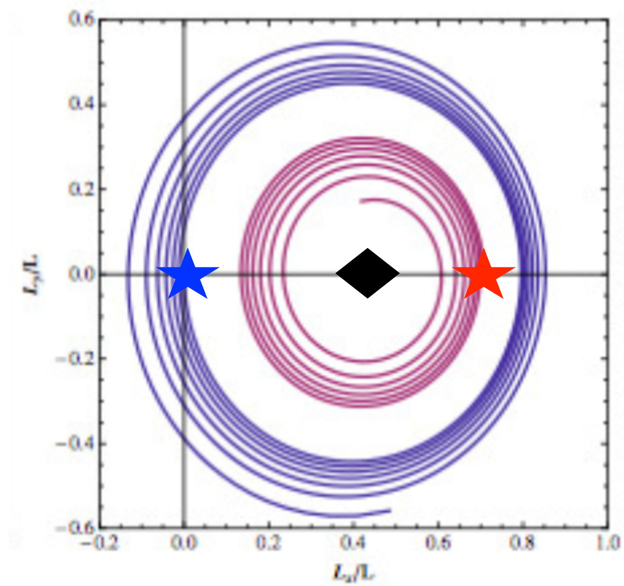


Precession of a spinning binary







GW Polarization of spinning binary

$$h_+ = -\frac{2G\mu}{c^2 R} x [C_+ \cos 2\psi + S_+ \sin 2\psi]$$

$$h_\times = -\frac{2G\mu}{c^2 R} x [C_\times \cos 2\psi + S_\times \sin 2\psi]$$

non spinning

$$h_+ = -\frac{2G\mu}{c^2 R} x (1 + \cos^2\Theta) \cos 2\psi$$

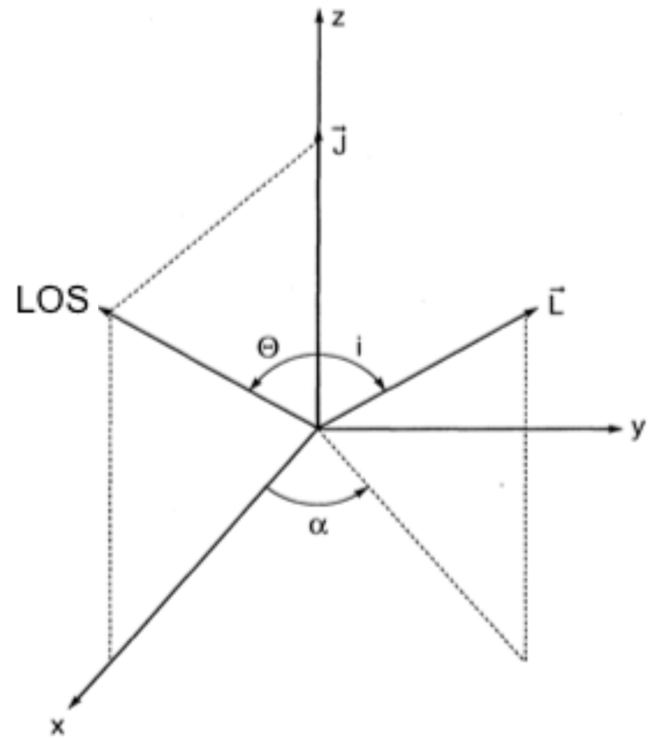
$$h_\times = -\frac{2G\mu}{c^2 R} x (2\cos\Theta) \sin 2\psi$$

$$C_+ = \tfrac{1}{2}\cos^2\Theta\,(\sin^2\alpha - \cos^2 i \cos^2\alpha) + \tfrac{1}{2}(\cos^2 i \sin^2\alpha - \cos^2\alpha) - \tfrac{1}{2}\sin^2\Theta \sin^2 i - \tfrac{1}{4}\sin 2\Theta \sin 2i \cos\alpha,$$

$$S_+ = \tfrac{1}{2}(1 + \cos^2\Theta) \cos i \sin 2\alpha + \tfrac{1}{2}\sin 2\Theta \sin i \sin\alpha,$$

$$C_\times = -\tfrac{1}{2}\cos\Theta \sin 2\alpha\,(1 + \cos^2 i) - \tfrac{1}{2}\sin\Theta \sin 2i \sin\alpha,$$

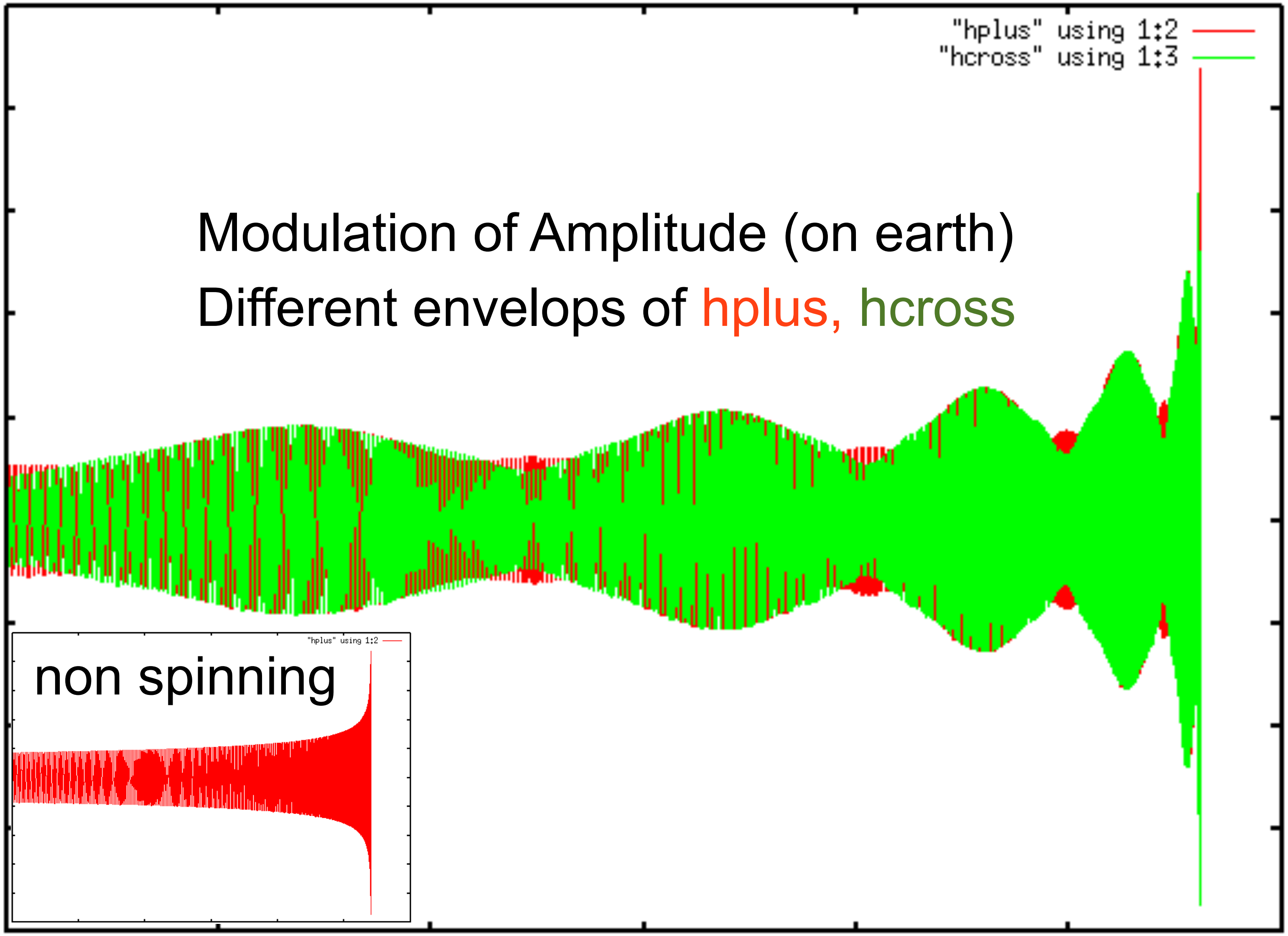
$$S_\times = -\cos\Theta \cos i \cos 2\alpha - \sin\Theta \sin i \cos\alpha,$$



