아인슈타인 방정식과 블랙홀

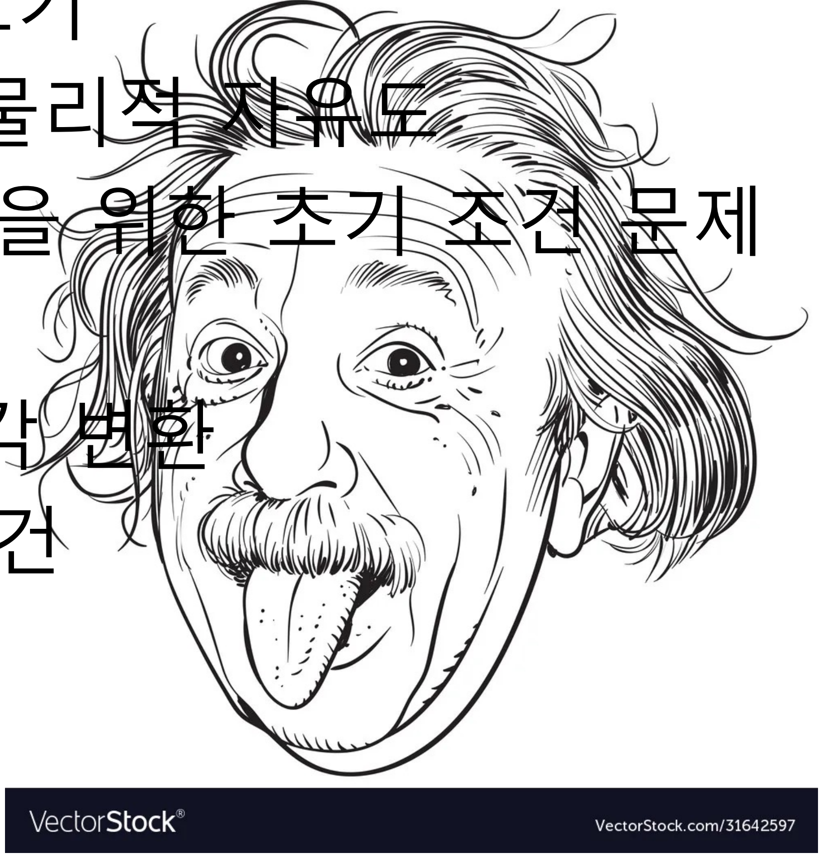
2023년 수치상대론 및 중력파 겨울학교

2023.02.01.Wed. 3PM @UNIST

한국천문연구원 현영환

수업내용

- 아인슈타인 방정식 형태 알아보기
- 아인슈타인 방정식 해의 실제 물리적 자유도
- 아인슈타인 방정식의 수치계산을 위한 초기 조건 문제
- 아인슈타인 방정식의 3+1 분리
- 초기 조건 문제의 자유도와 등각 변환
- 슈바르츠쉴트 블랙홀의 초기조건



아인슈타인 방정식

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\begin{cases} c \\ G \\ G_{\mu\nu} \\ T_{\mu\nu} \end{cases}$$

아인슈타인 방정식... 몇 개?

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

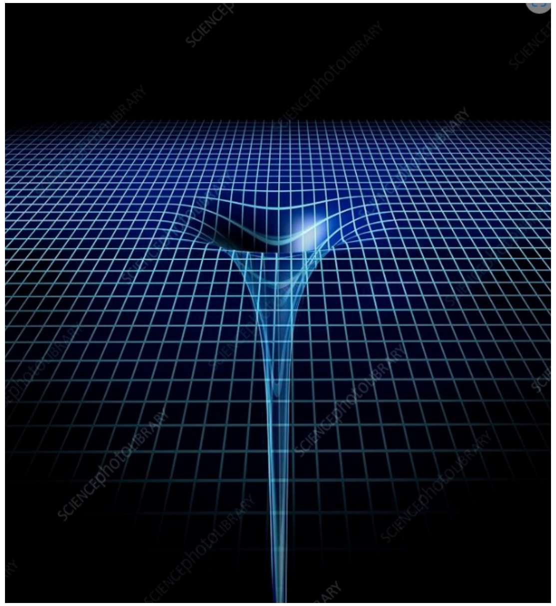
$$\mu = 0, 1, 2, 3 \leftrightarrow x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z)$$

$$G_{\mu\nu} = (G_{00},\ G_{01},\ G_{02}, ...)$$

$$G_{01} = G_{10},\ G_{02} = G_{20}, \cdots$$

# of eq's?, # of unknowns?

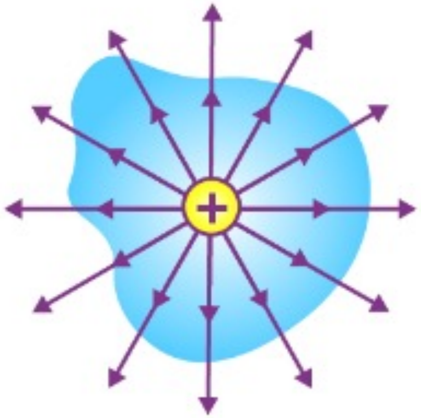
아인슈타인 방정식... 장론?



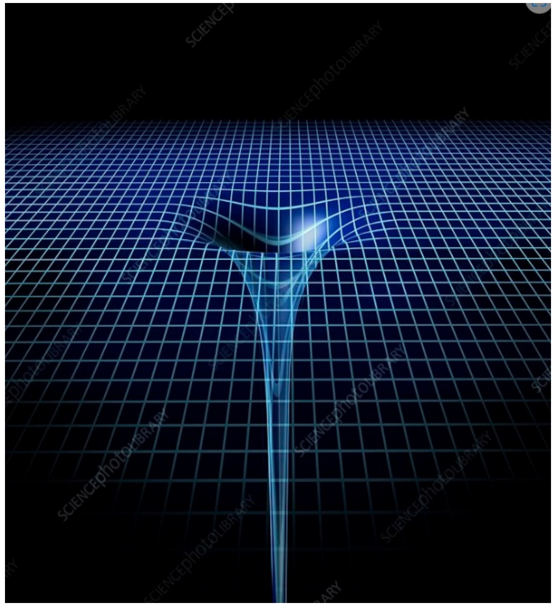
$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$



아인슈타인 방정식… 장론… 계량텐서?



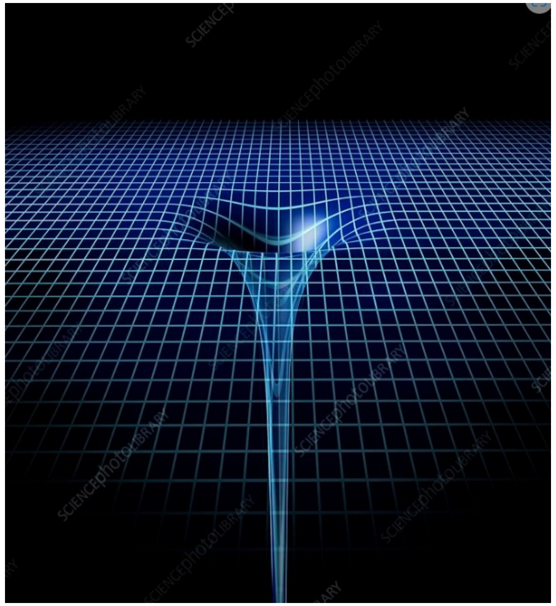
$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\begin{aligned} \boldsymbol{g} = ds^2 &= g_{\mu\nu} \boldsymbol{dx}^\mu \otimes \boldsymbol{dx}^\nu \\ &= g_{00} \boldsymbol{dx}^0 \otimes \boldsymbol{dx}^0 + \underbrace{g_{01} \boldsymbol{dx}^0 \otimes \boldsymbol{dx}^1 + g_{10} \boldsymbol{dx}^1 \otimes \boldsymbol{dx}^0}_{2 g_{01} \boldsymbol{dx}^0 \otimes \boldsymbol{dx}^1} + \cdots \\ &= g_{tt} \boldsymbol{dt} \otimes \boldsymbol{dt} + \underbrace{g_{tx} \boldsymbol{dt} \otimes \boldsymbol{dx} + g_{xt} \boldsymbol{dx} \otimes \boldsymbol{dt}}_{2 g_{tx} \boldsymbol{dt} \otimes \boldsymbol{dx}} + \cdots \end{aligned}$$

