# 수치적인 방법을 이용한 편미분 방정식 풀이

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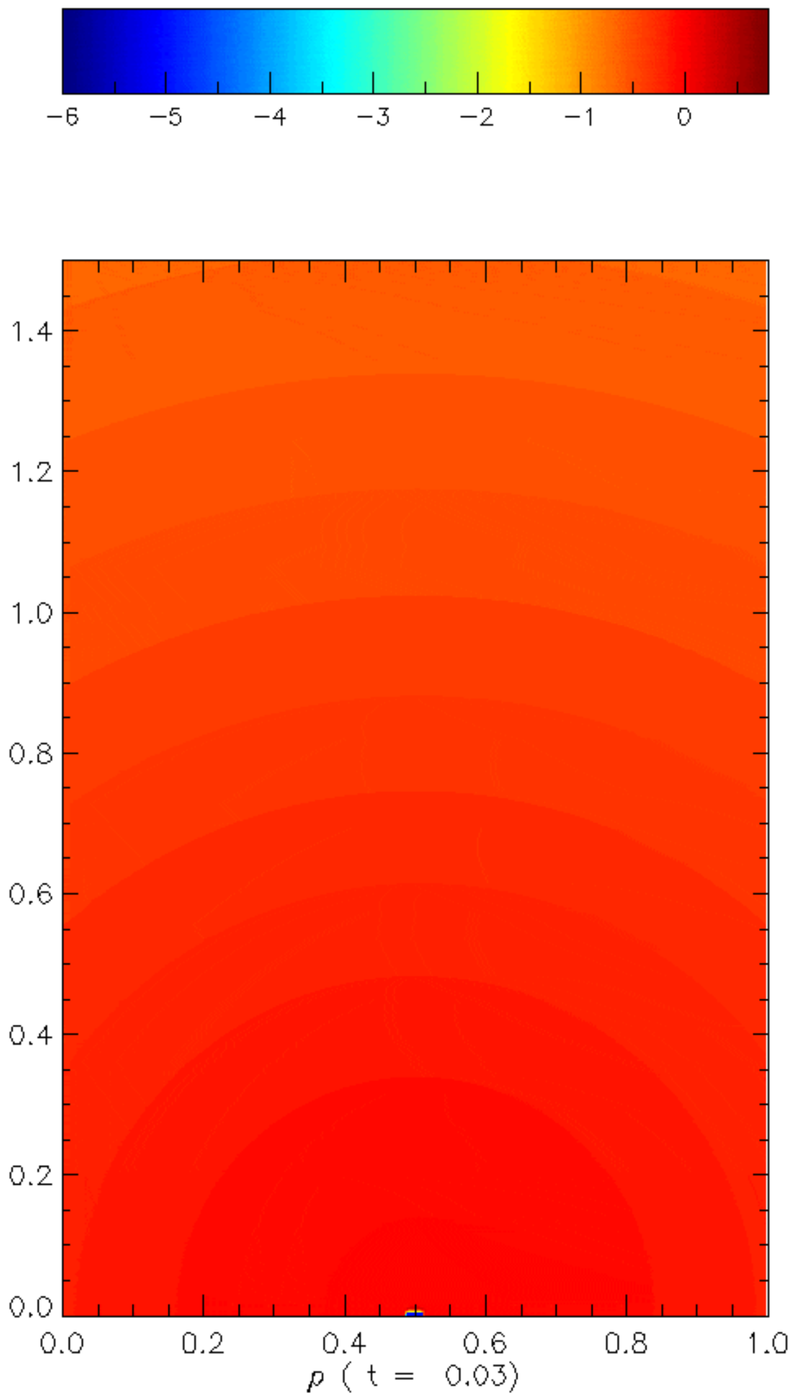
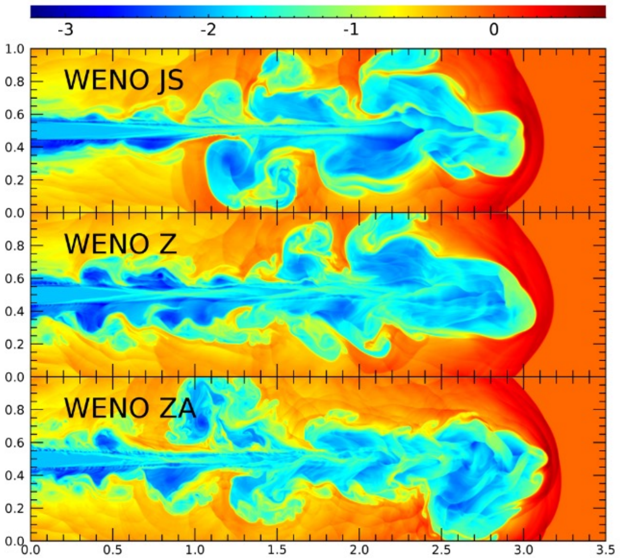
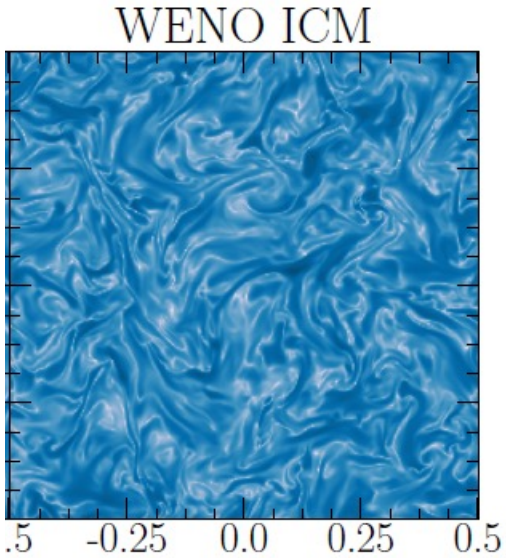



