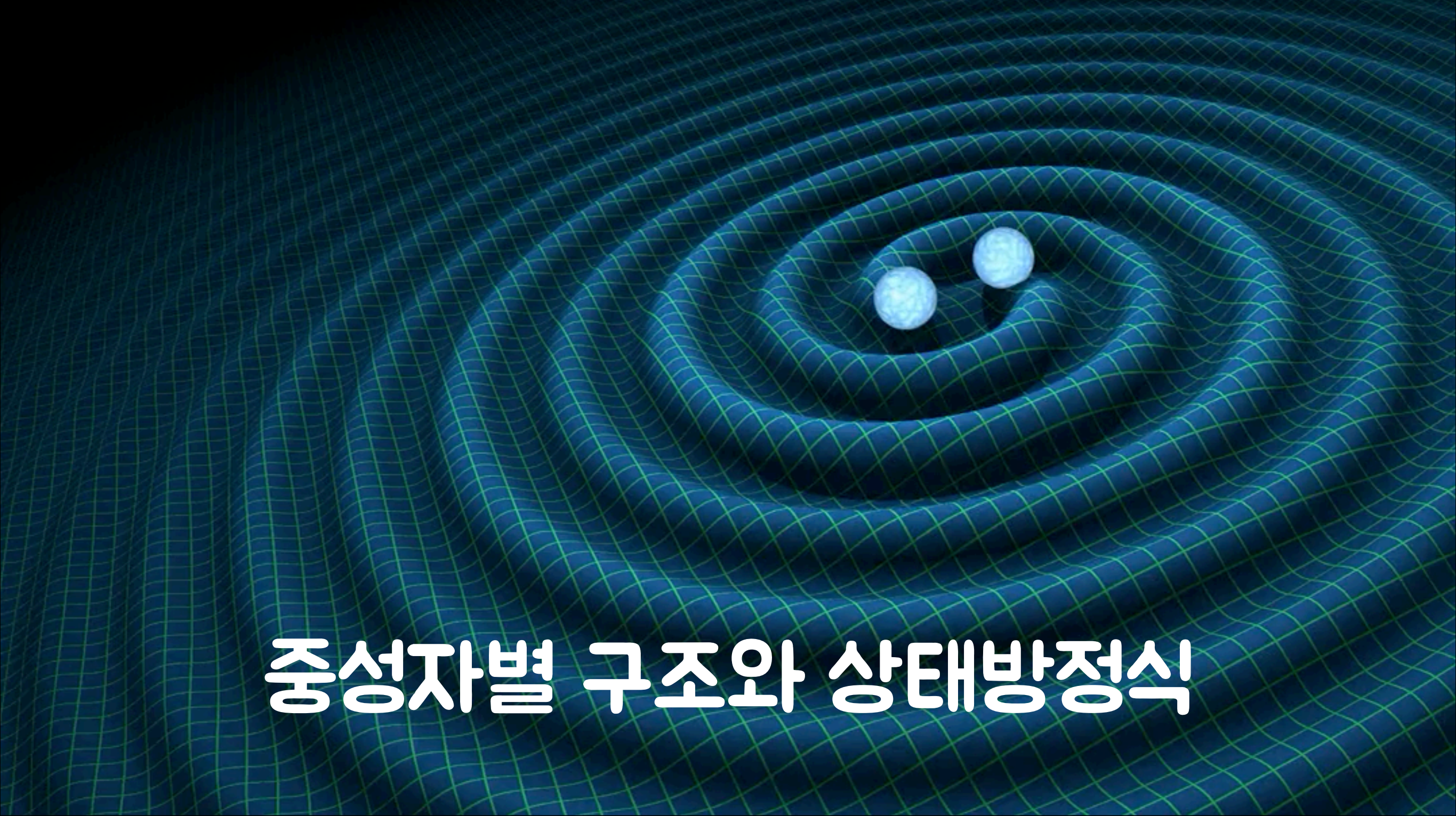
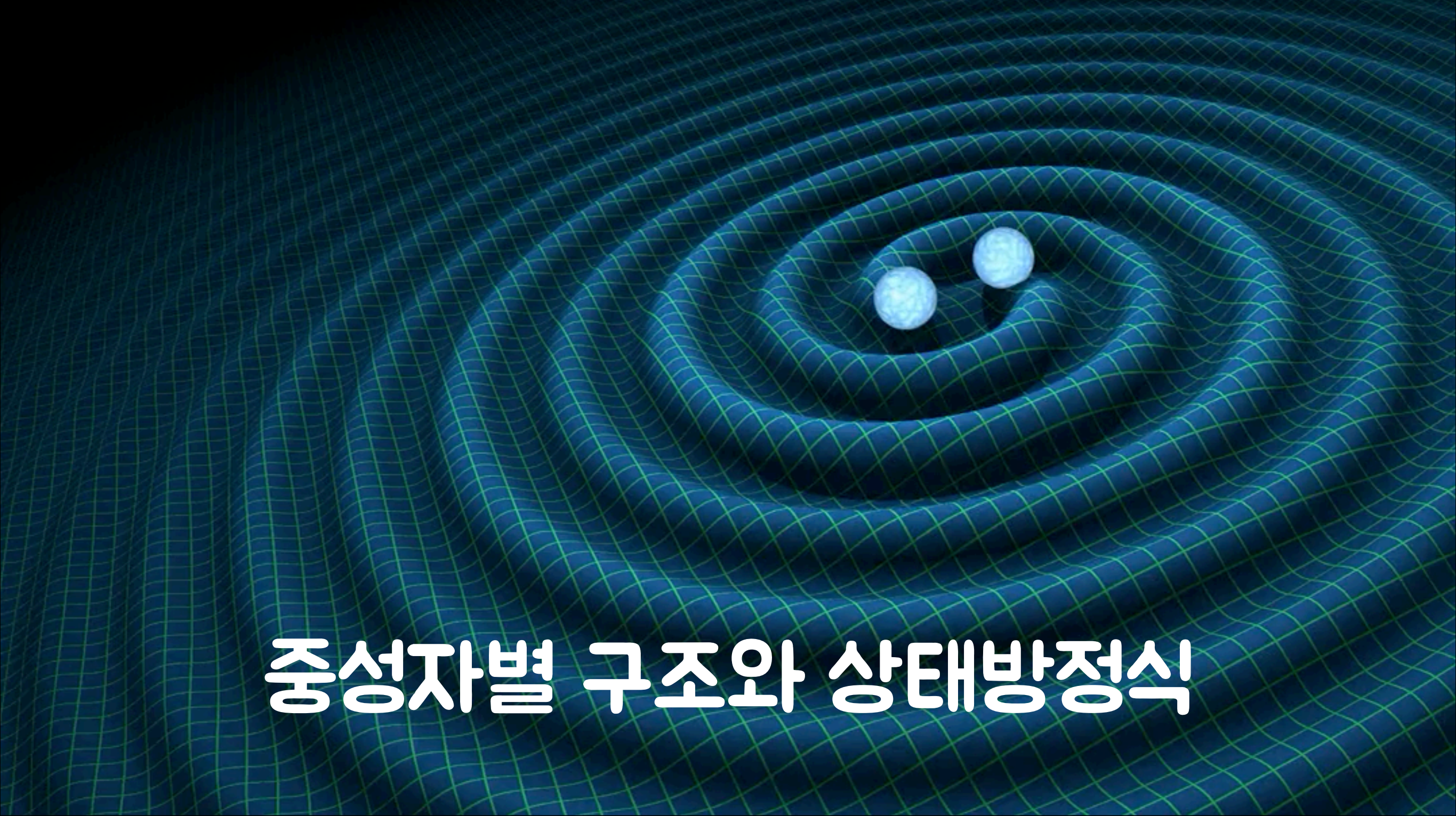




중성자별 구조와 상태방정식

김영민
한국천문연구원

연락처: ymkim@kasi.re.kr



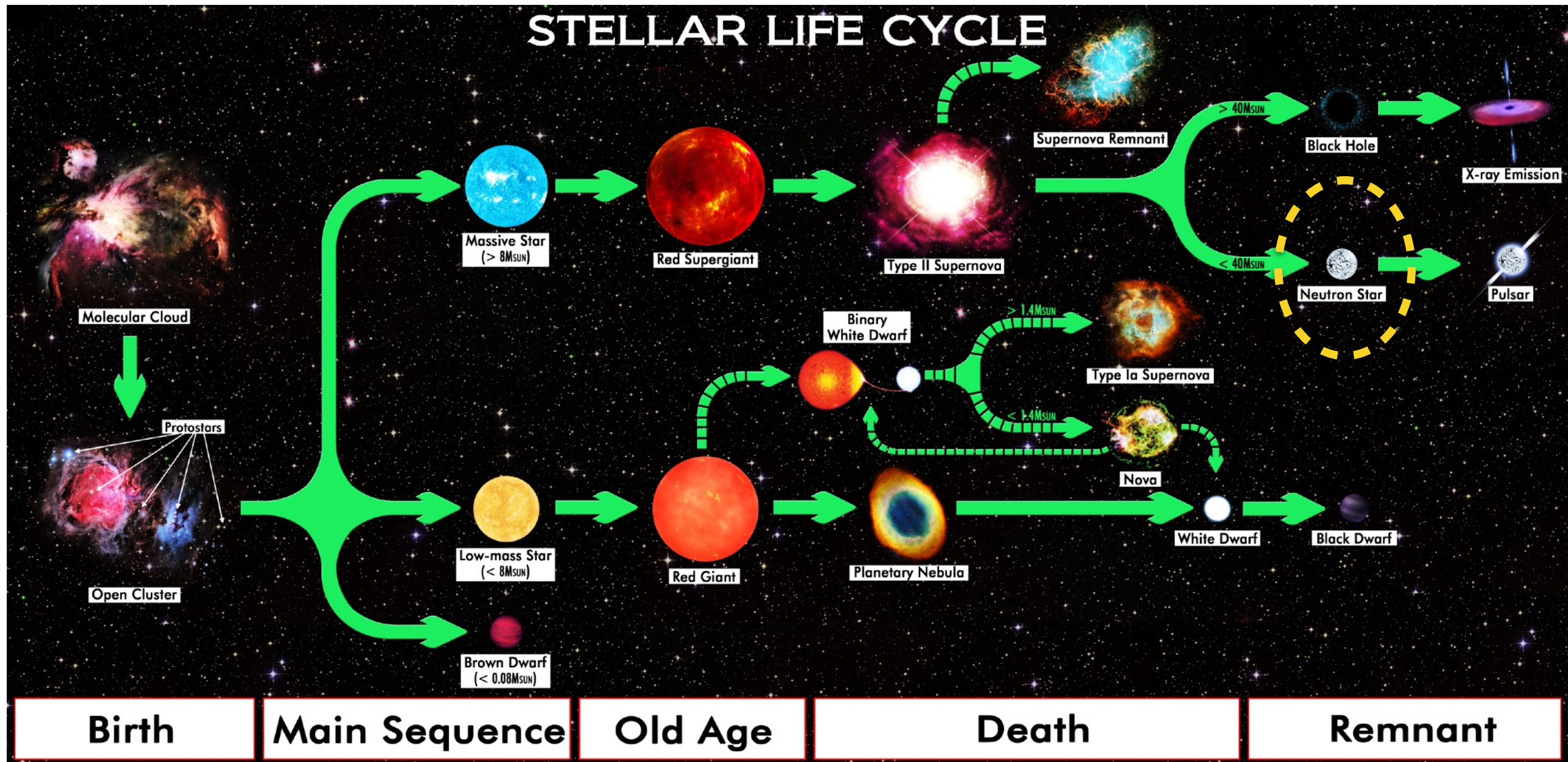
중성자별 구조와 상태방정식

김영민
한국천문연구원

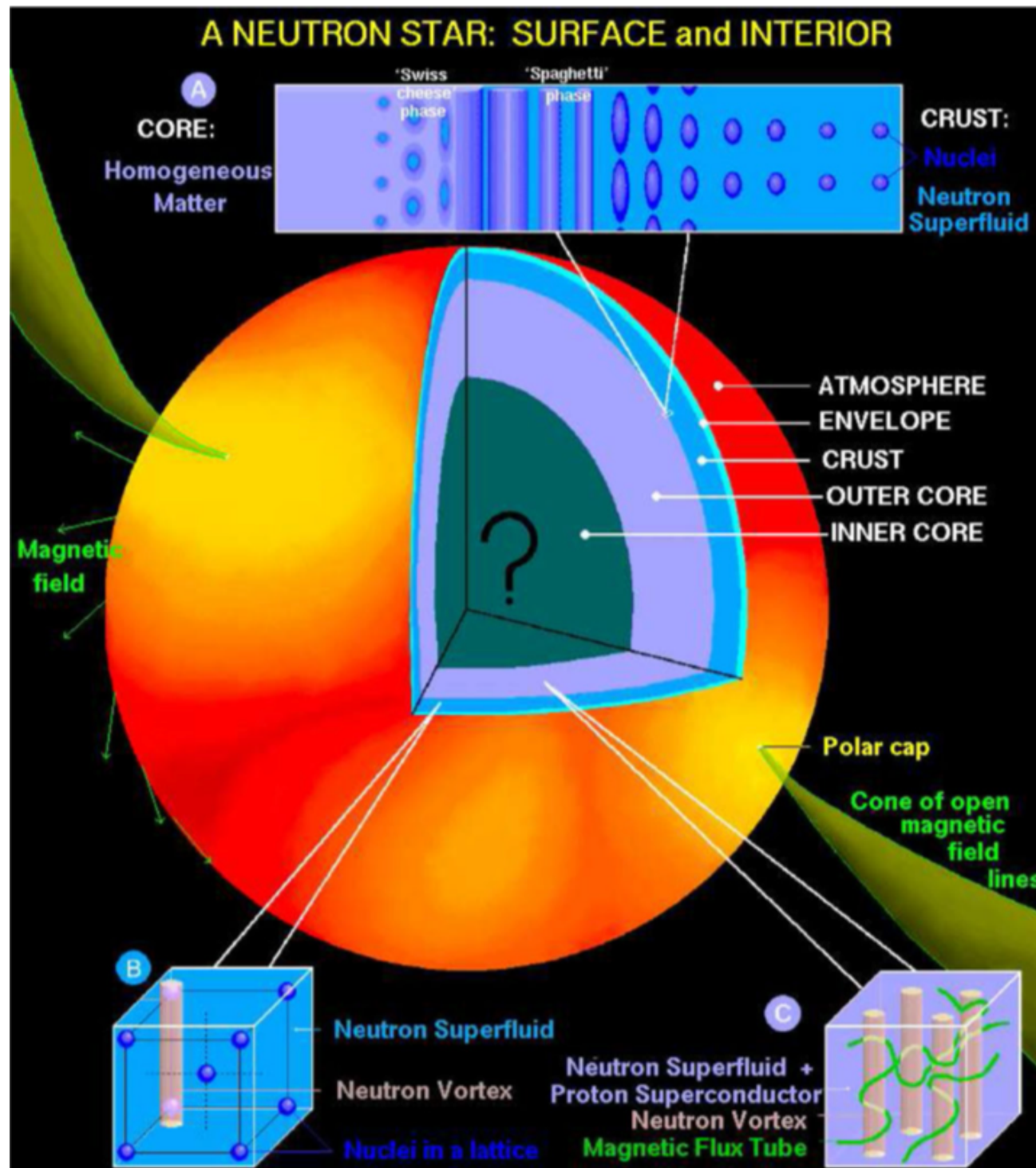
연락처: ymkim@kasi.re.kr

중성자별 (Neutron Star) 이란?

위키피디아/항성진화



Neutron Star



$$M = 1.4 \sim 2.0 M_{\odot}$$

$$R = 10 \sim 15 \text{ km}$$

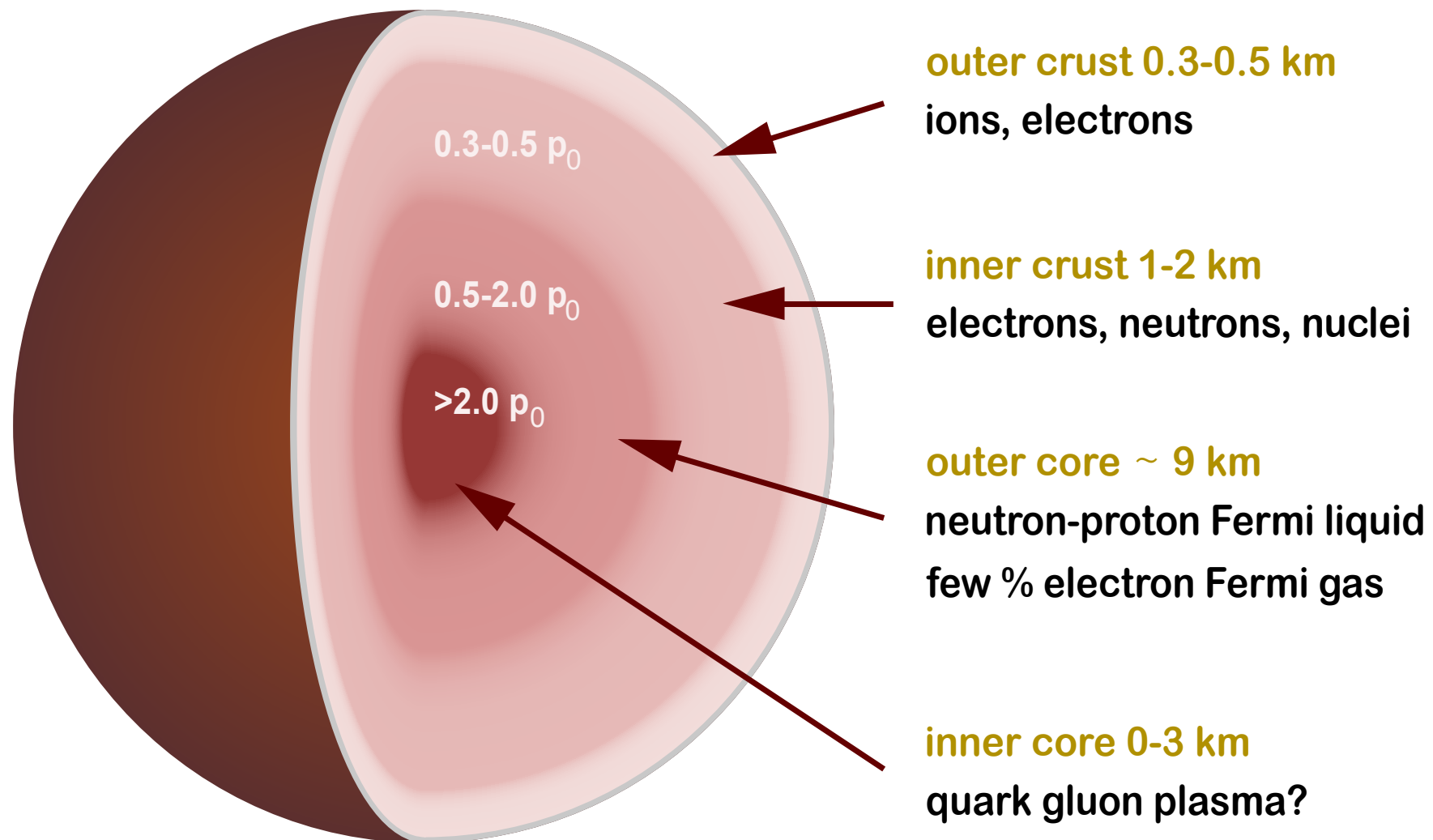
$$A \sim 10^{57} \text{ nucleons}$$

$$\rho_{\text{center}} \approx \text{several} \times \rho_0$$

$$n_0 \approx 0.16 \text{ fm}^{-3} \approx 1.6 \times 10^{44} \text{ m}^{-3}$$

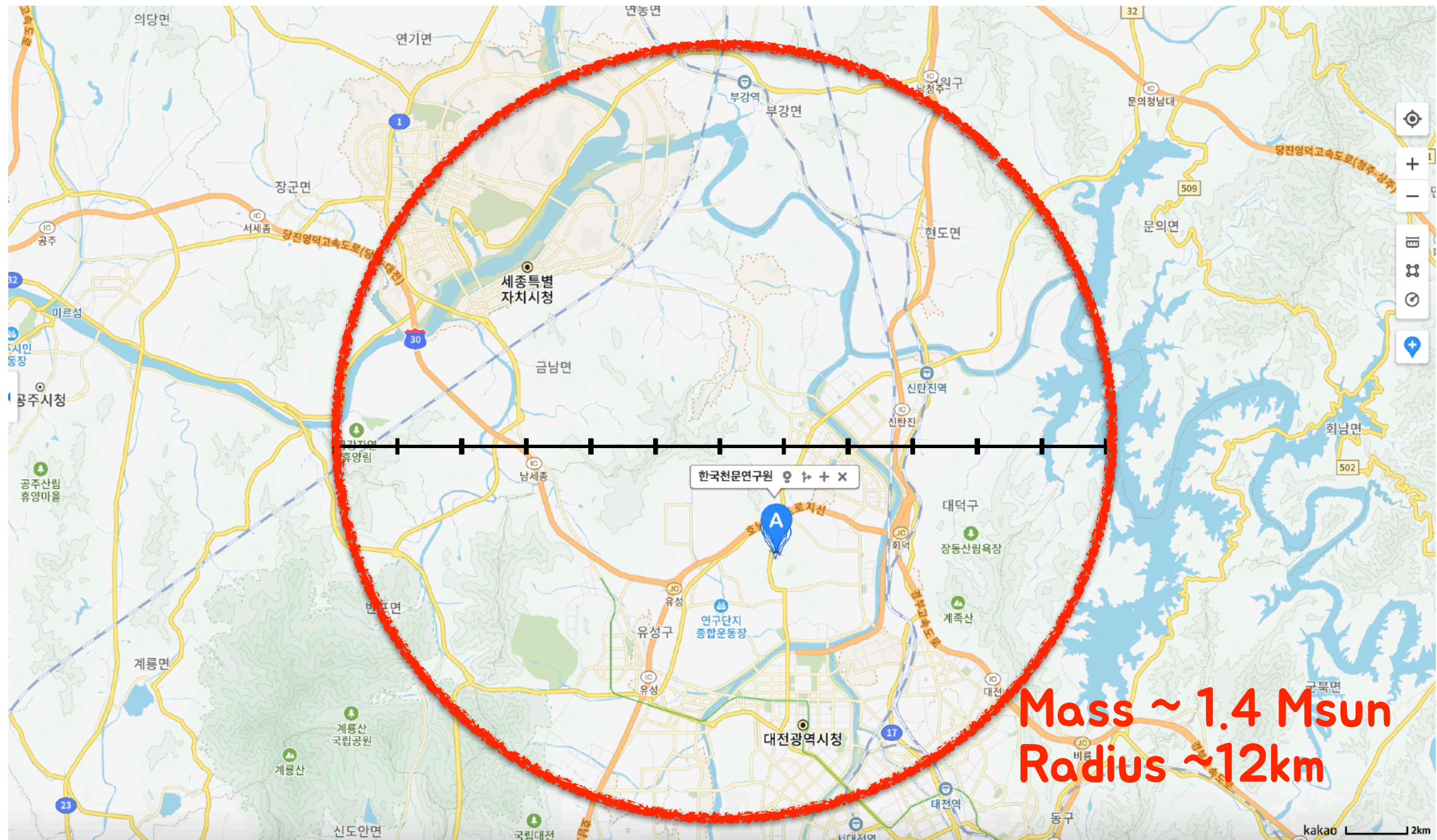
$$\rho_0 \approx 2.04 \times 10^{17} \text{ kg} \cdot \text{m}^{-3}$$

중성자별 (Neutron Star)