$$\boldsymbol{g}(\boldsymbol{\partial_t}, \boldsymbol{\partial_t}) = g_{tt} \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle \otimes \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle + g_{tx} \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle \otimes \langle \boldsymbol{dx} \cdot \boldsymbol{\partial_t} \rangle + \cdots$$

$$\boldsymbol{g}(\boldsymbol{\partial_t}, \boldsymbol{\partial_x}) = g_{tt} \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle \otimes \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_x} \rangle + g_{tx} \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle \otimes \langle \boldsymbol{dx} \cdot \boldsymbol{\partial_x} \rangle + \cdots$$

아인슈타인 방정식... 장론... 계량텐서?



$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\begin{aligned}
\boldsymbol{g} = ds^2 &= g_{\mu\nu} \boldsymbol{dx}^{\mu} \otimes \boldsymbol{dx}^{\nu} \\
&= g_{00} \boldsymbol{dx}^0 \otimes \boldsymbol{dx}^0 + \underbrace{g_{01} \boldsymbol{dx}^0 \otimes \boldsymbol{dx}^1 + g_{10} \boldsymbol{dx}^1 \otimes \boldsymbol{dx}^0}_{2g_{01}\boldsymbol{dx}^0 \otimes \boldsymbol{dx}^1} + \cdots \\
&= g_{tt} \boldsymbol{dt} \otimes \boldsymbol{dt} + \underbrace{g_{tx} \boldsymbol{dt} \otimes \boldsymbol{dx} + g_{xt} \boldsymbol{dx} \otimes \boldsymbol{dt}}_{2g_{tx}\boldsymbol{dt} \otimes \boldsymbol{dx}} + \cdots
\end{aligned}$$

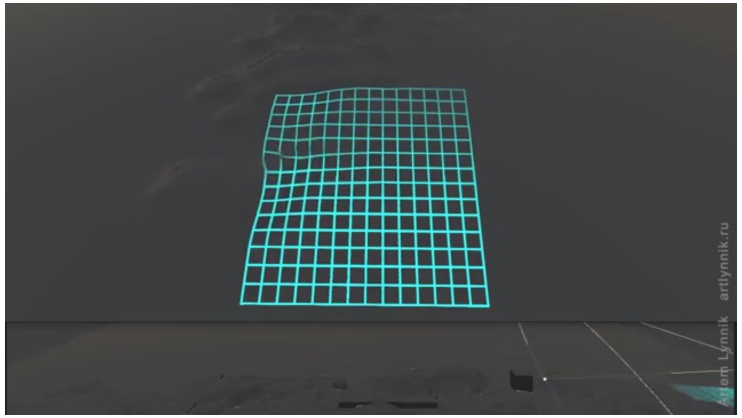
$$\boldsymbol{g}(\boldsymbol{\partial_t}, \boldsymbol{\partial_t}) = g_{tt} \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle \otimes \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle + g_{tx} \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle \otimes \langle \boldsymbol{dx} \cdot \boldsymbol{\partial_t} \rangle + \cdots$$

$$\boldsymbol{g}(\boldsymbol{\partial_t}, \boldsymbol{\partial_x}) = g_{tt} \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle \otimes \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_x} \rangle + g_{tx} \langle \boldsymbol{dt} \cdot \boldsymbol{\partial_t} \rangle \otimes \langle \boldsymbol{dx} \cdot \boldsymbol{\partial_x} \rangle + \cdots$$

아인슈타인 방정식… 계량텐서…

Diffeomorphism symmetry

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$



[Lumigrids Projects a Laser Grid In Front Of Your Bicycle To See Terrain Changes at Night (odditymall.com)]

$$\begin{aligned} \boldsymbol{g} &= g_{tt}\,\boldsymbol{dt} \otimes \boldsymbol{dt} + g_{tx}\,\boldsymbol{dt} \otimes \boldsymbol{dx} + \cdots \\ &= g_{tr}\,\boldsymbol{dt} \otimes \boldsymbol{dt} + g_{tr}\,\boldsymbol{dt} \otimes \boldsymbol{dr} + \cdots \end{aligned}$$

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

$$T_{\mu\nu}(x) \to T'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} T_{\rho\sigma}(x)$$

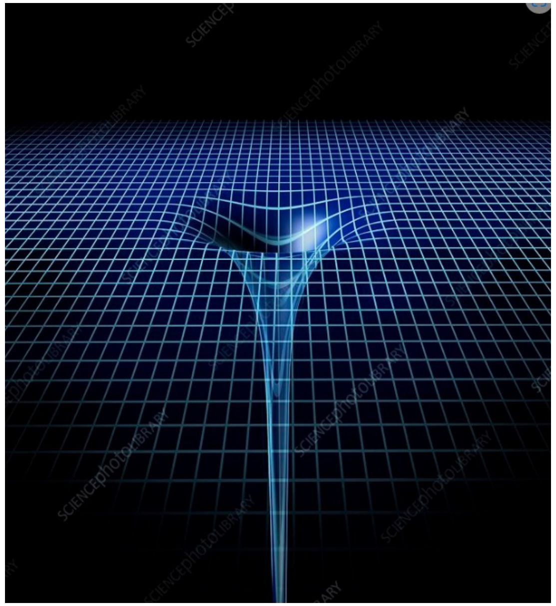
We can fix x^μ. $\rightarrow$ Gauge fixing

$$\nabla_\mu g^{\mu\nu} = 0 \;=\; \frac{1}{\sqrt{-g}}\underbrace{\partial_\mu(\sqrt{-g}g^{\mu\nu})}_{=0} + \underbrace{\Gamma^\nu_{\alpha\beta}g^{\alpha\beta}}_{=0} = 0 \quad \Leftrightarrow$$

$$\boxed{\Gamma^\nu_{\alpha\beta}g^{\alpha\beta} = 0 \Leftrightarrow \partial_\mu(\sqrt{-g}g^{\mu\nu}) \equiv \partial_\mu \mathfrak{g}^{\mu\nu} = 0}$$

$$(\Leftrightarrow \Box(x^\rho) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu(x^\rho)) = 0, \text{ harmonic gauge})$$

아인슈타인 방정식... 장론... 좌표와 기하



$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\begin{aligned} \boldsymbol{g} = ds^2 &= \boldsymbol{dt} \otimes \boldsymbol{dt} + \boldsymbol{dx} \otimes \boldsymbol{dx} + \boldsymbol{dy} \otimes \boldsymbol{dy} + \boldsymbol{dz} \otimes \boldsymbol{dz} \\ &\equiv \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \end{aligned}$$

$$\boldsymbol{g} = ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{2GM}{r}} dr^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$$

$$\begin{aligned} \boldsymbol{g} = ds^2 &= dr^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \\ &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \end{aligned}$$

아인슈타인 방정식... 다시 몇 개?