대학원에 오신(실)것을 환영합니다!

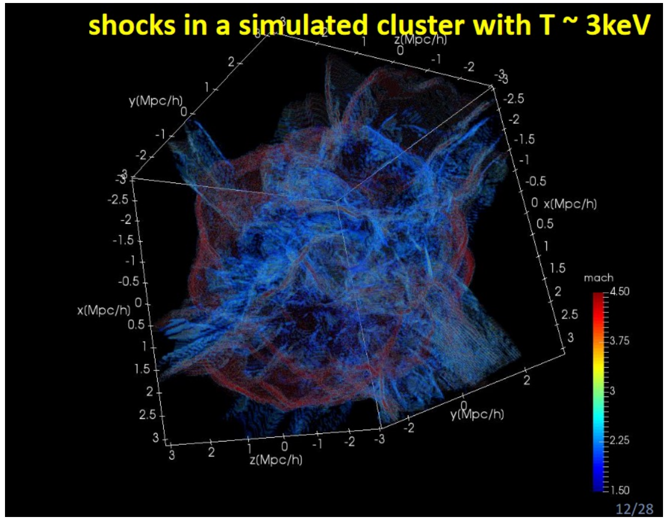
# AD - UNIST 천체물리 연구실 (**류동수 교수님**)



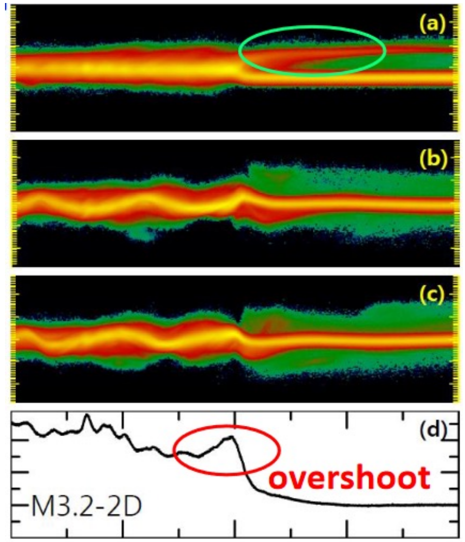

RHD




MHD




Cosmology




PIC

- 각종 시뮬레이션을 통해 천체물리학적 현상을 연구
- 고성능의 로컬 머신 보유 (한번에 512코어 활용 시뮬레이션 가능)

# 학습 목표

- 편미분 방정식의 종류에 대해 이해하고, 이류 방정식을 upwind scheme을 이용해 수치적 방법으로 풀수 있다.
- 수치 시뮬레이션의 원리에 대해 이해하고, Runge-Kutta 방법을 통해 시뮬레이션 할 수 있다.