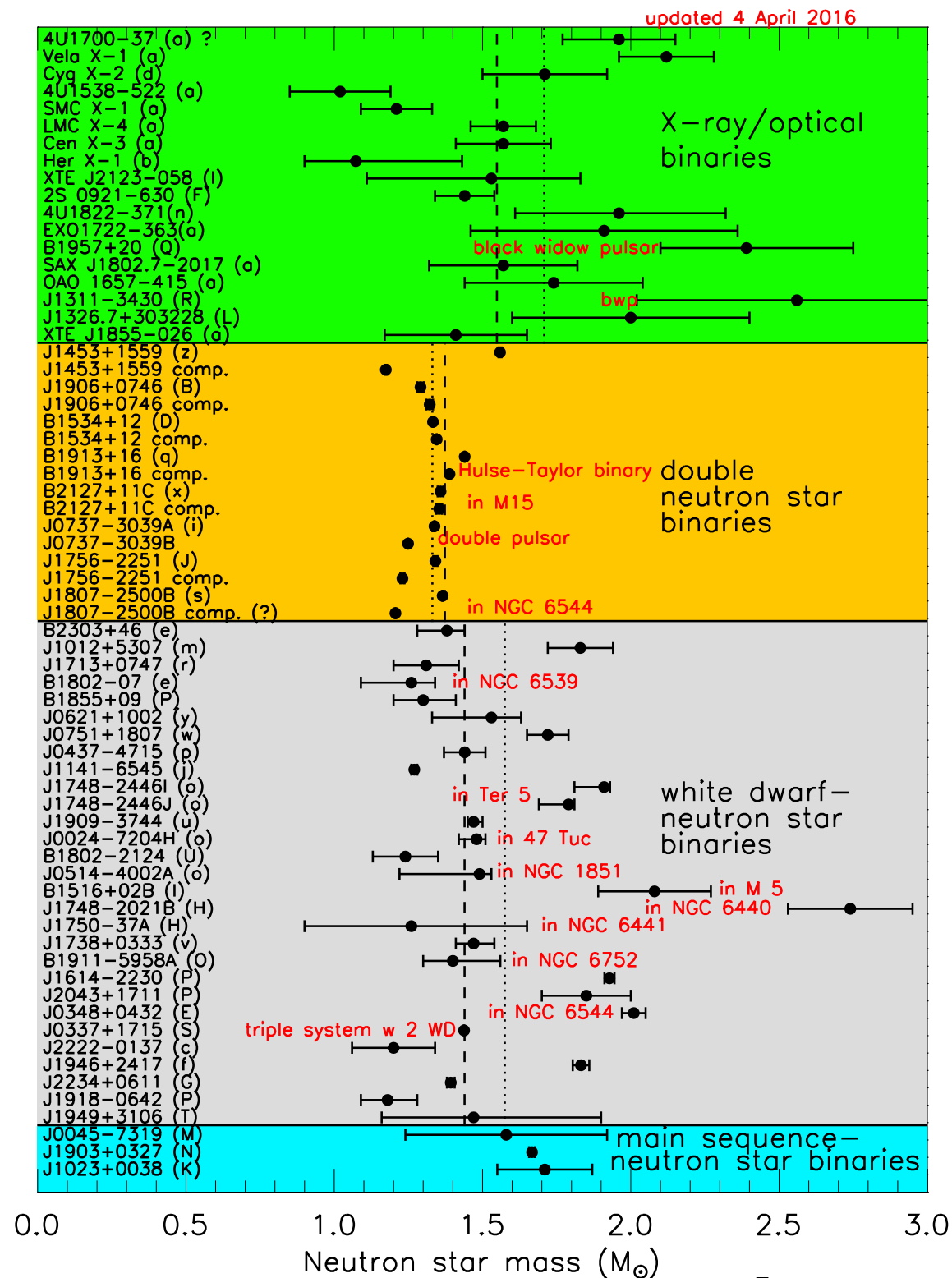


Wikipedia/Neutron_star

중성자별 (Neutron Star)

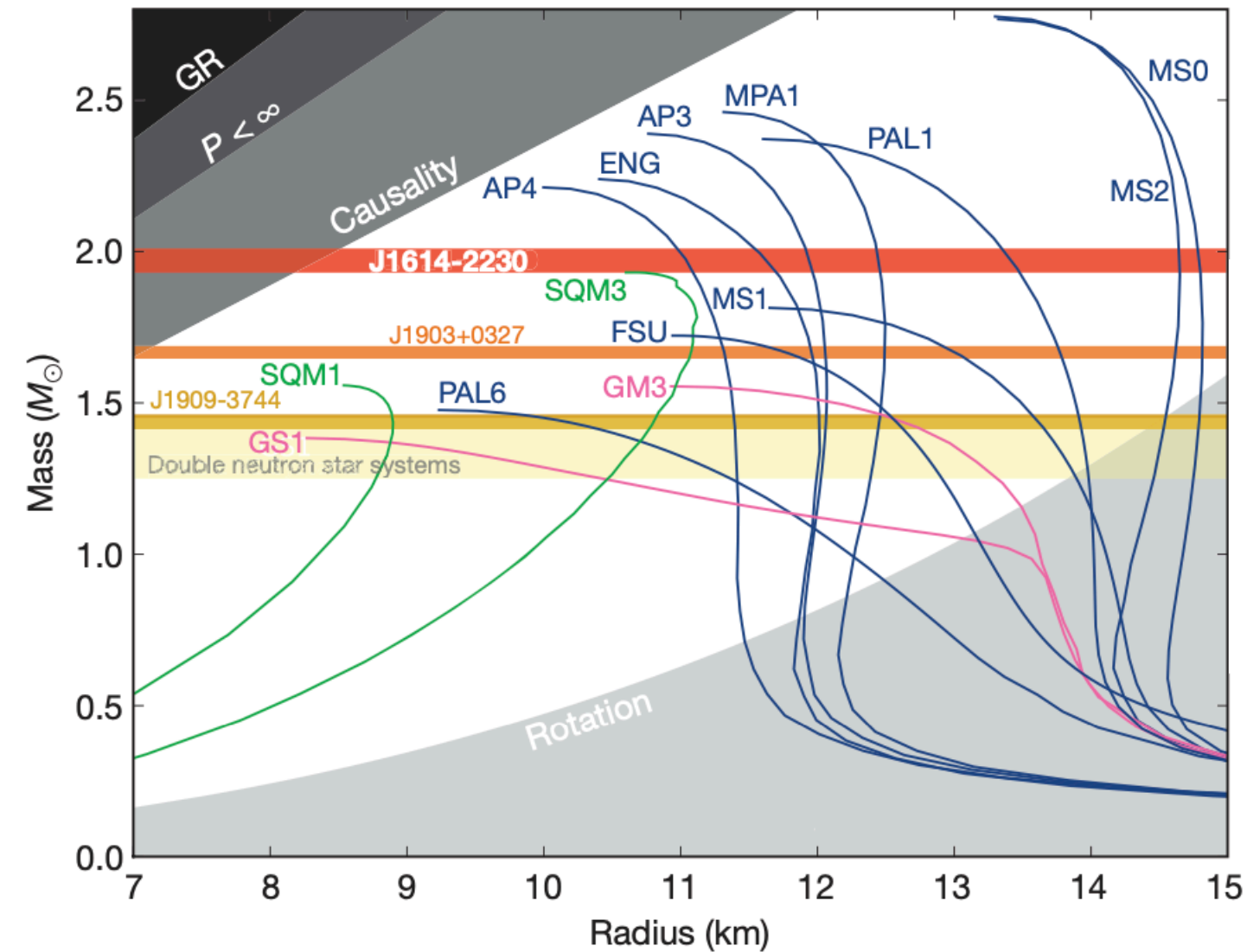


Neutron Star of Known Mass



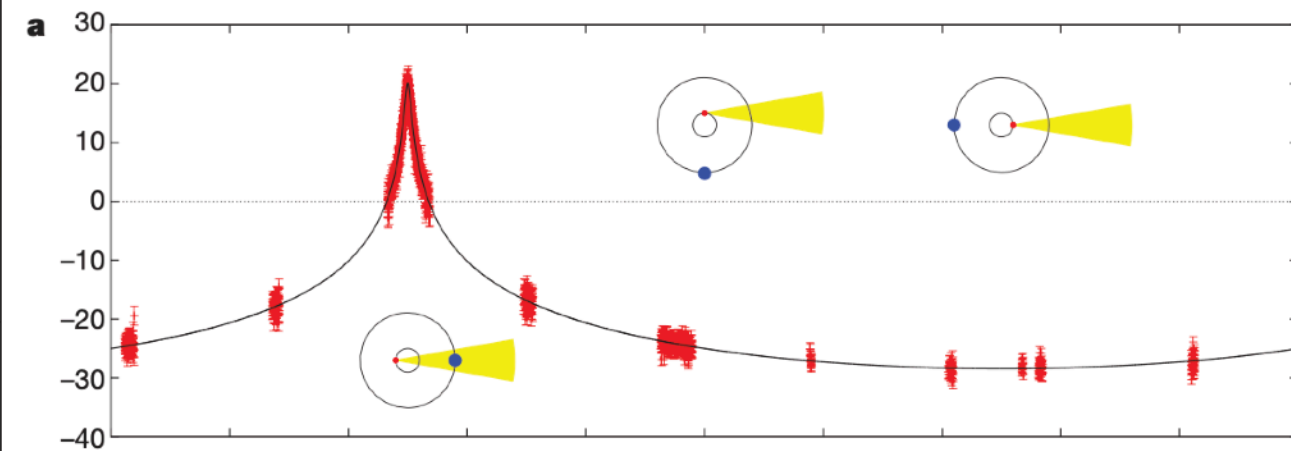
J. Lattimer,
 Annu.Rev.Nucl.Part.Sci.62,485(2012)
 and <https://stellarcollapse.org> by C. Ott

Maximum mass of Neutron Stars



Demorest et al. Nature (2010)

Shapiro delay



Low-Mass X-ray Binary

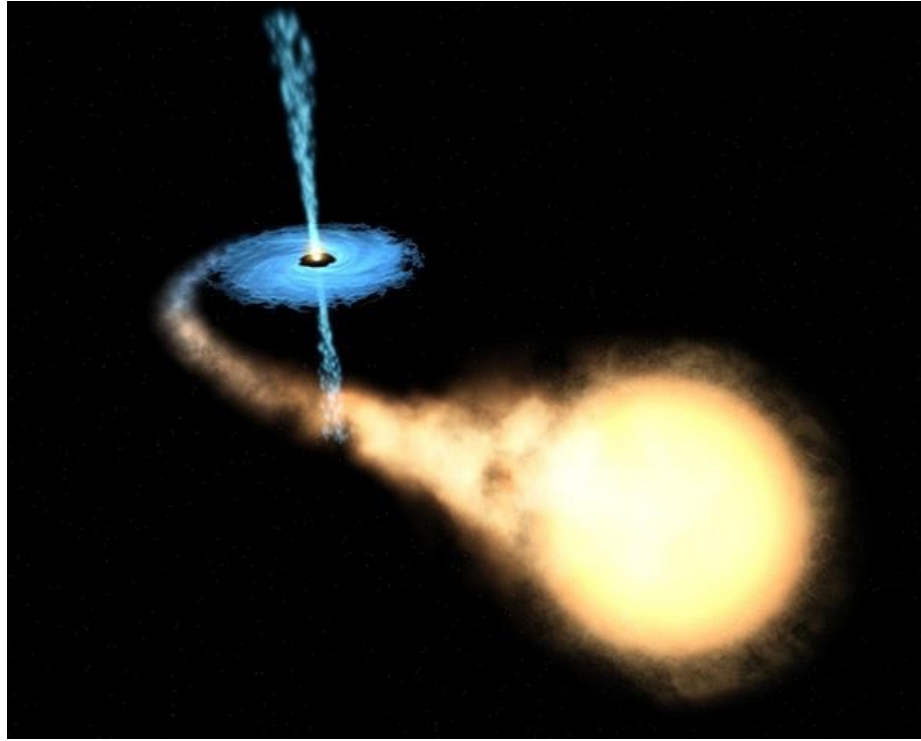
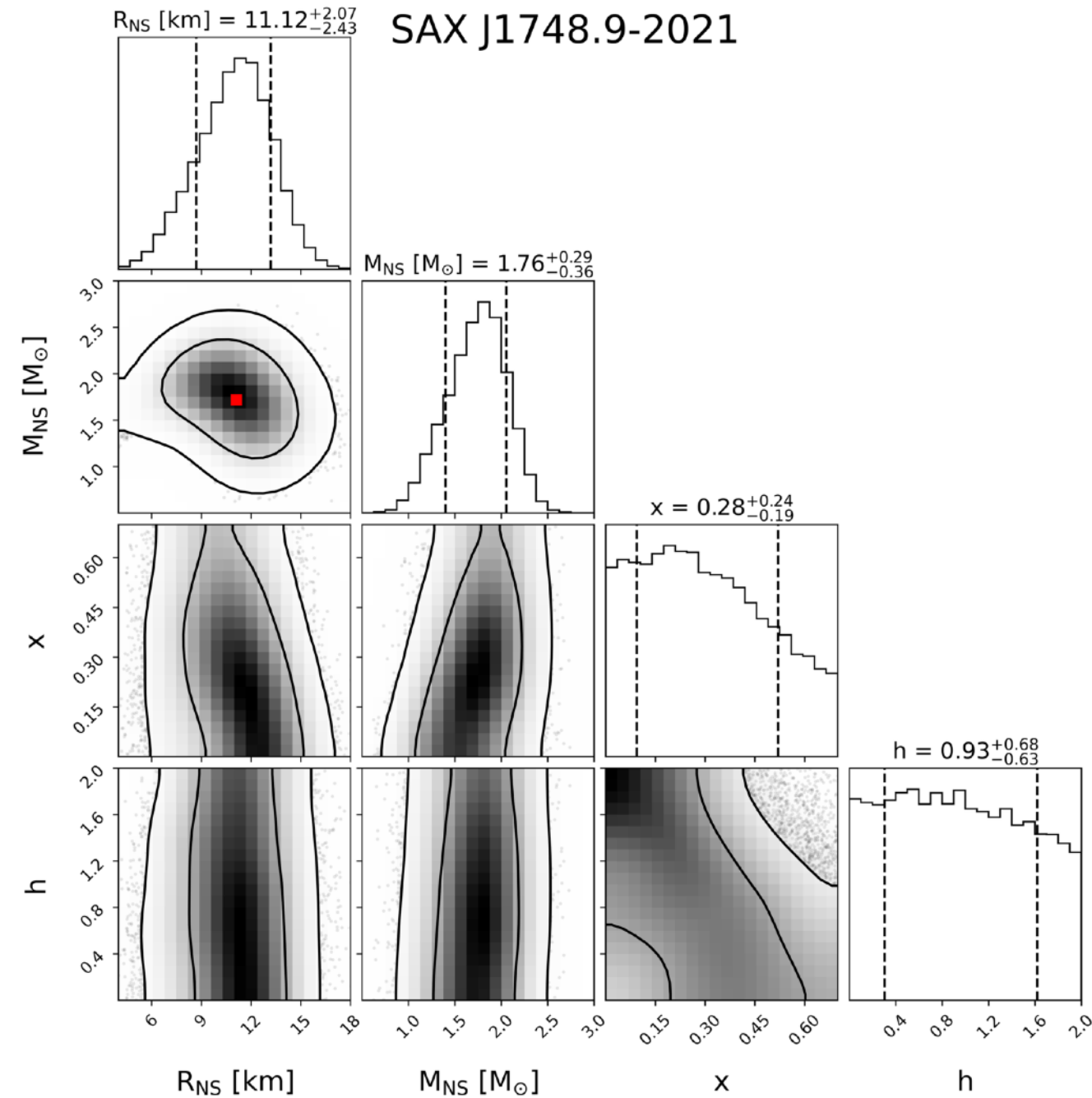


Table 9

Most Probable Values for Masses and Radii for Neutron Stars Constrained to Lie on One Mass Versus Radius Curve

Object	$r_{\text{ph}} = R$		$r_{\text{ph}} \gg R$	
	$M (M_{\odot})$	$R \text{ (km)}$	$M (M_{\odot})$	$R \text{ (km)}$
4U 1608–522	$1.52^{+0.22}_{-0.18}$	$11.04^{+0.53}_{-1.50}$	$1.64^{+0.34}_{-0.41}$	$11.82^{+0.42}_{-0.89}$
EXO 1745–248	$1.55^{+0.12}_{-0.36}$	$10.91^{+0.86}_{-0.65}$	$1.34^{+0.450}_{-0.28}$	$11.82^{+0.47}_{-0.72}$
4U 1820–30	$1.57^{+0.13}_{-0.15}$	$10.91^{+0.39}_{-0.92}$	$1.57^{+0.37}_{-0.31}$	$11.82^{+0.42}_{-0.82}$
M13	$1.48^{+0.21}_{-0.64}$	$11.04^{+1.00}_{-1.28}$	$0.901^{+0.28}_{-0.12}$	$12.21^{+0.18}_{-0.62}$
ω Cen	$1.43^{+0.26}_{-0.61}$	$11.18^{+1.14}_{-1.27}$	$0.994^{+0.51}_{-0.21}$	$12.09^{+0.27}_{-0.66}$
X7	$0.832^{+1.19}_{-0.051}$	$13.25^{+1.37}_{-3.50}$	$1.98^{+0.10}_{-0.36}$	$11.3^{+0.95}_{-1.03}$



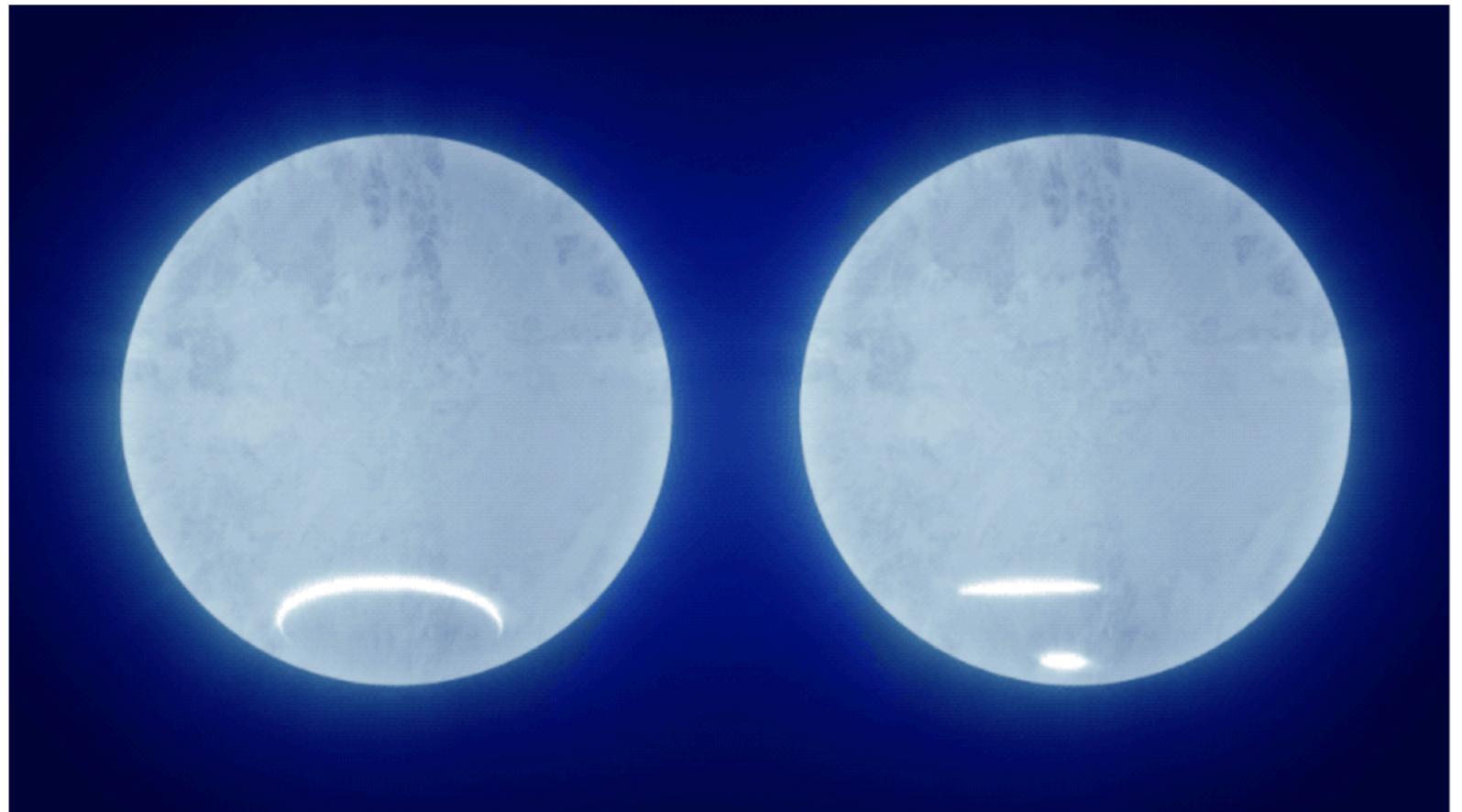
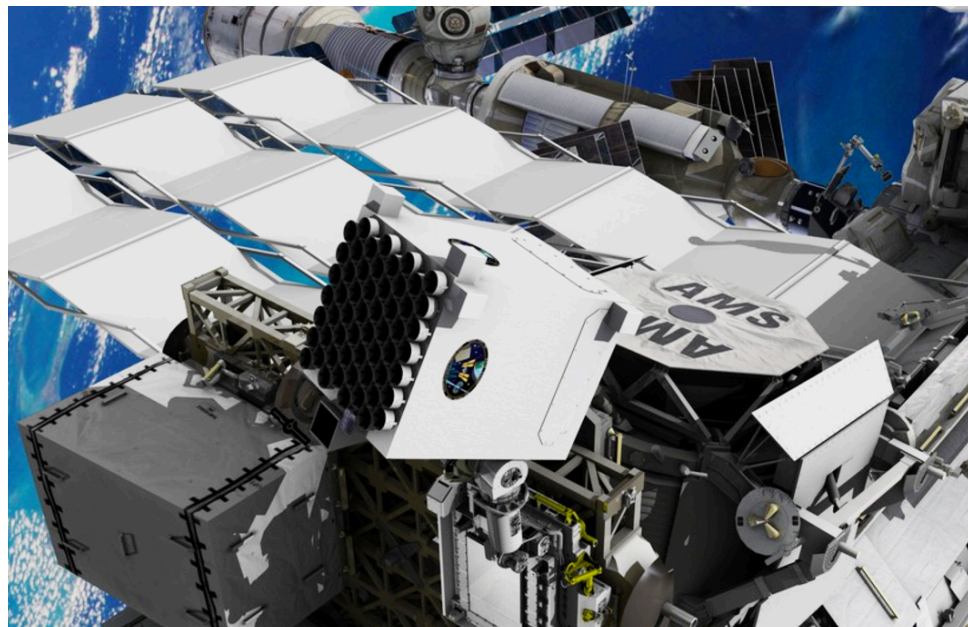
Kim et al., A&A 650, A139 (2021)

Observation by NICER

Focus on *NICER* Constraints on the Dense Matter Equation of State

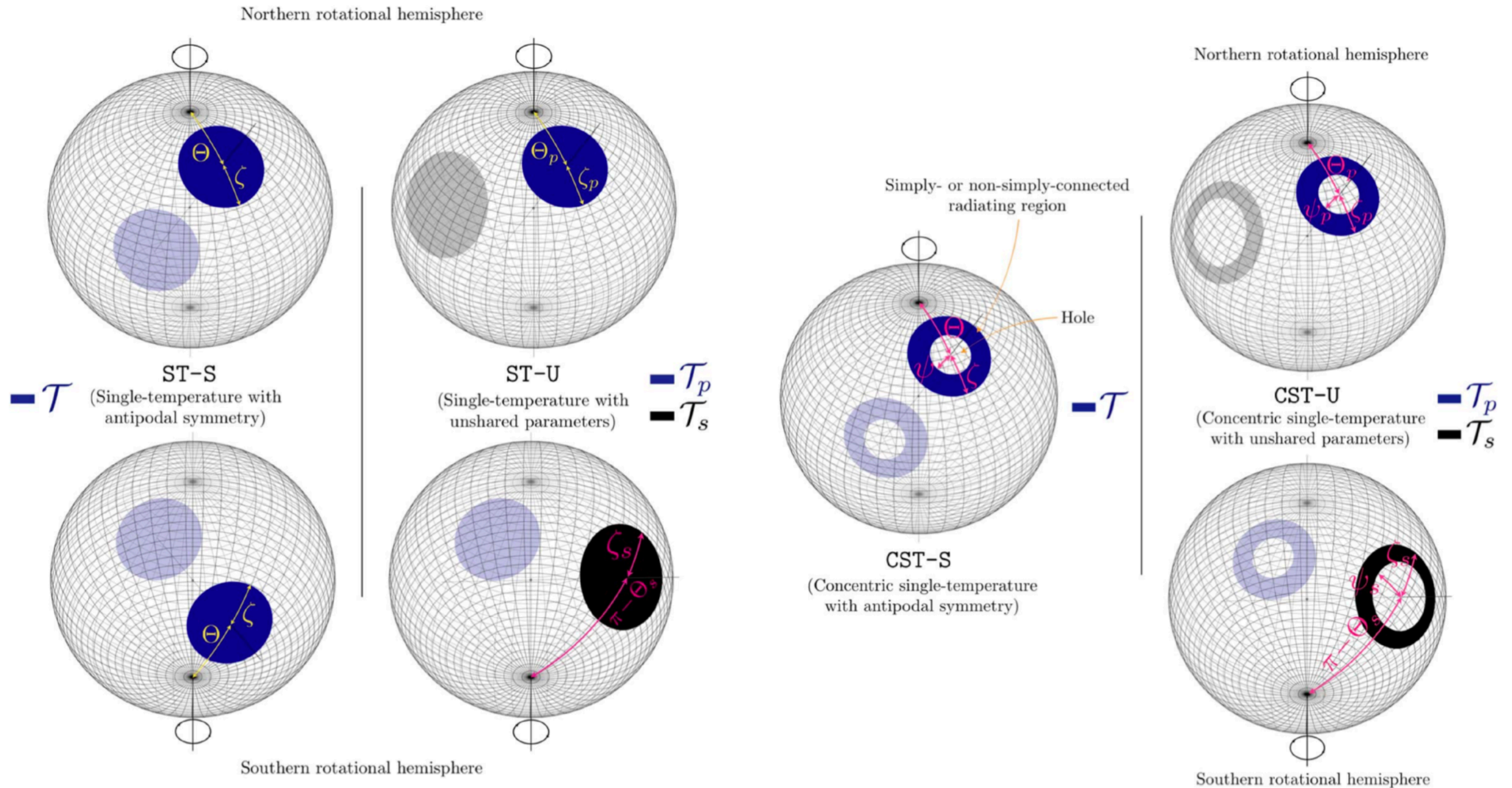
Zaven Arzoumanian & Keith C. Gendreau (NASA Goddard Space Flight Center)

