Numerical Analysis on Ordinary Differential Equations

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2024.01.30 @ KT HRD

Last Modification: 2024.01.30 AM 8:52

2024 School on Numerical Relativity and Gravitational Waves

Contents

- Numerical ODE Basics
- Predictor-Corrector Method
- Application to Newtonian Mechanics

Numerical ODE Basics

ODE as 1st-order Coupled Equations

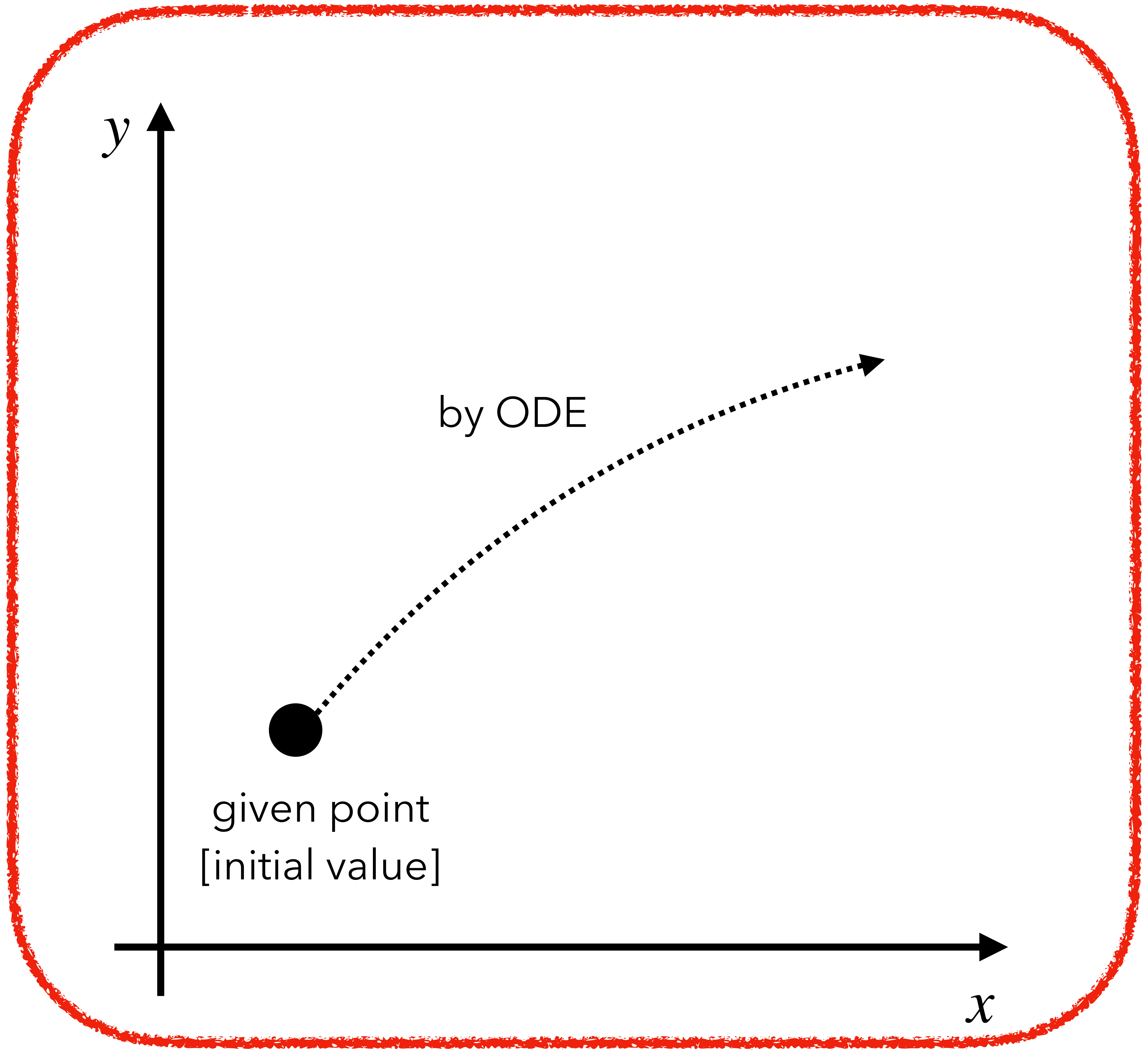
- **Newton's Law**
- $\mathbf{F} = m\ddot{\mathbf{r}}$
- by introducing $\mathbf{v}$
- $\frac{d}{dt}\begin{bmatrix}\mathbf{r}\\ \mathbf{v}\end{bmatrix} = \begin{bmatrix}\mathbf{v}\\ \mathbf{F}/m\end{bmatrix}$