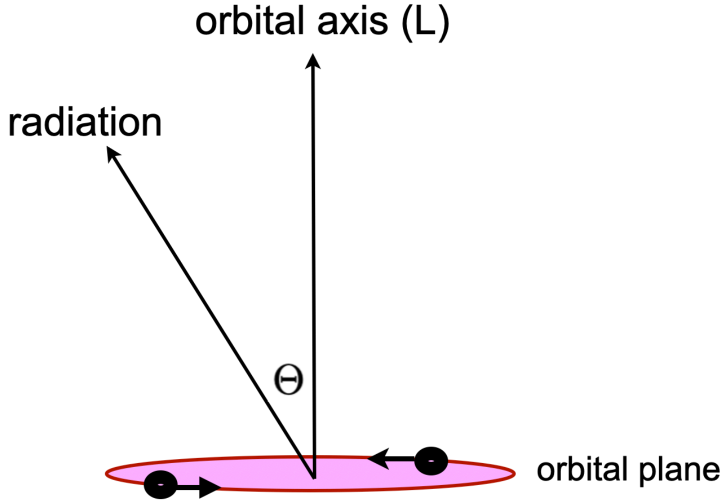
ι, α, Θ : vary in time

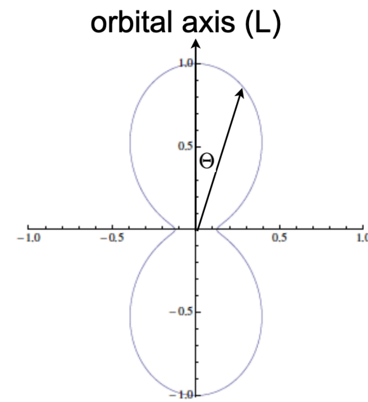
--> precessional motion

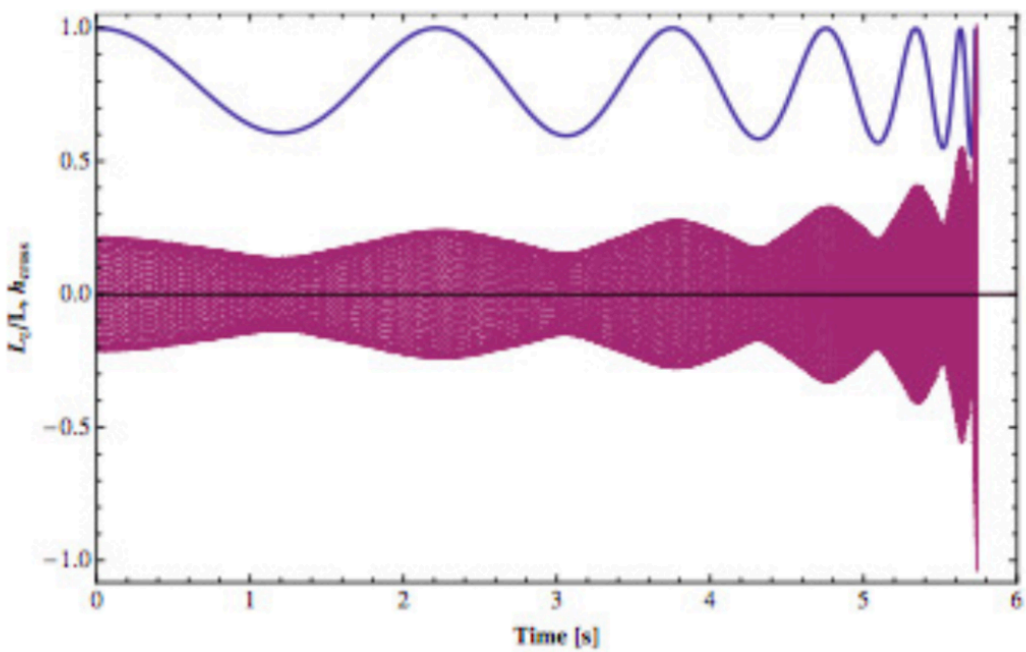
Waveform of spinning binary

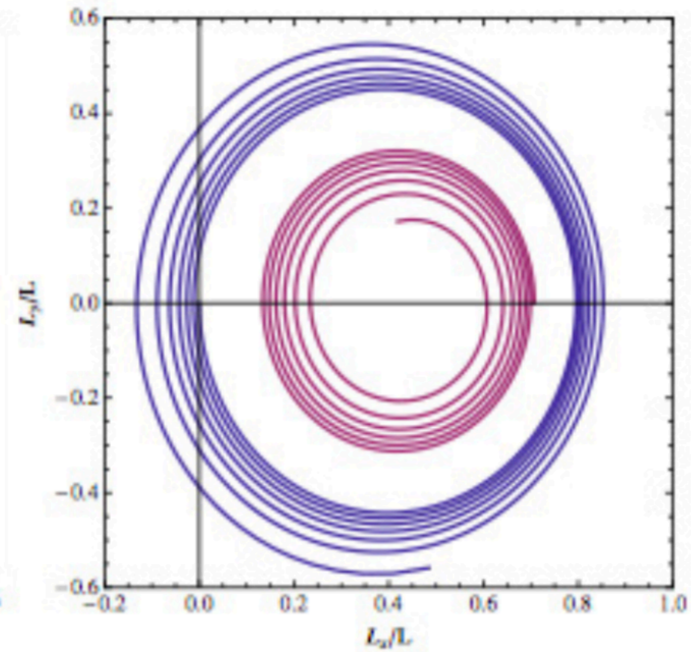


Radiation power (source frame)









Modulation magnitude

Variation of L_N depends on S, S dot L_N, S cross L_N

---> Precession effect depends on

1) Spin magnitude

2) Angle between L_N and S

$$\dot{\hat{\mathbf{L}}}_N = -\frac{(M\omega)^{1/3}}{\eta M^2}\dot{\mathbf{S}} = \frac{\omega^2}{2M}\left\{\left[\left(4+3\frac{m_2}{m_1}\right)\underline{\mathbf{S}_1} + \left(4+3\frac{m_1}{m_2}\right)\underline{\mathbf{S}_2}\right]\times\underline{\hat{\mathbf{L}}_N}\right.$$

$$\left. -\frac{3\,\omega^{1/3}}{\eta\,M^{5/3}}\left[\underline{(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)}\,\underline{\mathbf{S}_1} + \underline{(\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)}\,\underline{\mathbf{S}_2}\right]\times\underline{\hat{\mathbf{L}}_N}\right\}$$

Modulation magnitude

Variation of L_N depends on S, S dot L_N, S cross L_N

---> Precession effect depends on

1) Spin magnitude

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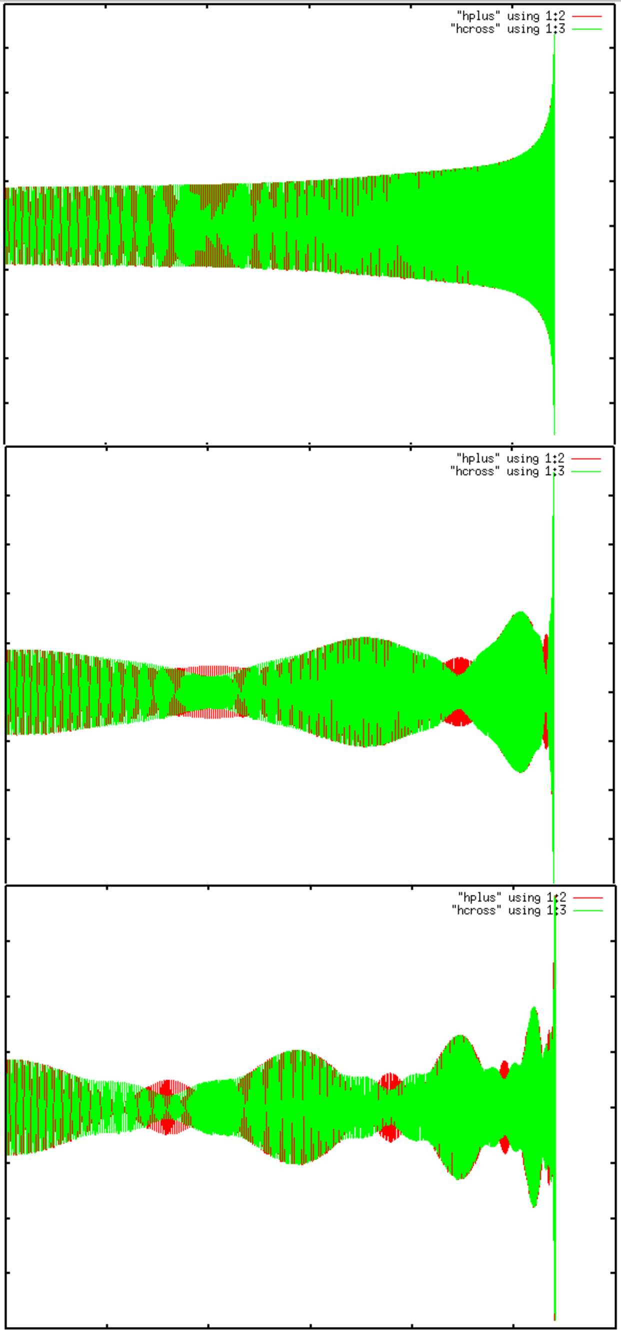
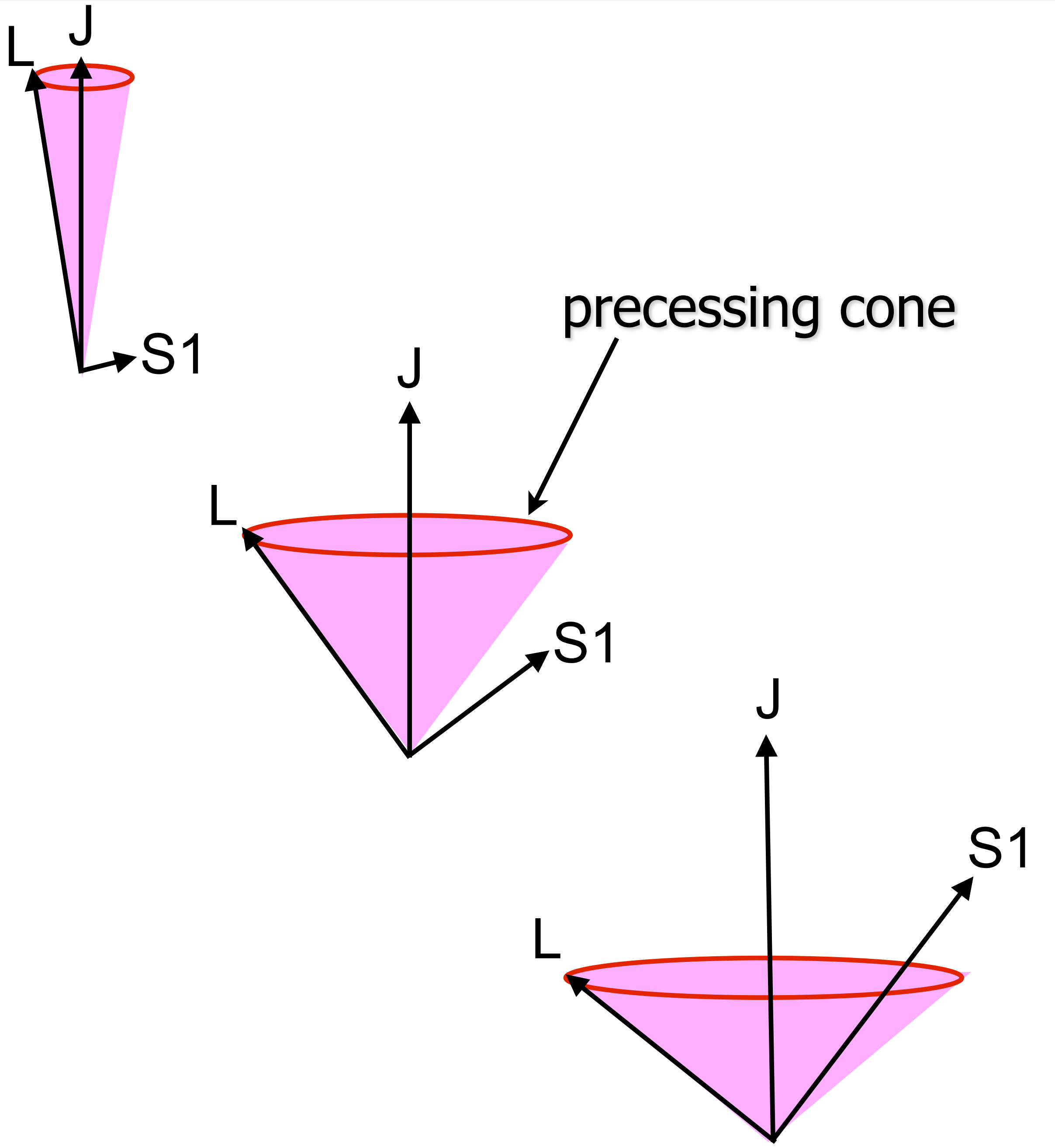
$$\dot{\hat{\mathbf{L}}}_N = -\frac{(M\omega)^{1/3}}{\eta M^2}\dot{\mathbf{S}} = \frac{\omega^2}{2M}\left\{\left[\left(4+3\frac{m_2}{m_1}\right)\mathbf{S}_1 + \left(4+3\frac{m_1}{m_2}\right)\mathbf{S}_2\right]\times\hat{\mathbf{L}}_N - \frac{3\,\omega^{1/3}}{\eta\, M^{5/3}}\left[(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_1 + (\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_2\right]\times\hat{\mathbf{L}}_N\right\}$$