December 2019



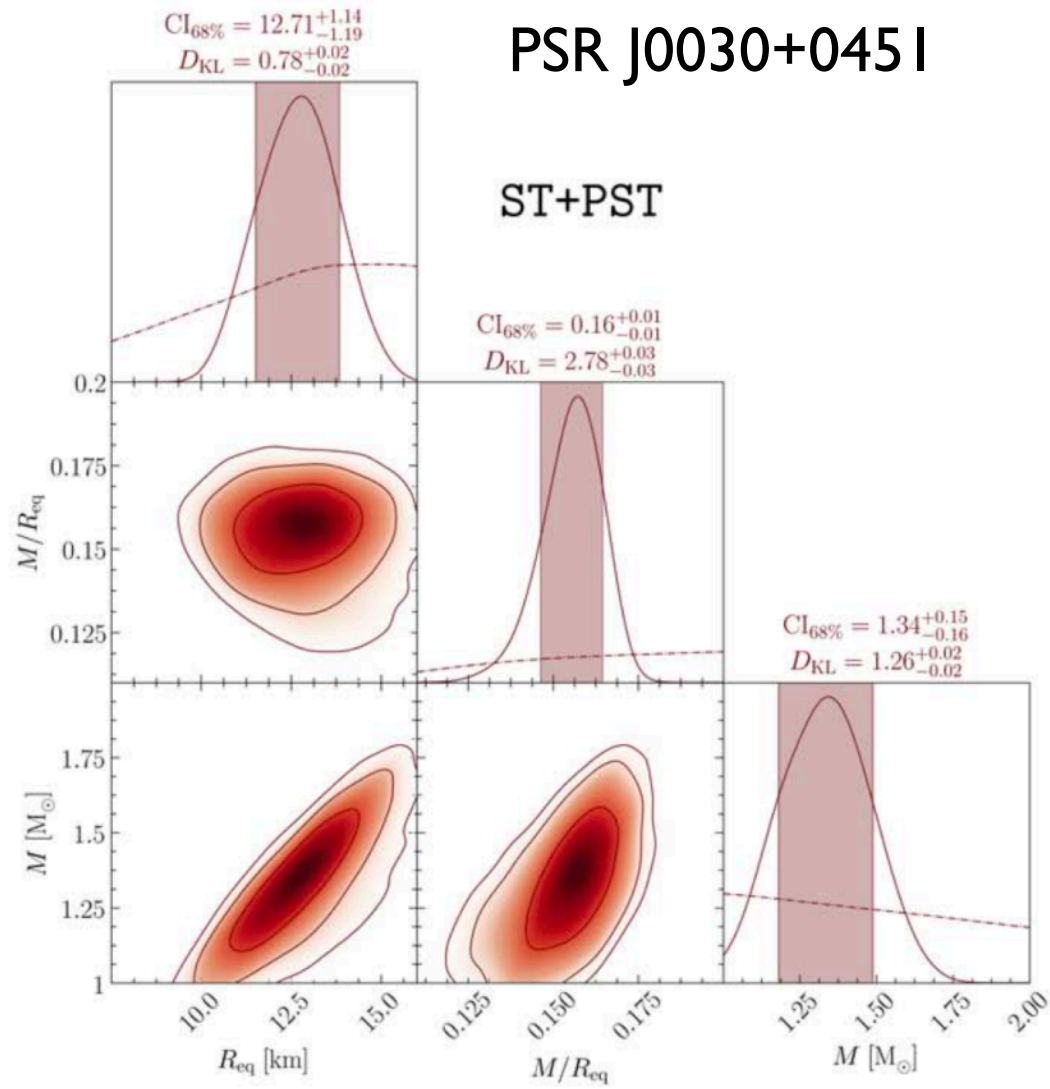
Hot spot region model in Riley 2019

Riley_2019_ApJL_887_L21



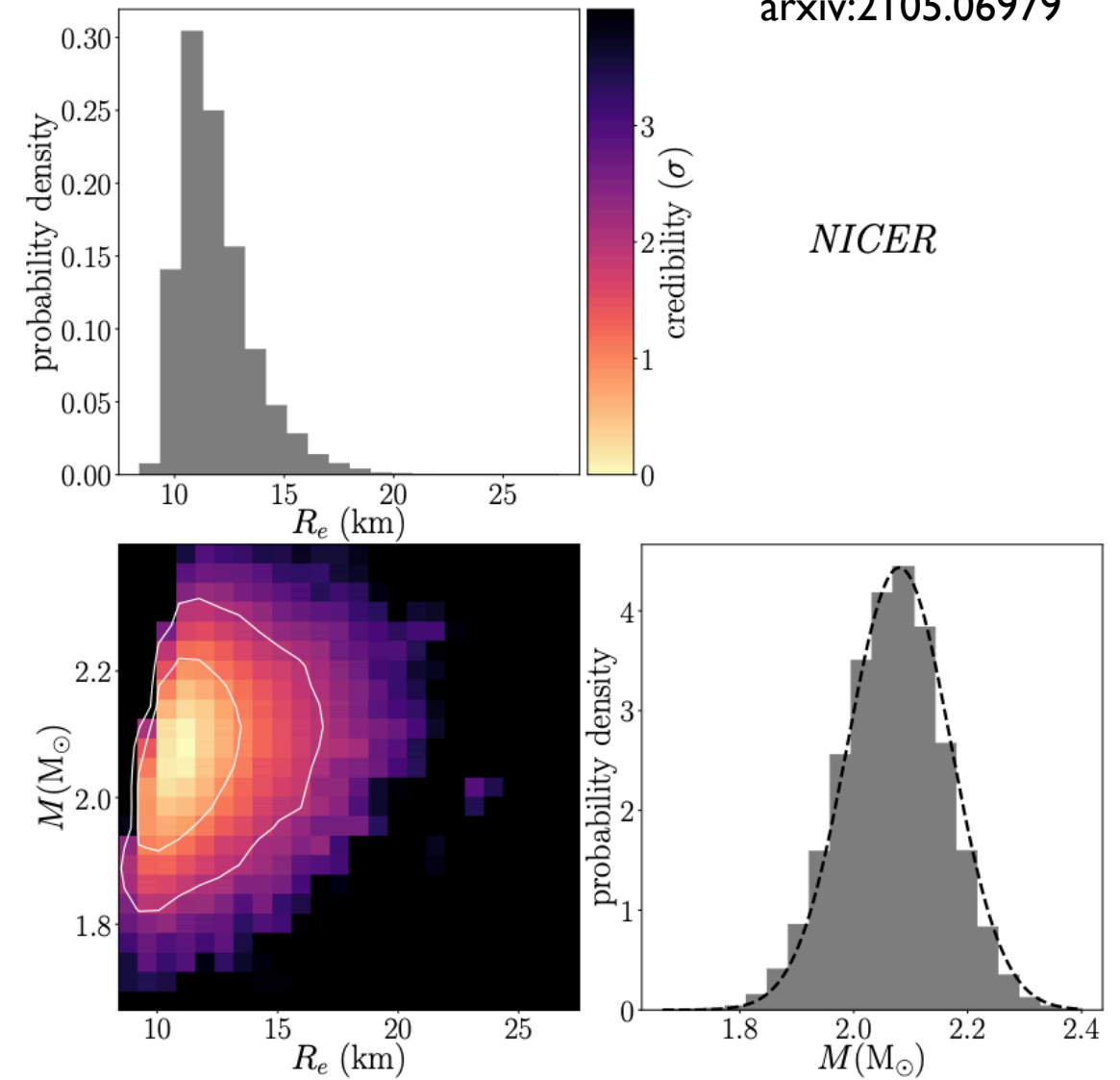
NICER observations

PSR J0030+045 I



PSR J0740+6620

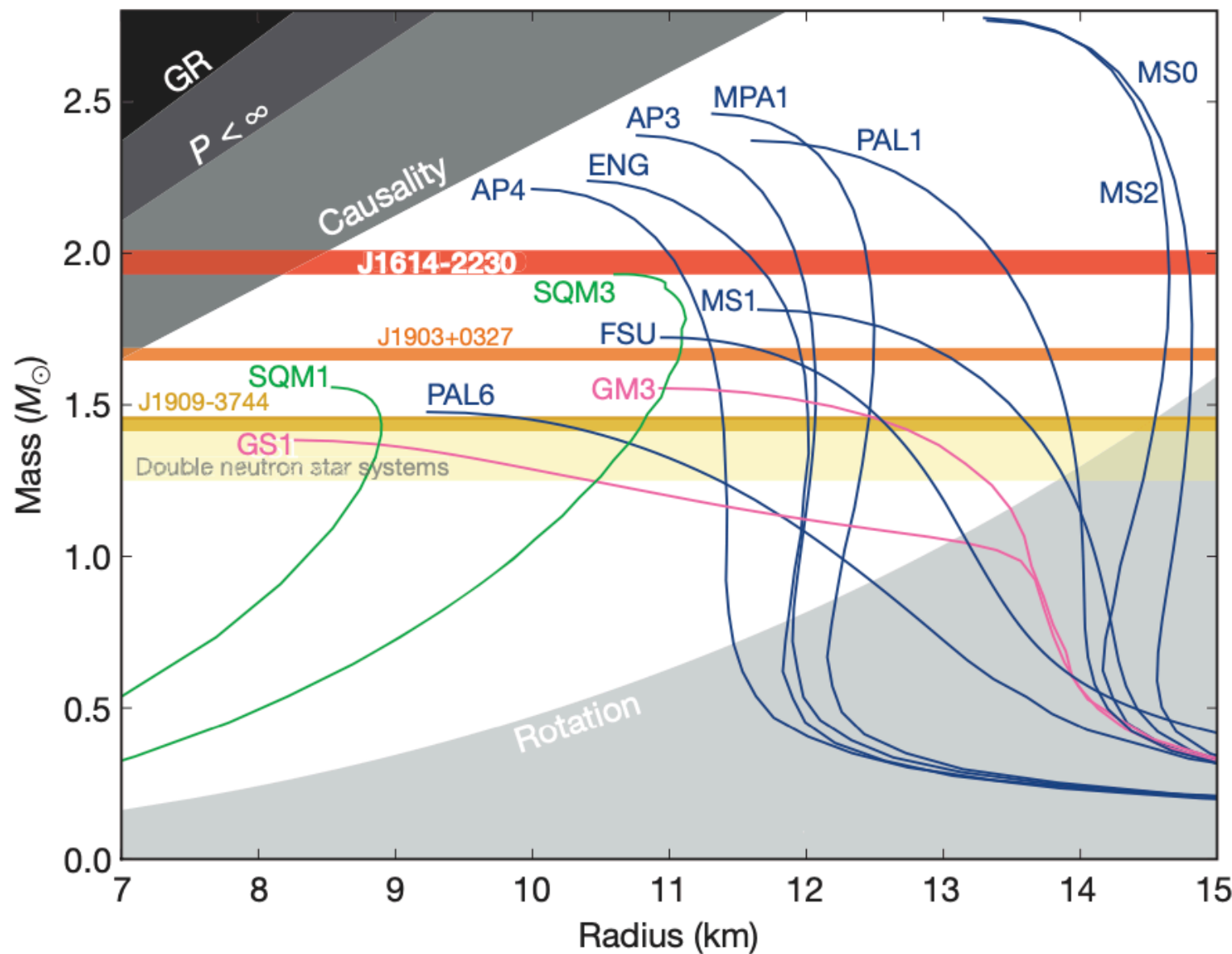
arxiv:2105.06979



Riley_2019_ApJL_887_L21

	Mass (M_{sun})	Radius (km)
Riley <i>et al.</i> ^[25]	$1.34^{+0.15}_{-0.16}$	$12.71^{+1.14}_{-1.19}$
Miller <i>et al.</i> ^[26]	$1.44^{+0.15}_{-0.14}$	$13.02^{+1.24}_{-1.06}$

Mass and Radius of Neutron Stars



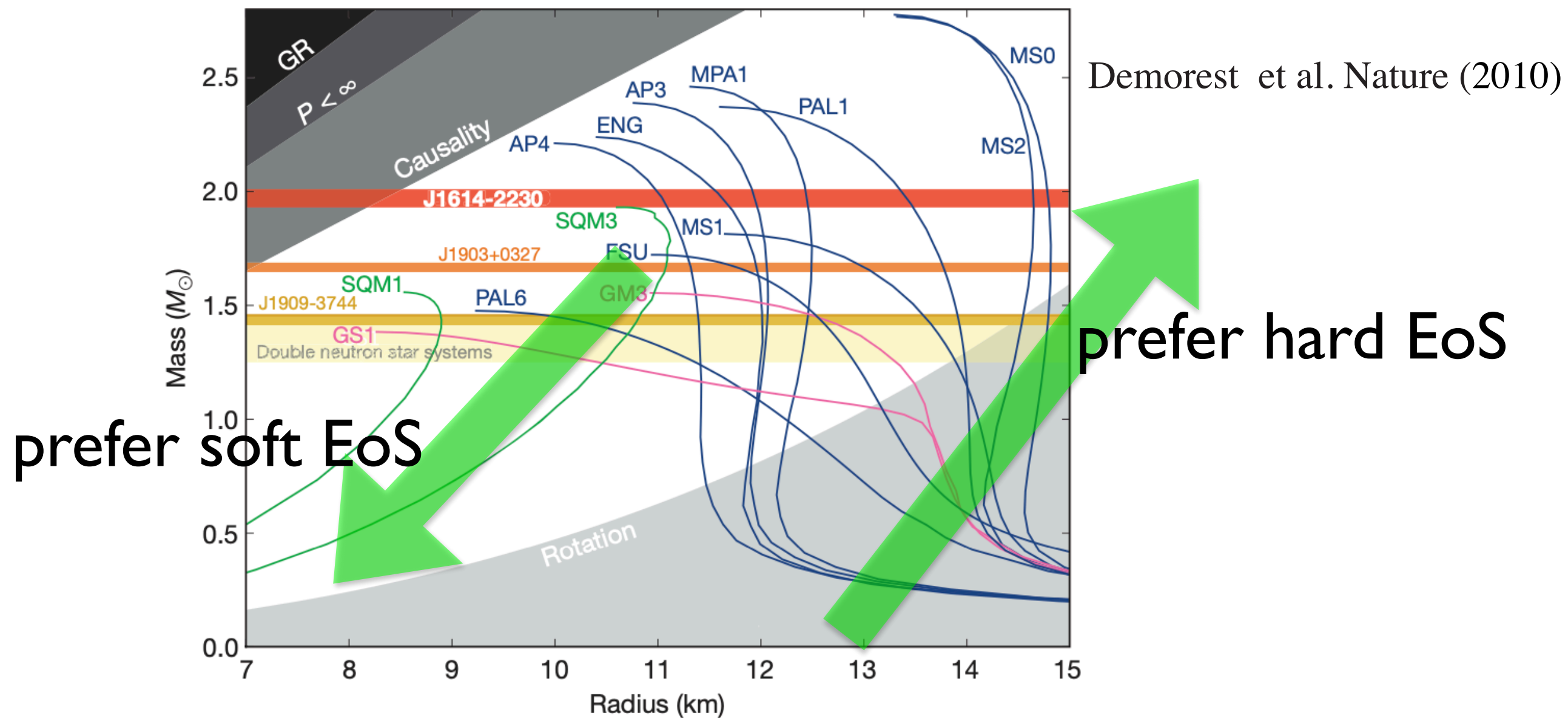
Demorest et al. Nature (2010)

TOV eq.

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \left(1 + \frac{P}{\rho c^2}\right) \left(1 + \frac{4\pi P r^3}{M c^2}\right) \left(1 - \frac{2GM}{rc^2}\right)$$

$$\frac{dM}{dr} = 4\pi r^2 \left(\frac{\epsilon}{c^2}\right) \quad ||$$

Mass and Radius of Neutron Stars



TOV eq.

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \left(1 + \frac{P}{\rho c^2}\right) \left(1 + \frac{4\pi P r^3}{M c^2}\right) \left(1 - \frac{2GM}{r c^2}\right)$$

$$\frac{dM}{dr} = 4\pi r^2 \left(\frac{\epsilon}{c^2}\right)$$

Deriving TOV eq.

Dervig TOV eq. (I)

MTW
ch. 23

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega^2$$

$$\rho(r), p(r), \eta(r)$$

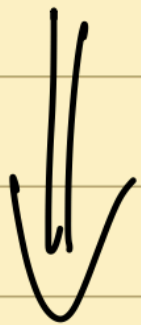
$u^\mu(r)$: 4-velocity of fluid

$$G = C = 1$$

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$$

Derving TOV eq. (2)

$$\nabla_\mu T^{\mu\nu} = 0$$



Energy-Momentum
conservation
in Hydrostatic
equilibrium

$$\frac{dp}{dr} + (p + \rho) \frac{d}{dr} \Phi(r) = 0.$$

Derviving TOV eq. (3)

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$G_{00} \rightarrow \frac{1}{r^2} \frac{d}{dr} [r(1 - e^{-2\Lambda})] = 8\pi \rho$$

$$2m \equiv r(1 - e^{-2\Lambda}), \quad e^{2\Lambda} = \left(1 - \frac{2m}{r}\right)^{-1}$$

Derving TOV eq. (4)

$$G_{00} \rightarrow \frac{2}{r^2} \frac{d m(r)}{dr} = 8\pi f$$

Integrator
→

$$m(r) = \int_0^r 4\pi \bar{r}^2 f d\bar{r} + m(0)$$

Derviving TOV eq. (5)

$$G_m \rightarrow -\frac{1}{r^2} + \frac{1}{r^2} e^{-2\Lambda} + \frac{2}{r} e^{-2\Lambda} \frac{d\Phi}{dr}$$

$$= 8\pi p$$

$$e^{-2\Lambda} = (1 - 2m/r)$$

$$\Rightarrow \frac{d\Phi}{dr}(r) = \frac{m + 4\pi r^3 p}{r(r - 2m)}$$

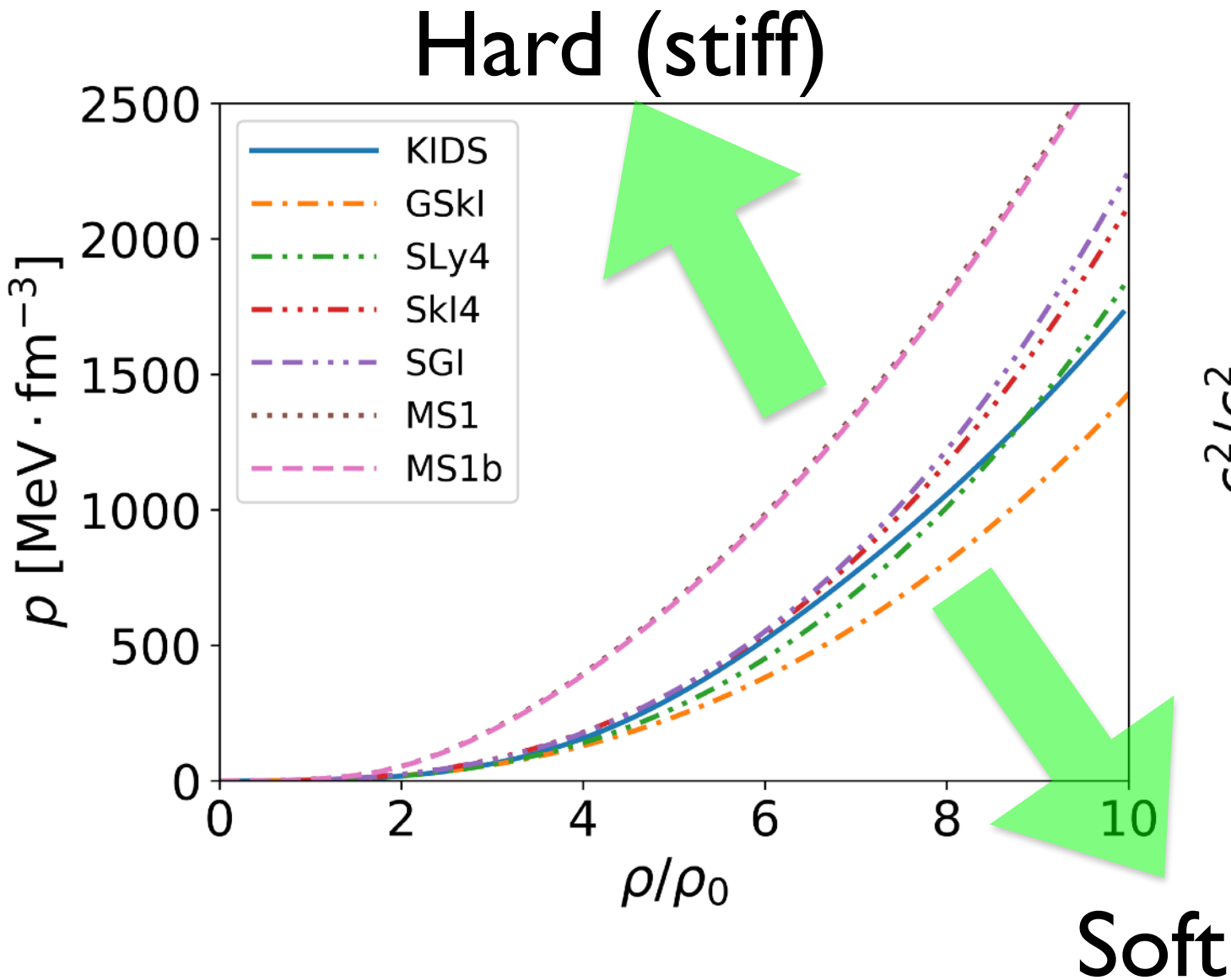
Derving TOV eq. (6)

$$\frac{d p(r)}{dr} = -(\rho + p) \frac{d \Phi(r)}{dr}$$

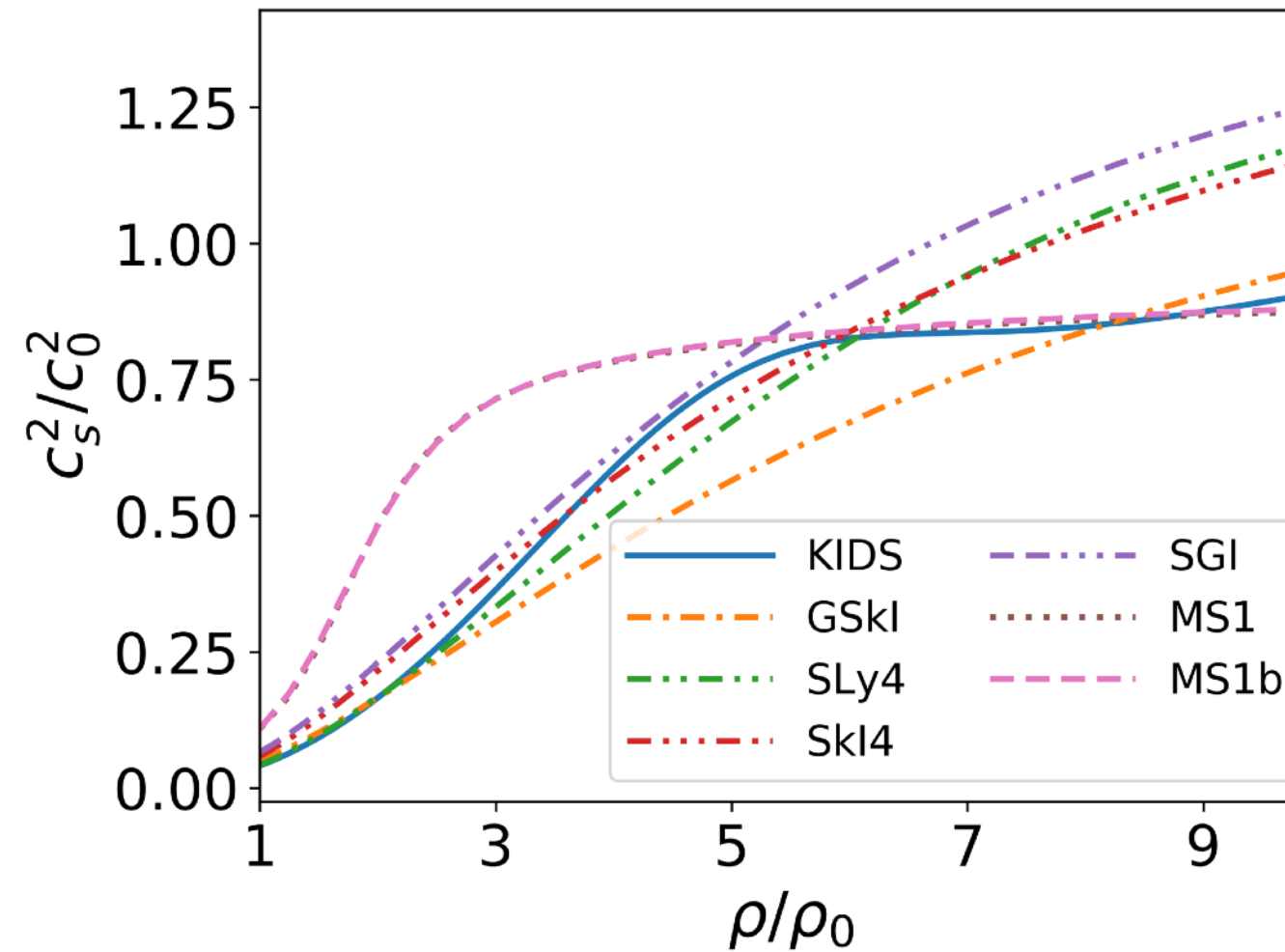
$$= -(\rho + p) \frac{m + 4\pi r^3 \rho}{r(r - 2m)}$$

$$\frac{d}{dr} m(r) = 4\pi r^2 \rho$$

Equation of State (EoS)