- **Einstein's Equations**
- $G_{ab} = 8\pi T_{ab}$
- by introducing K_{ab}
- $\mathscr{L}_n \begin{bmatrix}\gamma_{ab}\\ K_{ab}\end{bmatrix} = \begin{bmatrix}-2K_{ab}\\ R_{ab} + \cdots\end{bmatrix}$

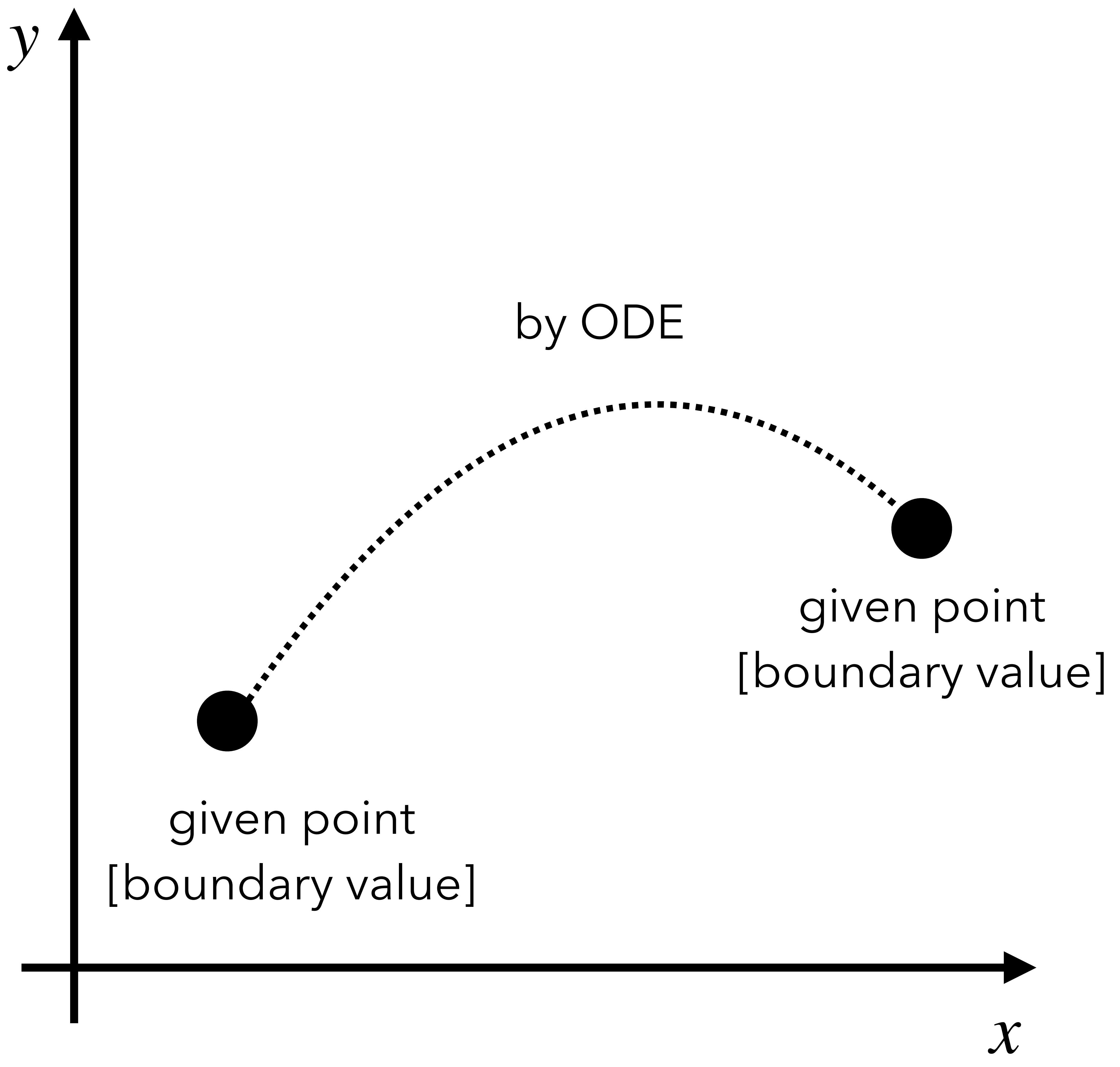
Coupled Ordinary Differential Equations

- $$\frac{d}{dx}\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$
- where
- $y_i(x)$: functions of x for $i = 1, \cdots, N$
- $f_i\left(x, y_1, \cdots, y_N\right)$: right-hand side