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

We can fix x^μ. $\rightarrow$ Gauge fixing

아인슈타인 방정식... 좀 더 자세히

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{2} g_{\mu\nu} R(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Riemann tensor :

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$
$$= \boxed{2\partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} + 2\Gamma^{\rho}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\sigma}}$$

Ricci tensor :

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu} = \boxed{2\partial_{[\rho}\Gamma^{\rho}_{\nu]\mu} + 2\Gamma^{\rho}_{[\rho|\lambda|}\Gamma^{\lambda}_{\nu]\mu}}$$

Ricci scalar :

$$R \equiv R^{\mu\nu}{}_{\mu\nu} = 2\partial_{[\mu}\Gamma^{\mu}_{\nu]\rho} g^{\rho\nu} + 2\Gamma^{\mu}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\rho} g^{\rho\nu}$$

Weyl tensor (Not Ricci in Riemann) :

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{4}{(n-2)} g_{[\rho[\mu}R_{\nu]\sigma]} + \frac{2}{(n-1)(n-2)} g_{\rho[\mu}g_{\nu]\sigma} R$$

아인슈타인 방정식... 좀 더 자세히, Pseudo-Riemannian geometry

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

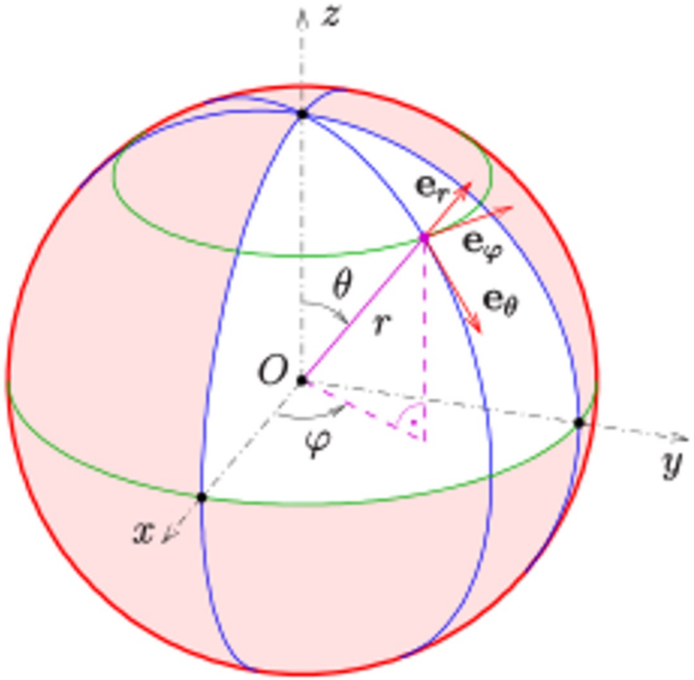
$$R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{2} g_{\mu\nu} R(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

For a sphere of radius r in the flat space,

we obtained,

$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$

$\widehat{R} = \frac{2}{r^2}, \; K_{\theta\theta} = r, \; K_{\phi\phi} = r\sin^2\theta, \; K = \frac{2}{r}$



아인슈타인 방정식… 좀 더 자세히 in terms of $g_{\mu\nu}$

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{2} g_{\mu\nu} R(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Riemann tensor :

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$
$$= \boxed{2\partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} + 2\Gamma^{\rho}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\sigma}}$$

Ricci tensor :

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu} = \boxed{2\partial_{[\rho}\Gamma^{\rho}_{\nu]\mu} + 2\Gamma^{\rho}_{[\rho|\lambda|}\Gamma^{\lambda}_{\nu]\mu}}$$

Ricci scalar :

$$R \equiv R^{\mu\nu}{}_{\mu\nu} = 2\partial_{[\mu}\Gamma^{\mu}_{\nu]\rho} g^{\rho\nu} + 2\Gamma^{\mu}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\rho} g^{\rho\nu}$$

Weyl tensor (Not Ricci in Riemann) :

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{4}{(n-2)} g_{[\rho[\mu}R_{\nu]\sigma]} + \frac{2}{(n-1)(n-2)} g_{\rho[\mu}g_{\nu]\sigma}R$$

Levi-Civita(affine) connection (Christoffel symbol) :

$$\boxed{\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})}$$

아인슈타인 방정식... 좀 더 자세히 in terms of $g_{\mu\nu}$

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{2} g_{\mu\nu} R(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

```
EinsteinCD[-μ, -ν] // ToRicci // RiemannToChristoffel // NoScalar // ChristoffelToGradMetric[#, g] & // Expand //
ToCanonical
```

확장

$$-\frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\gamma\delta} - \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\alpha g_{\mu\nu} + \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\mu g_{\nu\alpha} + \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\nu\delta} \partial_\gamma g_{\mu\alpha} +$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\delta g_{\beta\gamma} + \frac{1}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_\delta g_{\delta1\delta2} + \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\gamma g_{\mu\alpha} \partial_\delta g_{\nu\beta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta \partial_\beta g_{\alpha\gamma} +$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta \partial_\gamma g_{\alpha\beta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\delta g_{\beta\delta2} \partial_{\delta1} g_{\alpha\gamma} - \frac{3}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_{\delta1} g_{\alpha\gamma} \partial_{\delta2} g_{\beta\delta} +$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\beta g_{\alpha\gamma} \partial_{\delta2} g_{\delta\delta1} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_{\delta2} g_{\delta\delta1} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\mu g_{\nu\alpha} -$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\mu g_{\nu\alpha} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\mu g_{\alpha\gamma} \partial_\nu g_{\beta\delta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} \partial_\nu \partial_\mu g_{\alpha\beta}$$

아인슈타인 방정식… 좀 더 더 자세히 in terms of $g_{\mu\nu}$

```
EinsteinCD[-μ, -ν] // ToRicci // RiemannToChristoffel // NoScalar // ChristoffelToGradMetric[#, g] & // Expand //
ToCanonical
```

확장

$$
\begin{aligned}
&-\frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\gamma\delta} - \frac{1}{2} g^{\alpha\beta} \partial_\beta\partial_\alpha g_{\mu\nu} + \frac{1}{2} g^{\alpha\beta} \partial_\beta\partial_\mu g_{\nu\alpha} + \frac{1}{2} g^{\alpha\beta} \partial_\beta\partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\nu\delta} \partial_\gamma g_{\mu\alpha} + \\
&\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\delta g_{\beta\gamma} + \frac{1}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_\delta g_{\delta1\delta2} + \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\gamma g_{\mu\alpha} \partial_\delta g_{\nu\beta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta\partial_\beta g_{\alpha\gamma} + \\
&\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta\partial_\gamma g_{\alpha\beta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\delta g_{\beta\delta2} \partial_{\delta1} g_{\alpha\gamma} - \frac{3}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_{\delta1} g_{\alpha\gamma} \partial_{\delta2} g_{\beta\delta} + \\
&\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\beta g_{\alpha\gamma} \partial_{\delta2} g_{\delta\delta1} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_{\delta2} g_{\delta\delta1} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\mu g_{\nu\alpha} - \\
&\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\mu g_{\nu\alpha} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\mu g_{\alpha\gamma} \partial_\nu g_{\beta\delta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} \partial_\nu\partial_\mu g_{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\alpha\partial_\mu g_{\beta\nu} &= \frac{1}{2}g^{00}\partial_0\partial_\mu g_{0\nu} + \frac{1}{2}g^{01}\partial_0\partial_\mu g_{1\nu} + \frac{1}{2}g^{02}\partial_0\partial_\mu g_{2\nu} + \frac{1}{2}g^{03}\partial_0\partial_\mu g_{3\nu} \\
&+ \frac{1}{2}g^{10}\partial_1\partial_\mu g_{0\nu} + \frac{1}{2}g^{11}\partial_1\partial_\mu g_{1\nu} + \frac{1}{2}g^{12}\partial_1\partial_\mu g_{2\nu} + \frac{1}{2}g^{13}\partial_1\partial_\mu g_{3\nu} + \frac{1}{2}g^{20}\partial_2\partial_\mu g_{0\nu} + \frac{1}{2}g^{21}\partial_2\partial_\mu g_{1\nu} \\
&+ \frac{1}{2}g^{22}\partial_2\partial_\mu g_{2\nu} + \frac{1}{2}g^{23}\partial_2\partial_\mu g_{3\nu} + \frac{1}{2}g^{30}\partial_3\partial_\mu g_{0\nu} + \frac{1}{2}g^{31}\partial_3\partial_\mu g_{1\nu} + \frac{1}{2}g^{32}\partial_3\partial_\mu g_{2\nu} + \frac{1}{2}g^{33}\partial_3\partial_\mu g_{3\nu}
\end{aligned}
$$

- 2nd order PDE of $g_{\mu\nu}$

아인슈타인 방정식... 2^{nd} order PDF of $g_{\mu\nu}$, can be solved?