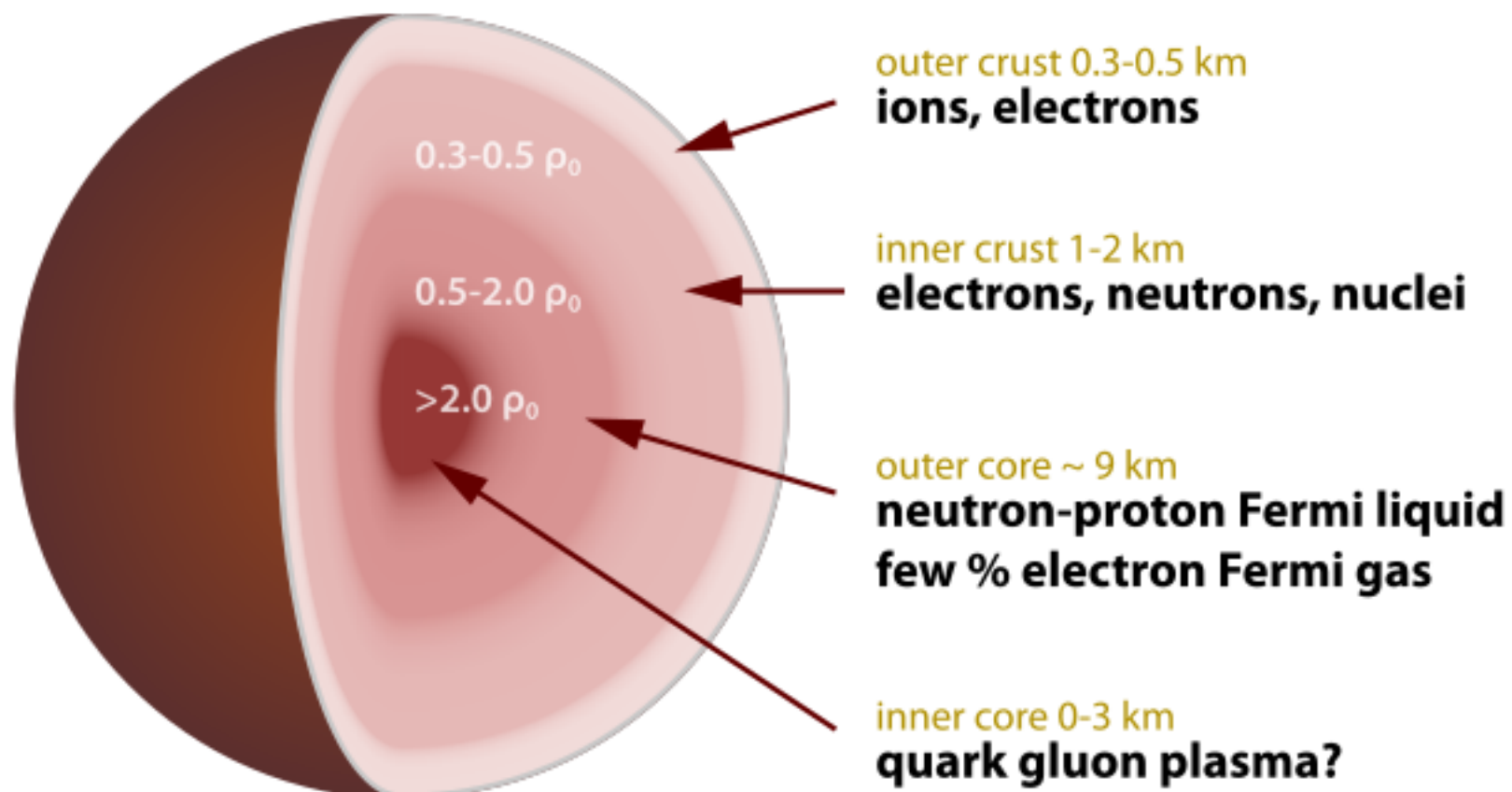
Y.-M. Kim et al. (EPJA 2020)



$$\frac{c_s^2}{c_0^2} = \frac{dp}{d\epsilon} = \Gamma \frac{p}{\epsilon + p}$$

Equation of State (EoS)

- I. Equation of state : a thermodynamic equation relating state variables (P,V,T, E) which describe the state of matter under a given set of physical conditions. - from Wikipedia
- NS is cold on the nuclear scale (1 MeV $\sim 10^{10}$ K) : frozen in the point of view of hadronic and leptonic compositions.

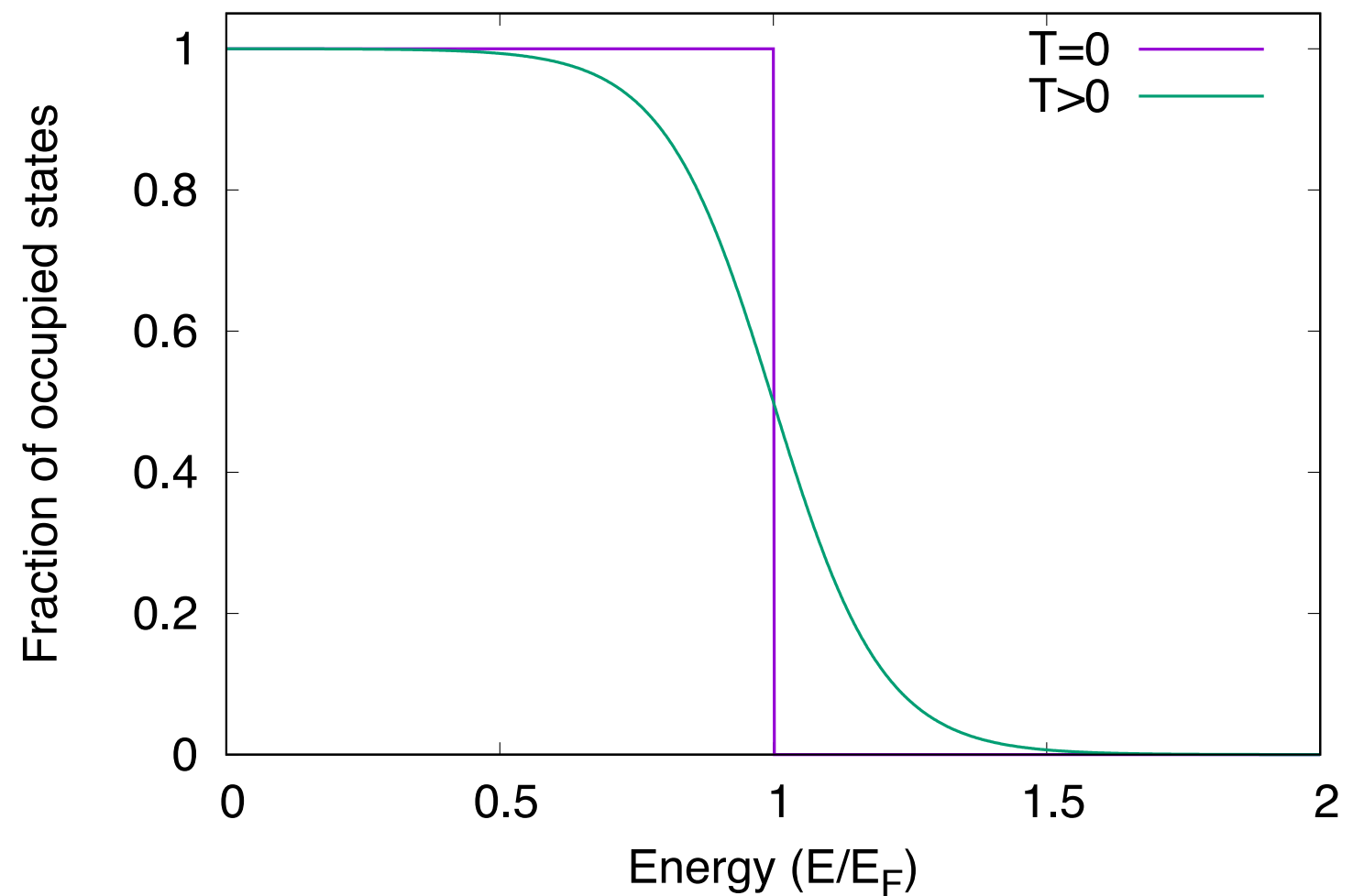


Degenerate Fermi Gas

$$f(E) = \frac{1}{e^{(E-\mu)/kT} + 1}$$

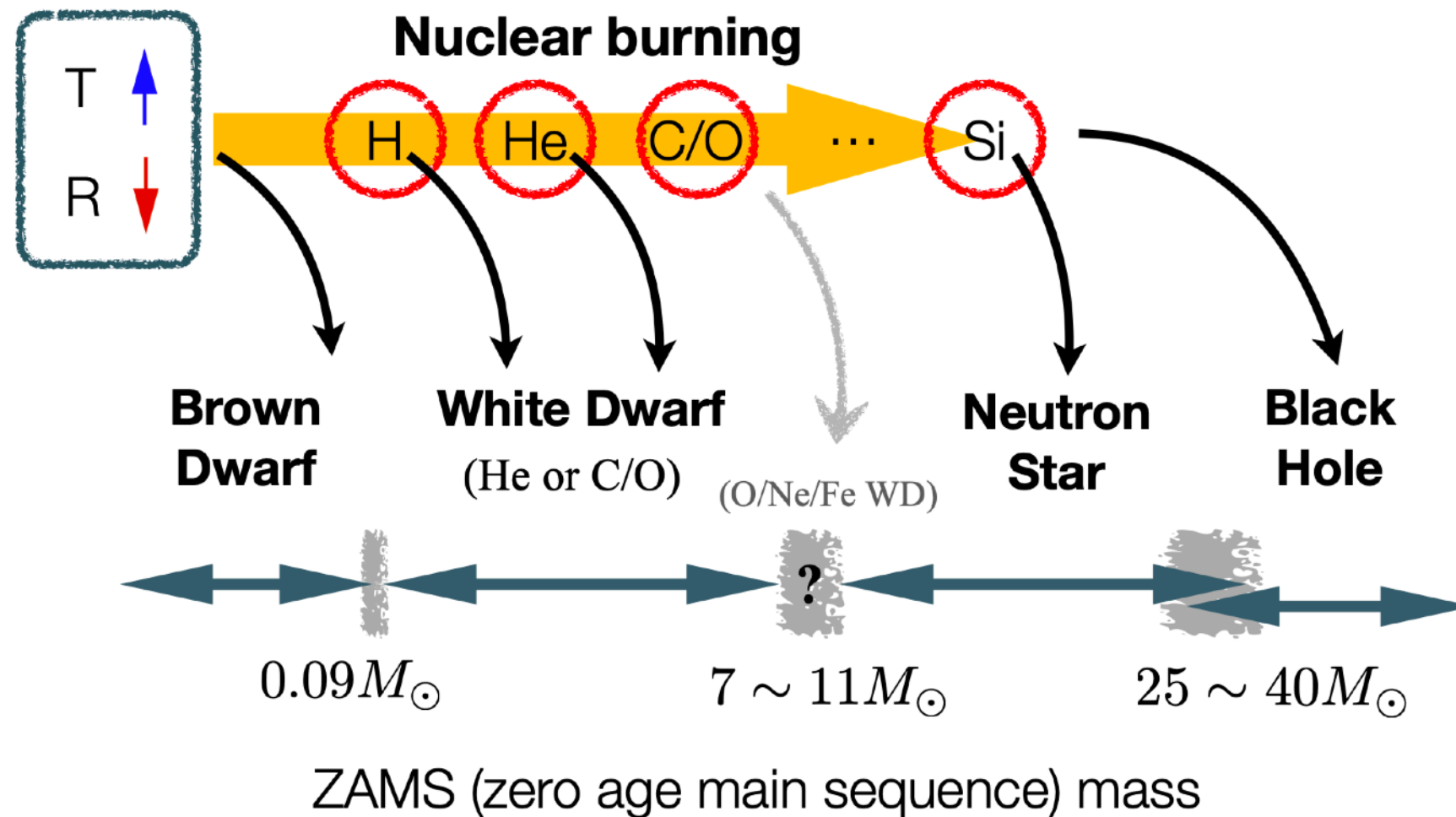
$$T \rightarrow 0$$

$$f(E) = \begin{cases} 1, & E \leq E_F \\ 0, & E > E_F \end{cases}$$



Degenerate pressure of Fermi gas

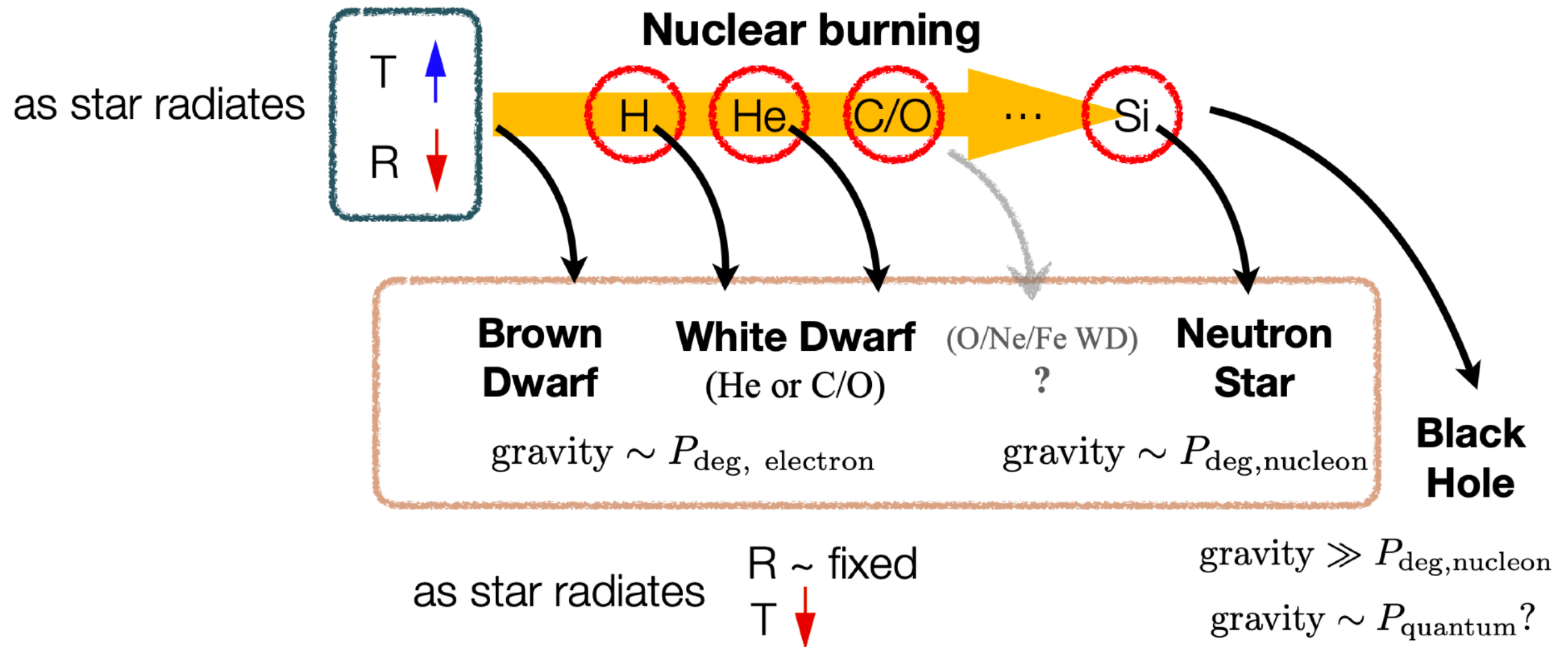
Initial mass of progenitor star (with solar metallicity)



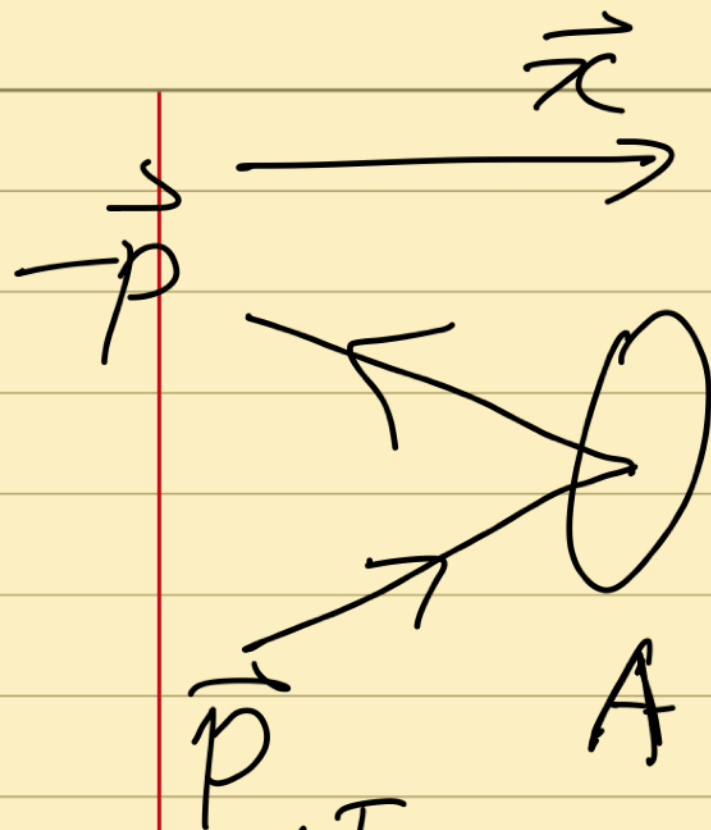
Slide from C.Lee (PNU)'s lecture

Degenerate pressure of Fermi gas

Evolution of star



Degenerate Pressure of electron gas (I)



$$\Delta P_x = 2 P_x$$

$$v_x = \frac{dx}{dt}$$

$$\frac{dF_x}{dA} = \frac{2P_x}{dA dt} = \frac{2P_x}{dA (dx / v_x)} = \frac{2P_x v_x}{dV}$$

Degenerate Pressure of electron gas (2)

pressure

$$P_e = \int_0^\infty \frac{dN(p)}{2} \frac{2p_x v_x}{dV} dp$$

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{3} v_x$$

$$p = \sqrt{3} p_x$$

$$v p = 3 p_x v_x$$

Degenerate Pressure of electron gas (3)

$$\Rightarrow P_e = \frac{1}{3} \int_0^{\infty} n_e(p) p v dp$$

$$dN(p) dp = 2 \times \frac{d^3 p dV}{h^3} \quad \text{if } |\vec{p}| \leq p_F$$

$$\Rightarrow n_e(p) dp = \frac{8\pi}{h^3} p^2 dp$$

Degenerate Pressure of electron gas (4)

$$\Rightarrow P_e = \frac{8\pi}{3h^3} \int_0^{P_F} v p^3 dp$$

$$p = \gamma m v, \quad E = \gamma m c^2$$

$$v = \frac{p}{m\gamma} = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{p^2 c^2 + m^2 c^4}}$$

$$\Rightarrow P_e = \frac{8\pi}{3h^3} \int_0^{P_F} \frac{p^4 c^2}{\sqrt{p^2 c^2 + m_e^2 c^4}} dp$$

Degenerate Pressure of electron gas (5)

$$\chi \equiv P/mc$$

$$P_e = \frac{8\pi m_e^4 c^5}{3h^3} \int_0^{\chi_F} \frac{\chi^4}{\sqrt{1+\chi^2}} d\chi$$

$$= \frac{\pi m_e^4 c^5}{h^3} \left[\chi_F (1+\chi_F^2)^{1/2} \left(\frac{2}{3} \chi_F^2 - 1 \right) + \ln \left[\chi_F + (1+\chi_F^2)^{1/2} \right] \right]$$

Degenerate Pressure of electron gas (6)

$$\chi_F \ll 1$$

$$P_e = \frac{8\pi m_e^4 c^5}{15 h^3} \left[\chi_F^5 - \frac{5}{14} \chi_F^7 + \frac{5}{24} \chi_F^9 - \dots \right]$$

$$\rightarrow P_e \sim \frac{8\pi m_e^4 c^5}{15 h^3} \chi_F^5 \quad \left(\chi_F = \frac{p_F}{m_e} \right)$$

non-relativistic

$$p_F \sim \rho^{1/3}$$

$$\Rightarrow P_e \sim \rho^{5/3} \Rightarrow P = K_1 \rho^{5/3}$$

Degenerate Pressure of electron gas (7)

$$\chi_F \gg 1$$

$$P_e = \frac{2\pi m_e^4 c^5}{3h^3} \left[\chi_F^4 - \chi_F^2 + \frac{3}{2} \ln(2\chi_F) + \dots \right]$$

$$\rightarrow P_e \sim \frac{2\pi m_e^4 c^5}{3h^3} \chi_F^4$$

like-wise

$$P_e \sim \rho^{4/3} \Rightarrow P = K_2 \rho^{4/3}$$

for fully relativistic.