# Eulerian VS Lagrangian

- Grid based
- Solving partial differential equation
- Ex) PLUTO, FLASH

- Particle based
- Solving Force equation
- Ex) Particle in Cell, SPH

# Types of partial differential equations

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

Example

$B^2 - AC > 0$ : Hyperbolic — Wave Eq

$B^2 - AC = 0$ : Parabolic — diffusion Eq

$B^2 - AC < 0$ : Elliptic — LaPlace Eq

Wave Eq

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

# Upwind scheme

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Advection equation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 \quad \text{for} \quad a > 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0 \quad \text{for} \quad a < 0$$

Upwind scheme

$$a^+ = \max(a, 0)\,, \qquad a^- = \min(a, 0)$$

$$u_x^- = \frac{u_i^n - u_{i-1}^n}{\Delta x}\,, \qquad u_x^+ = \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

$$u_i^{n+1} = u_i^n - \Delta t\left[a^+ u_x^- + a^- u_x^+\right]$$

# Basic concept of the
# Eulerian Hydrodynamic simulation

# Hydrodynamic conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0,$$

**Mass conservation**

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0,$$

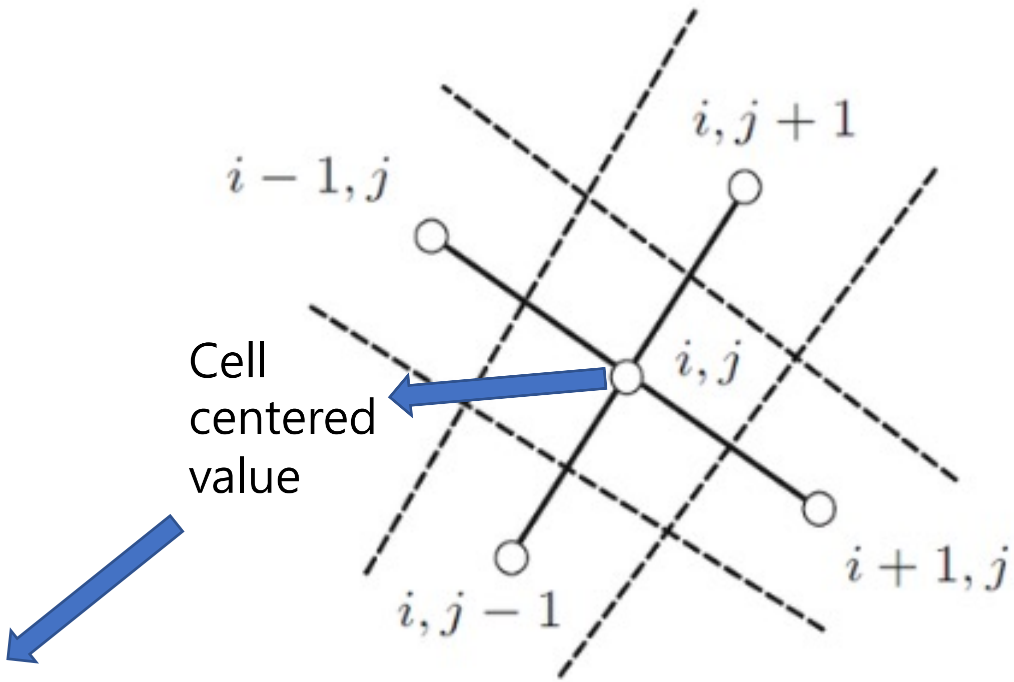
**Momentum conservation**

$$\frac{\partial(\rho e_{\mathrm{T}})}{\partial t} + \frac{\partial}{\partial x}(\rho u e_{\mathrm{T}} + pu) = \frac{\partial(\rho e_{\mathrm{T}})}{\partial t} + \frac{\partial}{\partial x}(\rho u h_{\mathrm{T}}) = 0,$$

