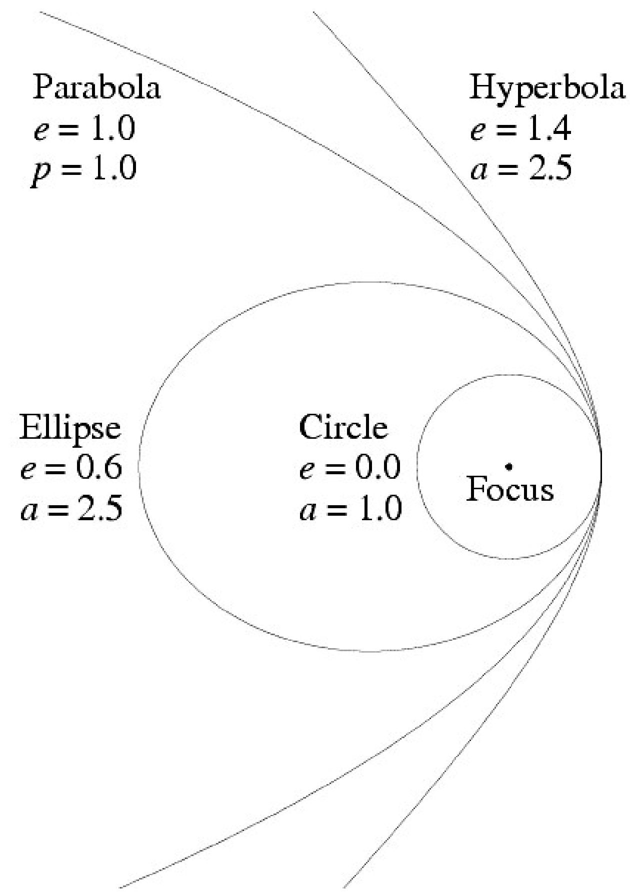
2024 Competition on Computational Astrophysics

2024.01.29

대전인재개발원

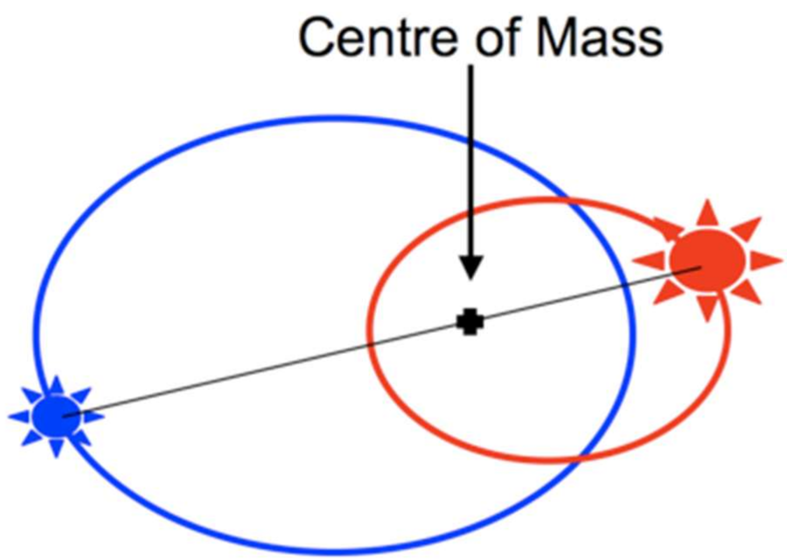
Orbits in a gravitational potential

- The shape depends on the sign of the total energy, $E_{tot} = E_{kin} + E_{pot}$:
  - $E_{tot} < 0$ – Ellipse
  - $E_{tot} = 0$ – Parabola
  - $E_{tot} > 0$ – Hyperbola
- For the elliptical orbits, the eccentricity depends on the angular momentum: circular orbits have the maximum angular momentum for a given energy.



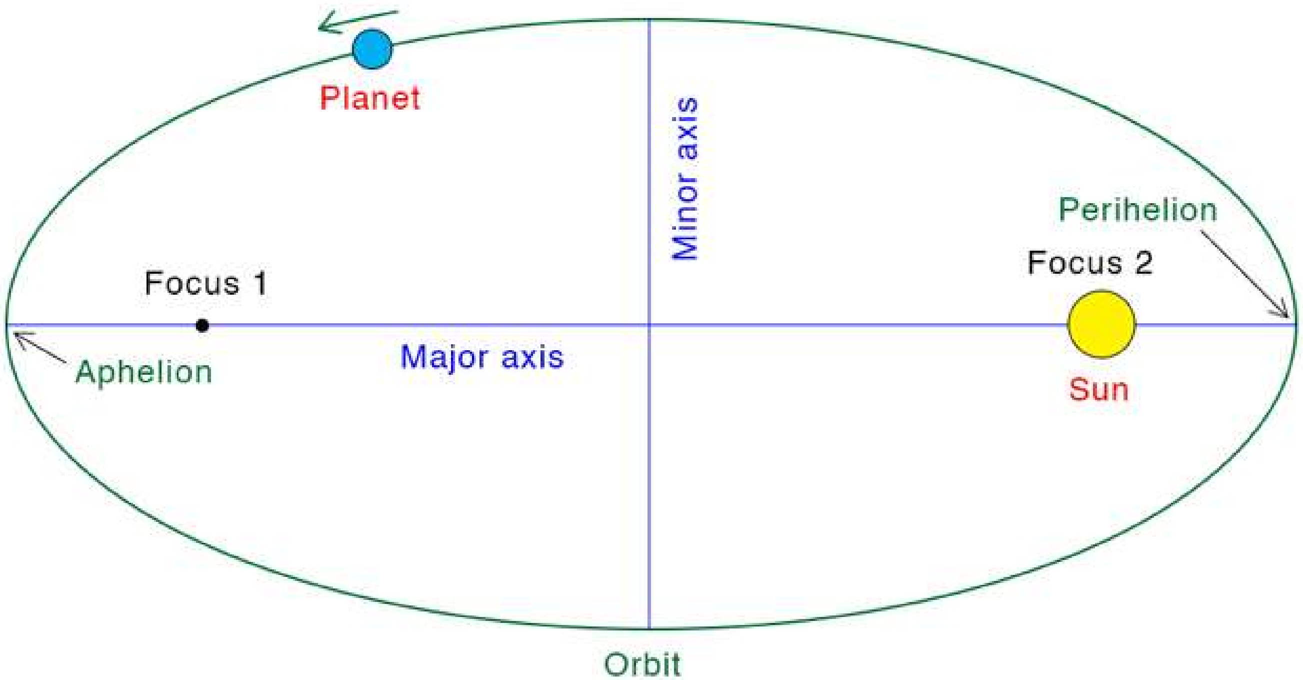
Binary motion

- Two masses orbiting each other by mutual gravity
- Two masses are gravitationally bound ($E_{tot} < 0$)