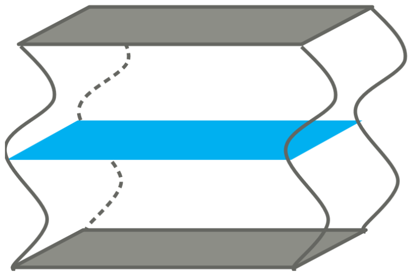
$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

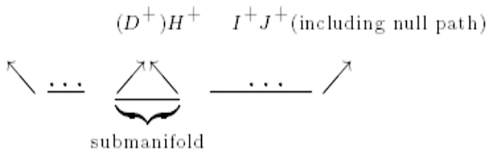
- **Cauchy surface**

  Σ in $\mathcal{M}$ of one-time intersections with each causal curve
  $(\Sigma|_{\text{d.o.d of } \Sigma = \mathcal{M}})$

- **Globally hyperbolic spacetime**

  $(\mathcal{M}, g)$ which has Σ_{Cauchy}
  $\xrightarrow{\text{topology}} (\mathcal{M}, g)|_{\mathcal{M}=\Sigma\times\mathbb{R}}$



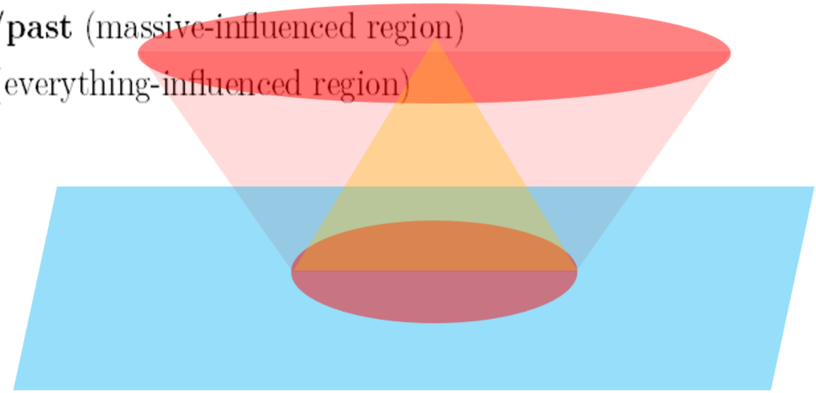


$D^{\pm}(S)$: future/past **domain** of dependence (determined region)

$H^{\pm}(S)$: future/past **Cauchy horizon** (determined region limit)

$I^{\pm}(S)$: chronological **future/past** (massive-influenced region)

$J^{\pm}(S)$: causal **future/past** (everything-influenced region)



Initial value formulation:

appropriate initial data > subsequent uniquely determined dynamical evolution

Appropriate initial data:

small changes in initial data > small change in solution

> predictable physics law

Any changes in initial data can not change solutions outside causal future. > "Initial value formulation" is well-posed.

아인슈타인 방정식… Initial value formulation: constraints

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

(1) $G_{\mu\nu} = 0$ (vacuum Einstein eq. $\rightarrow 10 g_{\mu\nu}$ dof.)

(2) $\nabla_\alpha G^{\alpha\beta} = 0$ (Bianchi id. 4 eq.)

$$\Rsh\ \partial_0 \underbrace{G^{0\beta}}_{\text{no 2nd time derivative.}} = \underbrace{-(\partial_i G^{i\beta} + \Gamma^\alpha{}_{\alpha\sigma} G^{\sigma\beta} + \Gamma^\beta{}_{\alpha\sigma} G^{\alpha\sigma})}_{\leftarrow\ G_{\mu\nu}:\ \text{at most 2nd time derivative.}}$$

$\Rsh\ G_{0\beta} = 0 \ \rightarrow$ no evolution eq. just constraint on initial data.

$\Rsh\ g_{\mu\nu}$ evolved by $G_{ij} = 0$ from $G_{0\beta} = 0|_{\Sigma_t}$

satisfies constraints in M by $\nabla_\alpha G^{\alpha\beta} = 0$

아인슈타인 방정식… 3+1 formulation:

constraints(initial data) + evolution???

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\begin{cases} G_{0\nu} = \frac{8\pi G}{c^4} T_{0\nu} \\ G_{ij} = \frac{8\pi G}{c^4} T_{ij} \end{cases}$$

아인슈타인 방정식... 3+1 formulation:

metric 3+1 decomposition

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Gaussian normal coordinates : $ds^2 = \sigma dz^2 + \gamma_{ij} dy^i dy^j$

$\Lsh\ ds^2 = g_{\mu\nu}(\mathbf{d}x^\mu) \otimes (\mathbf{d}x^\nu)$

$= -dt^2 + \gamma_{ij} dy^i dy^j \ \Leftrightarrow\ -(N^2 - \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j$

$\varsigma\ \mathbf{d}t = -N^{-1}\boldsymbol{n} \xrightarrow{N=1} -\boldsymbol{n} = -n_\mu(\mathbf{d}x^\mu)$ $\qquad \Lsh\ N = 1,\ \beta^i = 0$

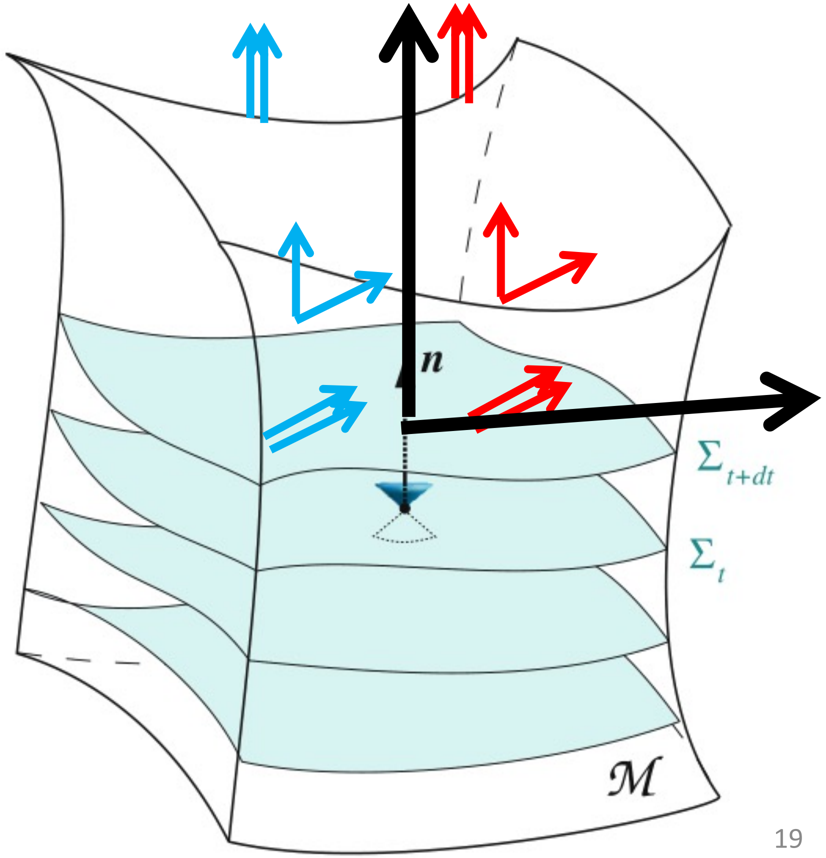
$= -n_\mu n_\nu dx^\mu dx^\nu + \gamma_{\mu\nu} dx^\mu dx^\nu$