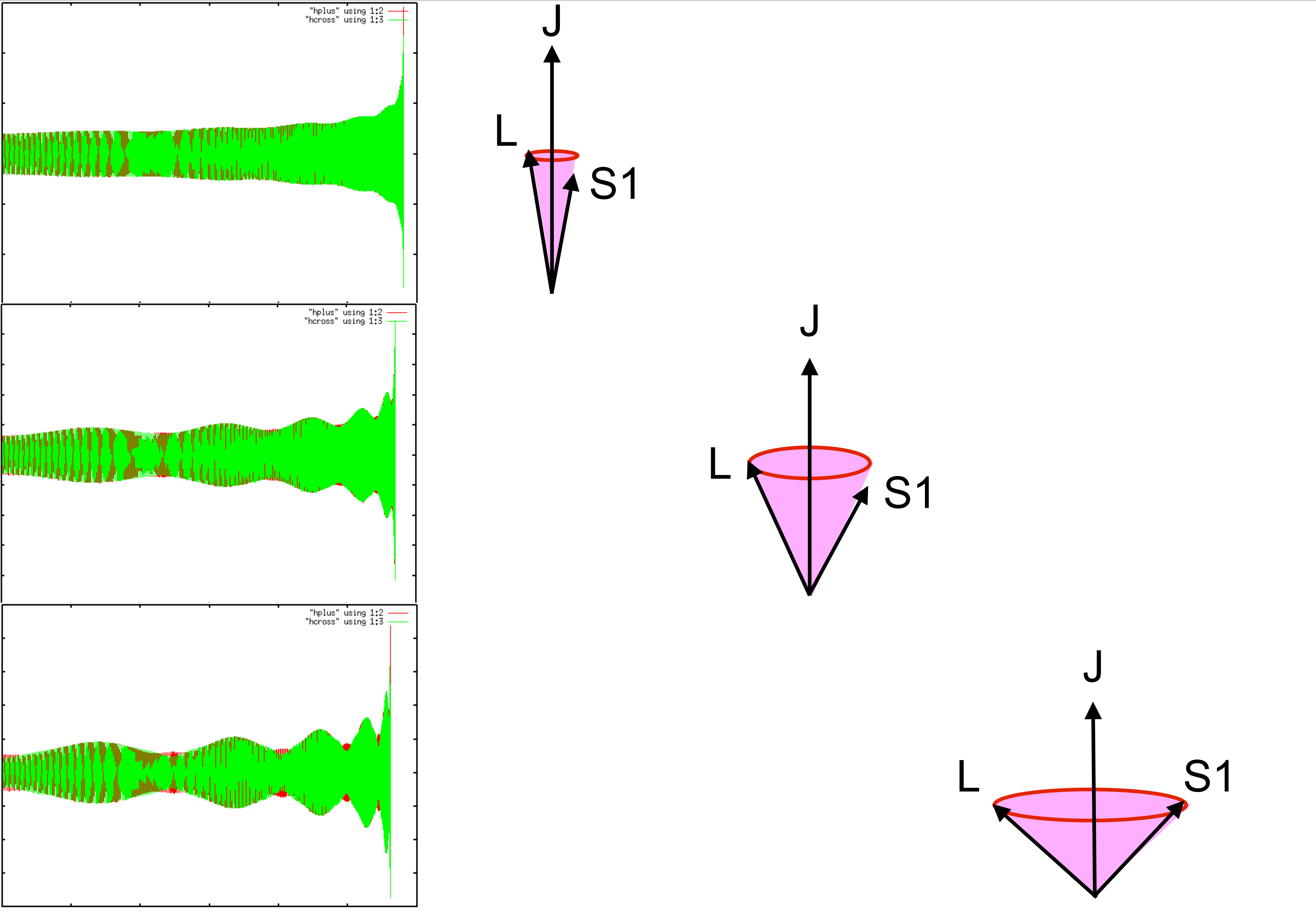
aligned-spin: no precession

Amplitude modulation with Spin (Angle=pi/2, S2=0)