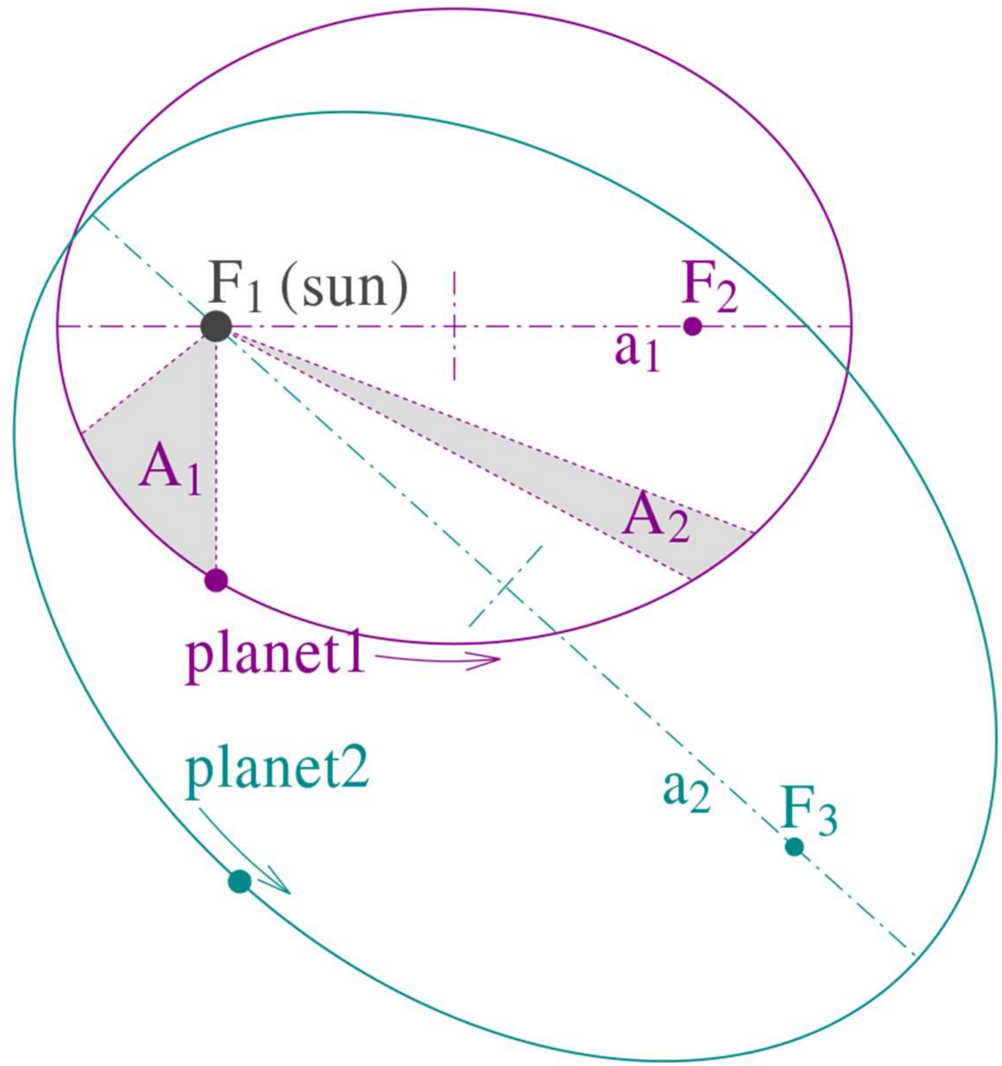
Kepler's first law

- Planets move in elliptical orbits with the Sun at one focus of the ellipse



Kepler's second law

- A line from the Sun to the planet sweeps out equal areas in equal times, *i.e.* planets don't move at constant speed.



Kepler's third law

- (Period of orbit)^2 proportional to (semi-major axis of orbit)^3.
- In symbolic form: $P^2 \propto a^3$.

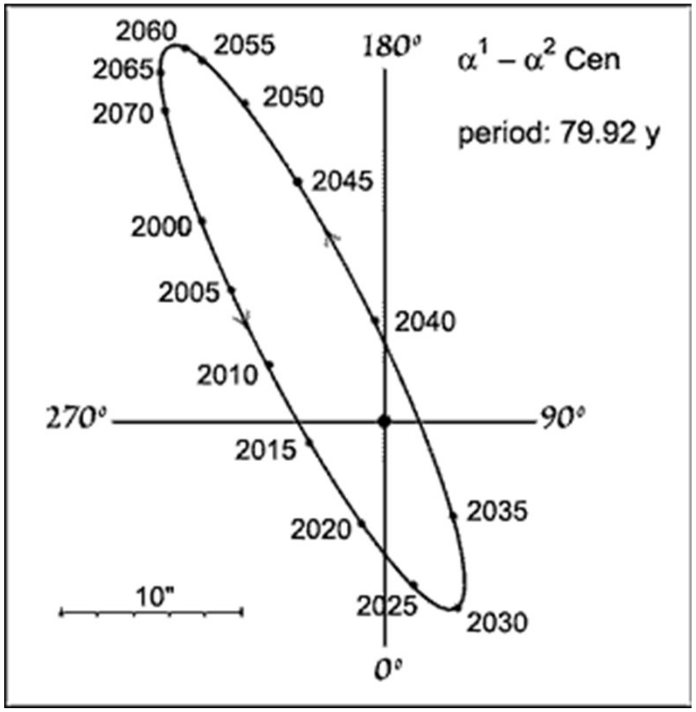
- If two quantities are proportional, we can insert a proportionality constant, k, which depends on the units adopted for P and a, and get an equation:
- $P^2 = ka^3$.
- For the solar system, k=1 with semi-major axis in AU, period in year.

Binary stars in Astronomy

- Visual binary
  - Close enough from us so we can track their motion in a long time
  - Ex) alpha Centauri A and B

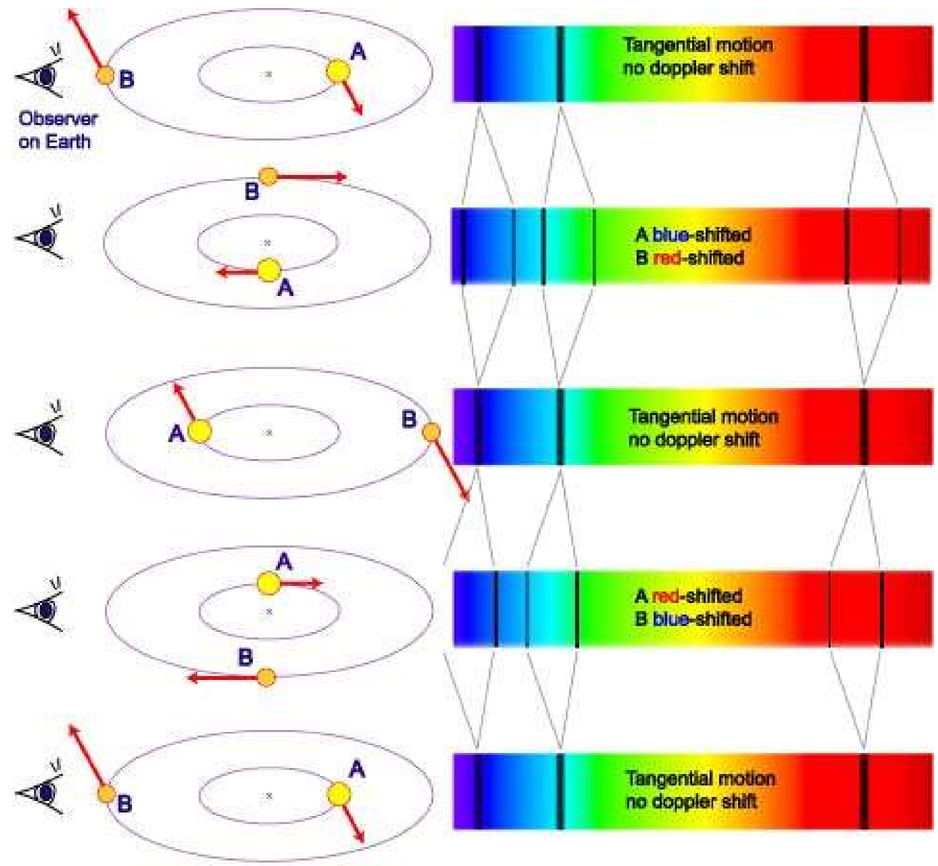
Mass A (M1)	1.0788 $M_\odot$
Mass B (M2)	0.9092 $M_\odot$