$= (-n_\mu n_\nu + \gamma_{\mu\nu}) dx^\mu dx^\nu$

$\therefore\ \gamma_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu$

$= P_{\mu\nu}$ (projection tensor)

> Projection tensor = metric on hypersurface



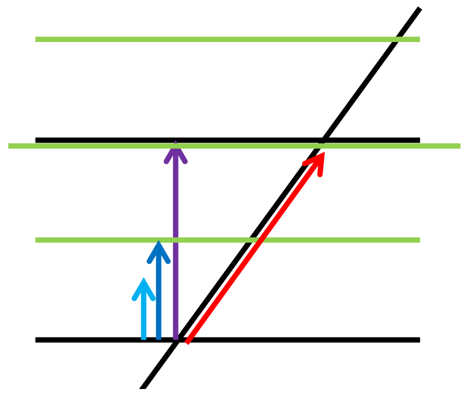
아인슈타인 방정식… 3+1 formulation:

metric 3+1 decomposition

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu, \; g^{\mu\nu} = (\mathbf{d}x^\mu) \cdot (\mathbf{d}x^\nu)$$

$$\downarrow \begin{cases} \boldsymbol{t} = \boldsymbol{m} + \boldsymbol{\beta} & \\ N \equiv \alpha & \text{(lapse function)} \\ N^\alpha \equiv \beta^\alpha = (0, \vec{\beta}) & \text{(shift vector)} \\ m^\alpha = N n^\alpha = (1, -\vec{\beta}) & \text{(evolution vector)} \end{cases}$$



$$\begin{aligned} ds^2 &= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -(N^2 - \beta_i \beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}t \mathrm{d}x^i + \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j \\ &= -N^2 \mathrm{d}t^2 + \gamma_{ij} \underbrace{(\mathrm{d}x^i + \beta^i \mathrm{d}t)}_{\mathrm{d}\hat{x}^i} \underbrace{(\mathrm{d}x^j + \beta^j \mathrm{d}t)}_{\mathrm{d}\hat{x}^j} \end{aligned}$$

Note that $\mathrm{d}x^i$ is not on Σ_t when $\vec{\beta} \neq 0$, but $\mathrm{d}\hat{x}^i$ is on Σ_t.

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

아인슈타인 방정식… 3+1 formulation: $G_{\mu\nu}$

3+1 decomposition

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Summary….

Scalar Gauss relation : $\widehat{R} = R$ …

Contracted Gauss' equation : $\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}}$ …

$\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2K_{\rho[\mu}K_{\nu]\sigma}$ **(Gauss eq.)**

$2\widehat{\nabla}_{[\mu}\widehat{\nabla}_{\nu]}V^{\rho} \equiv \widehat{R}^{\rho}{}_{\sigma\mu\nu}V^{\sigma}$

$2\nabla_{[\widehat{\mu}}\nabla_{\widehat{\nu}]}n^{\widehat{\rho}} = R^{\widehat{\rho}}{}_{\lambda\widehat{\mu}\widehat{\nu}}n^{\lambda}$

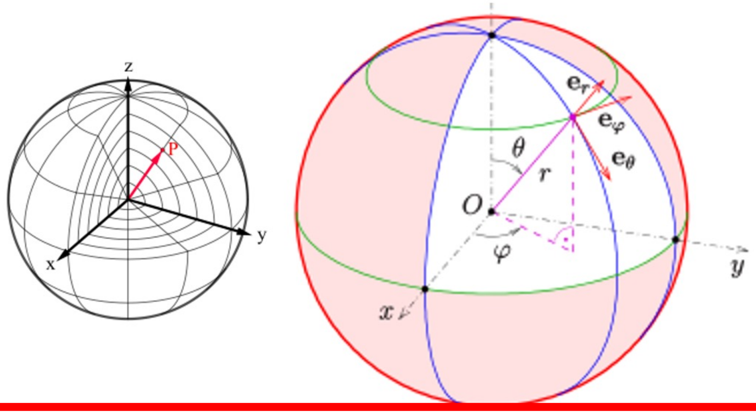
Codazzi's equation : $2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}}$

Contracted Codazzi's equation : $2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}}$

For a sphere of radius r in the flat space, we obtained,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

$$\widehat{R} = \frac{2}{r^2},\ K_{\theta\theta} = r,\ K_{\phi\phi} = r\sin^2\theta,\ K = \frac{2}{r}$$



The contracted Gauss equation in the flat space:

$$\widehat{R} = \cancel{R} - \sigma 2\cancel{R_{nn}} + \overbrace{\sigma}^{=1}(K^2 - K_{\mu\nu}K^{\mu\nu})$$

$$\widehat{R} = K^2 - K_{\mu\nu}K^{\mu\nu} = K^2 - K_{\theta\theta}K^{\theta\theta} - K_{\phi\phi}K^{\phi\phi}$$

$$K^{\theta\theta} = g^{\theta\theta}g^{\theta\theta}K_{\theta\theta},\ K^{\phi\phi} = g^{\phi\phi}g^{\phi\phi}K_{\phi\phi}$$

$$= K^2 - (g^{\theta\theta})^2(K_{\theta\theta})^2 - (g^{\phi\phi})^2(K_{\phi\phi})^2$$

$$= \left(\frac{2}{r}\right)^2 - \frac{1}{r^4}r^2 - \frac{1}{r^4\sin^4\theta}r^2\sin^4\theta$$

$$= \frac{4}{r^2} - \frac{1}{r^2} - \frac{1}{r^2}$$

$$= \frac{2}{r^2}$$

Dimensional analysis of K and R

아인슈타인 방정식... 3+1 formulation: $T_{\mu\nu}$

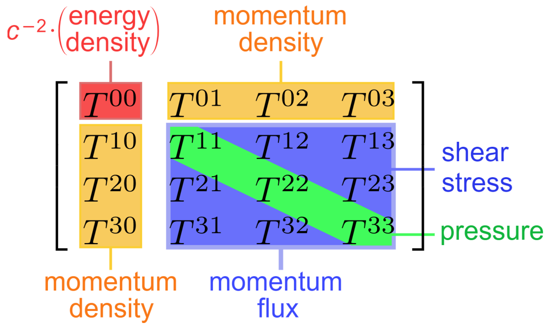
3+1 decomposition

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

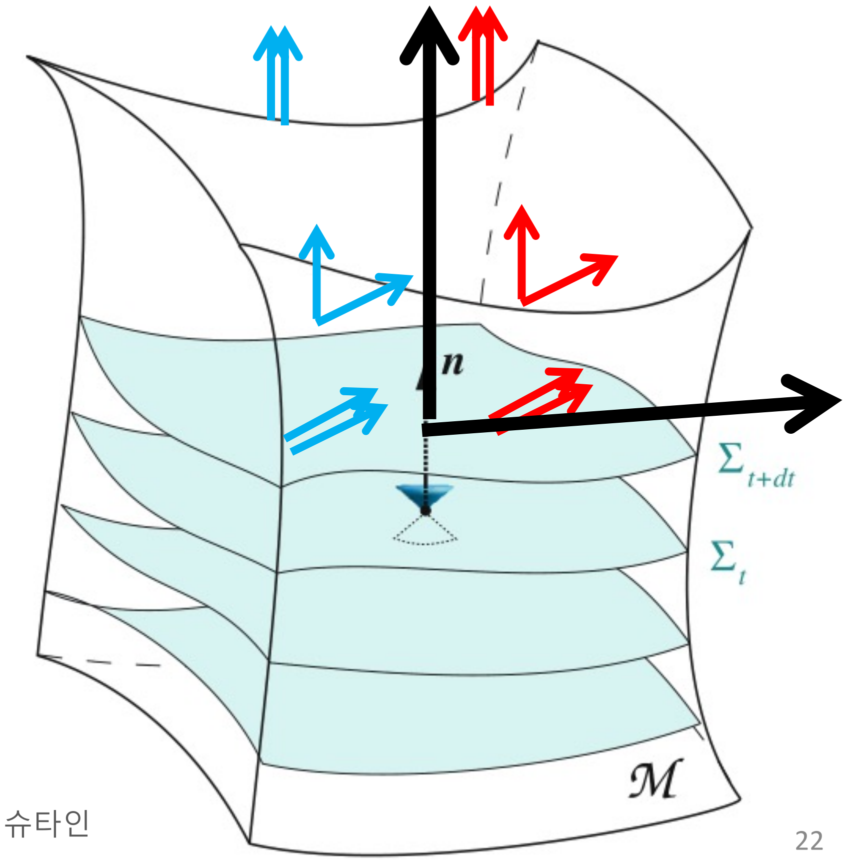
Eulerian observer (observer moving along a normal vector)