Initial Value Problem vs Boundary Value Problem



We will focus on the initial value problem

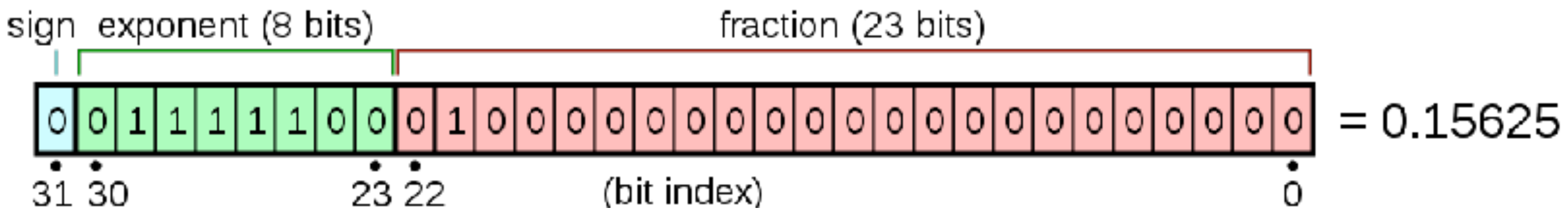


Euler's Method

- $y(x + \Delta x) = y(x) + \dfrac{dy}{dx}\Delta x + O\left(\Delta x^2\right)$
- $\dfrac{dy}{dx} = f\left(x, y\right) = \dfrac{y(x + \Delta x) - y(x)}{\Delta x} + O\left(\Delta x^2\right)$
- $y_{n+1} \leftarrow y_n + f\left(x_n, y_n\right) h$ where $h \equiv x_{n+1} - x_n$
- This method is not recommended for practical use.

Errors

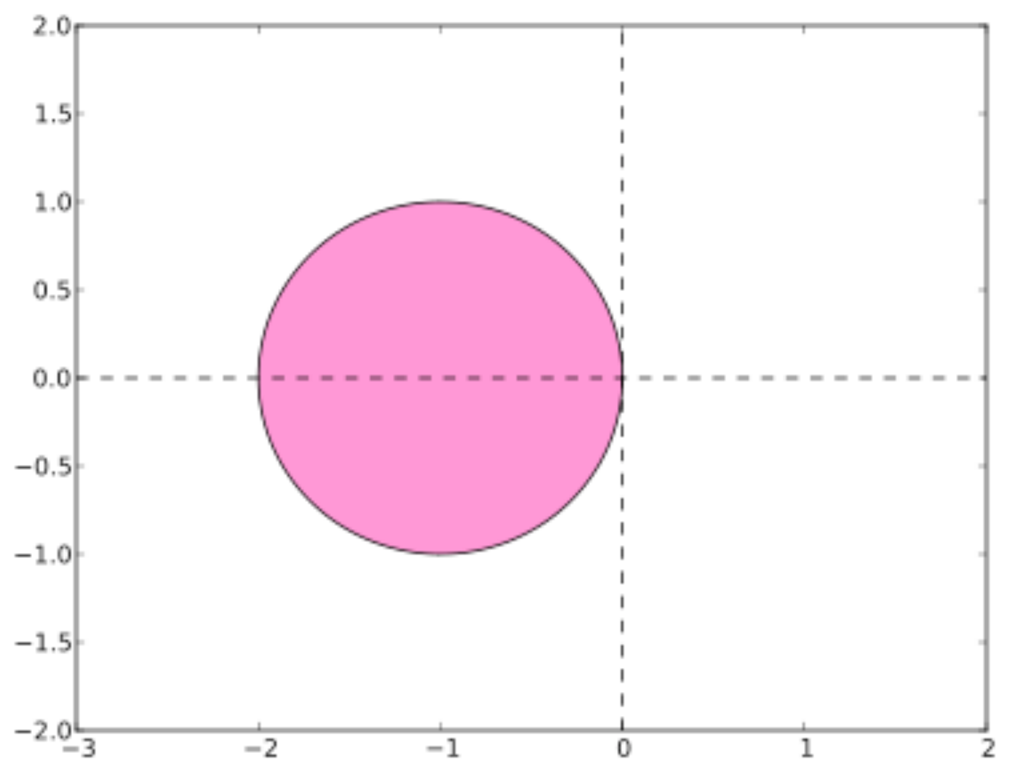
- Local Truncation Error = Real Solution - Numerical Solution
  - $\text{LTE} = y\left(x_{n+1}\right) - y_{n+1} = O\left(h^2\right)$ for Euler's method
- Global Truncation Error = Accumulated Local Truncation Errors
  - $\text{LTE} \times \frac{x_N - x_0}{h} = O\left(h\right)$ for Euler's method
- Rounding Errors
  - Single Precision (4 bytes) ~ 7 decimal digits
  - Double Precision (8 bytes) ~ 15 decimal digits