Period (P)	79.762 year
Semi-major axis (a)	22.765 AU
Eccentricity (e)	0.5194



Binary stars in Astronomy

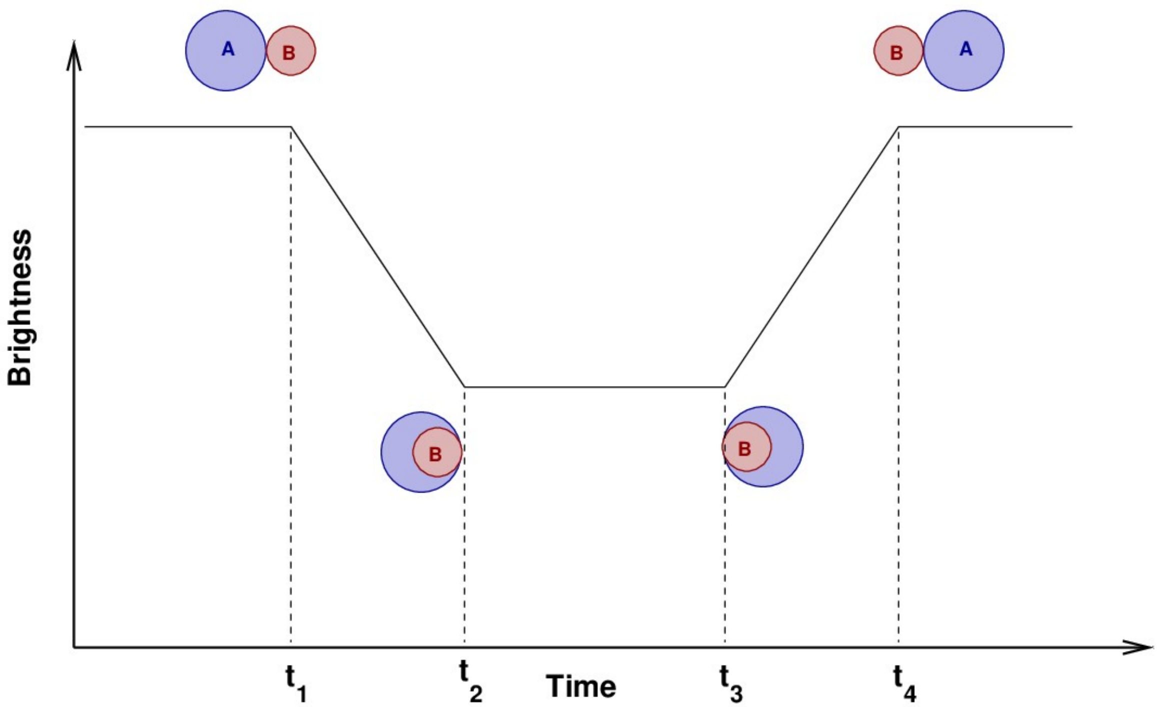
- Spectroscopic binaries
  - Doppler effect during binary motion



A Spectroscopic Binary System
High-mass star A and lower-mass B orbit around a common centre of mass. The observed combined spectrum shows periodic splitting and shifting of spectral lines. The amount of shift is a function of the alignment of the system relative to us and the orbital speed of the stars.