$$\boldsymbol{T} = E\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{p} + \boldsymbol{p}\otimes\boldsymbol{n} + \boldsymbol{S}$$

$$\begin{aligned} T &= T^{\mu}{}_{\mu} \\ &= g^{\mu\nu}T_{\mu\nu} \\ &= (\sigma n^{\mu}n^{\nu} + P^{\mu\nu})T_{\mu\nu} \\ &= -n^{\mu}n^{\nu}T_{\mu\nu} + P^{\mu\nu}T_{\mu\nu} \\ &= -E + S \end{aligned}$$



$$\begin{cases} (\text{energy density}) & \rho_e = T_{nn} \\ (\text{momentum density}) & p_\alpha = -T_{n\hat{\alpha}} \\ (\text{stress tensor}) & S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}} \end{cases}$$



아인슈타인 방정식… 3+1 formulation:

constraints(initial data) + evolution

- Intrinsic eq.
- Hamiltonian constraint 1
- Momentum constraint 3
- Unknown K,P 12
- Degree to choose coordinates 4
- Dynamic d.o.f (12-4-4) = 4
- Gravitational field d.o.f 4/2 =2
- We can choose 8 independent variable for initial data and solve constraint eq's.

$$ {}^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu} $$

$$ \hookrightarrow \begin{cases} (1)\ {}^{(4)}G_{nn} = 8\pi G T_{nn} & \rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ {}^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} & \rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} & \rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\ & \qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ & \qquad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} & \rightarrow\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases} $$

아인슈타인 방정식… 3+1 formulation:
conformal decomposition

$$^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho$$
$$^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu$$

- 4 constraint equations
- To i,j indices: K's 6 dof's
- 어떤 값을 초기 조건으로

Degree of Freedom of the Weyl tensor : 10

Riemann(20) − Ricci(10) = Weyl(10)
in the d-dimensional spacetime,

$$R^\mu_{\nu\rho\sigma}\left[\frac{n^2(n^2-1)}{12}\right] - R_{\mu\nu}\left[\frac{n(n+1)}{2}\right] = C^\mu{}_{\nu\rho\sigma}\left[\frac{n(n+1)(n+2)(n-3)}{12}\right]$$

therefore, in 3-dimensional spacetime, Weyl tensors vanish.

아인슈타인 방정식... 3+1 formulation:
conformal decomposition

$$^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho$$

$$^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu$$

$$ds^2 = -(\alpha^2 - \psi^4\beta^2)dt^2 + 2\psi^4\beta dt dr + \psi^4 \underbrace{(dr^2 + r^2 d^2\Omega)}_{=\bar{\gamma}_{ij} d\bar{x}^i d\bar{x}^j}$$

$\therefore$ Any spherically symmetric metric is spatially conformally flat.

$$R = \psi^{-n}\bar{R} - 2n\psi^{-1-n}\bar{D}^2\psi + \underbrace{\frac{(4-n)n}{2}\psi^{-2-n}\bar{D}^i\psi\bar{D}_i\psi}_{\to 0 \text{ when } n=4}$$

(in flat case, since $\bar{\gamma}_{ij} = \eta_{ij}$ and $\bar{R}_{ij} = R = 0$,)

when $\gamma_{ij} = \psi^n\eta_{ij}$, that is , $n = 4$,

$$R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi$$

아인슈타인 방정식... 3+1 formulation:
conformal decomposition

trace/traceless and conformal decomposition of K_{ij}:

(Note that we didn't fix $\bar{\gamma}_{ij} = \eta_{ij}$ yet.

We increased number of unknowns, γ_{ij} to $\psi, \bar{\gamma}_{ij}$ for convenience.)

$$K_{ij} \equiv A_{ij} + \frac{1}{3}\gamma_{ij}K \quad (A_{ij} : \text{traceless spatial extrinsic curvature})$$

$$A^{ij} \equiv \psi^{-10}\bar{A}^{ij}, \;\; A_{ij} = \psi^{-2}\bar{A}_{ij}$$

$$\therefore \;\; K_{ij} \rightarrow \psi^{-2}\bar{A}_{ij} + \frac{1}{3}\gamma_{ij}K$$

further decomposition of traceless $\bar{A}^{ij} = \bar{A}^{ij}_{\mathrm{T}}$

into transverse/longitudinal components:

$$\bar{A}^{ij} \equiv \bar{A}^{ij}_{\mathrm{TT}} + \bar{A}^{ij}_{\mathrm{L}} \quad \text{where } \bar{D}_j\bar{A}^{ij}_{\mathrm{TT}} = 0$$

$$\bar{A}^{ij}_{\mathrm{L}} = \bar{D}^iW^j + \bar{D}^jW^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_kW^k \equiv (\bar{L}W)^{ij}$$

$$\bar{D}_j\bar{A}^{ij} = (\bar{\Delta}_{\mathrm{L}}W)^i$$

아인슈타인 방정식… 3+1 formulation:
conformal decomposition

$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\left\{\begin{aligned}
&(1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} &&\rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\
&(2)\ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} &&\rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i \\
&(3)\ ^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} &&\rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\
& && \quad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\
& && \quad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \tfrac{1}{2}\gamma_{ij}(S - E)] \\
&K_{\mu\nu} = \tfrac{1}{2}\mathcal{L}_n P_{\mu\nu} &&\rightarrow\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i
\end{aligned}\right.$$

↳

$$\left\{\begin{aligned}
&G_{nn}:\ 8\bar{D}^2\psi - \psi\bar{R} - \tfrac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho \\
&G_{ni}:\ (\bar{\Delta}_{\rm L} W)^j - \tfrac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_i K = 8\pi G\psi^{10} p^j \\
&G_{ij}:\ \partial_t K_{ij} = \alpha(R_{ij} + K K_{ij} - 2K_{ik}K^k{}_j) - D_i D_j\alpha - \alpha 8\pi G(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S - \rho)) \\
&\qquad + \beta^k D_k K_{ij} + K_{ik} D_j\beta^k + K_{kj} D_i \beta^k \\
&\quad (\text{trace})\ \partial_t K = -D^2\alpha + \alpha(K_{ij}K^{ij} + 4\pi(\rho + S)) + \beta^i D_i K \\
&K_{ij}:\ \partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\
&\quad \hookrightarrow 4\bar{\gamma}_{ij}\psi^3\partial_t\psi = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\
&\quad (\text{trace})\ \tfrac{1}{2}\gamma^{ij}\partial_t\gamma_{ij} = \partial_t \ln\gamma^{1/2} = -\alpha K + D_i\beta^i
\end{aligned}\right.$$

- free data:
  - conformal metric $\tilde{\boldsymbol{\gamma}}$;
  - symmetric traceless and transverse tensor $\hat{A}^{ij}_{\rm TT}$ (traceless and transverse are meant with respect to $\tilde{\boldsymbol{\gamma}}$: $\tilde{\gamma}_{ij}\hat{A}^{ij}_{\rm TT} = 0$ and $\tilde{D}_j\hat{A}^{ij}_{\rm TT} = 0$);
  - scalar field K;
  - conformal matter variables: $(\tilde{E}, \tilde{p}^i)$;
- constrained data (or "determined data"):
  - conformal factor Ψ, obeying the non-linear elliptic equation (9.19) (Lichnerowicz equation)
  - vector $\boldsymbol{X}$, obeying the linear elliptic equation (9.20).