Another way to obtain P_e

Partition function

$$Q = \ln(1 + e^{-\beta(E-\mu)})$$

Ideal Fermi Gas

$$PV = k_B T \sum_i g_i \ln[1 + e^{-\beta(E_i - \mu)}]$$

where $\beta \equiv 1/k_B T$
 $\mu \equiv$ chemical potential

$$N \equiv \sum_i \langle n_i \rangle = \sum_{E_i} \frac{1}{1 + e^{\beta(E_i - \mu)}}$$

$$V \rightarrow h^3$$

$$P = k_B T \int_0^\infty \frac{4\pi p^2 dp}{h^3} \ln[1 + e^{-\beta(E - \mu)}]$$

$$N = \int_0^\infty \frac{4\pi V p^2 dp}{h^3} \frac{1}{1 + e^{\beta(E - \mu)}}$$

$$P = \frac{1}{3} \frac{N}{V} \langle p u \rangle$$

where $u \equiv \frac{dE}{dp}$

$$\equiv P_e = \frac{1}{3} \int_0^\infty n_e(p) p u dp$$

$$P = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{(p^2/m) p^2 dp}{[1 + (p/mc)^2]^{1/2}}$$

$$E = mc^2 [1 + (p/mc)^2]^{1/2}$$

$$p = mc \sinh \theta$$

$$u = c \cosh \theta$$

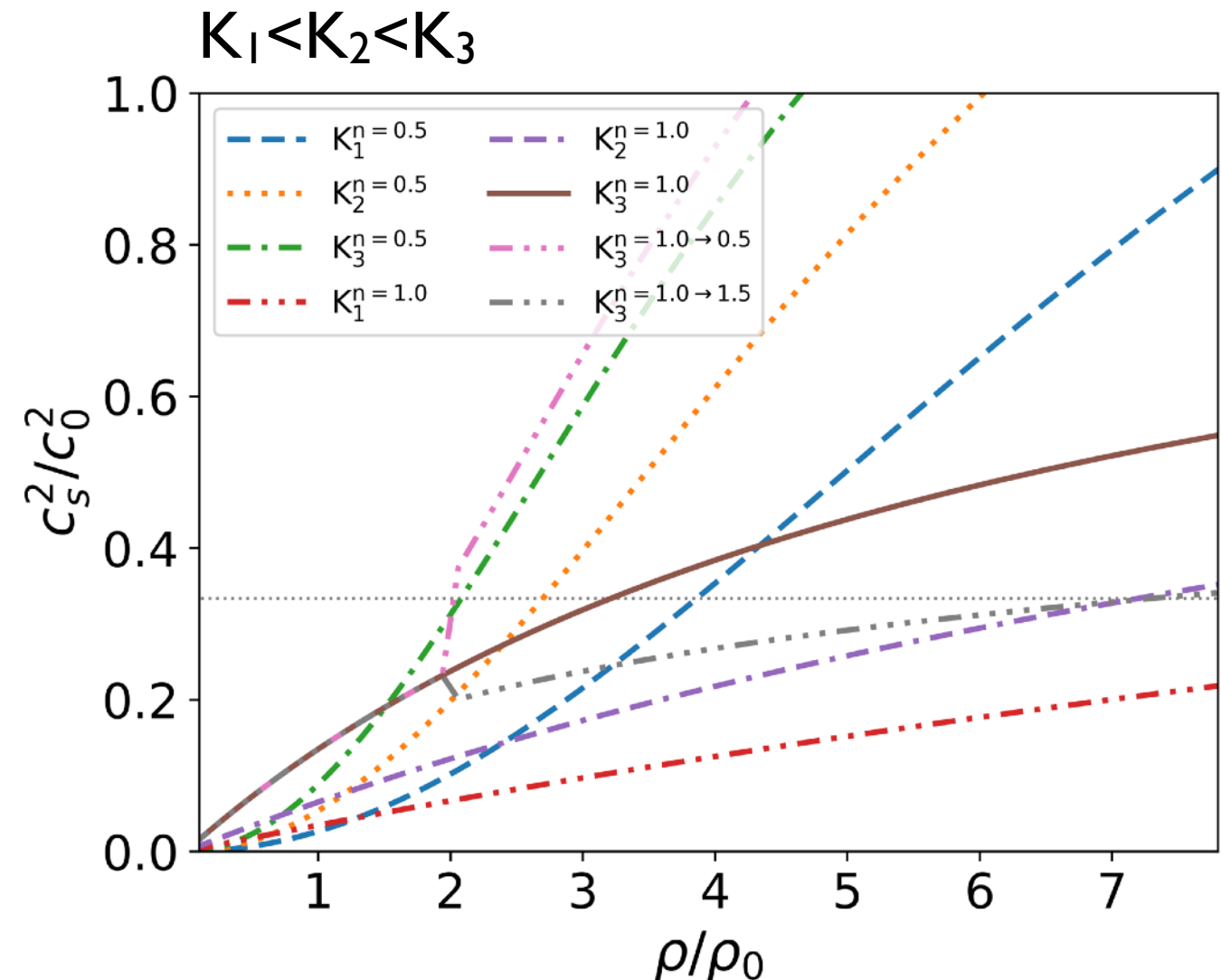
Polytropic equation of state

$$p = K\rho^\Gamma = K\rho^{1+1/n}$$

$$\begin{aligned}\epsilon &= (1+a)\rho c_0^2 + \frac{p}{\Gamma-1} \\ &= (1+a)\rho c_0^2 + \frac{K}{\Gamma-1}\rho^\Gamma,\end{aligned}$$

(a=0)

$$\frac{c_s^2}{c_0^2} = \frac{dp}{d\epsilon} = \Gamma \frac{p}{\epsilon + p}$$



M. Kim et al., JKPS, 78, 932-941 (2021)

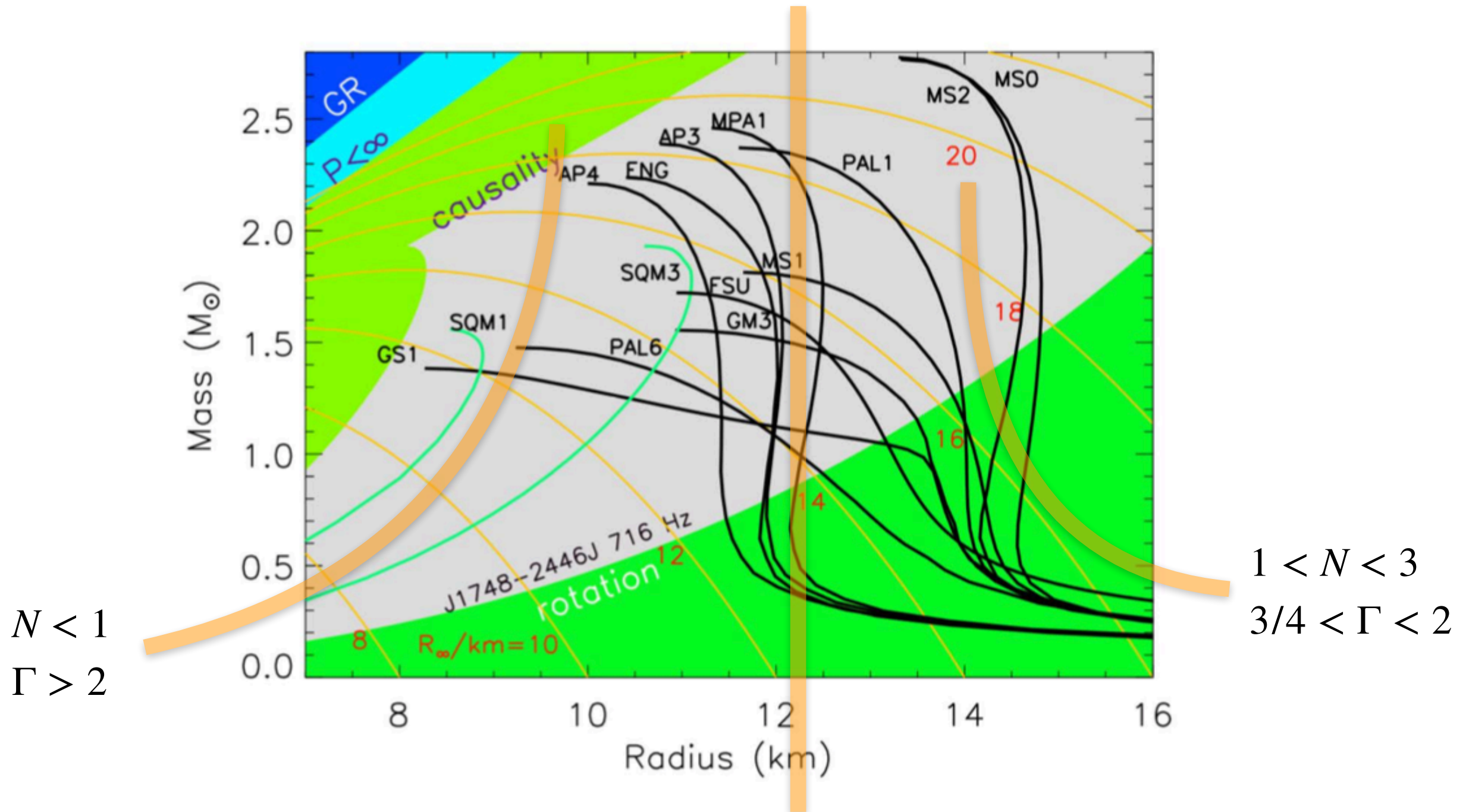
<https://link.springer.com/article/10.1007/s40042-021-00084-4>

M-R Relation in Various EoSs

Recall that

$$M(R) = 4\pi R^{\frac{3-N}{1-N}} \left[\frac{(N+1)K}{4\pi G} \right]^{\frac{N}{N-1}} \xi_s^{\frac{5-3N}{1-N}} \left| \theta'(\xi_s) \right|$$

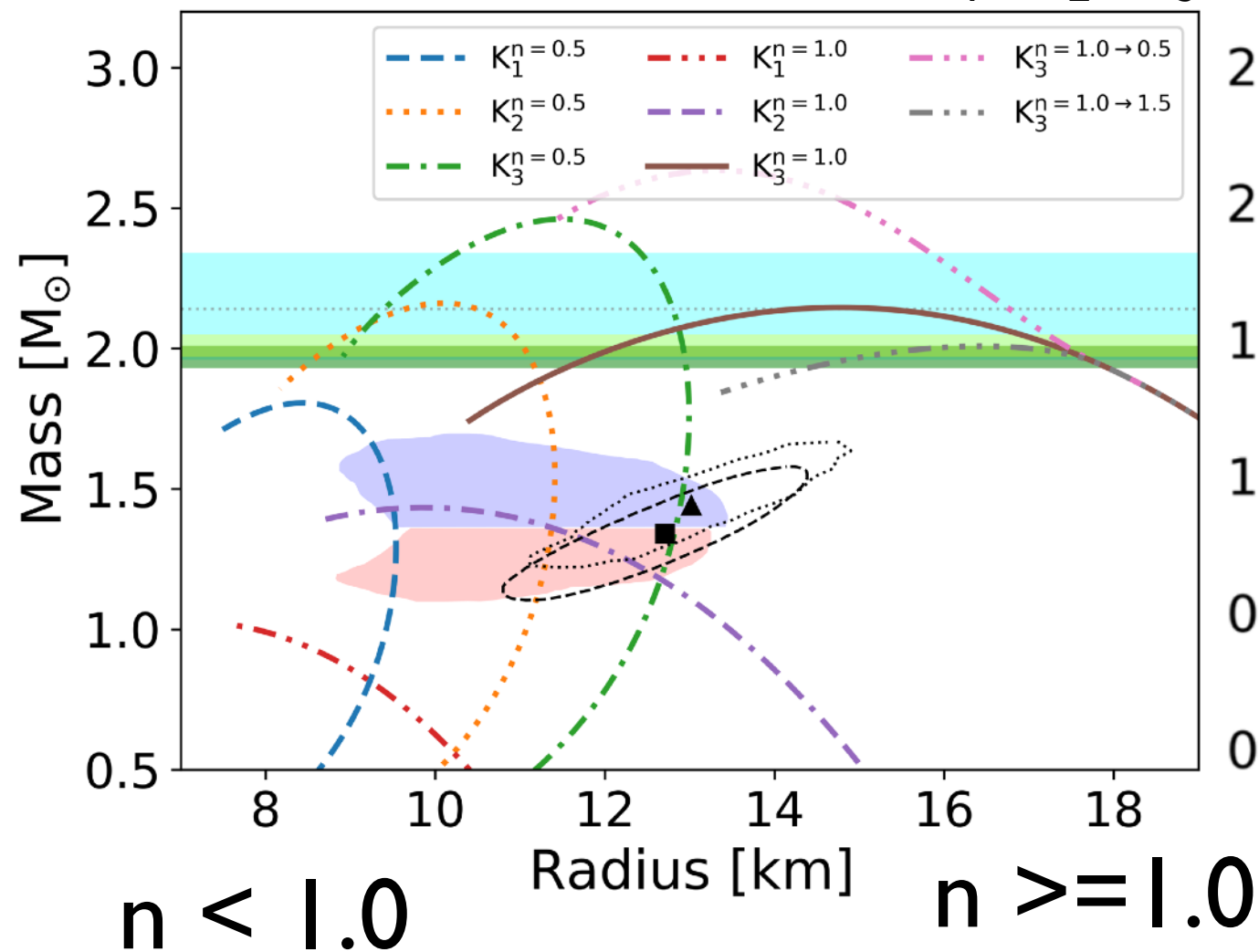
Vertical line follows N=1 polytope
in Lane-Emden Solution



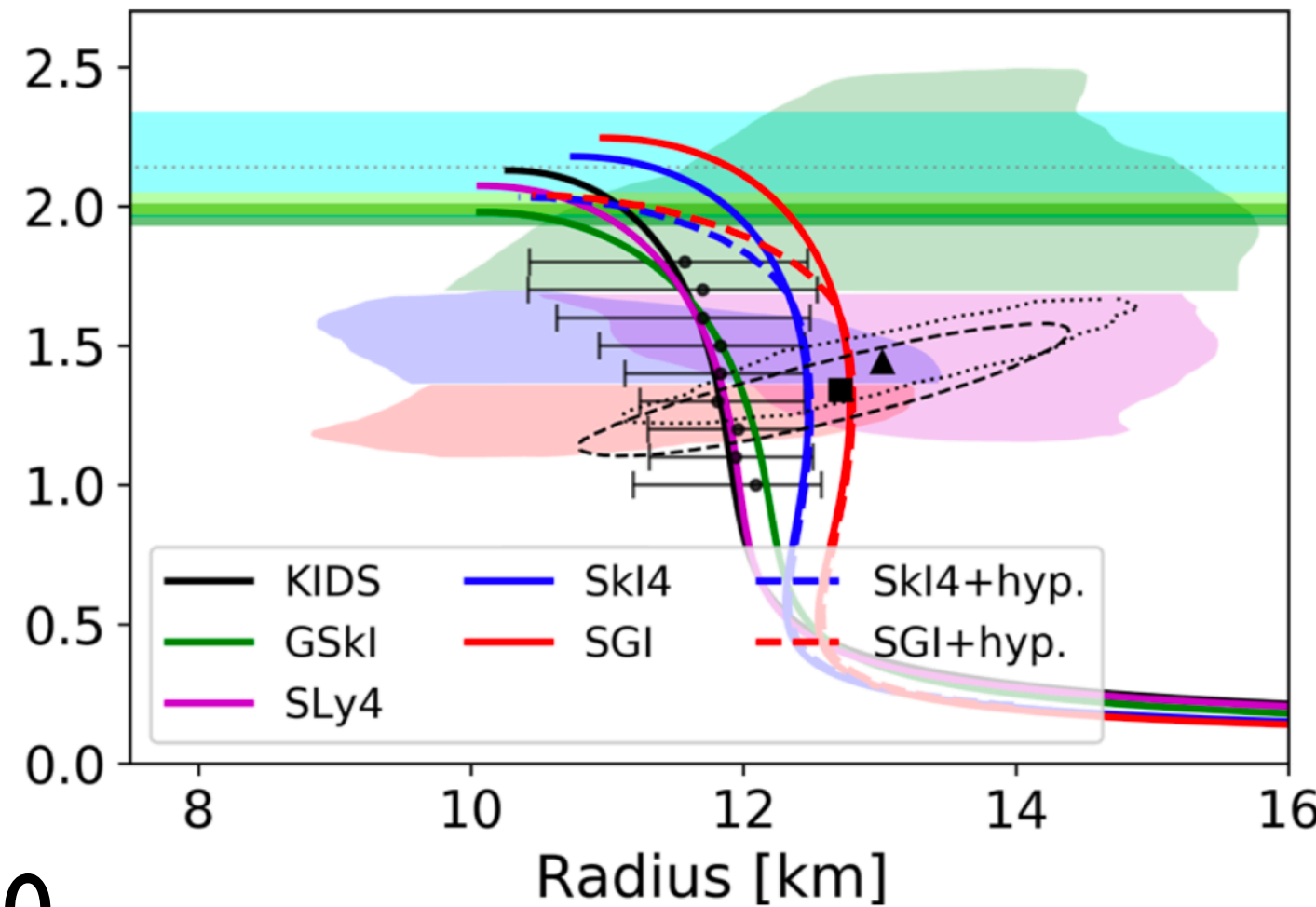
Polytropic EoSs vs. Realistic EoSs

M. Kim et al. (JKPS 2021)

$$K_1 < K_2 < K_3$$



M. Kim et al. (IJMPE 2020)



Nucleus and Strong force

Strong Force

The strong force holds together things that have the same charge. It's stronger than the electromagnetic force, so that's why atoms with multiple protons and neutrons don't fly apart. But most importantly, it holds together the quarks that make up protons and neutrons themselves. Each proton contains two up quarks and one down. Each neutron contains two down quarks and one up.

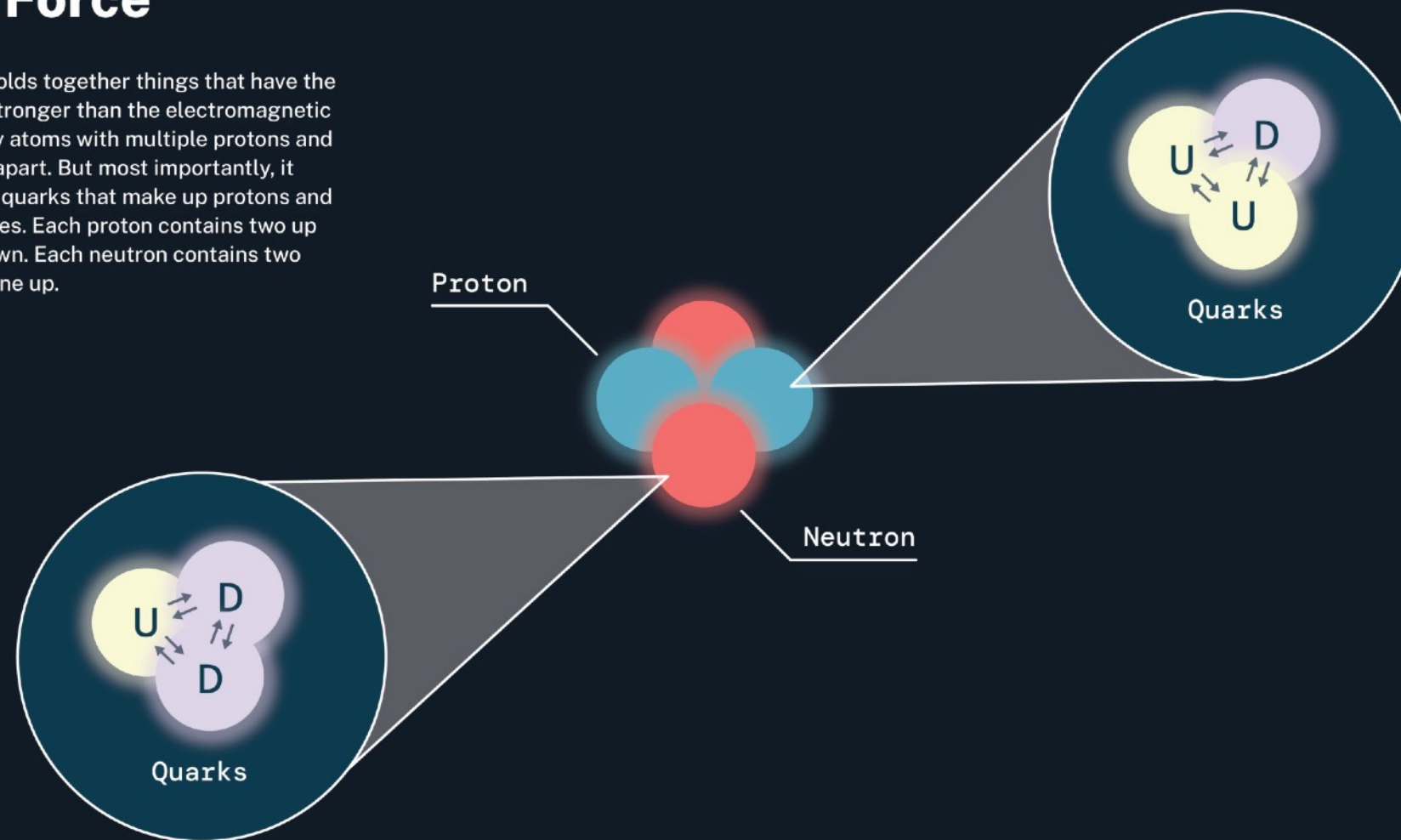
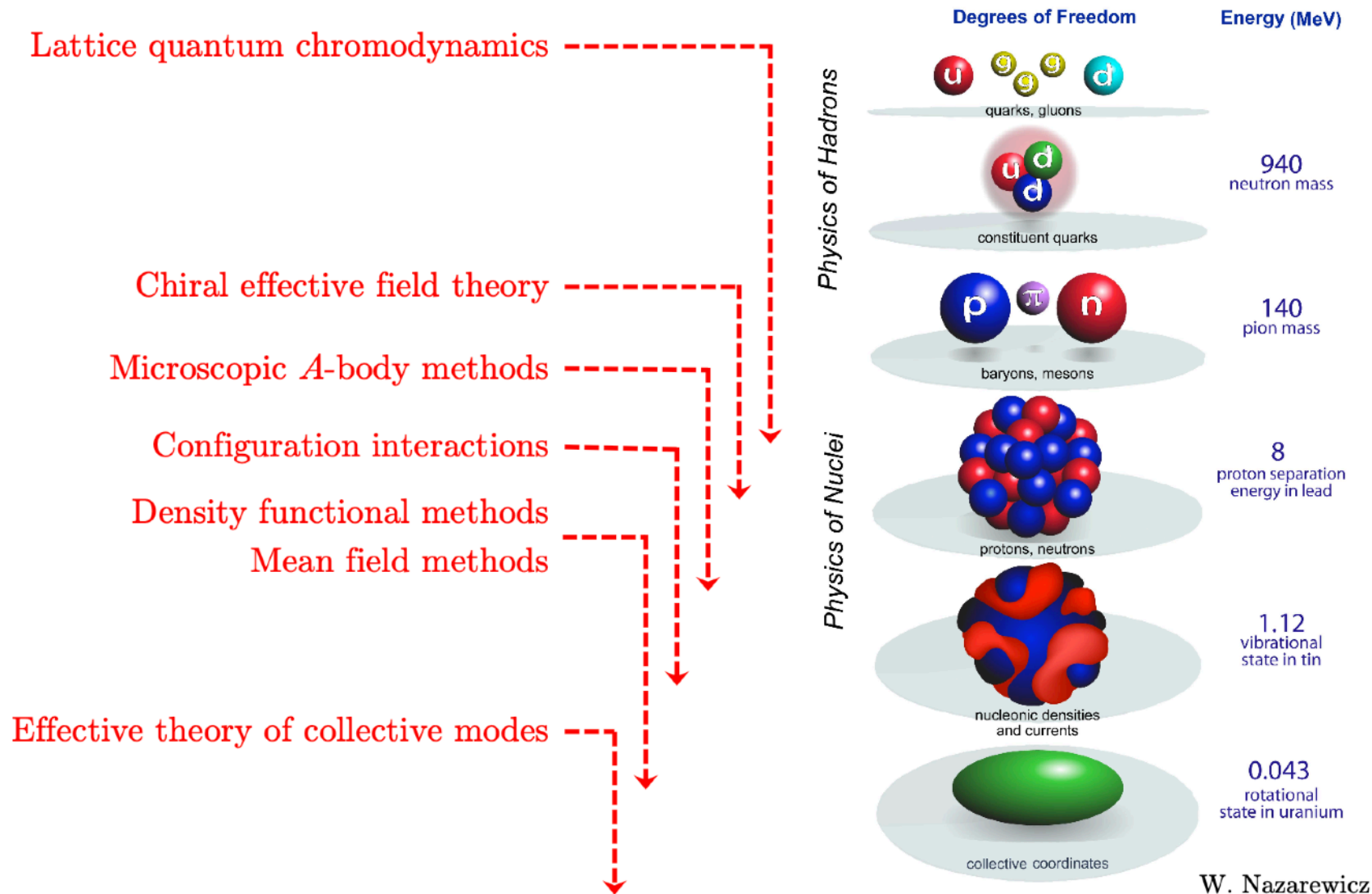