**Energy conservation**

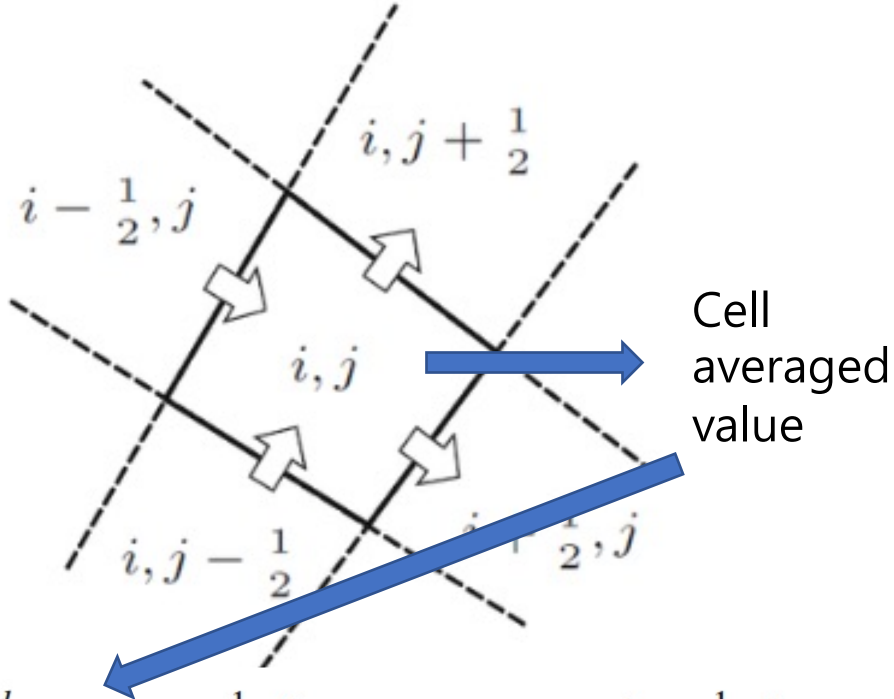
# Finite difference VS Finite volume

**(a)**




$$\boldsymbol{q}_{i,j,k}{}' = \boldsymbol{q}_{i,j,k} - \frac{\Delta t}{\Delta x}(\boldsymbol{F}_{i+\frac{1}{2},j,k} - \boldsymbol{F}_{i-\frac{1}{2},j,k}) - \frac{\Delta t}{\Delta y}(\boldsymbol{G}_{i,j+\frac{1}{2},k} - \boldsymbol{G}_{i,j-\frac{1}{2},k})$$

**(b)**




$$\frac{d}{dt}Q_{i,j}(t) = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2},j}(t) - F_{i-\frac{1}{2},j}(t)\right) - \frac{1}{\Delta y}\left(G_{i,j+\frac{1}{2}}(t) - G_{i,j-\frac{1}{2}}(t)\right),$$

# Conservation equation to characteristic equation

$$\frac{\partial \mathbf{u}}{\partial t} + A \frac{\partial \mathbf{u}}{\partial x} = 0, \quad \Rightarrow \quad Q^{-1} A Q = \Lambda \quad \Rightarrow \quad Q^{-1} \frac{\partial \mathbf{u}}{\partial t} + Q^{-1} A \frac{\partial \mathbf{u}}{\partial x} = 0.$$