Accordingly the general strategy to get valid initial data for the Cauchy problem is to choose $(\tilde{\gamma}_{ij}, \hat{A}^{ij}_{\rm TT}, K, \tilde{E}, \tilde{p}^i)$ on Σ_0 and solve the system (9.19)–(9.20) to get Ψ and X^i. Then one constructs

2012_Gourgoulhon_31 Formalism in General Relativity_Springer.

아인슈타인 방정식... 3+1 formulation:
Schwarzschild BH.

in the conformally flat, time-symmetric vacuum solution

whereby $(K_{ij} = -K^{ij} = 0,\ \rho = 0)$

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho$$

$$\Lsh\ \bar{D}^2\psi = \Delta^{\text{flat}}\psi = 0 \xrightarrow{\psi(\infty)=1} \psi = 1 + \frac{C}{r}$$

since the ADM mass is given by

$$M = -\frac{1}{2\pi G}\oint_\infty \bar{D}^i\left(1+\frac{C}{r}\right)\mathrm{d}^2S_i = -\frac{1}{2\pi G}\lim_{r\to\infty}(4\pi r^2) \times \left(-\frac{C}{r^2}\right) = \frac{2C}{G}$$

$$\therefore\ C = \frac{MG}{2}$$

we get $\psi = 1 + \dfrac{GM}{2r}$

$$ds^2 = -(\alpha^2 - \psi^4\beta^2)\mathrm{d}t^2 + 2\psi^4\beta\mathrm{d}t\mathrm{d}r + \psi^4\underbrace{(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)}_{=\bar{\gamma}_{ij}\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j}$$

$$ds^2 = -\left(\frac{1-\frac{m}{2r}}{2+\frac{m}{2r}}\right)^2\mathrm{d}t^2 + \left(1+\frac{m}{2r}\right)^4(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2)$$

$$\begin{cases}
G_{nn}:\ 8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho \\
G_{ni}:\ (\bar{\Delta}_{\mathrm{L}}W)^j - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_iK = 8\pi G\psi^{10}p^j \\
G_{ij}:\ \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_iD_j\alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)) \\
\qquad\qquad + \beta^kD_kK_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k \\
\quad (\text{trace})\ \partial_t K = -D^2\alpha + \alpha(K_{ij}K^{ij} + 4\pi(\rho+S)) + \beta^iD_iK \\
K_{ij}:\ \partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\
\ \Lsh\ 4\bar{\gamma}_{ij}\psi^3\partial_t\psi = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\
\quad (\text{trace})\ \frac{1}{2}\gamma^{ij}\partial_t\gamma_{ij} = \partial_t \ln\gamma^{1/2} = -\alpha K + D_i\beta^i
\end{cases}$$

아인슈타인 방정식... 3+1 formulation: Schwarzschild BH.

in the conformally flat, time-symmetric vacuum solution

whereby $(K_{ij} = -K^{ij} = 0,\ \rho = 0)$

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho$$

$$\hookrightarrow \bar{D}^2\psi = \Delta^{\text{flat}}\psi = 0 \xrightarrow{\psi(\infty)=1} \psi = 1 + \frac{C}{r}$$

since the ADM mass is given by

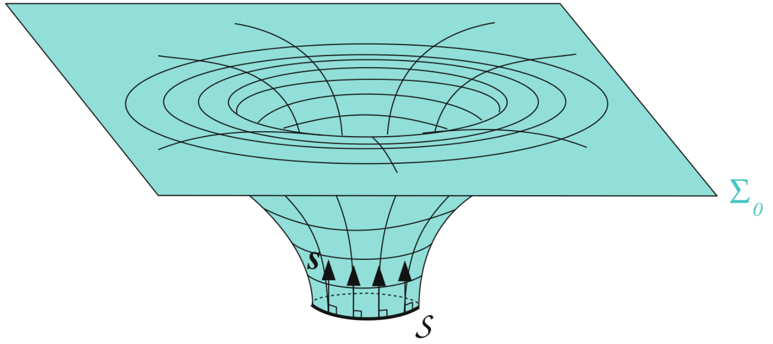
$$M = -\frac{1}{2\pi G}\oint_\infty \bar{D}^i\left(1+\frac{C}{r}\right)\mathrm{d}^2S_i = -\frac{1}{2\pi G}\lim_{r\to\infty}\left(4\pi r^2 \times \left(-\frac{C}{r^2}\right)\right) = \frac{2C}{G}$$

$$\therefore\ C = \frac{MG}{2}$$

we get $\psi = 1 + \frac{GM}{2r}$

$$ds^2 = -(\alpha^2 - \psi^4\beta^2)\mathrm{d}t^2 + 2\psi^4\beta\mathrm{d}t\mathrm{d}r + \psi^4\underbrace{(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)}_{=\bar{\gamma}_{ij}\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j}$$

$$ds^2 = -\left(\frac{1-\frac{m}{2r}}{2+\frac{m}{2r}}\right)^2\mathrm{d}t^2 + \left(1+\frac{m}{2r}\right)^4(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2)$$



- Momentarily static, K=0 (maximal slicing)
- S: minimal surface $\mathcal{D}_i s^i\Big|_{\mathscr{S}} = 0 \qquad \left(\frac{\partial\Psi}{\partial r} + \frac{\Psi}{2r}\right)\Big|_{r=a} = 0$

아인슈타인 방정식… 3+1 formulation:
Schwarzschild BH.