Binary stars in Astronomy

- Eclipsing binaries
  - The orbital plane is aligned in a line of sight



Binary stars in Astronomy

- Eclipsing binaries
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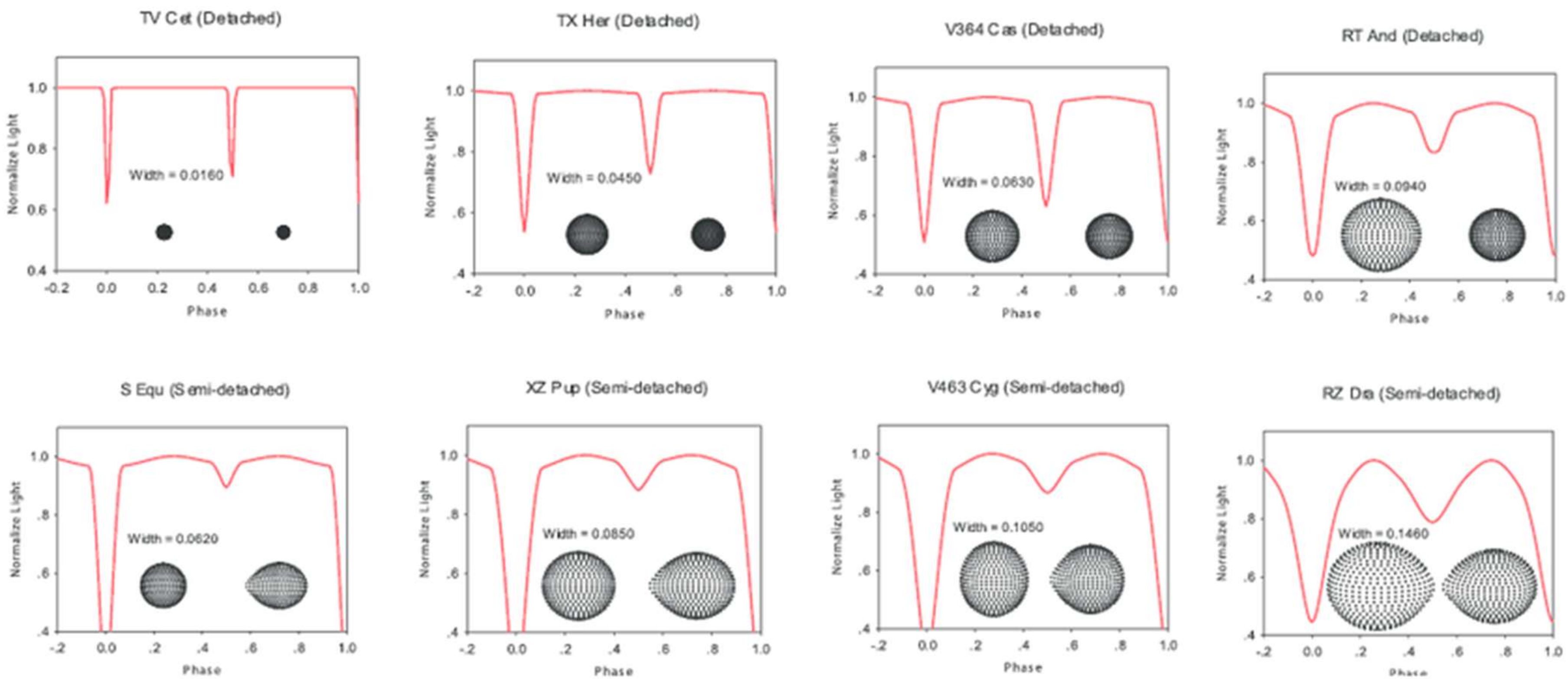
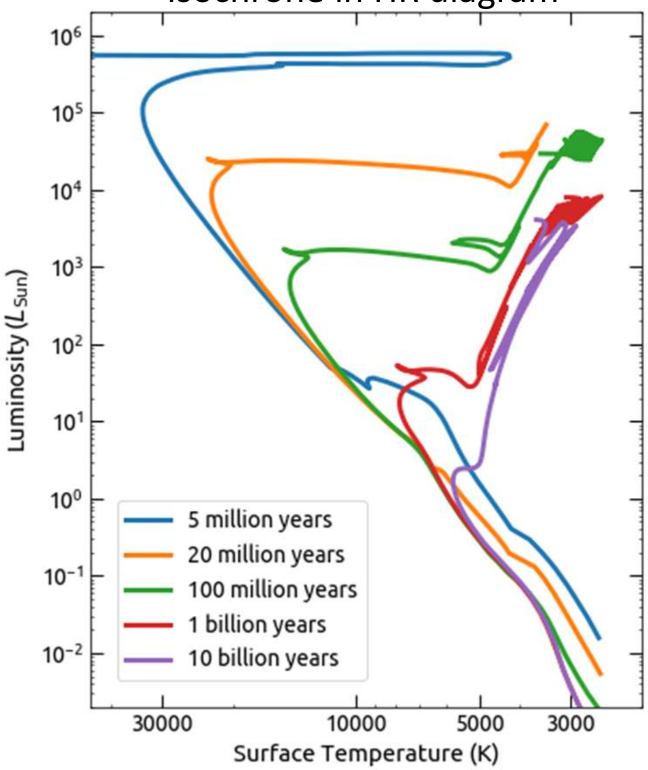
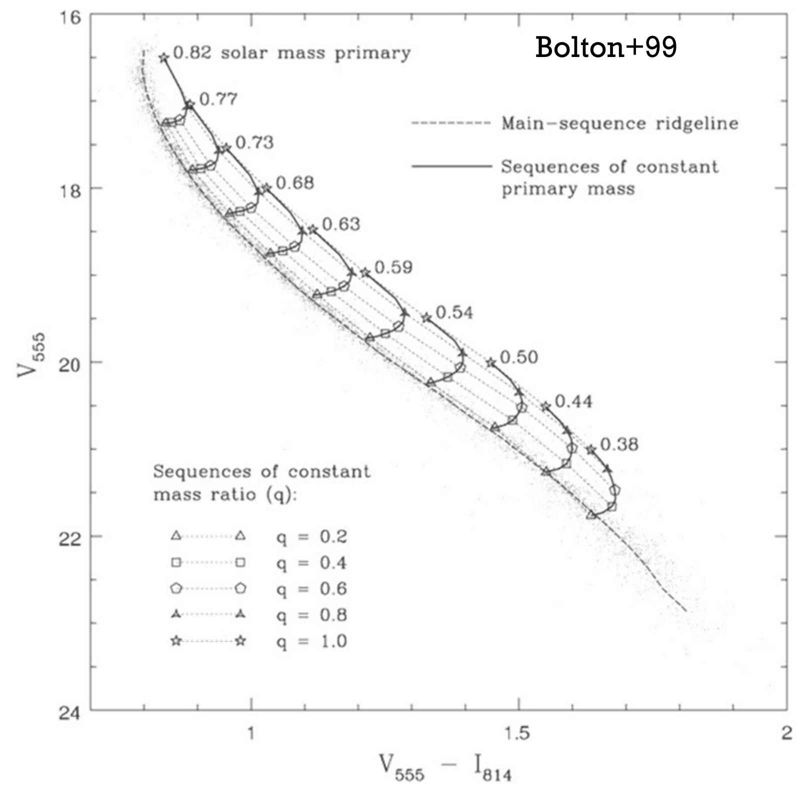