Jacobian matrix

Diagonalize

$$\Rightarrow \quad d\mathbf{v} = Q^{-1} d\mathbf{u}.$$

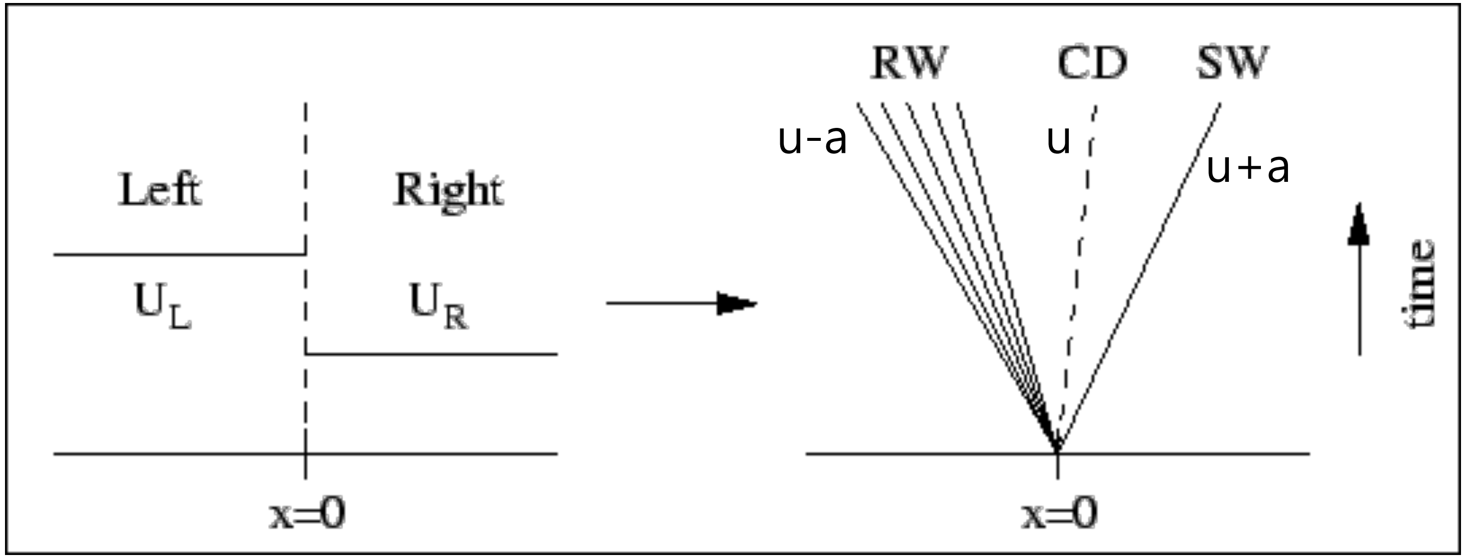
$$\Rightarrow \quad \frac{\partial \mathbf{v}}{\partial t} + Q^{-1} A Q \frac{\partial \mathbf{v}}{\partial x} = 0$$

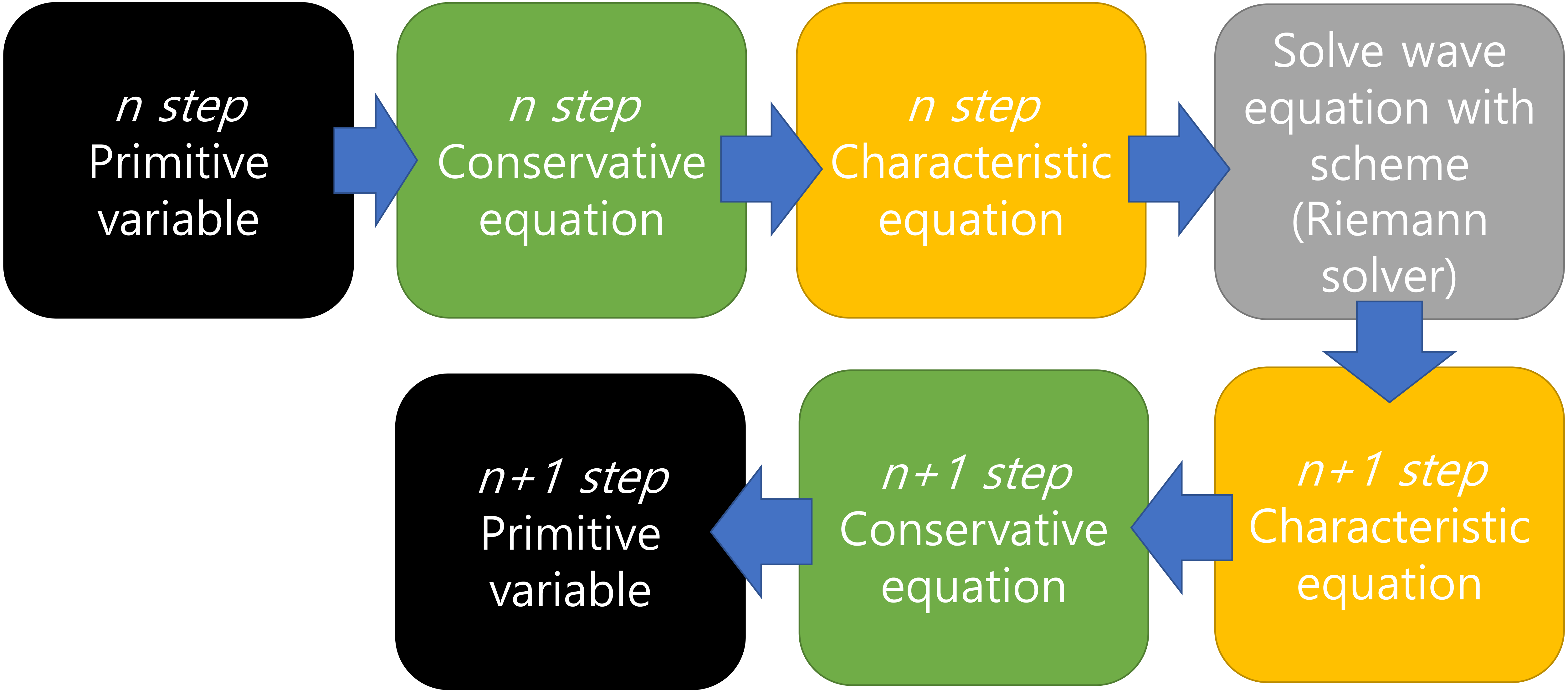
$$\Rightarrow \quad \frac{\partial \mathbf{v}}{\partial t} + \Lambda \frac{\partial \mathbf{v}}{\partial x} = 0. \qquad \Lambda = \begin{bmatrix} u & 0 & 0 \\ 0 & u+a & 0 \\ 0 & 0 & u-a \end{bmatrix}.$$