Image: NASA

Effective field theories and energy scales



Slide from Dean Lee' lecture in Nuclear Physics School 2020

EoS for a Neutron Star

Energy per nucleon

$$\frac{E}{A}(\rho, \delta) = \frac{V}{A} \epsilon(\rho, \delta) = \epsilon(\rho, \delta) / \rho$$

ρ : baryon density [fm^{-3}]

$$\delta = \frac{\rho_n - \rho_p}{\rho}$$

Pressure

$$P = -\frac{\partial E}{\partial V}(\rho, \delta)|_A = \frac{A}{V^2} \frac{\partial E}{\partial \rho}(\rho, \delta)|_A = \rho^2 \frac{\partial E/A(\rho, \delta)}{\partial \rho} |_A$$

Beta Equilibrium

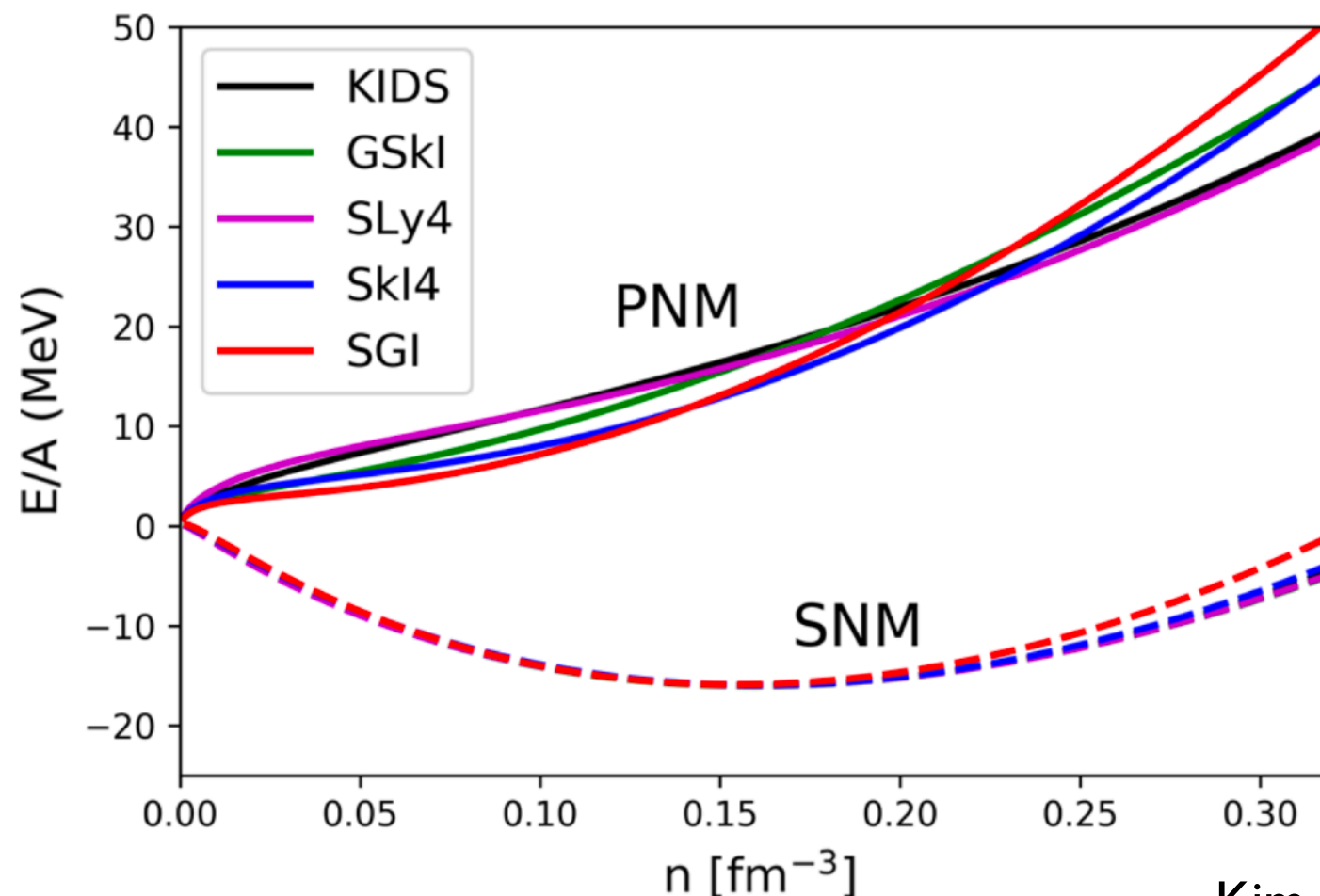
$$\rho_p = \rho_e + \rho_\mu (+\rho_K)$$

$$\mu_n = \mu_p + \mu_e \qquad \mu_e = \mu_\mu (= \mu_K)$$

Nuclear Equation of States

$$\frac{E}{A}(\rho, \delta) = \frac{V}{A} \epsilon(\rho, \delta) = \epsilon(\rho, \delta) / \rho \equiv \mathcal{E}(\rho, \delta)$$

$$= E(\rho, \delta = 0) + E_{sym}(\rho) \delta^2 + \mathcal{O}(\delta^4) + \dots,$$



$$\delta = \frac{\rho_n - \rho_p}{\rho}$$

$\rho_0 \sim 0.16$: saturation density

Nuclear Equation of States

$$\mathcal{E}(\rho, 0) = E_0 + \frac{1}{2}K_0x^2 + \frac{1}{6}Q_0x^3 + O(x^4),$$

$$\mathcal{S}(\rho) = J + Lx + \frac{1}{2}K_{\text{sym}}x^2 + \frac{1}{6}Q_{\text{sym}}x^3 + \frac{1}{24}R_{\text{sym}}x^4 + O(x^5),$$

$$K_0 \equiv 9\rho_0^2 \left. \frac{d^2}{d\rho^2} \frac{\mathcal{E}(\rho, 0)}{\rho} \right|_{\rho=\rho_0}, \quad x = \frac{(\rho - \rho_0)}{3\rho_0}$$

$$Q_0 \equiv 27\rho_0^3 \left. \frac{d^3}{d\rho^3} \mathcal{E}(\rho, 0) \right|_{\rho=\rho_0}.$$

Incompressibility	$K_0 \simeq 230 \text{ MeV}$
Skewness	$K'_0 \sim -2000 \text{ MeV}$

Nuclear Symmetry Energy

$$\mathcal{E}(\rho, 0) = E_0 + \frac{1}{2}K_0x^2 + \frac{1}{6}Q_0x^3 + O(x^4),$$

$$\mathcal{S}(\rho) = J + Lx + \frac{1}{2}K_{\text{sym}}x^2 + \frac{1}{6}Q_{\text{sym}}x^3 + \frac{1}{24}R_{\text{sym}}x^4 + O(x^5),$$

$$S \equiv E_{\text{sym}}(\rho_0)$$

$$L \equiv 3\rho_0 \left. \frac{d}{d\rho} \mathcal{S}(\rho) \right|_{\rho=\rho_0},$$

$$Q_{\text{sym}} \equiv 27\rho_0^3 \left. \frac{d^3}{d\rho^3} \mathcal{S}(\rho) \right|_{\rho=\rho_0},$$

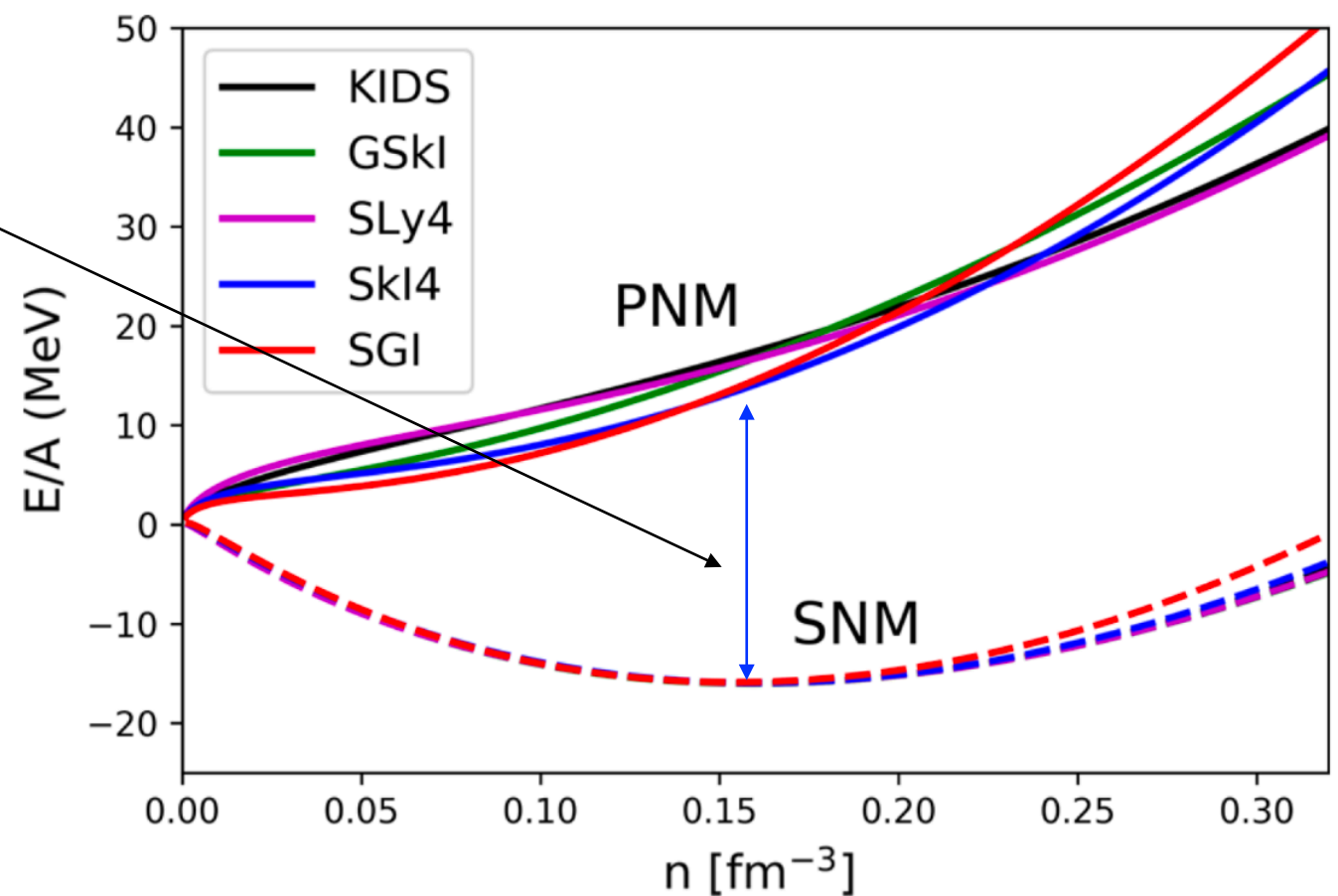
$$K_{\text{sym}} \equiv 9\rho_0^2 \left. \frac{d^2}{d\rho^2} \frac{\mathcal{S}(\rho)}{\rho} \right|_{\rho=\rho_0}.$$

$$R_{\text{sym}} \equiv 81\rho_0^4 \left. \frac{d^4}{d\rho^4} \mathcal{S}(\rho) \right|_{\rho=\rho_0}.$$

Nuclear Symmetry Energy

$$\frac{E}{A} = E(\rho, \delta = 0) + E_{sym}(\rho)\delta^2 + \mathcal{O}(\delta^4) + \dots,$$

$$E_{sym}(\rho) = E_{sym}(\rho_0) + \frac{L}{3} \left(\frac{\rho - \rho_0}{\rho_0} \right) + \frac{K_{sym}}{18} \left(\frac{\rho - \rho_0}{\rho_0} \right)^2$$

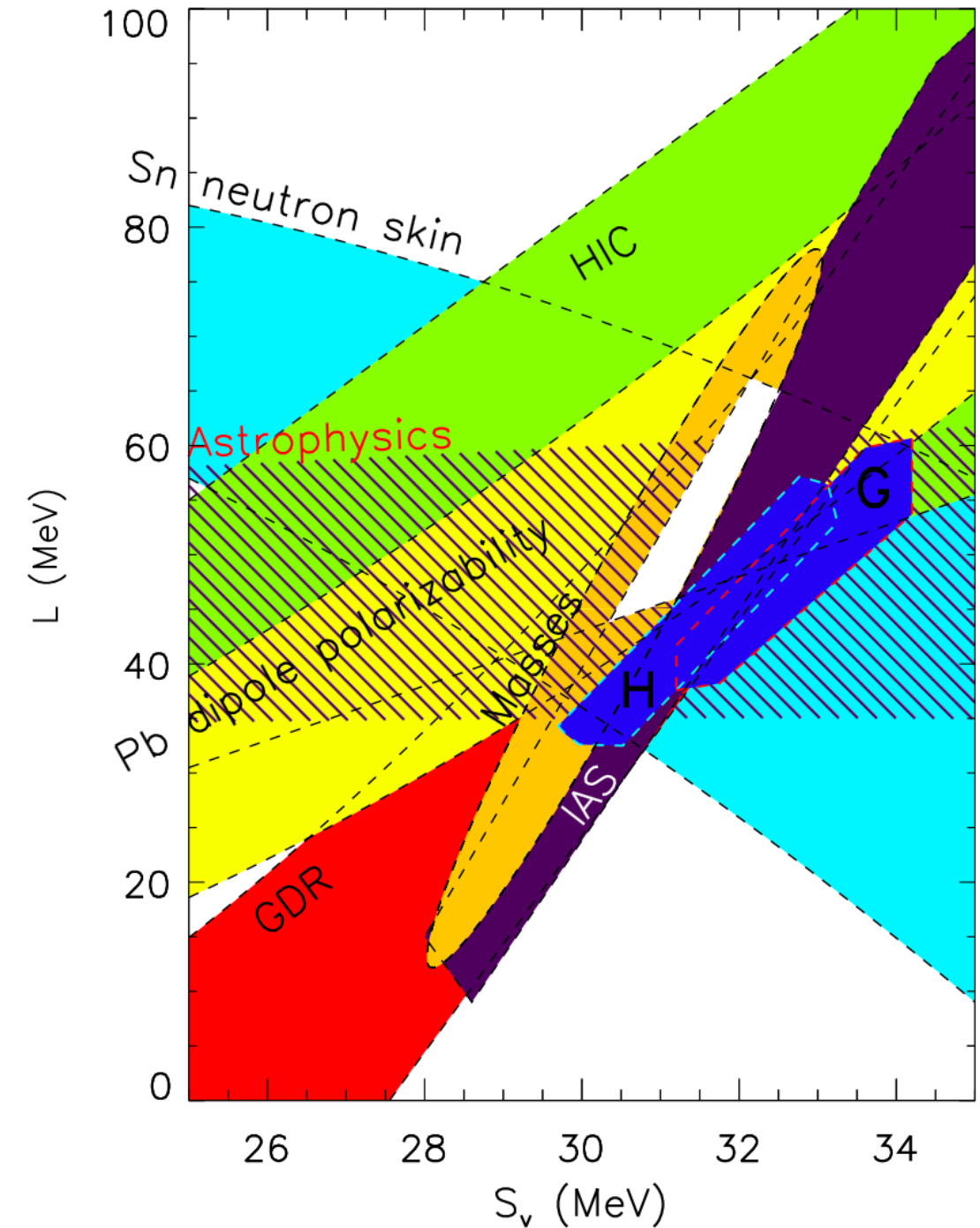
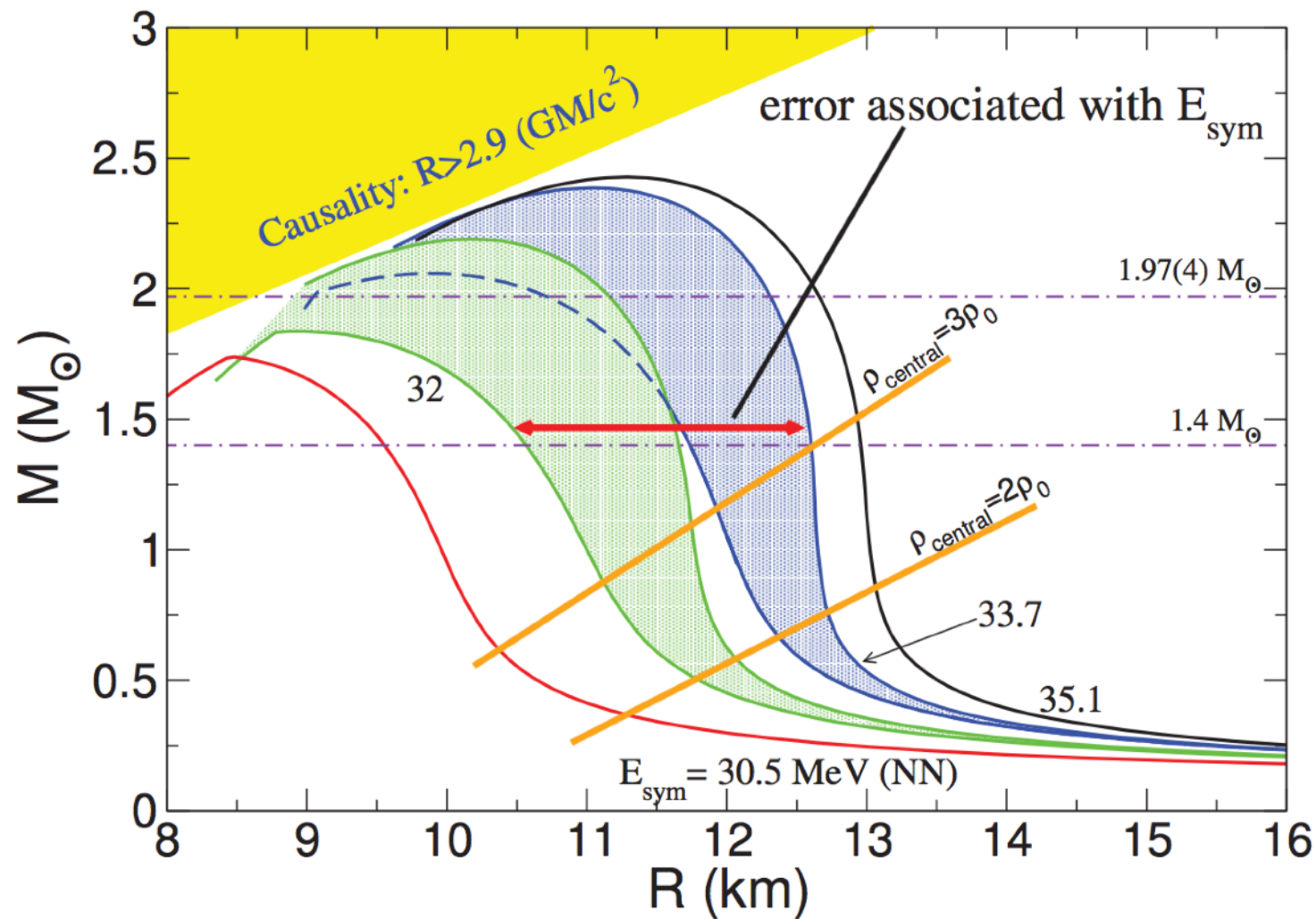


Kim et al. (IJMPE 2020)

Nuclear Symmetry Energy

J.M. Lattimer / Nuclear Physics A 928 (2014) 276–295

Gandolfi and Steiner (2016)



Constraints on Symmetry Energy

Model	ρ_0	E_0	K_0	$-Q_0$	J	L	$-K_\tau$	$M_{\max}$
Exp/Emp	$\simeq 0.16$	$\simeq 16.0$	$200 \sim 260$	$200 \sim 1200$	$30 \sim 35$	$40 \sim 76$	$372 \sim 760$	$\geq 1.93 \sim 2.05$
CSkP	-	-	$202.0 \sim 240.3$	$362.5 \sim 425.6$	$30.0 \sim 35.5$	$48.6 \sim 67.1$	$360.1 \sim 407.1$	-
GSkI	0.159	16.02	230.2	405.6	32.0	63.5	364.2	1.98
SLy4	0.160	15.97	229.9	363.1	32.0	45.9	322.8	2.07
SkI4	0.160	15.95	248.0	331.2	29.5	60.4	322.2	2.19
SGI	0.154	15.89	261.8	297.9	28.3	63.9	362.5	2.25
KIDS	0.160	16.00	240.0	372.7	32.8	49.1	375.1	2.14

Kim et al., PhysRevC.98.065805

Constraints on EoS

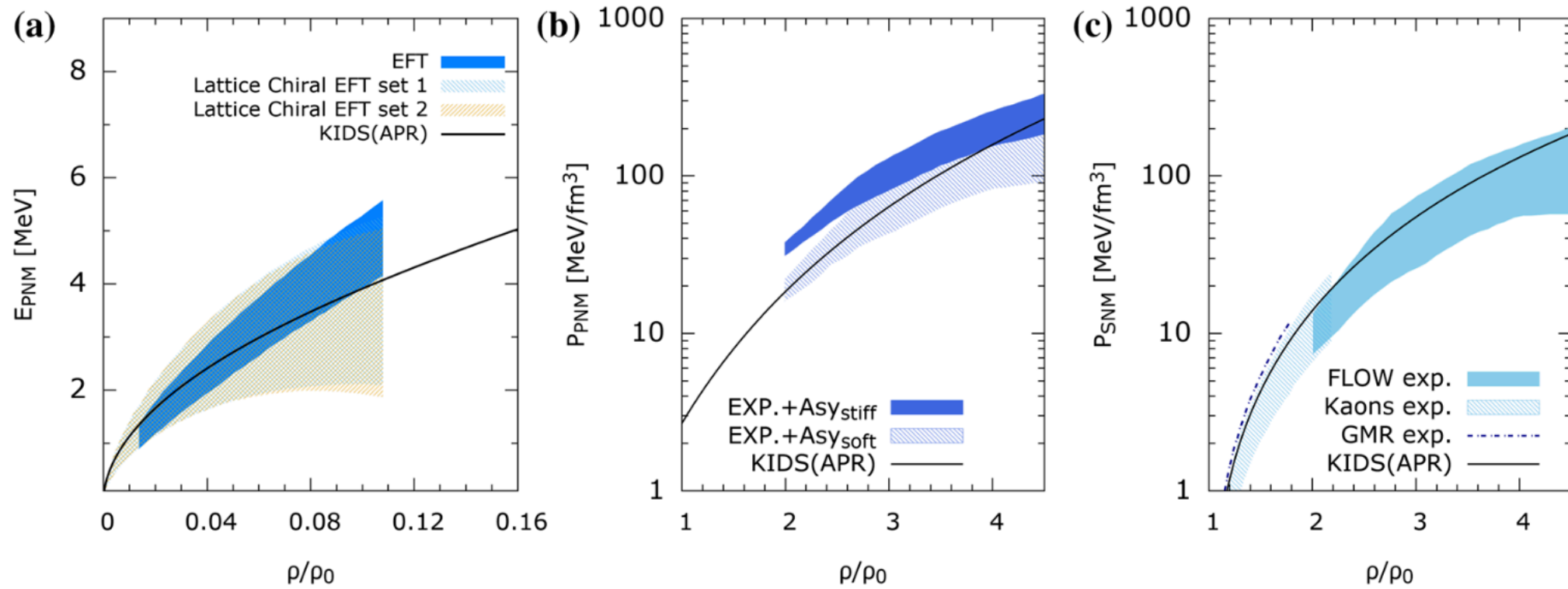
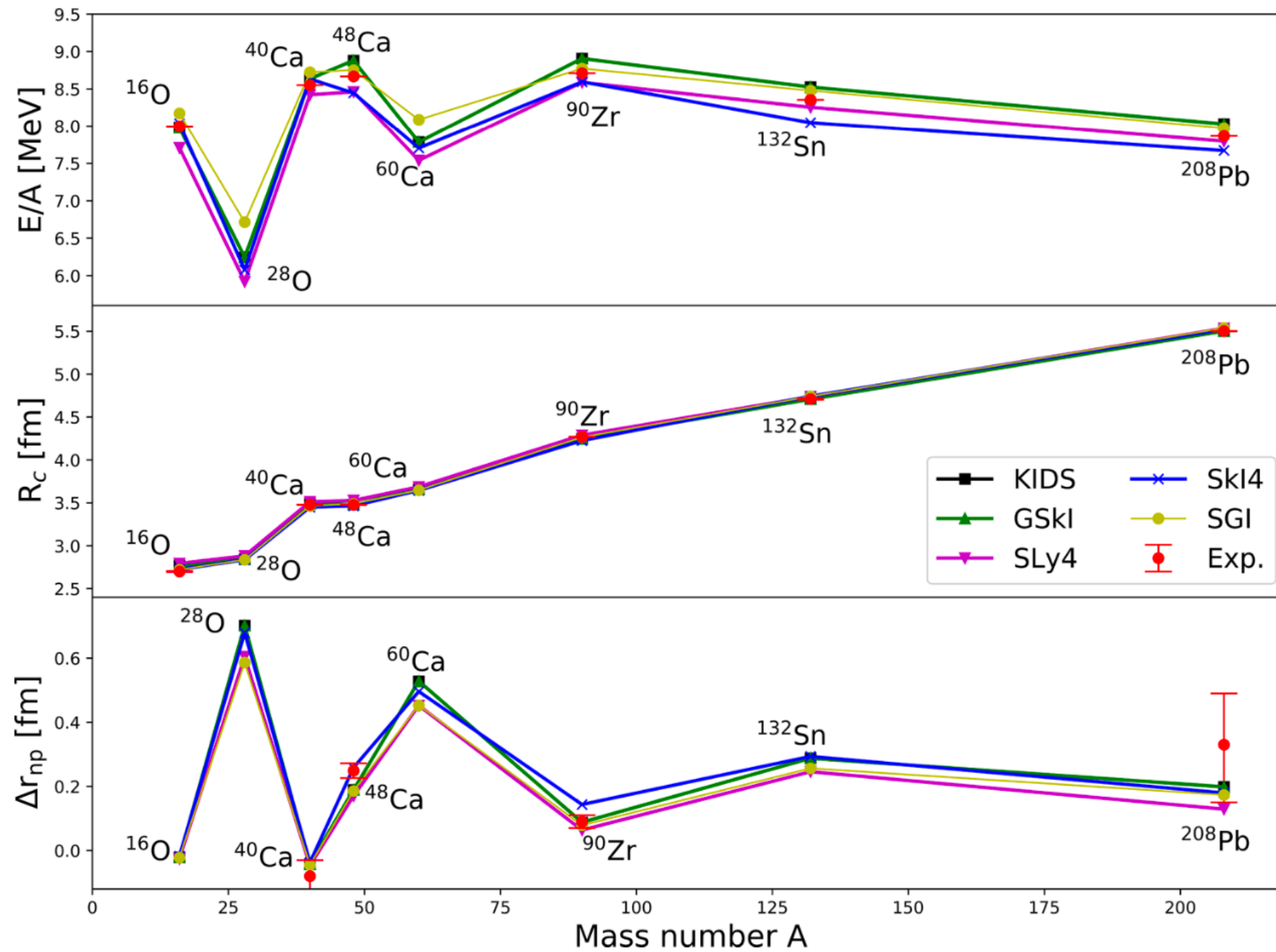