Binary stars in Astronomy

- Searching unresolved binaries in group of stars

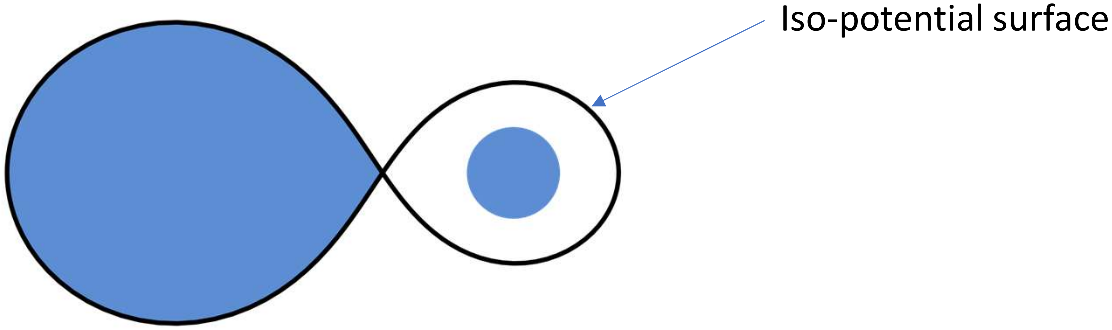
Isochrone in HR diagram



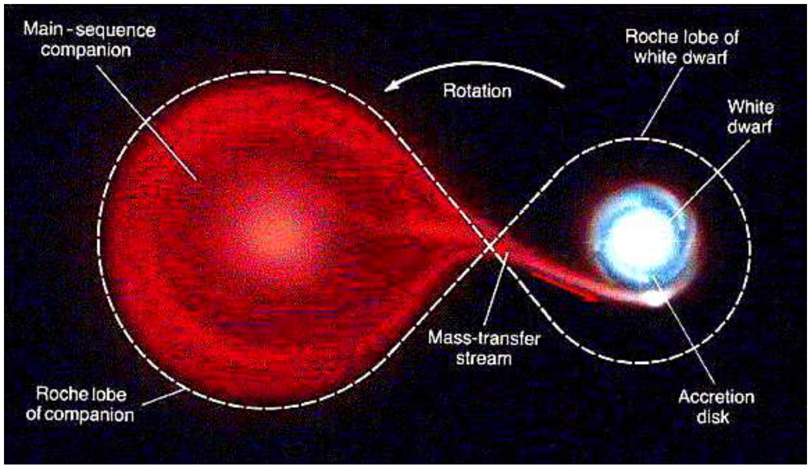


Binary stars in Astronomy

- Roche-lobe overflow



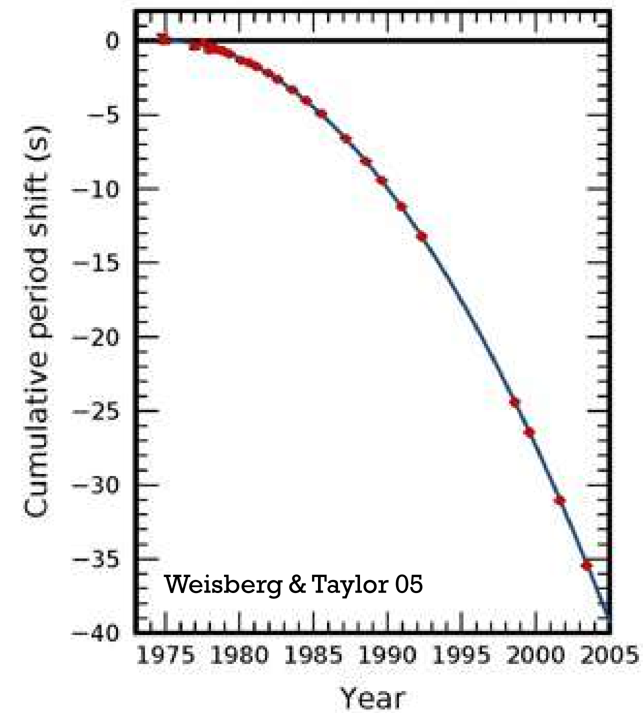
- Mass transfer and X-ray emission from the accretion disk

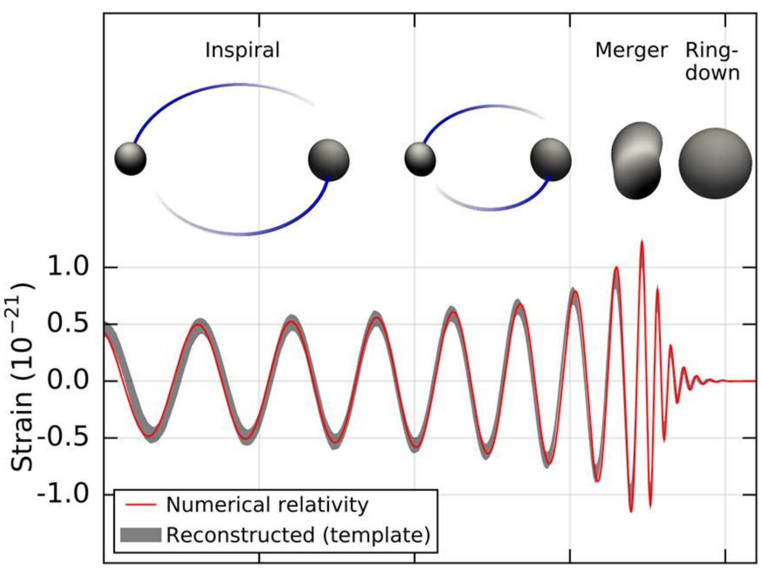


Gravitational waves

- Propagation of ripples in space-time
  - Predicted by general relativity (Einstein 1916)
  - Indirect evidence PSR 1913+16

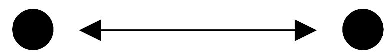
    (Hulse & Taylor 74; Weisberg & Taylor 05)





- Compact binary coalescences
  - The first detection – GW150914 (Abbott et al. 2016)
  - BH-BH, NS-NS, BH-NS binary mergers detected so far

Two body problem



- When N is 2, you have a ***2-Body Problem***: exactly 2 particles, each exerting a force that acts on the other.
- The relationship between the 2 particles can be expressed as a differential equation that can be solved analytically, producing a closed-form solution.
- So, given the particles' initial positions and velocities, you can trivially calculate their positions and velocities at any later time.
- Simplified as a reduced body and center of mass

Force integration schemes

1. Euler's method

- Simplest integrator
  - 1^{st} order time complexity

- given position and velocity (x_0, v_0)

$a_0 = - m/r^3 * x_0$

$x_1 = x_0 + v_0*dt + a_0*dt^2/2$

$v_1 = v_0 + a_0*dt$

Force integration schemes

2. Second-order Euler's method

- Add time derivative of acceleration

$a_0 = - m/r^3 * x_0$

$a'_0 = -m/r^3 * v_0 + 3mx_0 \cdot v_0/r^5 * x_0$

$x_1 = x_0 + v_0 * dt + a_0 * dt^2/2 + a'_0 * dt^3/6$

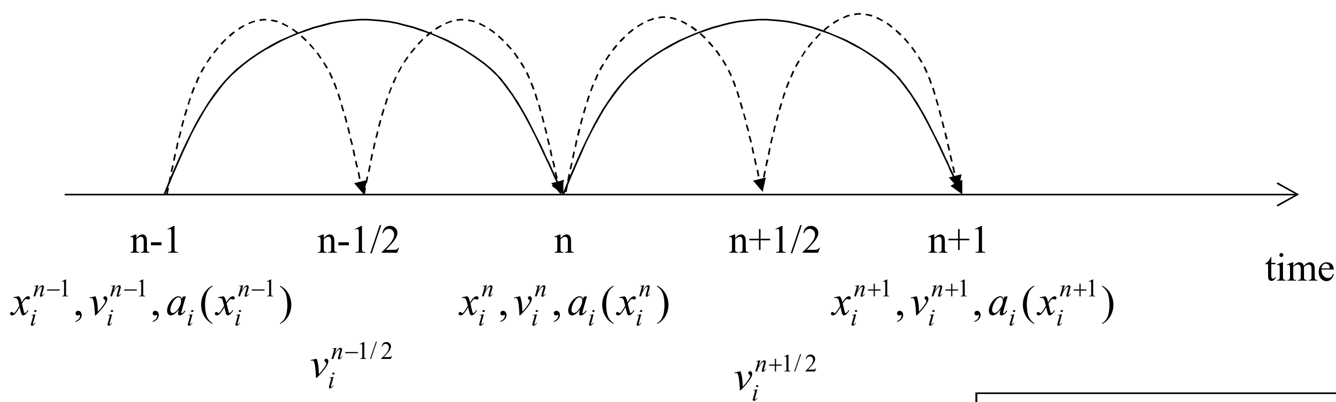
$v_1 = v_0 + a_0 * dt + a'_0 * dt^2/2$

- Euler's method is not time symmetrized.