3 wave equation
(for hydrodynamics)

# Riemann Problem

$$\Lambda = \begin{bmatrix} u & 0 & 0 \\ 0 & u+a & 0 \\ 0 & 0 & u-a \end{bmatrix}.$$




*n step*
Primitive variable
*n step*
Conservative equation
*n step*
Characteristic equation
Solve wave equation with scheme (Riemann solver)
*n+1 step*
Characteristic equation
*n+1 step*
Conservative equation
*n+1 step*
Primitive variable

# Time integration: Runge-Kutta method

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h,$$

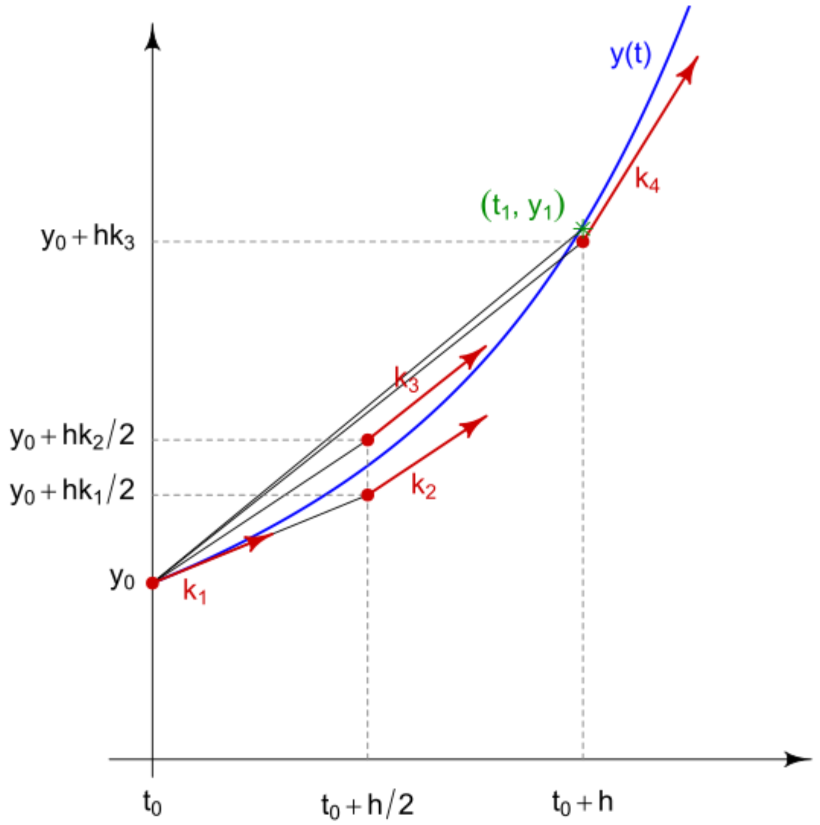
$$t_{n+1} = t_n + h$$

$$k_1 = f(t_n, y_n),$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right),$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right),$$