in the conformally flat, time symmetric vacuum solution

where

$8\bar{D}^2\psi$

↳ $\bar{D}^2\psi$

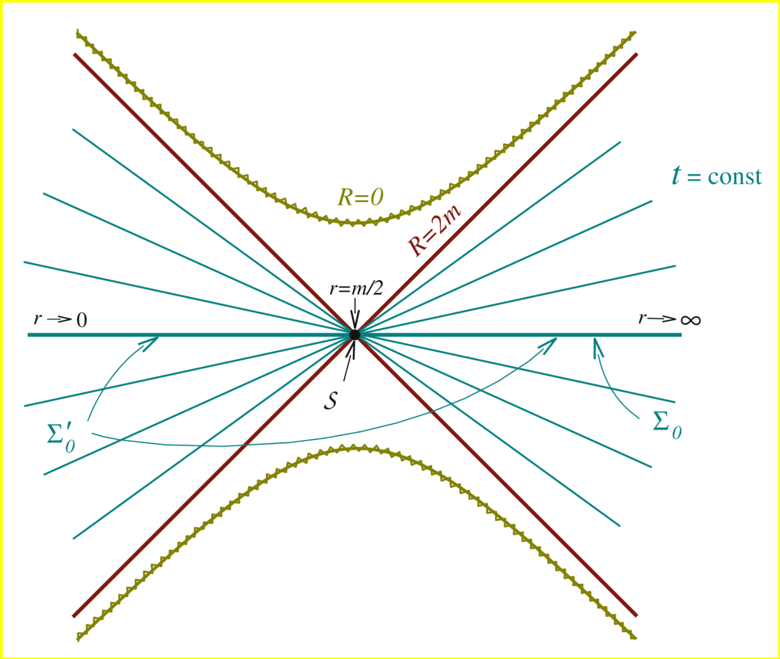
since

M =

$\pi r^2 \times \left(-\frac{C}{r^2}\right) = \frac{2C}{G}$

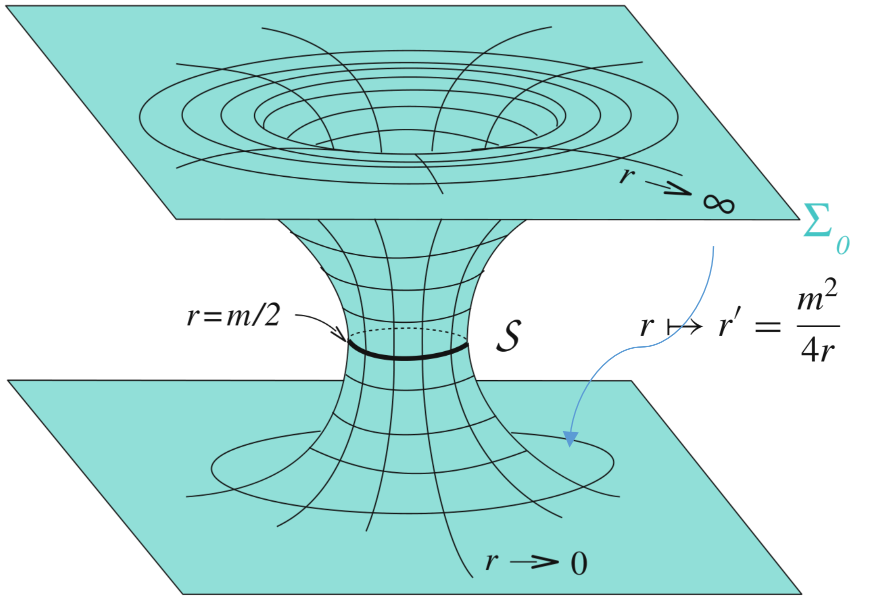
∴ C

we get



$$ds^2 = -(\alpha^2 - \psi^4\beta^2)dt^2 + 2\psi^4\beta \mathrm{d}t\mathrm{d}r + \psi^4 \underbrace{(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)}_{=\bar{\gamma}_{ij}\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j}$$

$$ds^2 = -\left(\frac{1-\frac{m}{2r}}{2+\frac{m}{2r}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{m}{2r}\right)^4 (\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2)$$



- Einstein-Rosen bridge
- Kruskal diagram

아인슈타인 방정식... 3+1 formulation: Numerical calculation

- Minimal distortion (elimination purely coordinate-related fluctuation in Yij)
- Locating horizon
- Constructing initial data

$$^{(4)}G_{nn} \to 8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho \quad \text{(H. constraint)}$$

$$\hookrightarrow \Delta^{\text{flat}}\psi = 0 \text{ (Laplace eq.)} \xrightarrow[\psi|_{r\to\infty} \to 1]{\text{assuming asymptotic flatness}} \psi = 1 + \frac{m}{2r}$$

$$\hookrightarrow \text{linear eq.} \to \boxed{\psi = 1 + \sum_\alpha \frac{m_\alpha}{2r_{C_\alpha}}}$$

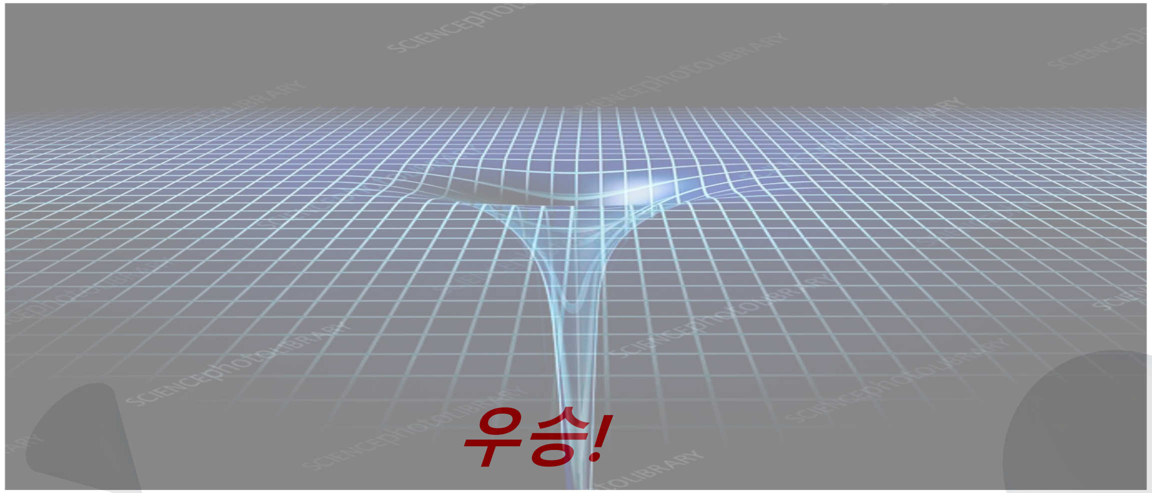
where $r_{C_\alpha} = |x^i - C^i_\alpha|$ and

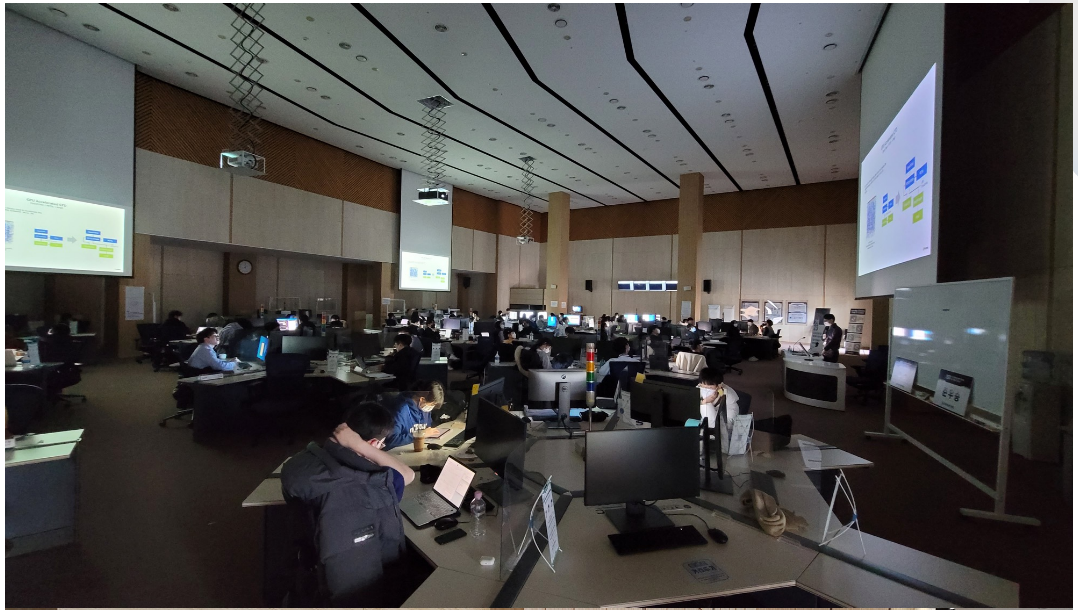
C^i_α is the coordinate location of the αth black hole

- Bowen-York approach
- Evolution

$$\text{initial data constructed using} \begin{cases} \text{the conformal transverse-traceless decomposition} \\ \text{maximal slicing } (K = 0, \text{not } K_{ij} = 0) \\ \text{conformal flatness } (\bar{\gamma}_{ij} = \eta_{ij}) \end{cases}$$

즐거운 겨울학교 되세요.


우승!

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