Fig. 1 Energy below (a) and pressure above (b, c) the saturation density. The result of an EFT and lattice chiral EFT in a is taken from [16] and [17], respectively. The results from experiment in b are from [21,22], and those in c are from [22–25]

Properties of Nuclei



Kim et al. (IJMPE 2020)

Standard Skyrme effective interaction

E. Chabanat et al./Nuclear Physics A 627 (1997) 710–746

713

NILO Skyrme pseudopotential

$$\begin{aligned}
 V(\mathbf{r}_1, \mathbf{r}_2) = & t_0 (1 + x_0 P_\sigma) \delta(\mathbf{r}) && \text{central term} \\
 & + \frac{1}{2} t_1 (1 + x_1 P_\sigma) \left[\mathbf{P}'^2 \delta(\mathbf{r}) + \delta(\mathbf{r}) \mathbf{P}^2 \right] && \text{2nd order derivative} \\
 & + t_2 (1 + x_2 P_\sigma) \mathbf{P}' \cdot \delta(\mathbf{r}) \mathbf{P} && \text{non-local terms} \\
 & + \frac{1}{6} t_3 (1 + x_3 P_\sigma) [\rho(\mathbf{R})]^\sigma \delta(\mathbf{r}) && \text{density-dependent term} \\
 & + iW_0 \boldsymbol{\sigma} \cdot [\mathbf{P}' \times \delta(\mathbf{r}) \mathbf{P}] && \text{spin-orbit term.}
 \end{aligned} \tag{2.1}$$

with:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{R} = \frac{1}{2} (\mathbf{r}_1 + \mathbf{r}_2),$$

$$\mathbf{P} = \frac{1}{2i} (\nabla_1 - \nabla_2), \quad \mathbf{P}' \text{ cc of } \mathbf{P} \text{ acting on the left}$$

and

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2, \quad P_\sigma = (1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) / 2.$$

Skyrme-type Energy Density Functional

$$\begin{aligned}\mathcal{E} = & \frac{\hbar^2}{2m}\tau + \frac{3}{8}t_0\rho - \frac{1}{8}(t_0 + 2y_0)\rho\delta^2 + \frac{1}{16}\sum_{n=1}^{N-1}t_{3n}\rho^{1+n/3} \\ & - \frac{1}{48}\sum_{n=1}^{N-1}(t_{3n} + 2y_{3n})\rho^{1+n/3}\delta^2 + \frac{1}{64}(9t_1 - 5t_2 - 4y_2) \\ & \times \frac{(\nabla\rho)^2}{\rho} - \frac{1}{64}(3t_1 + 6y_1 + t_2 + 2y_2)\frac{(\nabla\rho\delta)^2}{\rho} \\ & + \frac{1}{8}(2t_1 + y_1 + 2t_2 + y_2)\tau - \frac{1}{8}(t_1 + 2y_1 - t_2 - 2y_2) \\ & \times \sum_q \frac{\rho_q\tau_q}{\rho} + \frac{1}{2}W_0\left(\frac{\mathbf{J}\cdot\nabla\rho}{\rho} + \sum_q \frac{\mathbf{J}_q\cdot\nabla\rho_q}{\rho}\right), \quad (15)\end{aligned}$$

Relativistic Mean-Field (RMF) theory

$$\begin{aligned}
 \mathcal{L} = & \sum_B \bar{\psi}_B (i\gamma_\mu \partial^\mu - m_B + g_{\sigma B} \sigma - g_{\omega B} \gamma_\mu \omega^\mu - \frac{1}{2} g_{\rho B} \gamma_\mu \boldsymbol{\tau} \cdot \boldsymbol{\rho}^\mu) \psi_B \\
 & + \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2) - \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu \\
 & - \frac{1}{4} \rho_{\mu\nu} \cdot \rho^{\mu\nu} + \frac{1}{2} m_\rho^2 \rho_\mu \cdot \rho^\mu - \frac{1}{3} b m_n (g_\sigma \sigma)^3 - \frac{1}{4} c (g_\sigma \sigma)^4 \\
 & + \sum_\lambda \bar{\psi}_\lambda (i\gamma_\mu \partial^\mu - m_\lambda) \psi_\lambda .
 \end{aligned} \tag{5.38}$$

σ - ω - ρ model

$$\partial^2 \sigma + \frac{\partial U}{\partial \sigma} = -g_\sigma \rho_S$$

$$(\partial^2 + m_\omega^2) \omega^\nu = g_\omega j_b^\nu$$

$$(\partial^2 + m_\rho^2) \boldsymbol{\rho}^\nu = g_\rho \mathbf{j}_I^\nu$$

$$U = \frac{m_\sigma^2}{2} \sigma^2 + \frac{g_2}{3} \sigma^3 + \frac{g_3}{4} \sigma^4.$$

$$\rho_S = g \sum_{a=p,n,\bar{p},\bar{n}} \int \frac{d^3 p}{(2\pi)^3 E_a^*} m_a^* f_a(x, \mathbf{p}_a)$$

$$j_b^\mu = g \sum_{a=p,n,\bar{p},\bar{n}} (-1)^a \int \frac{d^3 p}{(2\pi)^3 E_a^*} p_a^\mu f_a(x, \mathbf{p}_a)$$

$$j_I^{3\mu} = g \sum_{a=p,n,\bar{p},\bar{n}} \tau_a^3 \int \frac{d^3 p}{(2\pi)^3 E_a^*} p_a^\mu f_a(x, \mathbf{p}_a)$$

Relativistic Mean-Field (RMF) theory

$$T^{\mu\nu} = -g^{\mu\nu} \mathcal{L} + \sum_{\phi} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \partial^{\nu} \phi,$$

TABLE I. Parameter sets.

Parameter	Set I	Set II	NL3
f_{σ} (fm ²)	10.33	same	15.73
f_{ω} (fm ²)	5.42	same	10.53
f_{ρ} (fm ²)	0.95	3.15	1.34
f_{δ} (fm ²)	0.00	2.50	0.00
A (fm ⁻¹)	0.033	same	-0.01
B	-0.0048	same	-0.003

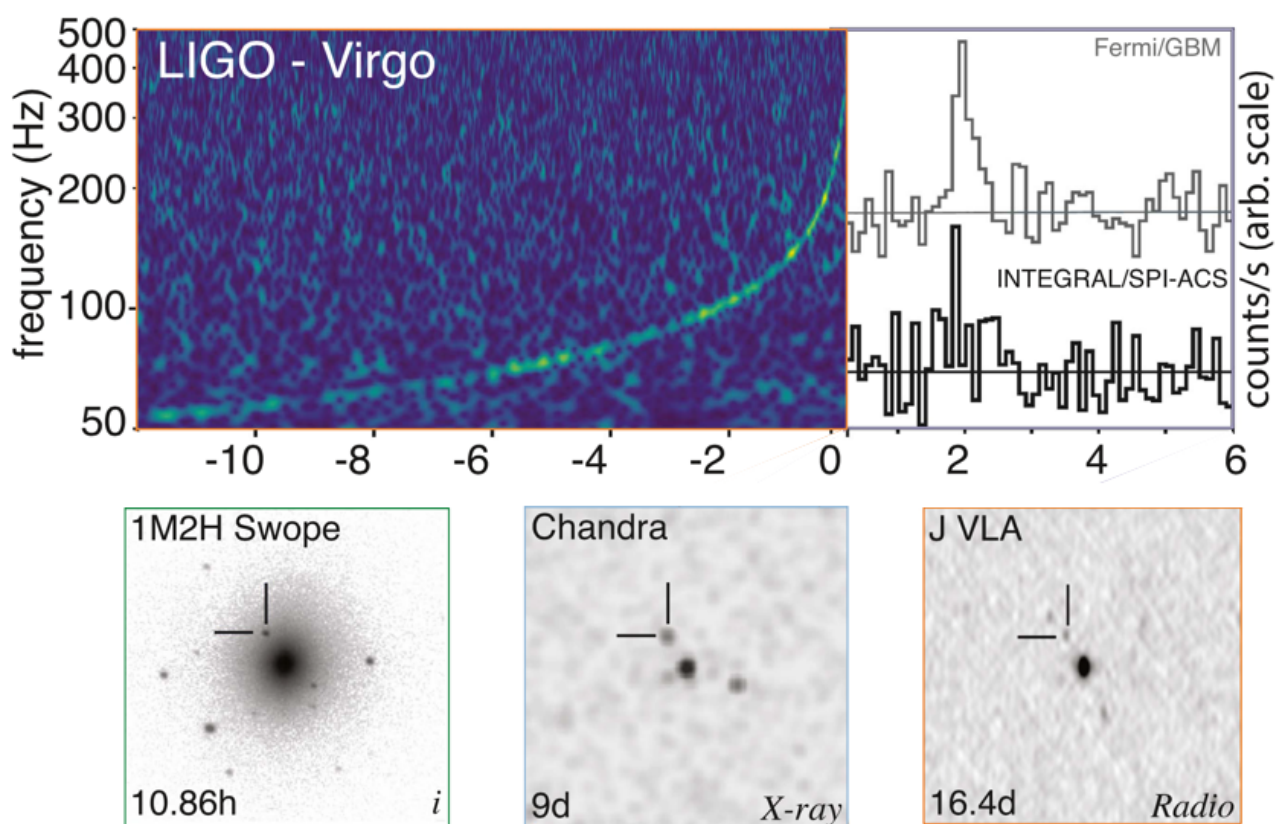
and compare with (3.26) to read the energy density equations to rewrite the energy density as

$$\begin{aligned} \epsilon = & \frac{1}{3} b m_n (g_{\sigma} \sigma)^3 + \frac{1}{4} c (g_{\sigma} \sigma)^4 + \frac{1}{2} m_{\sigma}^2 \sigma^2 + \frac{1}{2} m_{\omega}^2 \omega_0^2 + \frac{1}{2} m_{\rho}^2 \rho_{03}^2 \\ & + \sum_B \frac{2J_B + 1}{2\pi^2} \int_0^{k_B} \sqrt{k^2 + (m_B - g_{\sigma B} \sigma)^2} k^2 dk \\ & + \sum_{\lambda} \frac{1}{\pi^2} \int_0^{k_{\lambda}} \sqrt{k^2 + m_{\lambda}^2} k^2 dk. \end{aligned} \quad (5.50)$$

The pressure is given by

$$\begin{aligned} p = & -\frac{1}{3} b m_n (g_{\sigma} \sigma)^3 - \frac{1}{4} c (g_{\sigma} \sigma)^4 - \frac{1}{2} m_{\sigma}^2 \sigma^2 + \frac{1}{2} m_{\omega}^2 \omega_0^2 + \frac{1}{2} m_{\rho}^2 \rho_{03}^2 \\ & + \frac{1}{3} \sum_B \frac{2J_B + 1}{2\pi^2} \int_0^{k_B} k^4 dk / \sqrt{k^2 + (m_B - g_{\sigma B} \sigma)^2} \\ & + \frac{1}{3} \sum_{\lambda} \frac{1}{\pi^2} \int_0^{k_{\lambda}} k^4 dk / \sqrt{k^2 + m_{\lambda}^2}. \end{aligned} \quad (5.51)$$

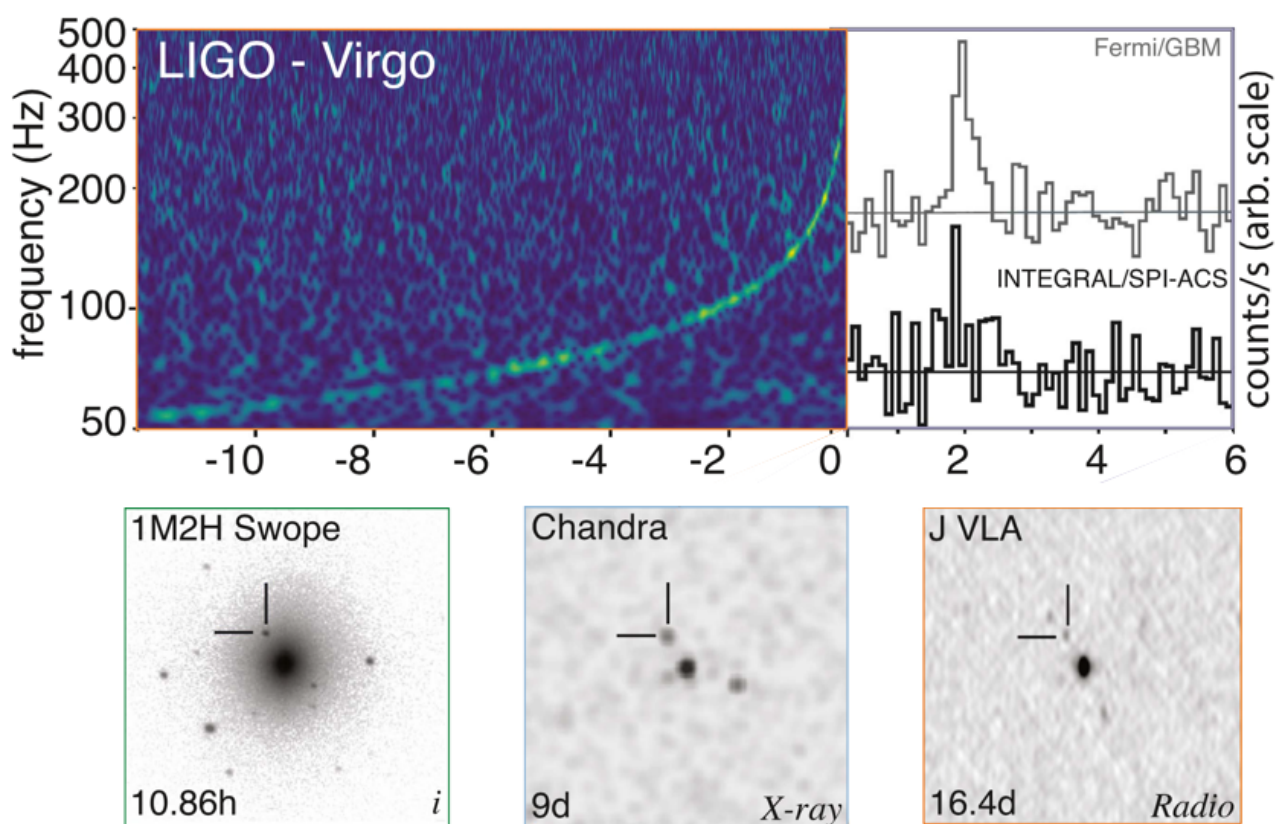
첫 번째 다중신호천문학 이벤트



GW170817 / GRB170817A / AT2017gfo
- PRL.119.161102, ApJL.848.L12

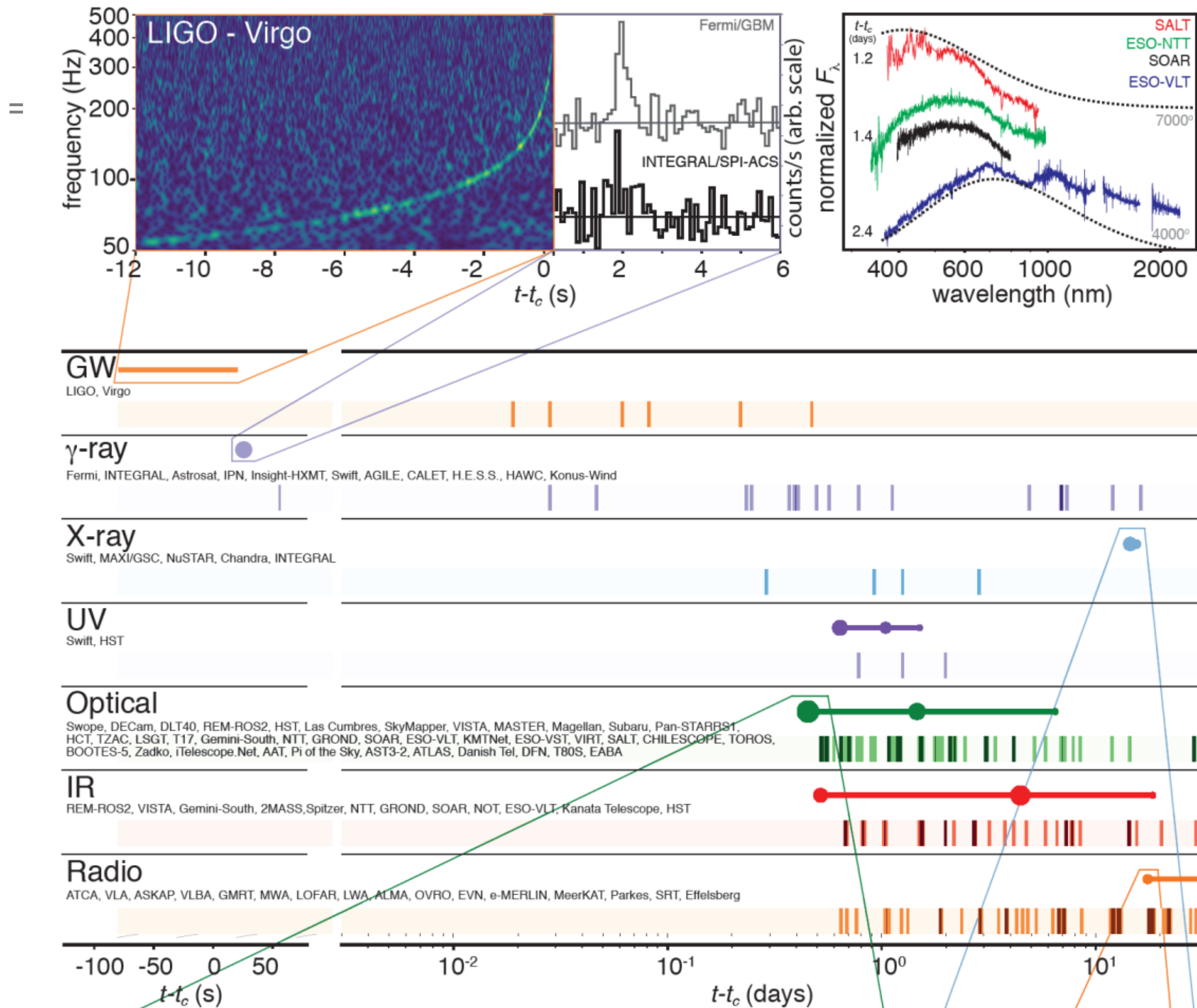
새로운 다중신호천문학의 시대가 옴!

첫 번째 다중신호천문학 이벤트



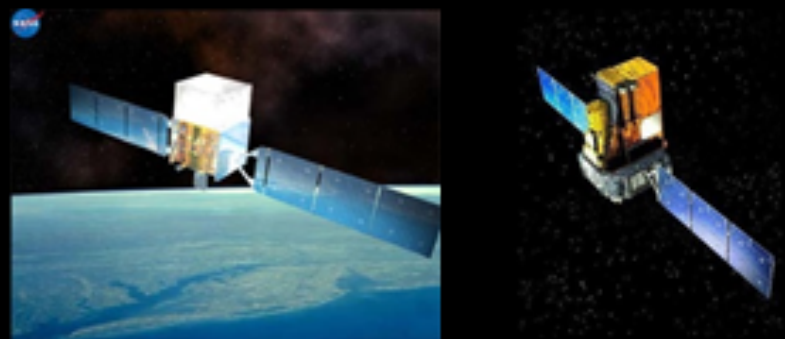
GW170817 / GRB170817A / AT2017gfo
- PRL.119.161102, ApJL.848.L12

새로운 다중신호천문학의 시대가 옴 !