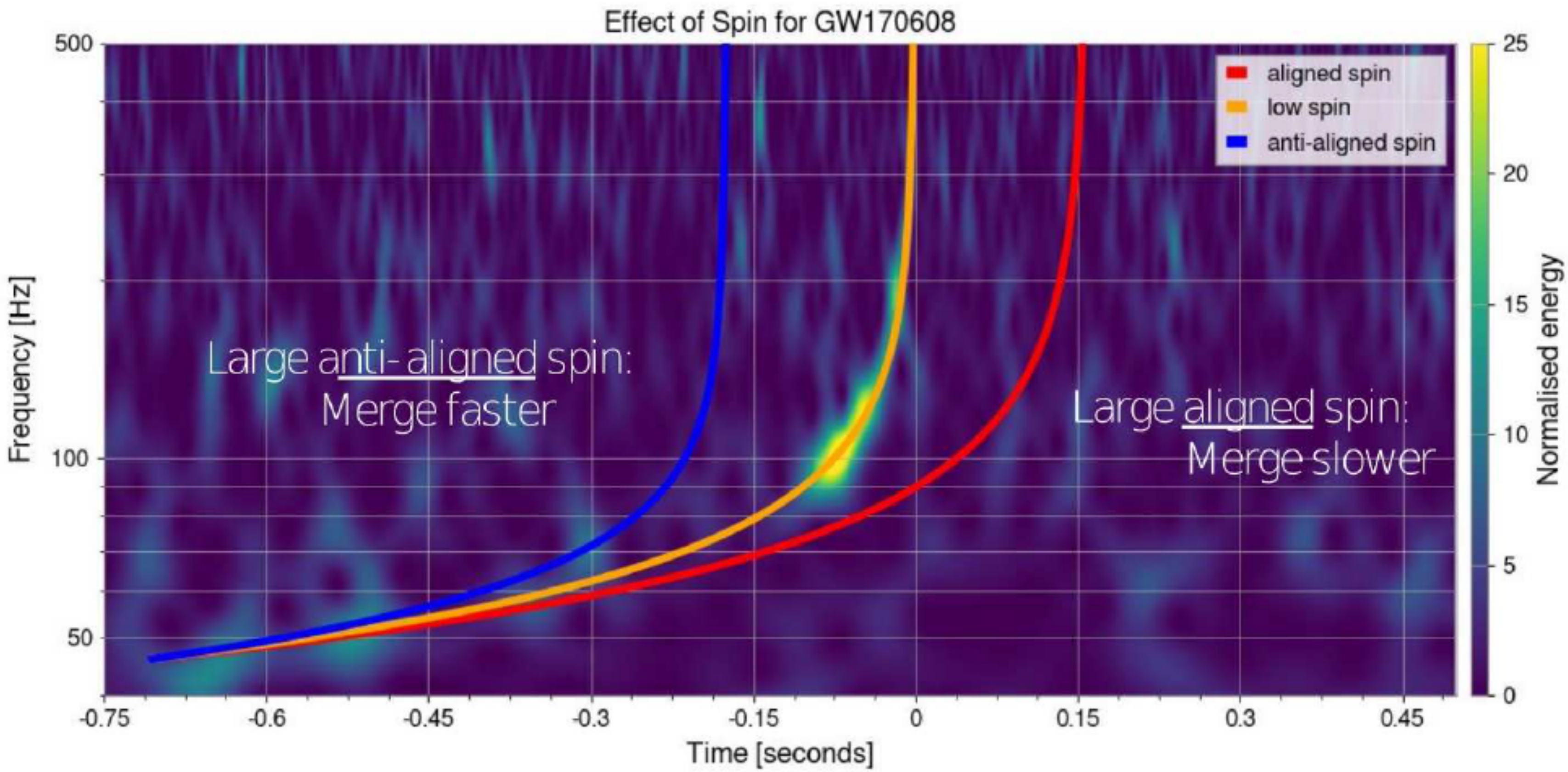




Amplitude modulation with Angle (S1=0.9, S2=0)

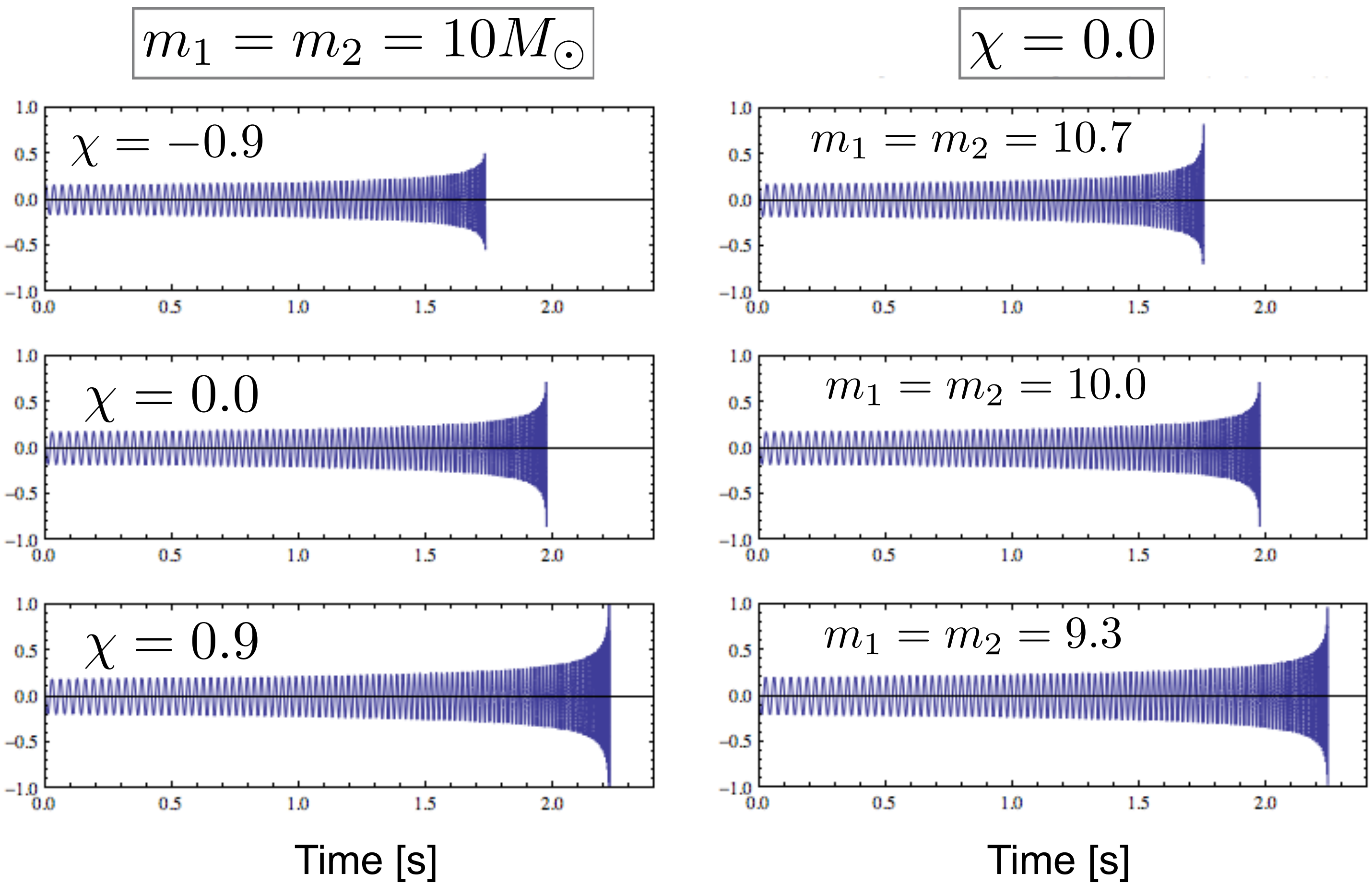


aligned-spin binaries



(see the `pyCBC` tutorial:
`http://pycbc.org/pycbc/latest/html/waveform.html#plotting-frequency-evolution-of-td-waveform`)

spin-mass degeneracy



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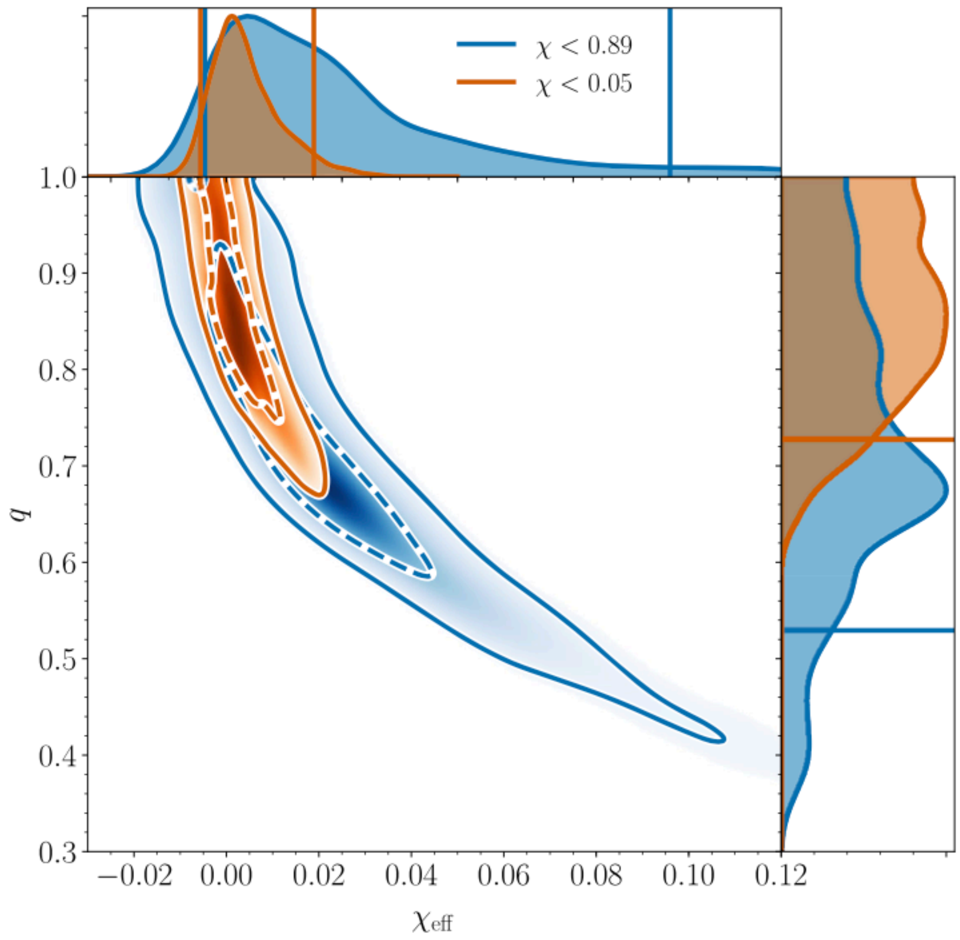


$\chi < 0.89$
$\chi < 0.05$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
q
−0.02
0.00
0.02
0.04
0.06
0.08
0.10
0.12
$\chi_{\rm eff}$

GW phase: nonspinning

$$h_+ = -\frac{2G\mu}{c^2 R} x (1 + \cos^2\Theta) \cos 2\underline{\psi}$$

$$h_\times = -\frac{2G\mu}{c^2 R} x (2\cos\Theta) \sin 2\underline{\psi}$$

Phase evolution from post Newtonian (PN)

Post Newtonian Energy & Flux

$$E = -\frac{\mu c^2 x}{2}\left\{1 + \left(-\frac{3}{4} - \frac{1}{12}\nu\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right)x^2 + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)x^3\right\} + \mathcal{O}\left(\frac{1}{c^8}\right).$$