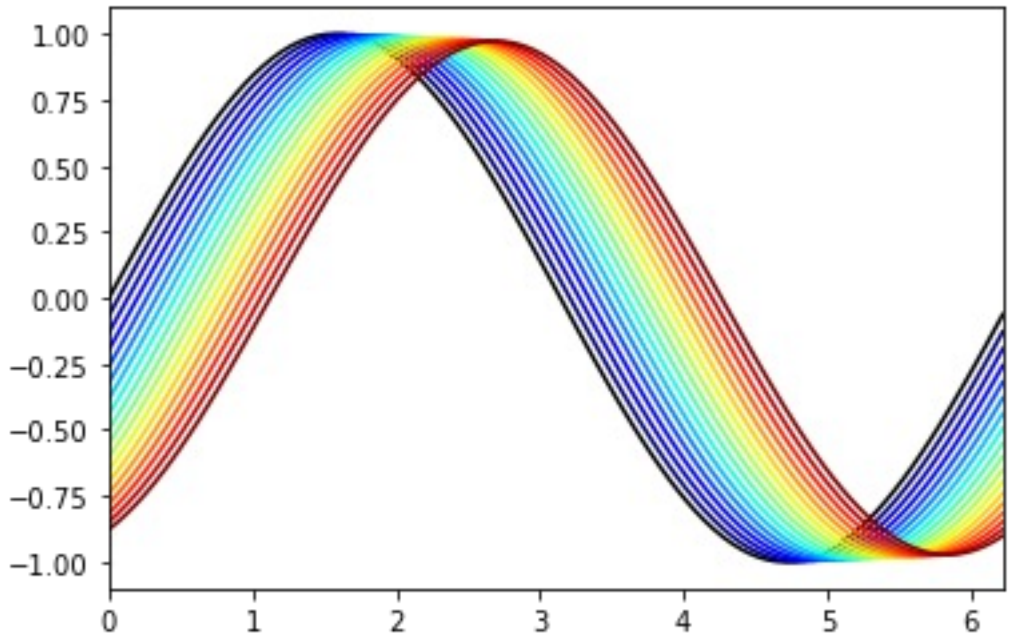
$$k_4 = f(t_n + h, y_n + hk_3).$$



# Time integration: Runge-Kutta method

$$q^{(0)} = q^n, \qquad q^{(1)} = q^{(0)} + \frac{\Delta t}{2}\mathcal{L}^{(0)}, \qquad q^{(2)} = q^{(0)} + \frac{\Delta t}{2}\mathcal{L}^{(1)},$$

$$q^{(3)} = q^{(0)} + \Delta t \mathcal{L}^{(2)}, \qquad q^{n+1} = \frac{1}{3}\left(-q^{(0)} + q^{(1)} + 2q^{(2)} + q^{(3)}\right) + \frac{\Delta t}{6}\mathcal{L}^{(3)}.$$



# Summary

- Widely adopted scheme for Hyperbolic PDE: upwind scheme

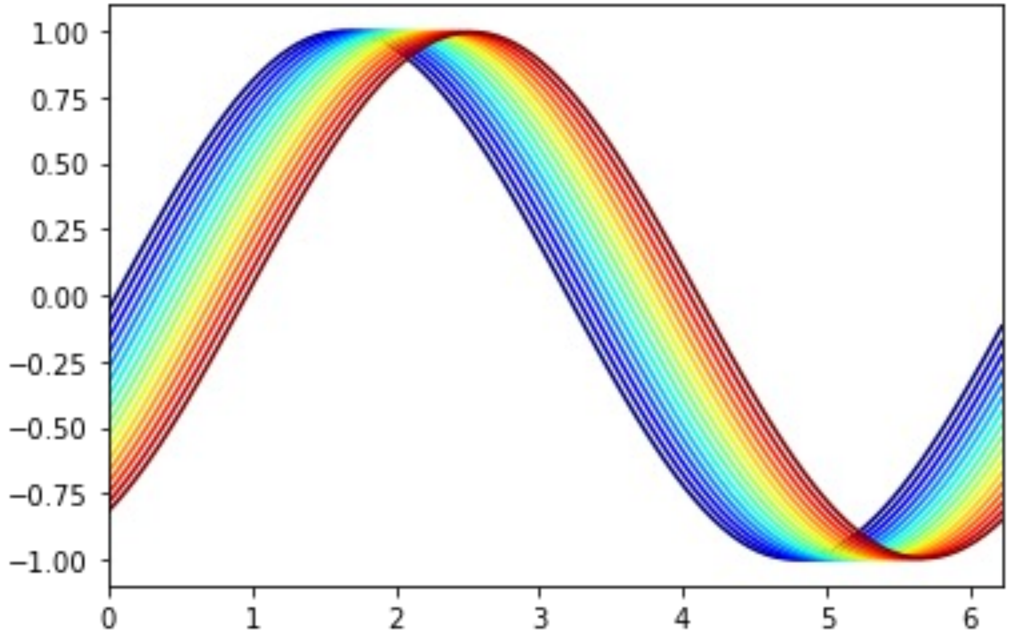
- Widely adopted scheme for time integration: RK4