Single Precision

Explicit Method vs Implicit Method

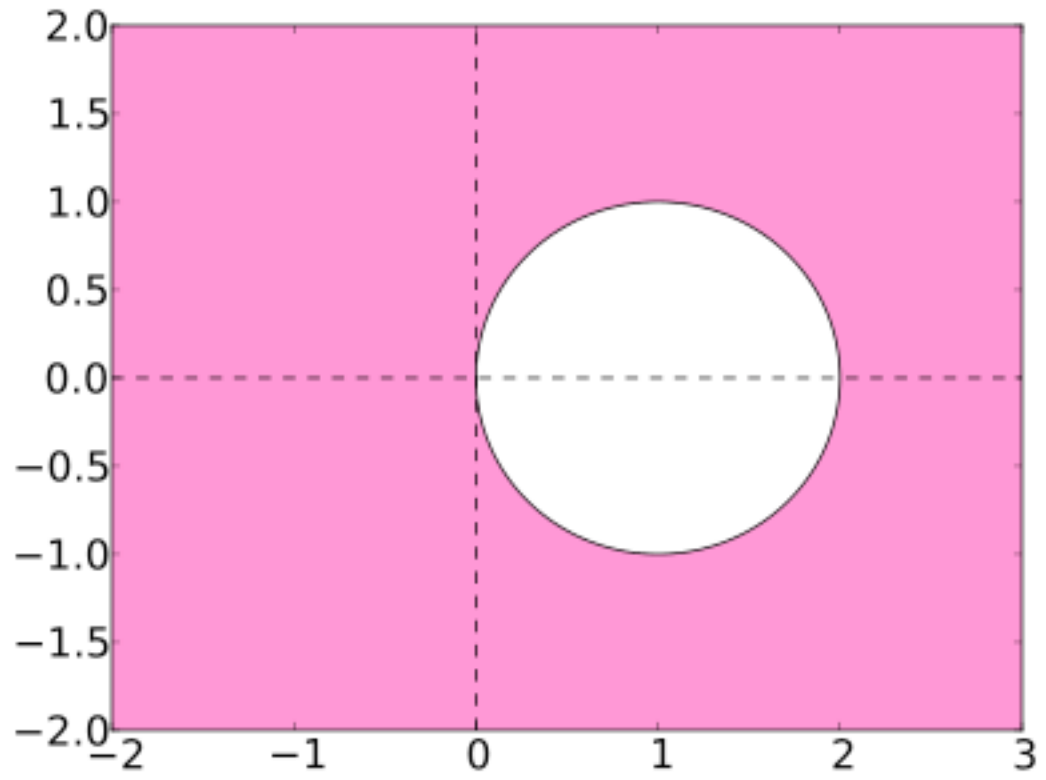
- **Euler's Method**
- $y_{n+1} \leftarrow y_n + f\left(x_n, y_n\right) h$
- Numerical stability is bad.



$z = hk$ for ODE $y' = ky$

We focus on the explicit method

- **Backward Euler's Method**
- $y_{n+1} \leftarrow y_n + f\left(x_{n+1}, y_{n+1}\right) h$
- Numerical stability is good.