Newtonian binding energy of a binary

$$\mathcal{L} = \frac{32c^5}{5G}\nu^2 x^5\left\{1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{856}{105}\ln(16x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}.$$

$$x \equiv \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3} \qquad m = m_1 + m_2 \qquad \mu = m_1 m_2/m \qquad \nu \equiv \frac{\mu}{m} \equiv \frac{m_1 m_2}{(m_1+m_2)^2}.$$

Phase evolution

Energy balance equation : orbital binding energy loss = GW emission energy

$$\frac{dE}{dt} = -L$$

$$\frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} \qquad \frac{dx}{dt} = -\frac{L}{(dE/dx)}$$

expand with x , $x \equiv \left(\frac{G\, m\, \omega}{c^3}\right)^{2/3}$

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\eta(M\omega)^{5/3}\left(1 - \frac{743+924\eta}{336}(M\omega)^{2/3} + \left(\frac{34\,103}{18\,144} + \frac{13\,661}{2016}\eta + \frac{59}{18}\eta^2\right)(M\omega)^{4/3} - \frac{1}{672}(4159+14\,532\eta)\pi(M\omega)^{5/3}\eta\right.$$
$$+\left[\left(\frac{16\,447\,322\,263}{139\,708\,800} - \frac{1712}{105}\gamma_E + \frac{16}{3}\pi^2\right) + \left(-\frac{273\,811\,877}{1\,088\,640} + \frac{451}{48}\pi^2 - \frac{88}{3}\hat{\theta}\right)\eta + \frac{541}{896}\eta^2 - \frac{5605}{2592}\eta^3\right.$$
$$\left.\left.- \frac{856}{105}\log[16(M\omega)^{2/3}]\right](M\omega)^2 + \left(-\frac{4\,415}{4\,032} + \frac{661\,775}{12\,096}\eta + \frac{149\,789}{3\,024}\eta^2\right)\pi(M\omega)^{7/3}\right),$$

$$w(t) = \int_0^t \dot{w}(w)dt, \qquad \psi(t) = \int_0^t w(t)dt$$ ---> Numerical integration

TaylorT4, T1,2,3,.. : time domain models

TaylorF2

$$h(f) = \frac{M_c^{5/6}}{\pi^{2/3} D_{\rm eff}} \sqrt{\frac{5}{24}} f^{-7/6} e^{i\Psi(f)},$$

stationary phase approx.

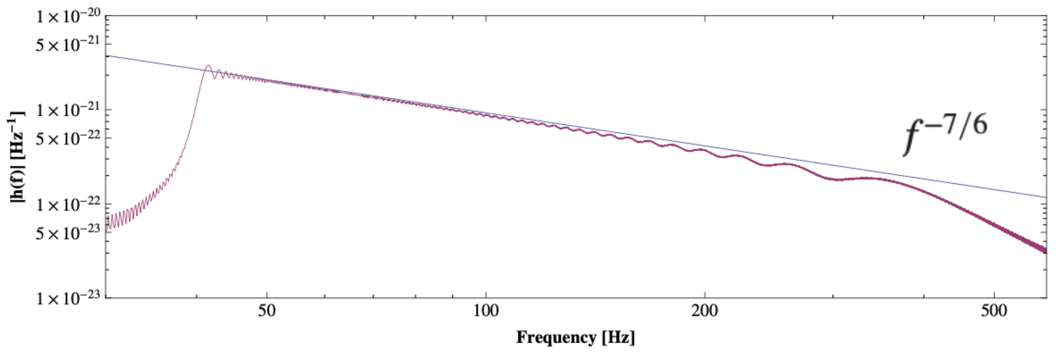


Figure 3.2: **Fourier domain TaylorT4 and SPA waveforms from a non-spinning binary.** We assume the same binary model as in figure 3.1. TaylorT4 (red) and SPA (blue) waveforms coincide at 40 Hz.

$$\begin{aligned}
\Phi_{\rm SPA}(f) &= 2\pi f t_c - \Phi_c - \frac{\pi}{4} + \frac{3}{128\nu v^5}\Bigg\{1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11}{4}\nu\right)v^2 - 16\pi v^3 \\
&+ 10\left(\frac{3058673}{1016064} + \frac{5429}{1008}\nu + \frac{617}{144}\nu^2\right)v^4 \qquad (3.20) \\
&+ \left(\frac{38645}{756} - \frac{65}{9}\nu\right)\left[1 + 3\log\left(\frac{v}{v_{\rm iso}}\right)\right]\pi v^5 \Bigg[\Bigg(\frac{11583231236531}{4694215680} \\
&- \frac{640}{3}\pi^2 - \frac{6848}{21}\gamma_E - \frac{6848}{21}\log(4v)\Bigg) + \left(\frac{2255}{12}\pi^2 - \frac{15737765635}{3048192}\right)\nu \\
&+ \frac{76055}{1728}\nu^2 - \frac{127825}{1296}\nu^3\Bigg]v^6 + \left(\frac{77096675}{254016} + \frac{378515}{1512}\nu - \frac{74045}{756}\nu^2\right)\pi v^7\Bigg\}
\end{aligned}$$

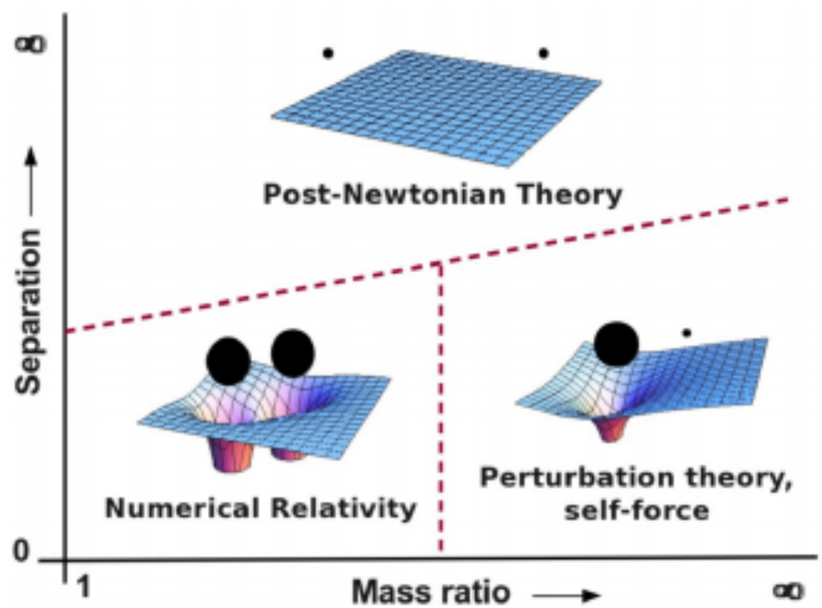
$$v \equiv [\pi f (m_1 + m_2)]^{1/3}$$

Waveform modeling

- **Post-Newtonian (PN) approximation**: slow motion approximation $(v/c << 1)$
- **Perturbative theory**: small mass ratio $(m/M \ll 1)$
- **Effective-One-Body approach**: combination of PN, Perturbative approach and NR

time-domain model !!

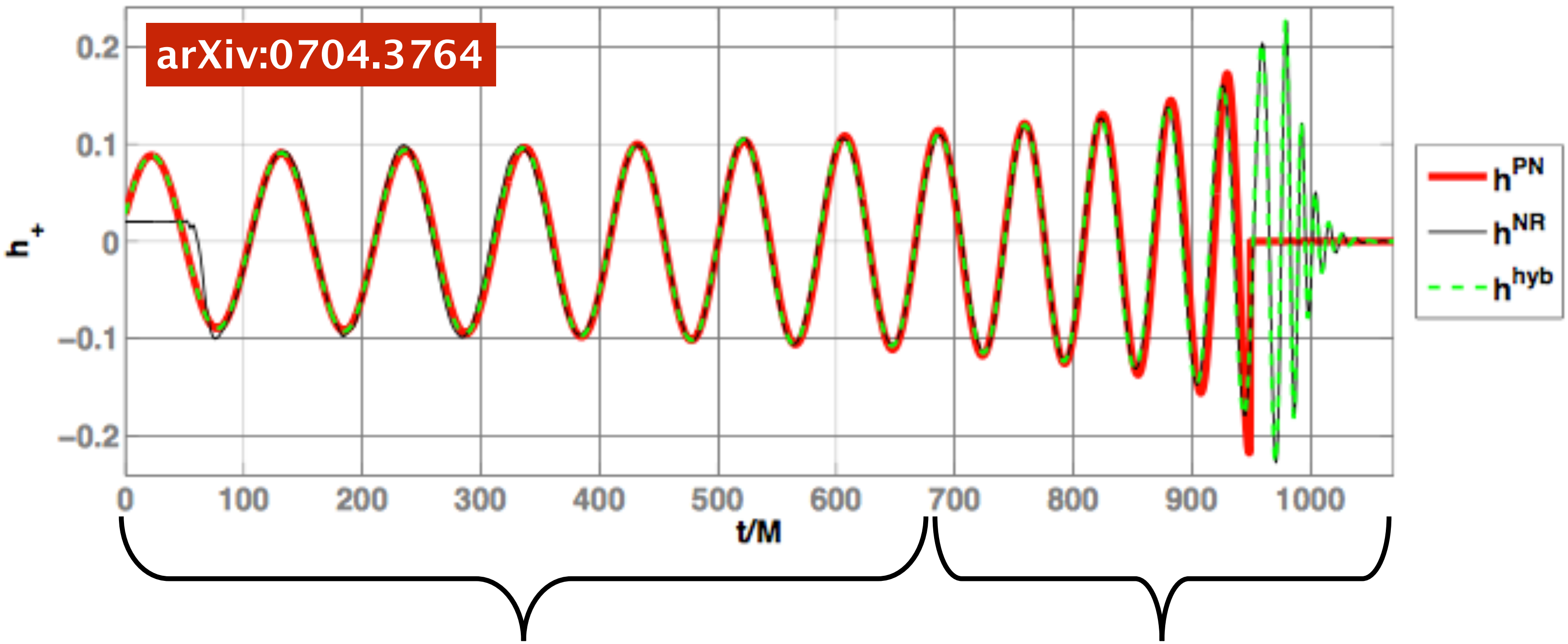
Binary parameter space



[Leor Barack]