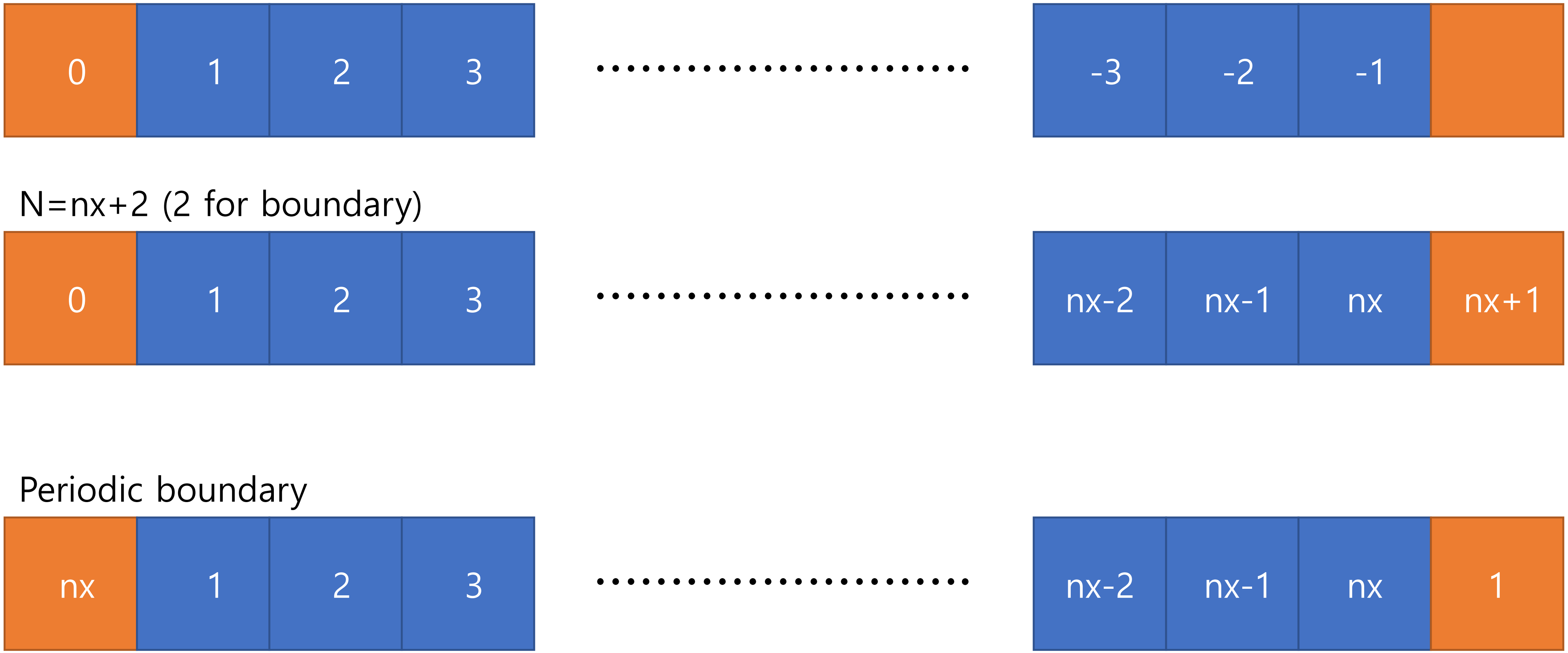
$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

$B^2 - AC > 0$ : Hyperbolic

$B^2 - AC = 0$ : Parabolic

$B^2 - AC < 0$ : Elliptic




0
1
2
3
-3
-2
-1
N=nx+2 (2 for boundary)
0
1
2
3
nx-2
nx-1
nx
nx+1
Periodic boundary
nx
1
2
3
nx-2
nx-1
nx
1