Force integration schemes

3. Leapfrog method

- Kick-Drift-Kick (KDK algorithm)
    - slightly different with 1st order Euler's method (time symmetry)



$$
\begin{aligned}
v_i^{n+1/2} &= v_i^n + a_i(x_i^n)\Delta t/2 \\
x_i^{n+1} &= x_i^n + v_i^{n+1/2}\Delta t \\
v_i^{n+1} &= v_i^{n+1/2} + a_i(x_i^{n+1})\Delta t/2
\end{aligned}
$$

4^{th} order Predictor & Corrector method

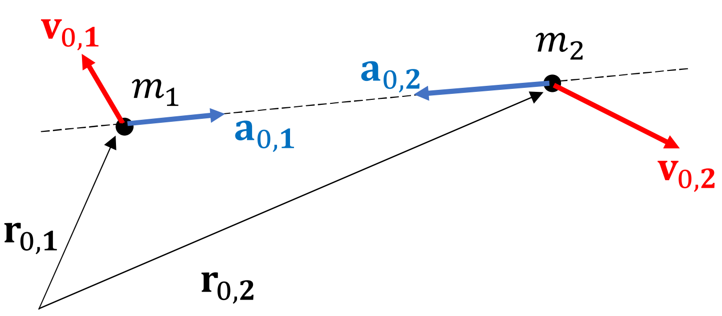
- Force and 1st derivative at current position (index "0")

$$\mathbf{a}_{0,i} = -\sum_{i \neq j} G m_j \frac{\mathbf{R}}{r^3}$$

$$\dot{\mathbf{a}}_{0,i} = -\sum_{i \neq j} G m_j \left[\frac{\mathbf{V}}{r^3} - \frac{3\dot{r}\mathbf{R}}{r^4}\right]$$

$$\mathbf{R} = (\mathbf{r}_{0,i} - \mathbf{r}_{0,j}), \mathbf{V} = (\mathbf{v}_{0,i} - \mathbf{v}_{0,j})$$

$$\dot{r} = (\mathbf{R} \cdot \mathbf{V})/r$$

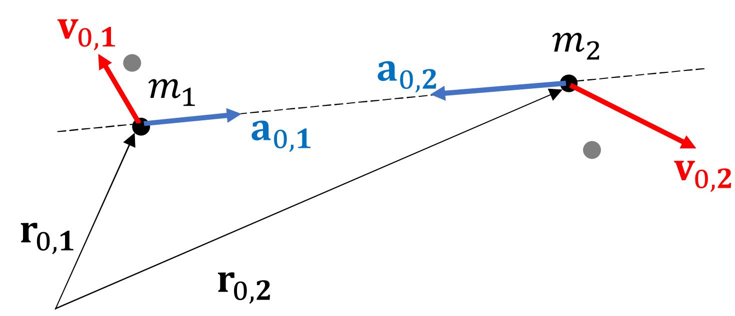


4th order Predictor & Corrector method

- Predict next position and velocity after time Δt (index "p") by Taylor series

$$\mathbf{r}_{p,i} = \mathbf{r}_{0,i} + \mathbf{v}_{0,i}\Delta t + \mathbf{a}_{0,i}\frac{\Delta t^2}{2} + \dot{\mathbf{a}}_{0,i}\frac{\Delta t^3}{6}$$

$$\mathbf{v}_{p,i} = \mathbf{v}_{0,i} + \mathbf{a}_{0,i}\Delta t + \dot{\mathbf{a}}_{0,i}\frac{\Delta t^2}{2}$$



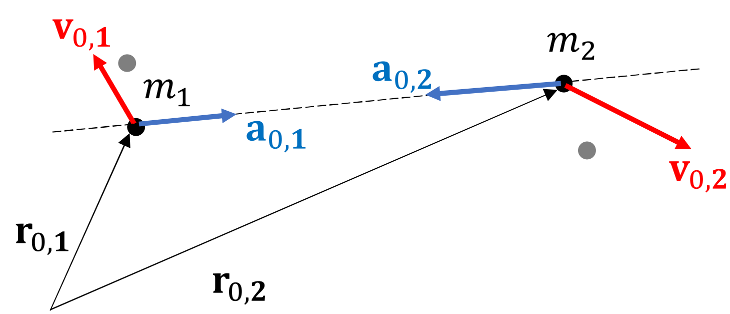
4^{th} order Predictor & Corrector method

- Taylor series expansion for the force and 1^{st} derivative at predicted position

$$\mathbf{a}_{p,i} = \mathbf{a}_{0,i} + \dot{\mathbf{a}}_{0,i}\Delta t + \cancel{\ddot{\mathbf{a}}_{0,i}\frac{\Delta t^2}{2}} + \dddot{\mathbf{a}}_{0,i}\frac{\Delta t^3}{6}$$

$$\dot{\mathbf{a}}_{p,i} = \dot{\mathbf{a}}_{0,i} + \cancel{\ddot{\mathbf{a}}_{0,i}\Delta t} + \dddot{\mathbf{a}}_{0,i}\frac{\Delta t^2}{2}$$

Multiplying $\Delta t/2$



- Subtract two equations

$$\dddot{\mathbf{a}}_{0,i} = 12\frac{\mathbf{a}_{0,i} - \mathbf{a}_{p,i}}{\Delta t^3} + 6\frac{\dot{\mathbf{a}}_{0,i} + \dot{\mathbf{a}}_{p,i}}{\Delta t^2}$$

- Substitute 3^{rd} derivative in equations

$$\ddot{\mathbf{a}}_{0,i} = -6\frac{\mathbf{a}_{0,i} - \mathbf{a}_{p,i}}{\Delta t^2} - 2\frac{2\dot{\mathbf{a}}_{0,i} + \dot{\mathbf{a}}_{p,i}}{\Delta t}$$

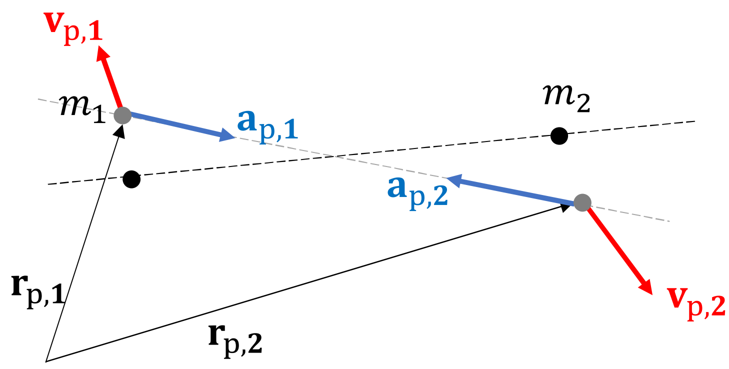
4^{th} order Predictor & Corrector method

- Calculate force and 1st derivative at predicted position

$$\mathbf{a}_{p,i} = -\sum_{i \neq j} G m_j \frac{\mathbf{R}}{r^3}$$

$$\dot{\mathbf{a}}_{p,i} = -\sum_{i \neq j} G m_j \left[\frac{\mathbf{V}}{r^3} - \frac{3\dot{r}\mathbf{R}}{r^4}\right]$$

$$\mathbf{R} = (\mathbf{r}_{p,i} - \mathbf{r}_{p,j}), \mathbf{V} = (\mathbf{v}_{p,i} - \mathbf{v}_{p,j})$$