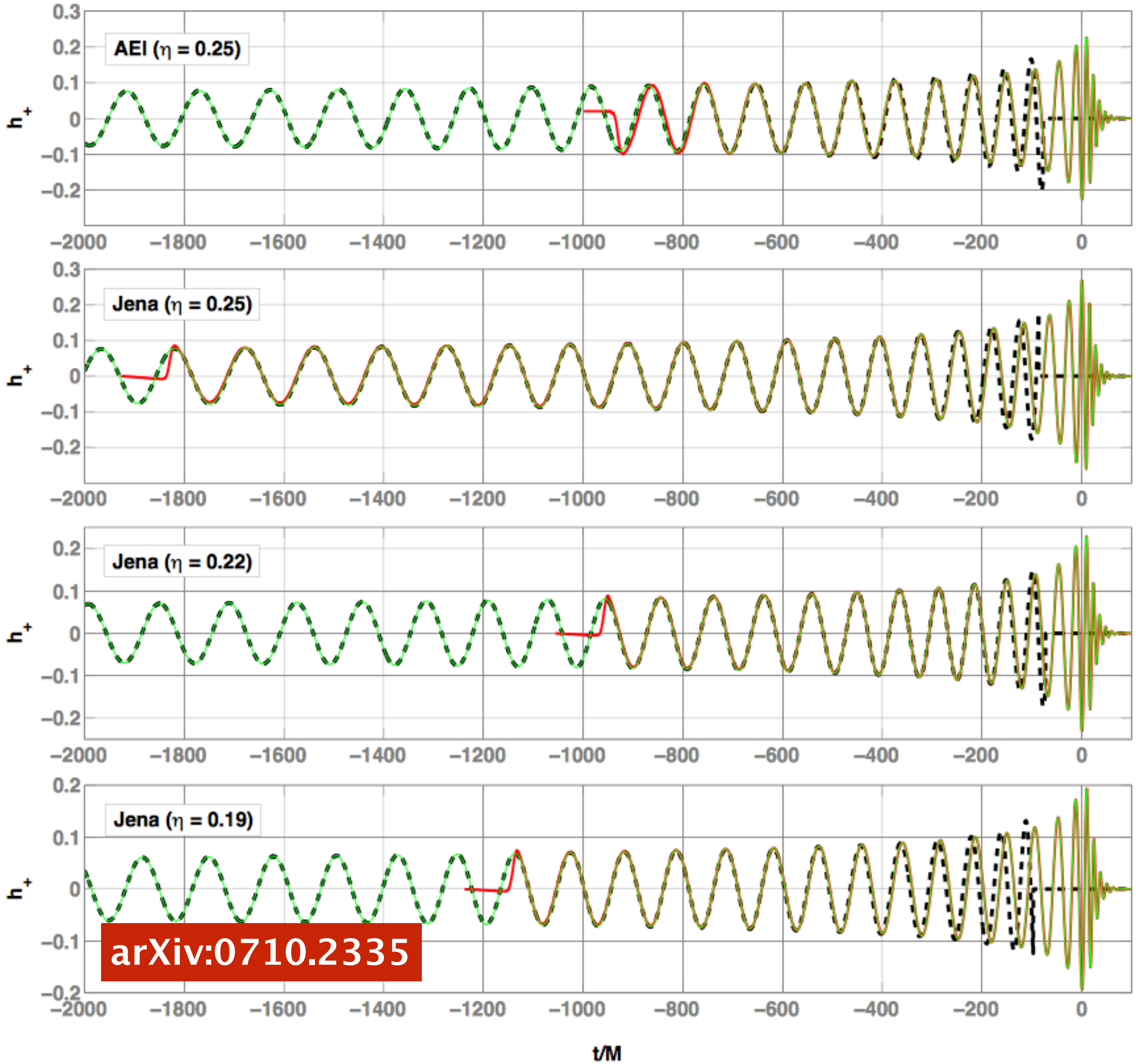
Phenomenological model : IMRPhenom

frequency-domain model !!