$z = hk$ for ODE $y' = ky$

Predictor-Corrector Method

Heun's Method

- **PEC version**
- Given: y_n and y_n'
- Predict: $\tilde{y}_{n+1} \leftarrow y_n + y_n' h$
- Evaluation: $y_{n+1}' \leftarrow f\left(x_n, \tilde{y}_{n+1}\right)$
- Correction:

$$y_{n+1} \leftarrow \boxed{\tilde{y}_{n+1}} + \boxed{\frac{1}{2}\left(-y_n' + y_{n+1}'\right) h}$$

Predictor　　Corrector

- $$= y_n + \frac{1}{2}\left(y_n' + y_{n+1}'\right)$$

- **PECE version**
- Given: y_n and y_n'
- Predict: $\tilde{y}_{n+1} \leftarrow y_n + y_n' h$
- Evaluation: $\tilde{y}_{n+1}' \leftarrow f\left(x_n, \tilde{y}_{n+1}\right)$
- Correction:

$$y_{n+1} \leftarrow \tilde{y}_{n+1} + \frac{1}{2}\left(-y_n' + \tilde{y}_{n+1}'\right) h$$

- Evaluation: $y_{n+1}' \leftarrow f\left(x_{n+1}, y_{n+1}\right)$

Application to Newtonian Mechanics

Taylor Series

- $\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_n h + \mathbf{a}_n h^2/2 + \dot{\mathbf{a}}_n h^3/3! + \ddot{\mathbf{a}}_n h^4/4! + \dddot{\mathbf{a}}_n h^5/5! + \cdots$
- $\mathbf{r}^{(0)}_{n+1} = \mathbf{r}^{(0)}_n + \mathbf{r}^{(1)}_n + \mathbf{r}^{(2)}_n + \mathbf{r}^{(3)}_n + \mathbf{r}^{(4)}_n + \mathbf{r}^{(5)}_n$
- $\mathbf{v}_{n+1} = \mathbf{v}_n + \mathbf{a}_n h + \dot{\mathbf{a}}_n h^2/2 + \ddot{\mathbf{a}}_n h^3/3! + \dddot{\mathbf{a}}_n h^4/4! + \cdots$
- $\mathbf{r}^{(1)}_{n+1} = \mathbf{r}^{(1)}_n + 2\mathbf{r}^{(2)}_n + 3\mathbf{r}^{(3)}_n + 4\mathbf{r}^{(4)}_n + 5\mathbf{r}^{(5)}_n + \cdots$
- where
- $h \equiv t_{n+1} - t_n$

$$\begin{bmatrix} \mathbf{r}^{(0)}_{n+1} \\ \mathbf{r}^{(1)}_{n+1} \\ \mathbf{r}^{(2)}_{n+1} \\ \mathbf{r}^{(3)}_{n+1} \\ \mathbf{r}^{(4)}_{n+1} \\ \mathbf{r}^{(5)}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{(0)}_{n} \\ \mathbf{r}^{(1)}_{n} \\ \mathbf{r}^{(2)}_{n} \\ \mathbf{r}^{(3)}_{n} \\ \mathbf{r}^{(4)}_{n} \\ \mathbf{r}^{(5)}_{n} \end{bmatrix}$$

Pascal's Triangle

4-variable Predictor-Corrector Method

- $$\begin{bmatrix} \mathbf{r}_{n+1}^{(0)} \\ \mathbf{r}_{n+1}^{(1)} \\ \mathbf{r}_{n+1}^{(2)} \\ \mathbf{r}_{n+1}^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{n}^{(0)} \\ \mathbf{r}_{n}^{(1)} \\ \mathbf{r}_{n}^{(2)} \\ \mathbf{r}_{n}^{(3)} \\ \mathbf{r}_{n}^{(4)} \\ \mathbf{r}_{n}^{(5)} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{n}^{(0)} \\ \mathbf{r}_{n}^{(1)} \\ \mathbf{r}_{n}^{(2)} \\ \mathbf{r}_{n}^{(3)} \end{bmatrix}}_{\text{Predictor}} + \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 5 \\ 6 & 10 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{n}^{(4)} \\ \mathbf{r}_{n}^{(5)} \end{bmatrix}}_{\text{Corrector}}$$