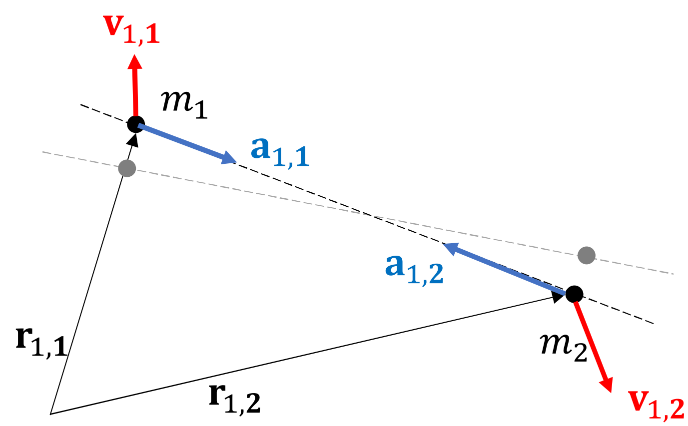


4th order Predictor & Corrector method

- Correct position and velocity up to 3rd derivative using Taylor series (index "1")

$$\mathbf{r}_{1,i}(t) = \mathbf{r}_{p,i} + \ddot{\mathbf{a}}_{0,i}\frac{\Delta t^4}{24} + \dddot{\mathbf{a}}_{0,i}\frac{\Delta t^5}{120}$$

$$\mathbf{v}_{1,i}(t) = \mathbf{v}_{p,i} + \ddot{\mathbf{a}}_{0,i}\frac{\Delta t^3}{6} + \dddot{\mathbf{a}}_{0,i}\frac{\Delta t^4}{24}$$

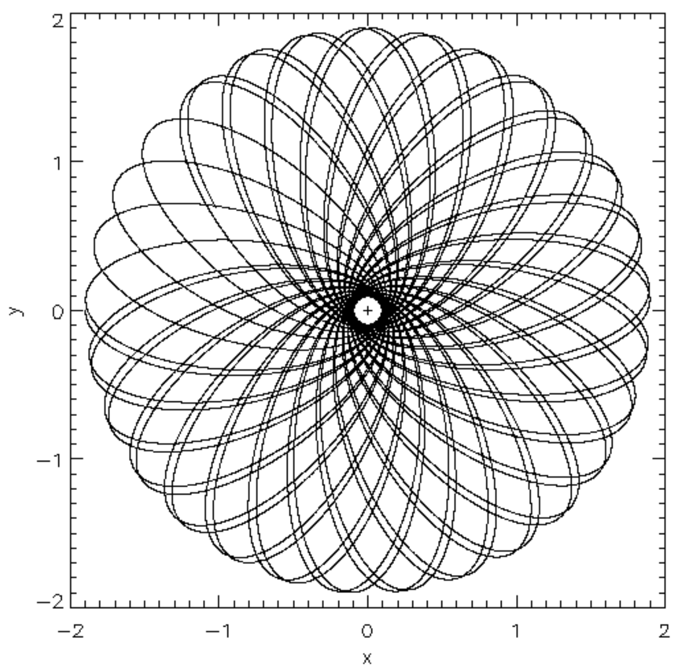


Post Newtonian Approximation

- Series expansion of relativistic equation of motion

  - Newtonian
  - +1 PN ($1/c^2$)
    - *Relativistic precession*
  - +1.5 PN ($1/c^3$)
    - *Spin-orbit coupling*
  - +2 PN ($1/c^4$)
    - *Spin-spin coupling*
    - *Higher order precession*
  - +2.5 PN ($1/c^5$)
    - *Gravitational radiation*

e=0.9, a=10^{-4}AU



Force integration with PN effect

- Force = Newtonian force + PN force

$$\mathbf{a} = \mathbf{a}_n + \mathbf{a}_{pn}, \qquad \mathbf{a}_{pn} = \frac{m_j}{r^2}\left(A\frac{\mathbf{R}}{r} + B\mathbf{V}\right)$$

$$\mathbf{R} = (\mathbf{r}_{0,i} - \mathbf{r}_{0,j}), \mathbf{V} = (\mathbf{v}_{0,i} - \mathbf{v}_{0,j})$$

$$\dot{\mathbf{a}} = \dot{\mathbf{a}}_n + \dot{\mathbf{a}}_{pn}$$

$$r = \left|\mathbf{r}_{0,i} - \mathbf{r}_{0,j}\right|, v = |\mathbf{v}_{0,i} - \mathbf{v}_{0,j}|$$

- PN coefficients A, B for 2.5 PN

$$A = \frac{8}{5}\eta\frac{M}{r}\dot{r}\left(\frac{17}{3}\frac{M}{r} + 3v^2\right)$$

$$B = -\frac{8}{5}\eta\frac{M}{r}\left(3\frac{M}{r} + v^2\right)$$

$$M = m_1 + m_2, \eta = \frac{m_1 m_2}{M^2}$$

Force integration with PN effect

- Derive $\dot{\mathbf{a}}_{pn}$ by yourself from $\mathbf{a}_{pn} = \frac{m_j}{r^2}\left(A\frac{\mathbf{R}}{r} + B\mathbf{V}\right)$

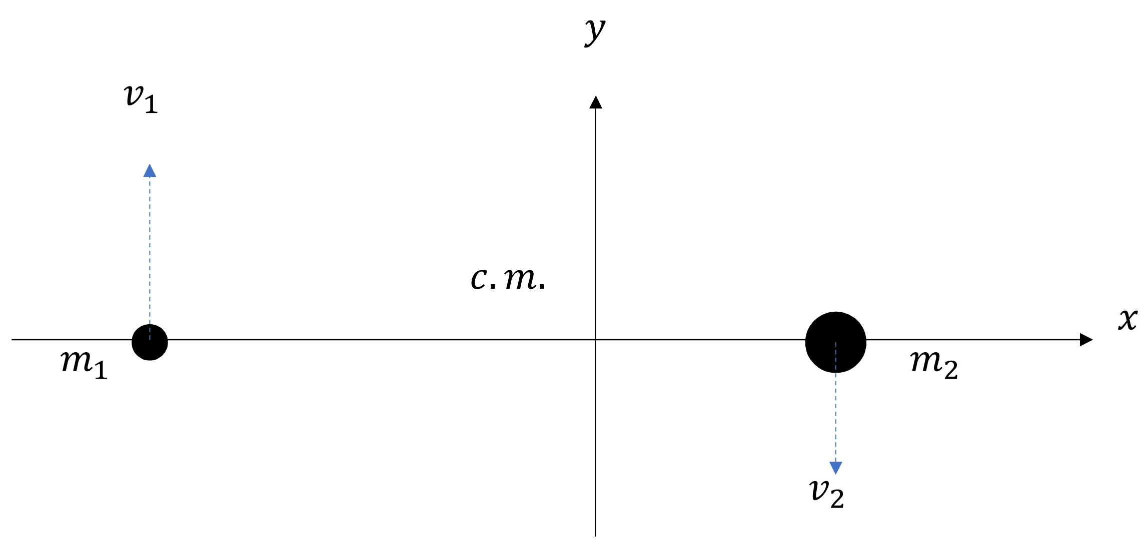
- Time dependent variables: $A, B, \mathbf{R}, \mathbf{V}, r$
  - $\dot{r} = (\mathbf{R} \cdot \mathbf{V})/r$
  - $\ddot{r} = \left(v^2 + \mathbf{R} \cdot \mathbf{a} - \dot{r}^2\right)/r$
  - $\frac{d\mathbf{R}}{dt} = \mathbf{V}$
  - $\frac{d\mathbf{V}}{dt} = \mathbf{a}$
  - $\dot{A} =$
  - $\dot{B} = \frac{8}{5}\eta\frac{M}{r^2}\dot{r}\left(3\frac{M}{r} + v^2\right) - \frac{8}{5}\eta\frac{M}{r}\left(-3\frac{M}{r^2}\dot{r} + 2\mathbf{V} \cdot \mathbf{a}\right)$