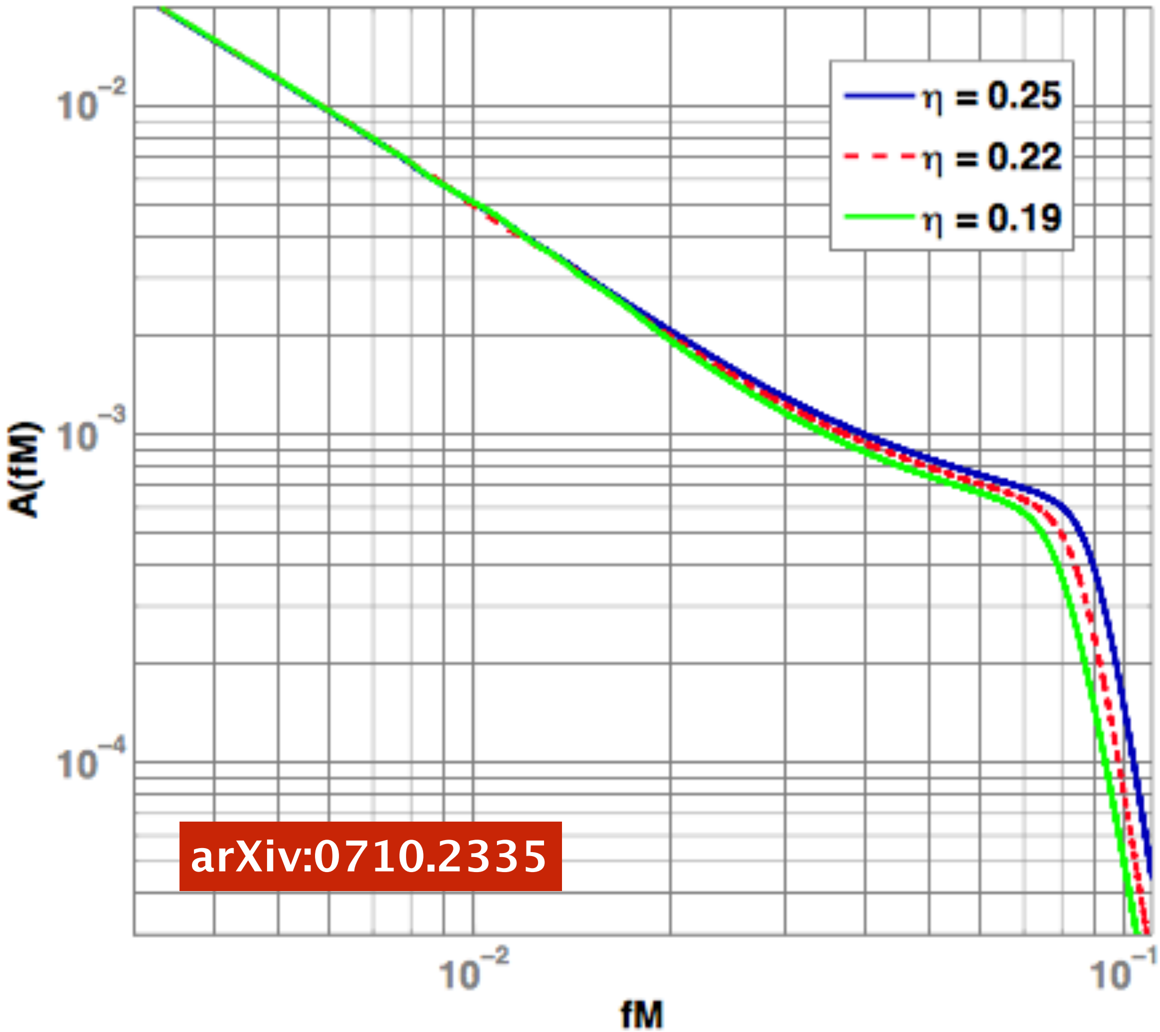


Hybrid = early inspiral:PN + late inspiral-M-R:NR

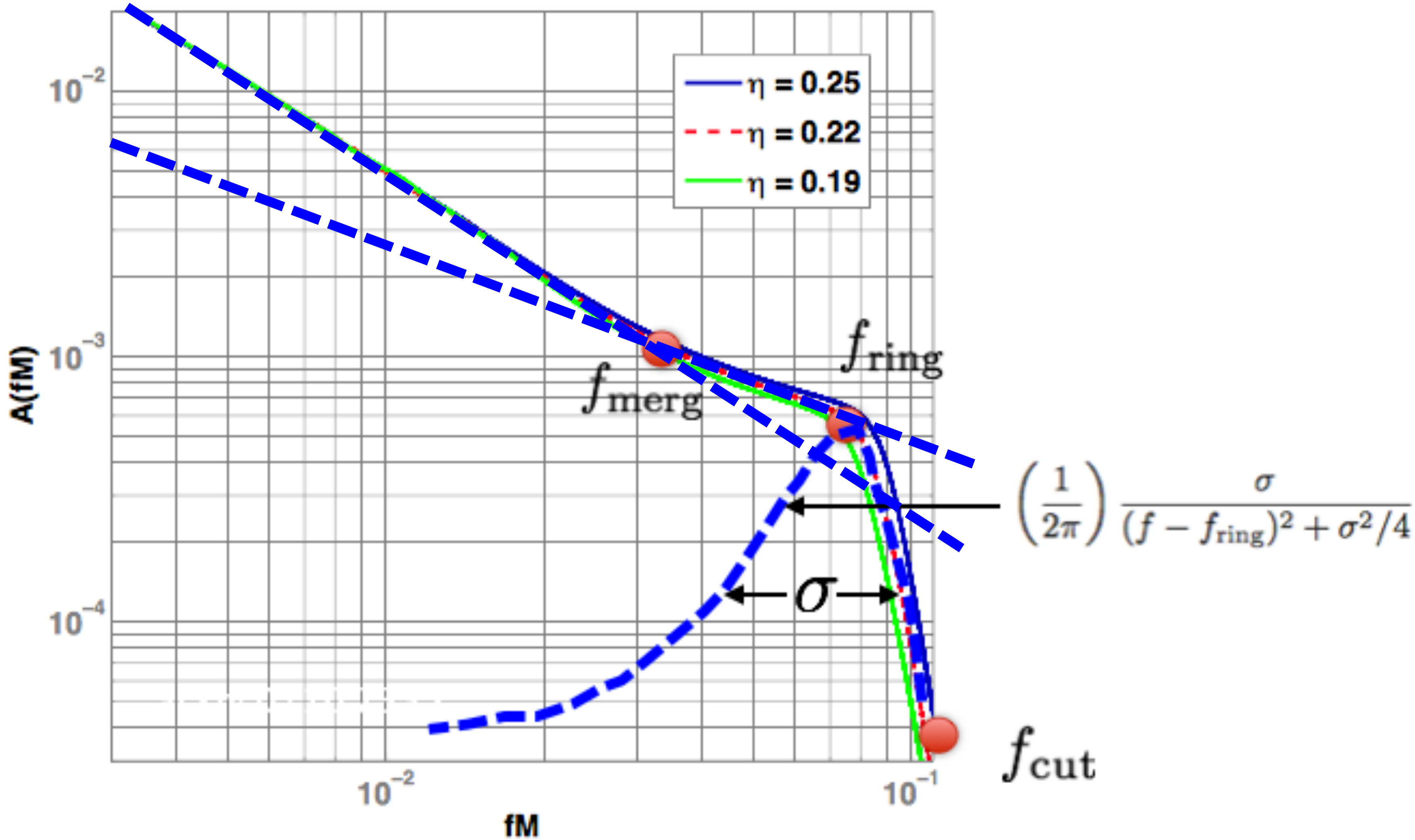
Constructing Hybrid waveforms



Fourier amplitudes of hybrid waveforms



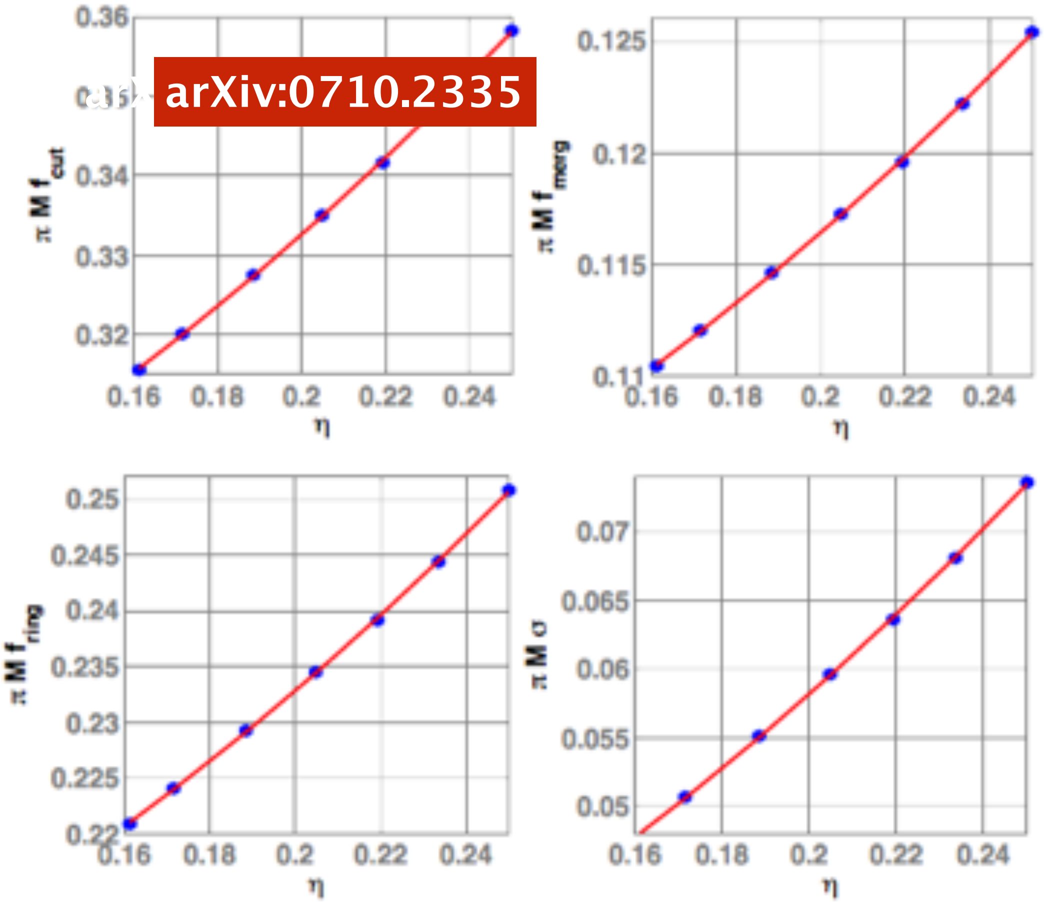
Fourier amplitudes of hybrid waveforms



$$A_{\rm eff}(f) \equiv C \begin{cases} (f/f_{\rm merg})^{-7/6} & \text{if } f < f_{\rm merg} \\ (f/f_{\rm merg})^{-2/3} & \text{if } f_{\rm merg} \leq f < f_{\rm ring} \\ w\,\mathcal{L}(f, f_{\rm ring}, \sigma) & \text{if } f_{\rm ring} \leq f < f_{\rm cut} \end{cases}$$

4 parameters

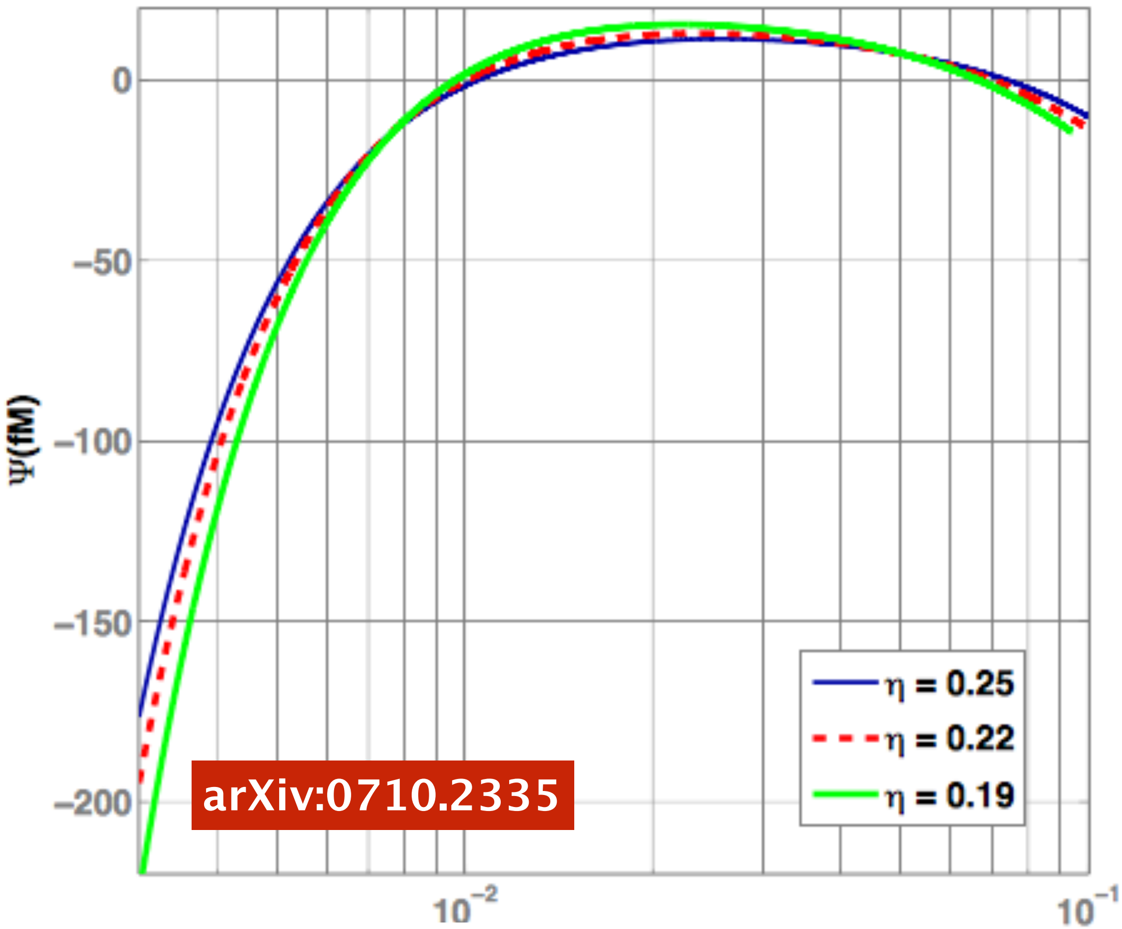
Best-match amplitude parameters



$$\alpha_{j\ \mathrm{int}} = \frac{a_j\,\eta^2 + b_j\,\eta + c_j}{\pi M}$$

Parameter	a_k	b_k	c_k
f_{merg}	2.9740×10^{-1}	4.4810×10^{-2}	9.5560×10^{-2}
f_{ring}	5.9411×10^{-1}	8.9794×10^{-2}	1.9111×10^{-1}
σ	5.0801×10^{-1}	7.7515×10^{-2}	2.2369×10^{-2}
f_{cut}	8.4845×10^{-1}	1.2848×10^{-1}	2.7299×10^{-1}