- $\mathbf{r}_{n}^{(4)}$, $\mathbf{r}_{n}^{(5)}$: have to be determined

Constraints by Equations of Motions

Predictor

- $$\begin{bmatrix} \tilde{\mathbf{r}}_{n+1}^{(0)} \\ \tilde{\mathbf{r}}_{n+1}^{(1)} \\ \tilde{\mathbf{r}}_{n+1}^{(2)} \\ \tilde{\mathbf{r}}_{n+1}^{(3)} \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{n}^{(0)} \\ \mathbf{r}_{n}^{(1)} \\ \mathbf{r}_{n}^{(2)} \\ \mathbf{r}_{n}^{(3)} \end{bmatrix}$$

- $$\begin{bmatrix} \mathbf{r}_{n+1}^{(0)} \\ \mathbf{r}_{n+1}^{(1)} \\ \mathbf{r}_{n+1}^{(2)} \\ \mathbf{r}_{n+1}^{(3)} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{r}}_{n+1}^{(0)} \\ \tilde{\mathbf{r}}_{n+1}^{(1)} \\ \tilde{\mathbf{r}}_{n+1}^{(2)} \\ \tilde{\mathbf{r}}_{n+1}^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 4 & 5 \\ 6 & 10 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{n}^{(4)} \\ \mathbf{r}_{n}^{(5)} \end{bmatrix}$$

Corrector

- Equations of Motions
- $\mathbf{r}_{n+1}^{(2)} = f\left(\mathbf{r}_{n+1}^{(0)}, \mathbf{r}_{n+1}^{(1)}\right)$
- $\mathbf{r}_{n+1}^{(3)} = g\left(\mathbf{r}_{n+1}^{(0)}, \mathbf{r}_{n+1}^{(1)}\right)$

Solving Constraints and PC Method

- Approximation (using predictors)
- $\begin{bmatrix} \mathbf{r}_{n+1}^{(2)} \\ \mathbf{r}_{n+1}^{(3)} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{r}}_{n+1}^{(2)} \\ \tilde{\mathbf{r}}_{n+1}^{(3)} \end{bmatrix} + \begin{bmatrix} 6 & 10 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{n}^{(4)} \\ \mathbf{r}_{n}^{(5)} \end{bmatrix}$
- $\simeq \begin{bmatrix} f\left(\tilde{\mathbf{r}}_{n+1}^{(0)}, \tilde{\mathbf{r}}_{n+1}^{(1)}\right) \\ g\left(\tilde{\mathbf{r}}_{n+1}^{(0)}, \tilde{\mathbf{r}}_{n+1}^{(1)}\right) \end{bmatrix}$
- $\begin{bmatrix} \mathbf{r}_{n}^{(4)} \\ \mathbf{r}_{n}^{(5)} \end{bmatrix} \leftarrow \begin{bmatrix} 6 & 10 \\ 4 & 10 \end{bmatrix}^{-1} \begin{bmatrix} f\left(\tilde{\mathbf{r}}_{n+1}^{(0)}, \tilde{\mathbf{r}}_{n+1}^{(1)}\right) - \tilde{\mathbf{r}}_{n+1}^{(2)} \\ g\left(\tilde{\mathbf{r}}_{n+1}^{(0)}, \tilde{\mathbf{r}}_{n+1}^{(1)}\right) - \tilde{\mathbf{r}}_{n+1}^{(3)} \end{bmatrix}$

- Then,
- $\begin{bmatrix} \mathbf{r}_{n+1}^{(0)} \\ \mathbf{r}_{n+1}^{(1)} \end{bmatrix} \leftarrow \begin{bmatrix} \tilde{\mathbf{r}}_{n+1}^{(0)} \\ \tilde{\mathbf{r}}_{n+1}^{(1)} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{n}^{(4)} \\ \mathbf{r}_{n}^{(5)} \end{bmatrix}$
- $\begin{bmatrix} \mathbf{r}_{n+1}^{(2)} \\ \mathbf{r}_{n+1}^{(3)} \end{bmatrix} \leftarrow \begin{bmatrix} f\left(\mathbf{r}_{n+1}^{(0)}, \mathbf{r}_{n+1}^{(1)}\right) \\ g\left(\mathbf{r}_{n+1}^{(0)}, \mathbf{r}_{n+1}^{(1)}\right) \end{bmatrix}$

Summary

- ODE can be rewritten as 1st-order coupled ODEs.
- Euler's method is the most simple method for initial value problem with ODE.
- Implicit method has better numerical stability than explicit method.
- Predictor-Corrector method has Prediction, Evaluation, and Correction steps.
- 4-variable predictor-corrector method can be applied to Newtonian equations of motions.
- Thank you for listening 😁