Initial conditions

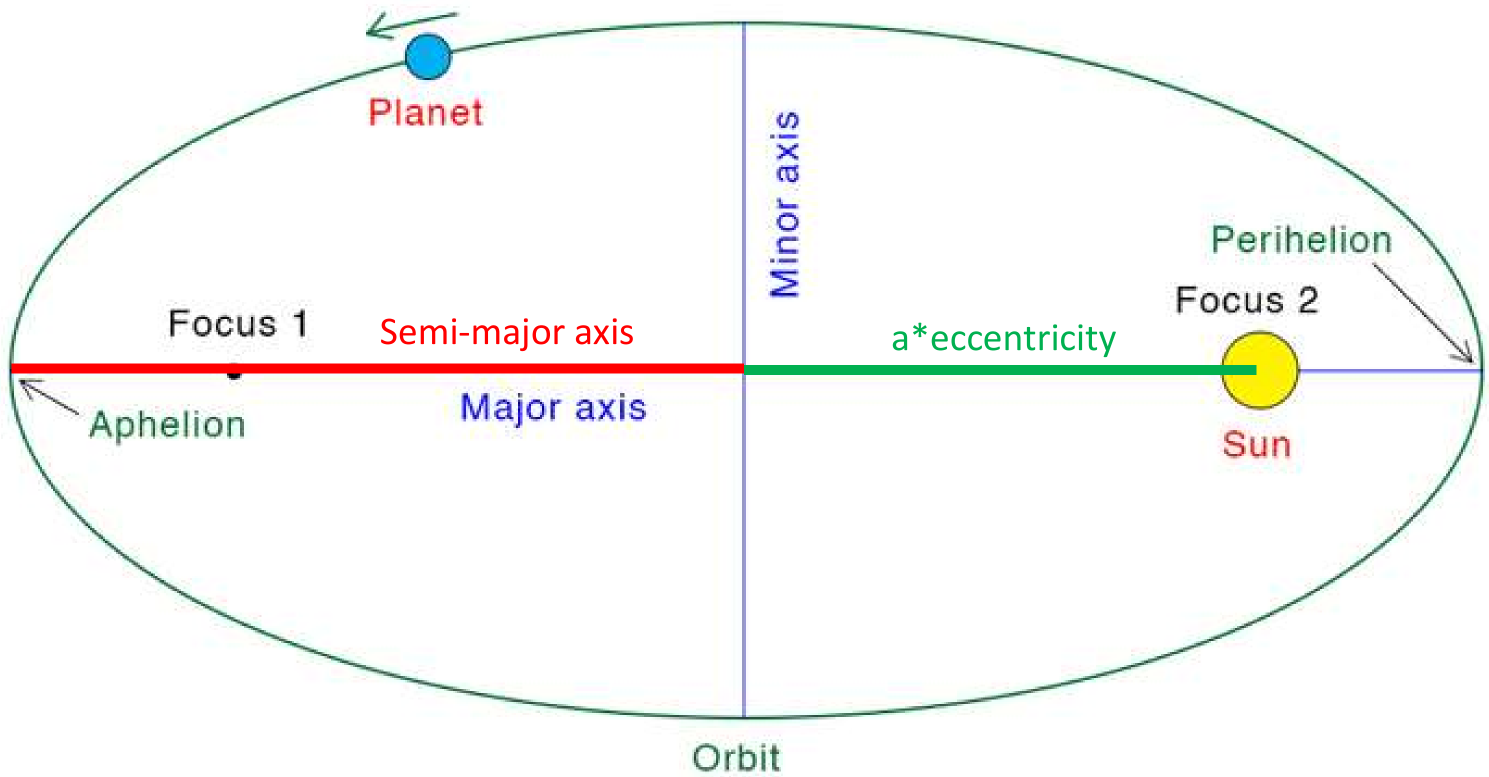
- 2 dimensional position and velocity $(x_1, y_1), (x_2, y_2), (vx_1, vy_1), (vx_2, vy_2)$
- Center of mass coordinate
- Derive initial conditions from semi-major and eccentricity
- Initial position @apocenter

- G=1, $\mathrm{m}/\mathrm{M}_{\odot}$, r/AU
- Use constants below

$\pi = 3.14159$
$\mathrm{G} = 6.67430 \times 10^{-8} cm^3 g^{-1} s^{-2}$
$\mathrm{c} = 2.99792 \times 10^{10} c\, m/s$
$1\mathrm{AU} = 1.49598 \times 10^{13} cm$
$\mathrm{M}_{\odot} = 1.98854 \times 10^{33} g$



Orbital parameters



Problem Set 1 – Circular orbit

- P1.1: get velocity scale and time scale when G=1, $\mathrm{m}/\mathrm{M}_{\odot}$, r/AU (10pts)

- P1.2: Integrate orbit with $m_1 = 1\mathrm{M}_{\odot}, m_2 = 1\mathrm{M}_{\odot}, a = 2\mathrm{AU}$ for 100 orbits ($N_{STEPS} \leq 20000$) (10pts)

- P1.3: Integrate orbit with $m_1 = 1\mathrm{M}_{\odot}, m_2 = 1\mathrm{M}_{\odot}, a = 2\mathrm{AU}$ for 100 orbits ($N_{STEPS} \leq 40000$) (10pts)

- P1.4 Integrate orbit with $m_1 = 10\mathrm{M}_{\odot}, m_2 = 2\mathrm{M}_{\odot}, a = 100\mathrm{AU}$ for 100 orbits ($N_{STEPS} \leq 20000$) (10pts)

Problem Set 2 – elliptical orbit

- P2.1: Integrate orbit with $m_1 = 1\mathrm{M}_\odot, m_2 = 1\mathrm{M}_\odot, a = 2\mathrm{AU}, \mathrm{e} = 0.5$ for 100 orbits ($N_{STEPS} \leq 40000$) (15pts)

- P2.2: Integrate the orbit of Halley comet for 50 orbits ($N_{STEPS} \leq 40000$) (15pts)
  - $a = 17.8\mathrm{AU}, \mathrm{e} = 0.967, m_1 = 1\mathrm{M}_\odot, m_2 = 2.22 \times 10^{14}\mathrm{g}$

- Design time-step by yourself
  - more steps near pericenter, less steps near apocenter

Problem Set 3 – PN calculation

- P3.1: Integrate orbit with $m_1 = 10\mathrm{M}_\odot, m_2 = 10\mathrm{M}_\odot, a = 2 \times 10^{-5}\mathrm{AU}, \mathrm{e} = 0$ until $r < r_{Sch}$ ($N_{STEPS} \leq 1000000$) (20pts)
  - $r_{Sch} = 2G(m_1 + m_2)/c^2$

- P3.2: Integrate orbit with $m_1 = 10\mathrm{M}_\odot, m_2 = 2\mathrm{M}_\odot, a = 2 \times 10^{-5}\mathrm{AU}, \mathrm{e} = 0.8$ until $r < r_{Sch}$ ($N_{STEPS} \leq 1000000$) (20pts)