IMR phase

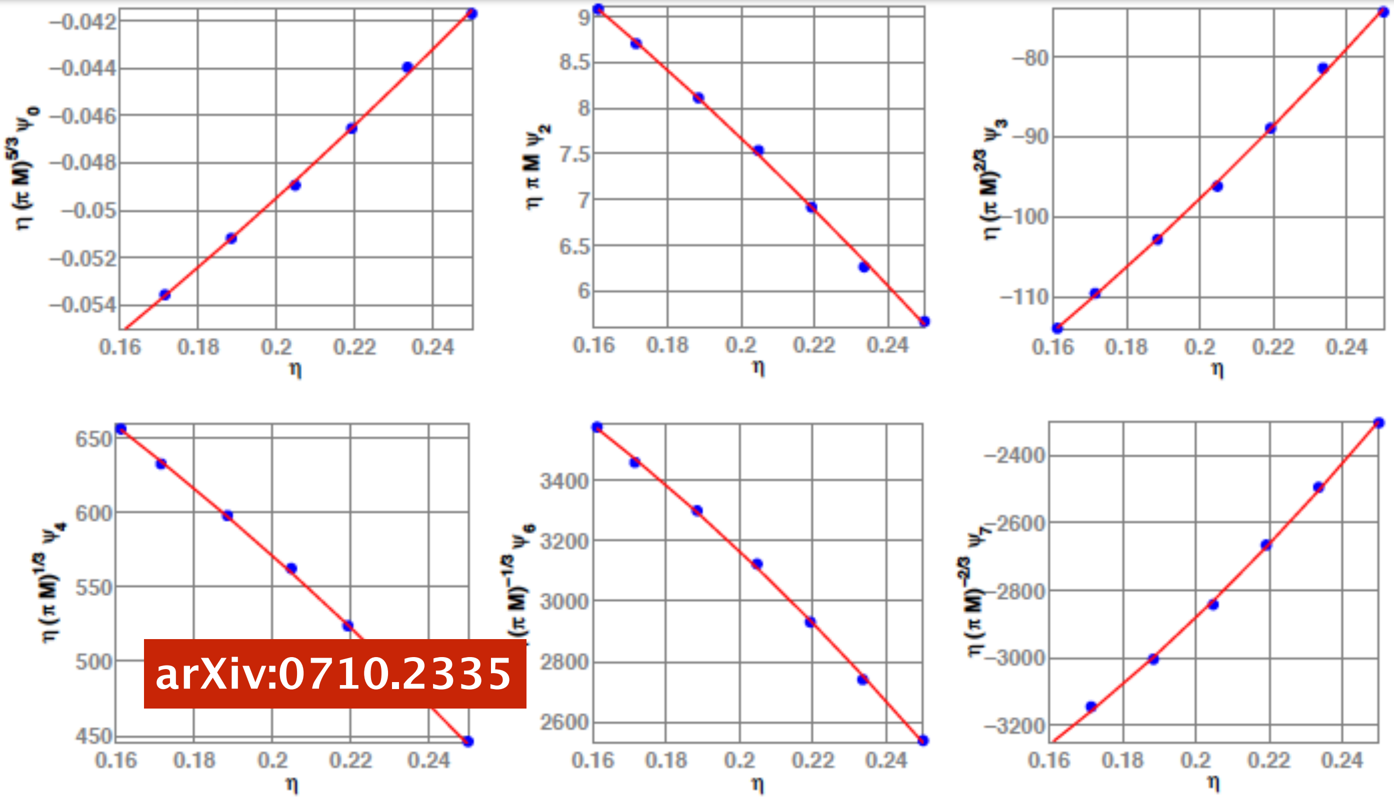


$$\Psi_{\rm eff}(f) = 2\pi f t_0 + \varphi_0 + \sum_{k=0}^{7} \psi_k \, f^{(k-5)/3} \,,$$

$$\boldsymbol{\psi} = \{\psi_0, \, \psi_2, \, \psi_3, \, \psi_4, \, \psi_6, \, \psi_7\}$$

6 parameters

Best-match phase parameters



$$\psi_{k\ \mathrm{int}} = \frac{x_k\,\eta^2 + y_k\,\eta + z_k}{\eta\,(\pi M)^{(5-k)/3}}$$

Parameter	x_k	y_k	z_k
ψ_0	1.7516×10^{-1}	7.9483×10^{-2}	-7.2390×10^{-2}
ψ_2	-5.1571×10^{1}	-1.7595×10^{1}	1.3253×10^{1}
ψ_3	6.5866×10^{2}	1.7803×10^{2}	-1.5972×10^{2}
ψ_4	-3.9031×10^{3}	-7.7493×10^{2}	8.8195×10^{2}
ψ_6	-2.4874×10^{4}	-1.4892×10^{3}	4.4588×10^{3}
ψ_7	2.5196×10^{4}	3.3970×10^{2}	-3.9573×10^{3}

Early inspiral amplitude : TaylorF2

TaylorF2

$$h(f) = \frac{M_c^{5/6}}{\pi^{2/3} D_{\text{eff}}} \sqrt{\frac{5}{24}} f^{-7/6} e^{i\Psi(f)},$$

IMRPhenom

$$u(f) \equiv \mathcal{A}_{\text{eff}}(f)\, e^{i\Psi_{\text{eff}}(f)}, \qquad \mathcal{A}_{\text{eff}}(f) \equiv \begin{cases} (f/f_{\text{merg}})^{-7/6} & \text{if} \quad f < f_{\text{merg}} \\ (f/f_{\text{merg}})^{-2/3} & \text{if} \quad f_{\text{merg}} \leqslant f < f_{\text{ring}} \\ w\mathcal{L}(f, f_{\text{ring}}, \sigma) & \text{if} \quad f_{\text{ring}} \leqslant f < f_{\text{cut}}. \end{cases}$$

$$\Psi_{\text{eff}}(f) = 2\pi f t_0 + \phi_0 + \psi_0 f^{-5/3} + \psi_2 f^{-1} + \psi_3 f^{-2/3} + \psi_4 f^{-1/3} + \psi_6 f^{1/3}$$

Model	PhenomA	PhenomB	PhenomC
Mass range $[M_\odot]$	$50 \leq M \leq 200$	$M \leq 400$	$M \leq 350$
Mass ratio range	$q \leq 4$	$q \leq 10$	$q \leq 4$
Detector	initial LIGO, Virgo, advanced LIGO	initial LIGO	advanced LIGO

BBH Models

Family	Short name	Full name	Precession	Multipoles $(\ell, \|m\|)$	Reference
EOBNR	EOBNR	SEOBNRv4_ROM	✗	(2, 2)	[57]
	EOBNR HM	SEOBNRv4HM_ROM	✗	(2, 2), (2,1), (3, 3), (4, 4), (5, 5)	[26,32]
	EOBNR P	SEOBNRv4P	✓	(2, 2), (2, 1)	[33,118,119]
	EOBNR PHM	SEOBNRv4PHM	✓	(2, 2), (2, 1), (3, 3), (4, 4), (5, 5)	[33,118,119]
Phenom	Phenom	IMRPhenomD	✗	(2, 2)	[120,121]
	Phenom HM	IMRPhenomHM	✗	(2, 2), (2, 1), (3, 3), (3, 2), (4, 4), (4, 3)	[22]
	Phenom P	IMRPhenomPv2/v3[a]	✓	(2, 2)	[23,122]
	Phenom PHM	IMRPhenomPv3HM	✓	(2, 2), (2, 1), (3, 3), (3, 2), (4, 4), (4, 3)	[24]

BH-NS Models

Full name (implemented in LAL) Short label (used in this work)	References Base model	 Corrections
SEOBNRv4_ROM_NRTidalv2 SEOBNR_T	[14–16]	[17, 18]
SEOBNRv4_ROM_NRTidalv2_NSBH [19] SEOBNR_NSBH	[14–16]	[17, 18, 20]
IMRPhenomPv2_NRTidalv2 IMRPhenomP_T	[21–23]	[17, 18]
IMRPhenomNSBH [24] IMRPhenom_NSBH	[25]	[18, 20]

TABLE I: Waveform models used in our analysis for NSBH systems.

Thanks !