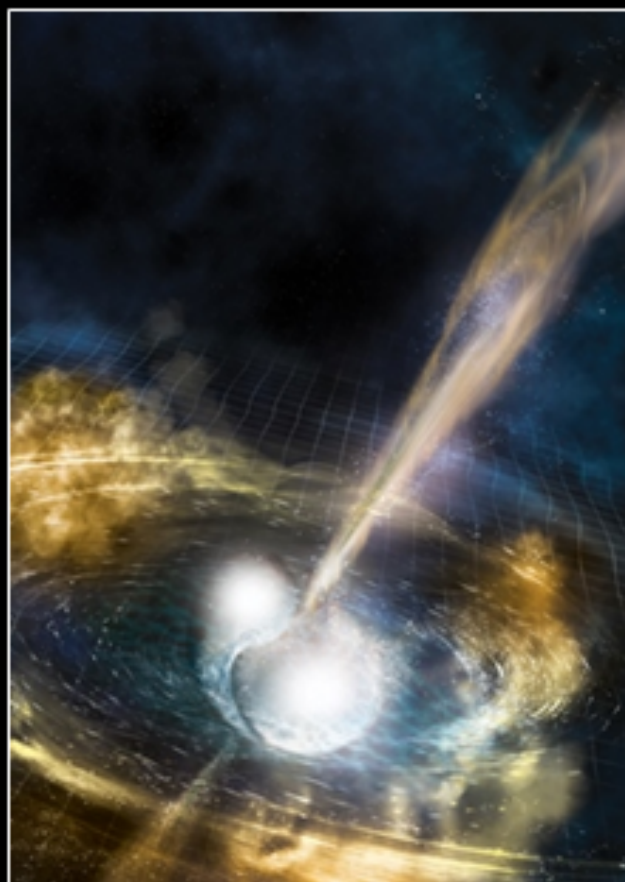




2017-08-17 12:41:04 UTC
라이고 및 비르고 중력파 신호 포착



+2초 후
페르미 및 인티그랄 감마선 신호 포착



+약 11시간 후
칠레 천문대 망원경들이
가시광선 신호 포착



+약 21시간 후
국내연구진 호주 이상각망원경으로
추적관측시작. 이후 약 4주간 추적관측
(KMTNet, BOOTES-5망원경 등)



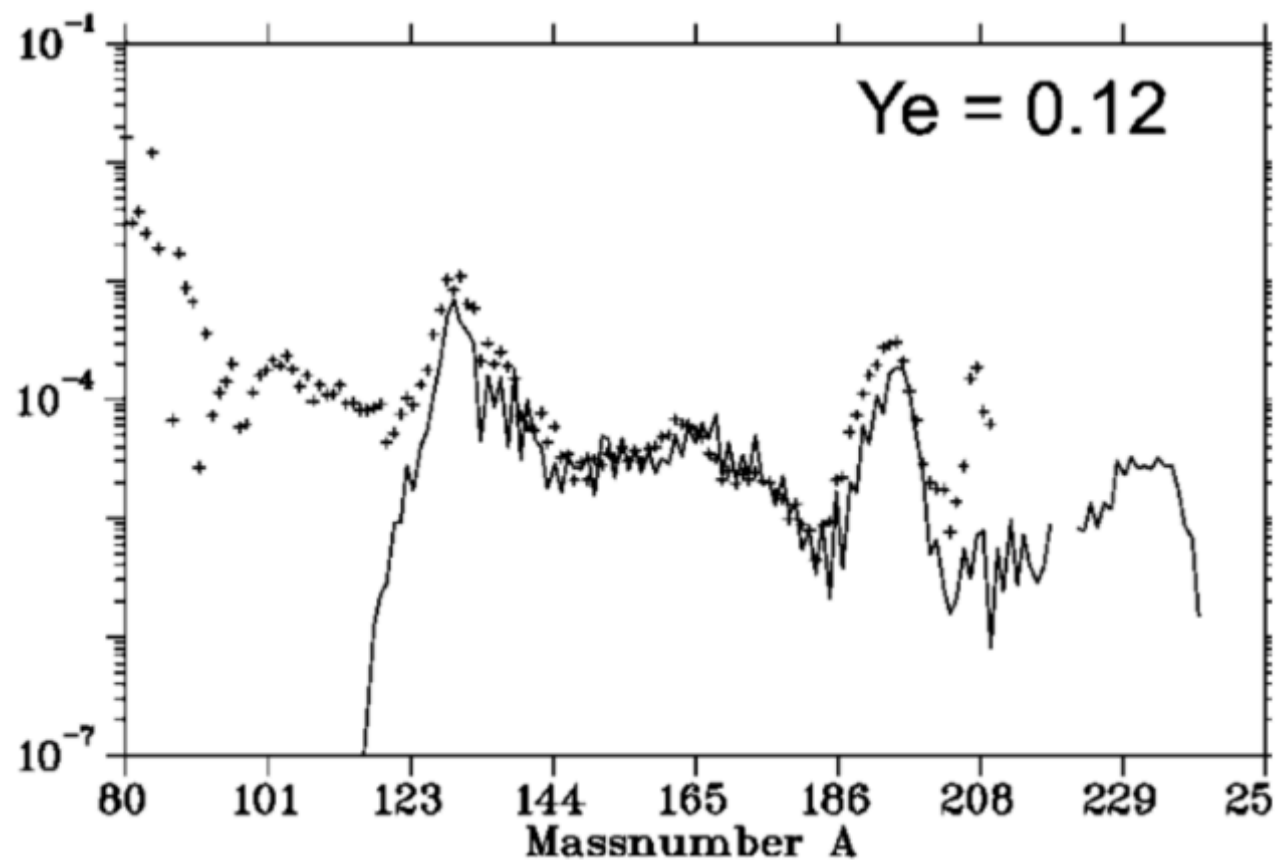
+약 9일 후
찬드라 우주망원경
X-선 신호 포착



+약 16일 후
지상 전파망원경
전파 신호 포착

Heavy Elements from BNS mergers

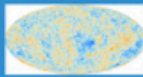
S Rosswog 2015



solar pattern vs NS-merger

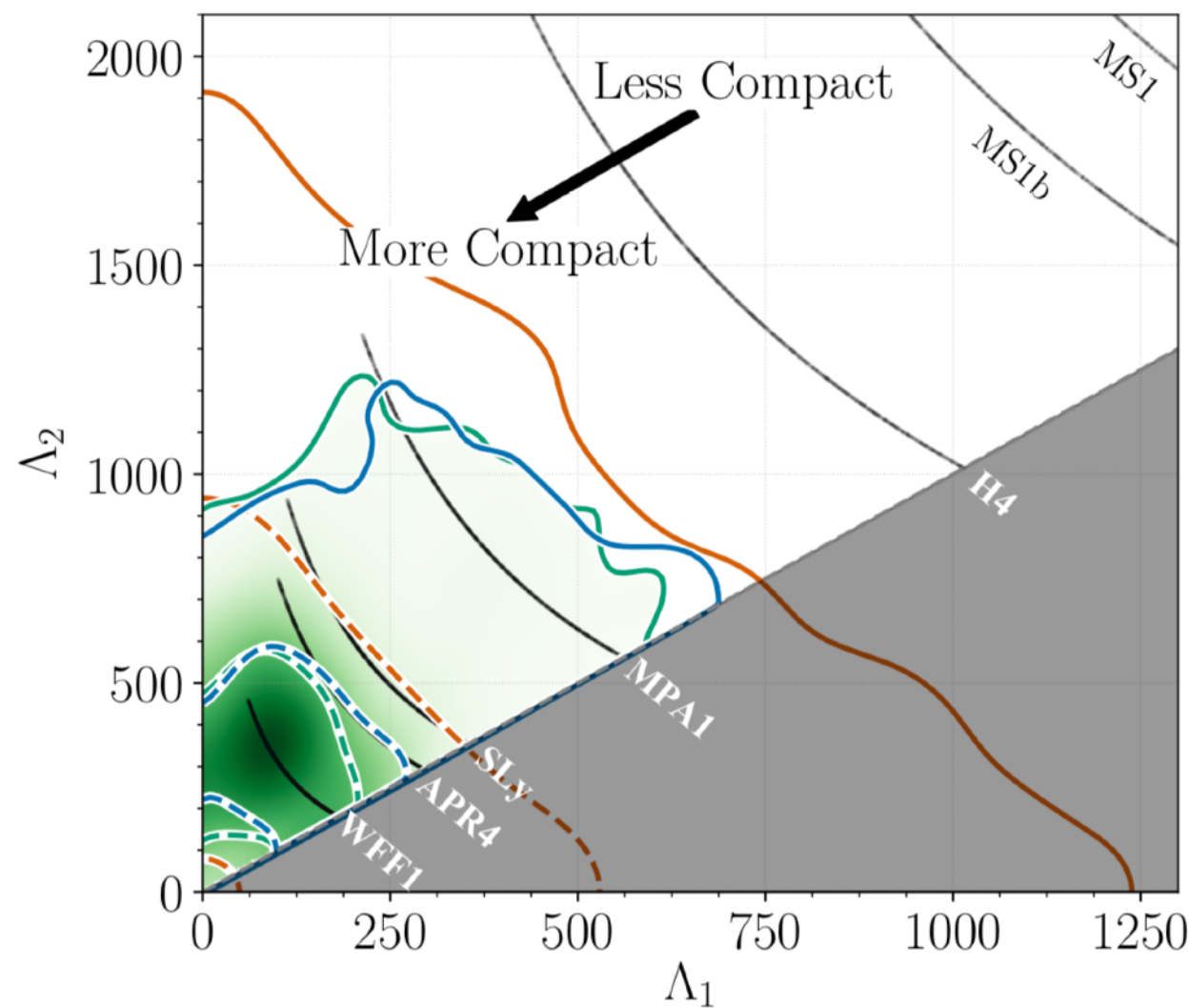
- **Supernovae:**
neutrino-driven wind
r-process **peak at $A \sim 130$**
- **NS mergers:**
r-process **peak at $A \sim 195$**

Heavy Elements from BNS mergers

1 H	big bang fusion 					cosmic ray fission 					2 He						
3 Li	4 Be	merging neutron stars 					exploding massive stars 					5 B	6 C	7 N	8 O	9 F	10 Ne
11 Na	12 Mg	dying low mass stars 					exploding white dwarfs 					13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
55 Cs	56 Ba		72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
87 Fr	88 Ra																
		57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
		89 Ac	90 Th	91 Pa	92 U												

A new constraint by GW Observation

$$\Lambda(1.4M_{\odot}) = 190^{+390}_{-120}$$

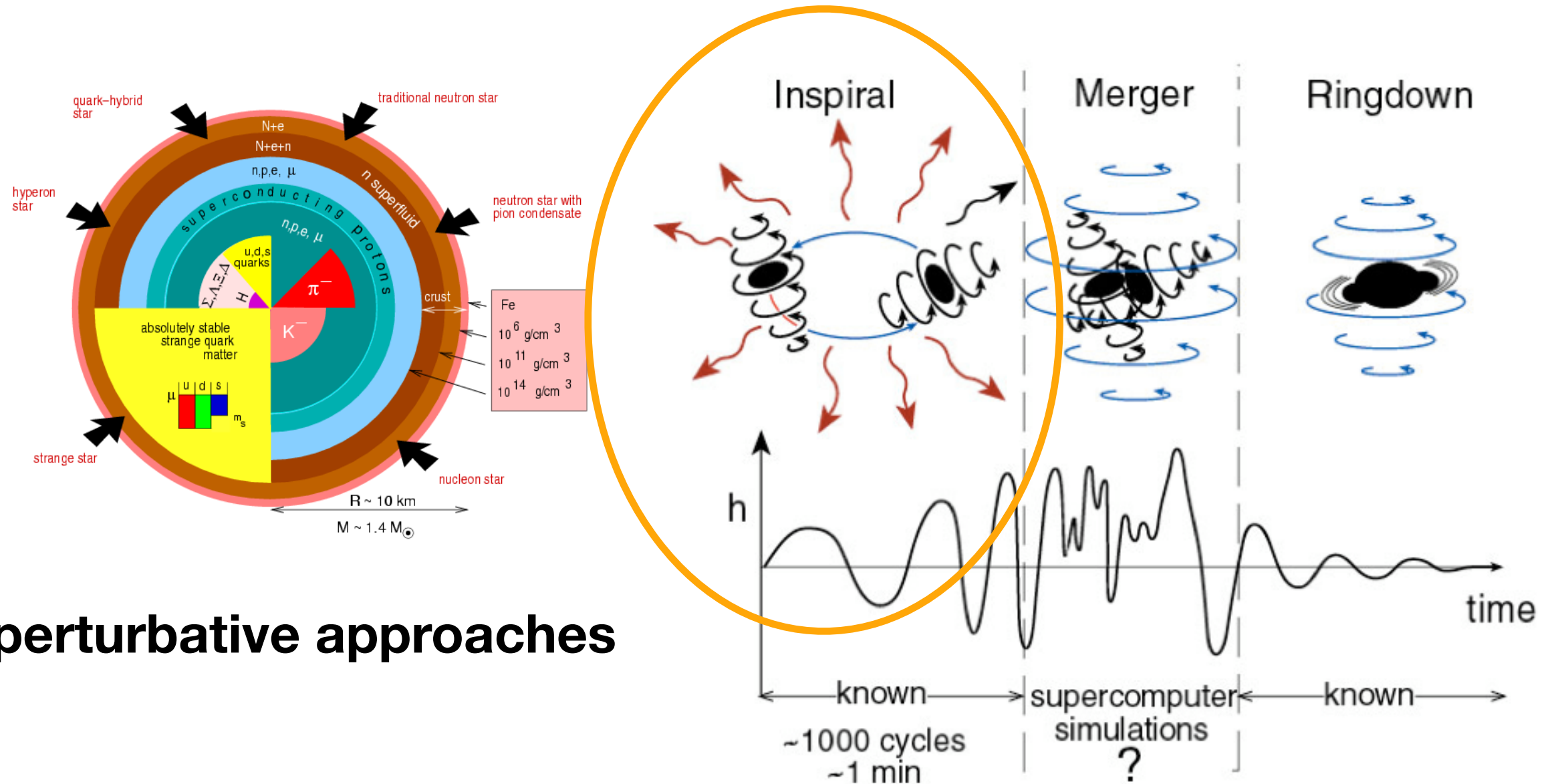


$$\tilde{h}_T(f) = \mathcal{A} f^{-7/6} e^{i\Psi_T(f)}$$

$$\Psi_T(f) = \varphi_c + 2\pi f t_c + \frac{3}{128\eta v^5} (\Delta\Psi_{3.5PN}^{\text{pp}} + \Delta\Psi_{3PN}^{\text{spin}} + \Delta\Psi_{2PN}^{\text{ecc.}} - \Delta\Psi_{6PN}^{\text{tidal}} + \Delta\Psi_{6PN}^{\text{tm}})$$

$$\Delta\Psi_{6PN}^{\text{tidal}} = -\frac{39}{2} \tilde{\Lambda} v^{10} + v^{12} \left(\frac{6595}{364} \delta\tilde{\Lambda} - \frac{3115}{64} \tilde{\Lambda} \right)$$

Response of NS to GW during inspiral



perturbative approaches

$$Q_{ij} = -\lambda \mathcal{E}_{ij}$$

$$\lambda = \frac{2}{3} \frac{R^5}{G} k_2$$

λ : Tidal deformability
 k_2 : Tidal Love number

Measurability of tidal deformability

Tidal term in GW waveform

$$\tilde{h}_T(f) = \mathcal{A} f^{-7/6} e^{i\Psi_T(f)}$$

M. Favata, PRL.112.101101 (2014)

$$\Psi_T(f) = \varphi_c + 2\pi f t_c + \frac{3}{128\eta v^5} (\Delta\Psi_{3.5\text{PN}}^{\text{pp}} + \Delta\Psi_{3\text{PN}}^{\text{spin}} + \Delta\Psi_{2\text{PN}}^{\text{ecc.}} + \Delta\Psi_{6\text{PN}}^{\text{tidal}} + \Delta\Psi_{6\text{PN}}^{\text{tm}}), \quad (1)$$

$$\Delta\Psi_{6\text{PN}}^{\text{tidal}} = -\frac{39}{2} \tilde{\Lambda} v^{10} + v^{12} \left(\frac{6595}{364} \delta\tilde{\Lambda} - \frac{3115}{64} \tilde{\Lambda} \right), \quad (4)$$

$$v = (\pi f M)^{1/3}$$

$$\tilde{\Lambda} = \frac{16}{13} \frac{(m_1 + 12m_2)m_1^4\Lambda_1 + (m_2 + 12m_1)m_2^4\Lambda_2}{(m_1 + m_2)^5}$$

$$\Lambda = \lambda/M^5 \rightarrow G \left(\frac{c^2}{GM} \right)^5 \lambda = \frac{2}{3} \left(\frac{Rc^2}{GM} \right)^5 k_2$$

Accumulated GW phase (I)

the number of wave cycles in frequency domain

$$\Delta N_{\text{cyc},\Psi} = \frac{1}{2\pi} \left[\Psi(f_2) - \Psi(f_1) + (f_1 - f_2) \frac{d\Psi}{df_1} \right], \quad (7.8)$$

$f_1 = 10$ Hz,
the low frequency cutoff for
Advanced LIGO
due to seismic noises

Waveform models:

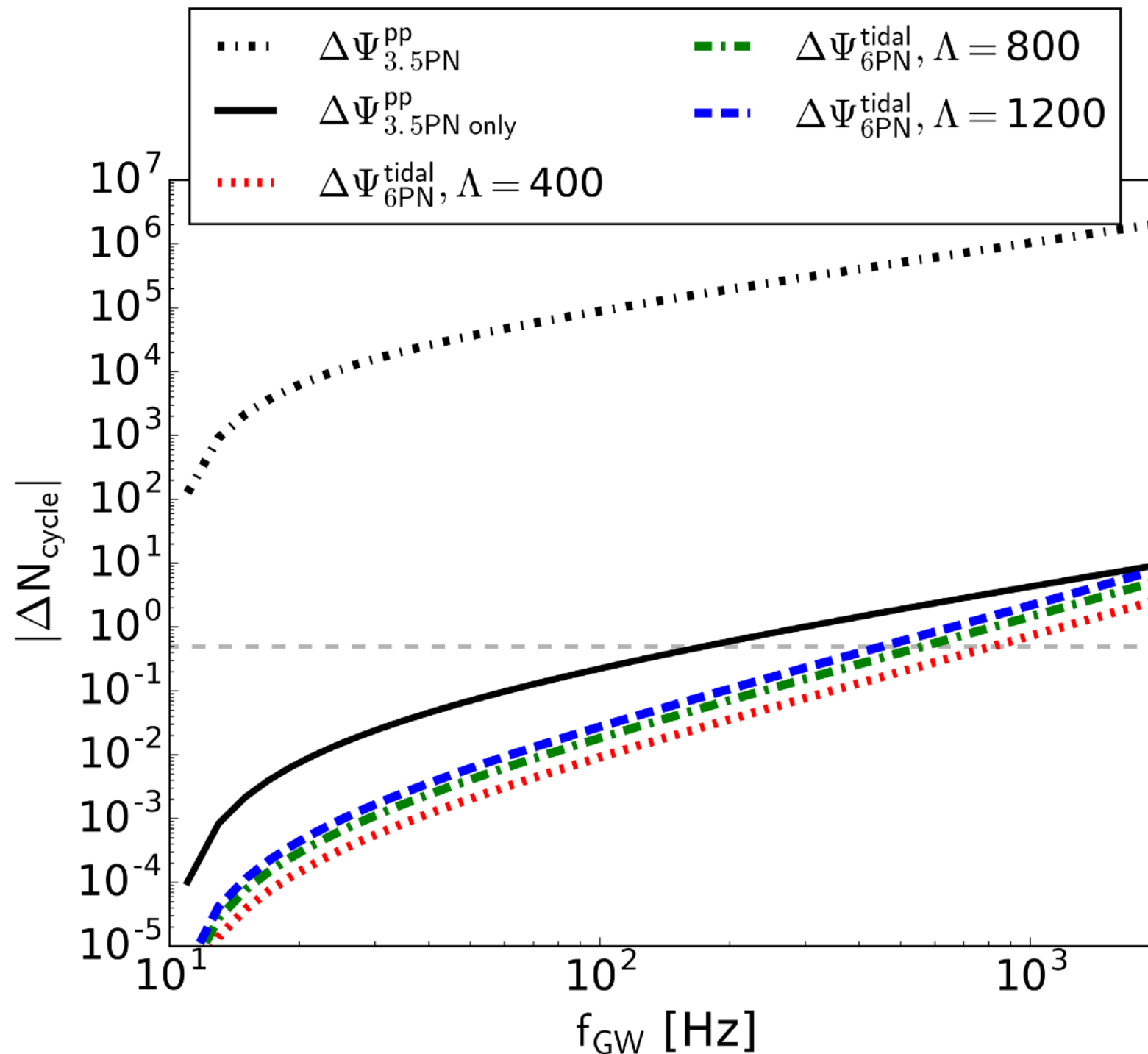
TaylorT2 for ΔN_{cyc}

TaylorF2(SPA) $\Delta N_{\text{cyc},\Psi}$

Moore et al., PRD.93.124061(2016)

	$1.4M_{\odot} + 1.4M_{\odot}, f_2 = 1000$ Hz		
PN order	ΔN_{cyc}	$\Delta N_{\text{cyc},\Psi}$	$\Delta N_{\text{useful}}^{\text{norm}}$
0PN(circ)	16 031	986 372	1821
0PN(ecc)	−463	−36 137	−6.37
1PN(circ)	439	21 743	125
1PN(ecc)	−15.8	−1193	−0.332
1.5PN(circ)	−208	−8520	−94.8
1.5PN(ecc)	1.67	103	0.113
2PN(circ)	9.54	294	6.70
2PN(ecc)	−0.215	−15.4	−0.008 17
2.5PN(circ)	−10.6	−218	−10.6
2.5PN(ecc)	0.0443	2.61	0.004 73
3PN(circ)	2.02	18.2	2.80
3PN(ecc)	0.002 00	0.119	−0.000 238
3.5PN(circ)	−0.662	−4.39	−0.977
Total	15 785	962 445	1843

Accumulated GW phase (2)

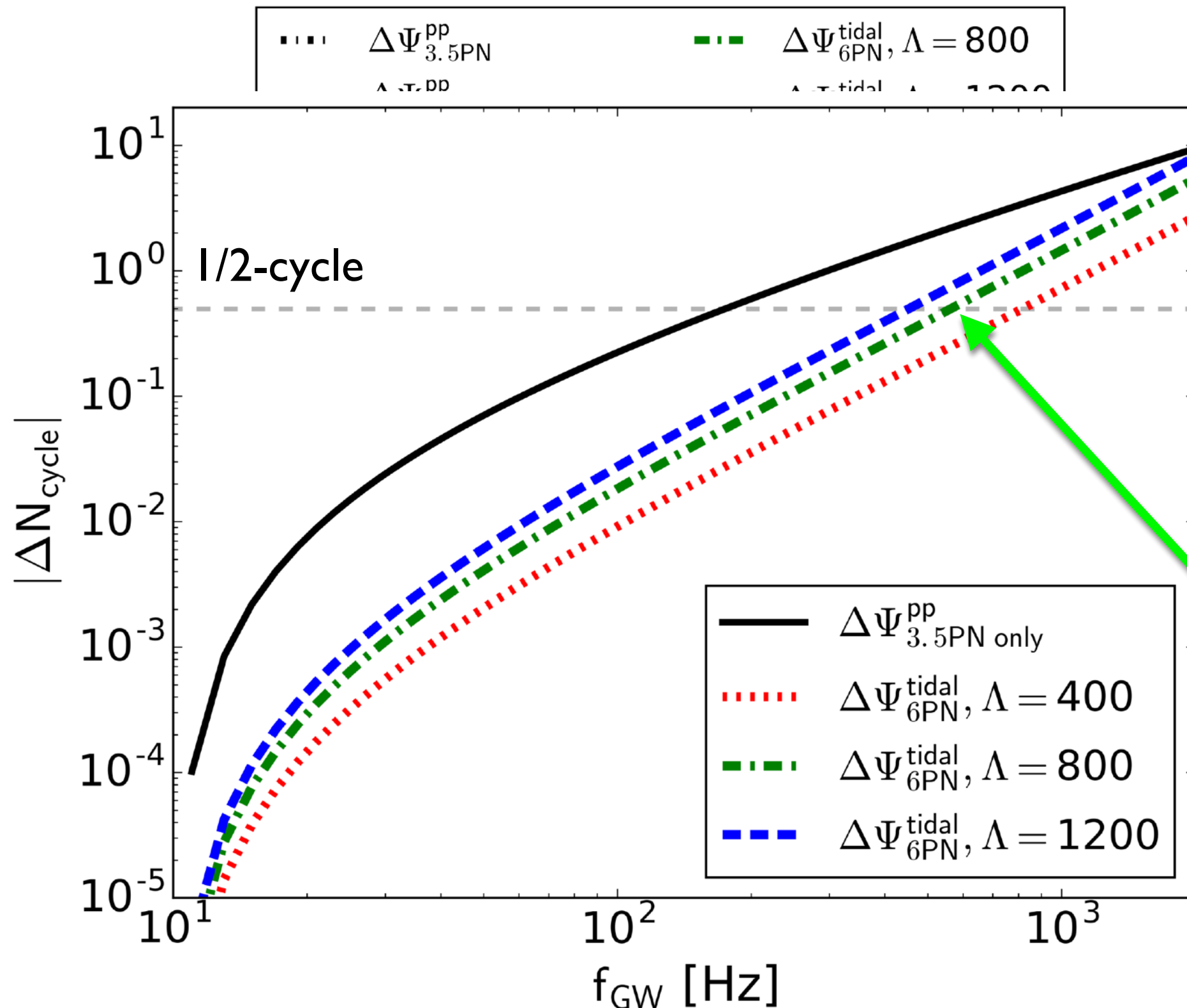


waveform model:
TaylorF2(SPA)

$M_{ch} = 1.188 M_{\odot}$

$M_1 = M_2 = 1.365 M_{\odot}$

Accumulated GW phase (2)



waveform model:
TaylorF2(SPA)

$M_{\text{ch}} = 1.188 M_{\odot}$

$M_1 = M_2 = 1.365 M_{\odot}$

Measurement Errors via Fisher Matrix

Fisher matrix

$$\Gamma_{ij} = \text{Re} \left\langle \frac{\partial h}{\partial \lambda_i} \middle| \frac{\partial h}{\partial \lambda_j} \right\rangle \quad \lambda_i = M_{\text{chirp}}, \eta, \phi_c, t_c, s_{1,2}, e_0, \tilde{\Lambda}, \dots$$

covariance matrix

$$\Sigma_{ij} = \Gamma_{ij}^{-1}$$

$$\tilde{h}_T(f) = \mathcal{A} f^{-7/6} e^{i\Psi_T(f)}$$

correlation matrix

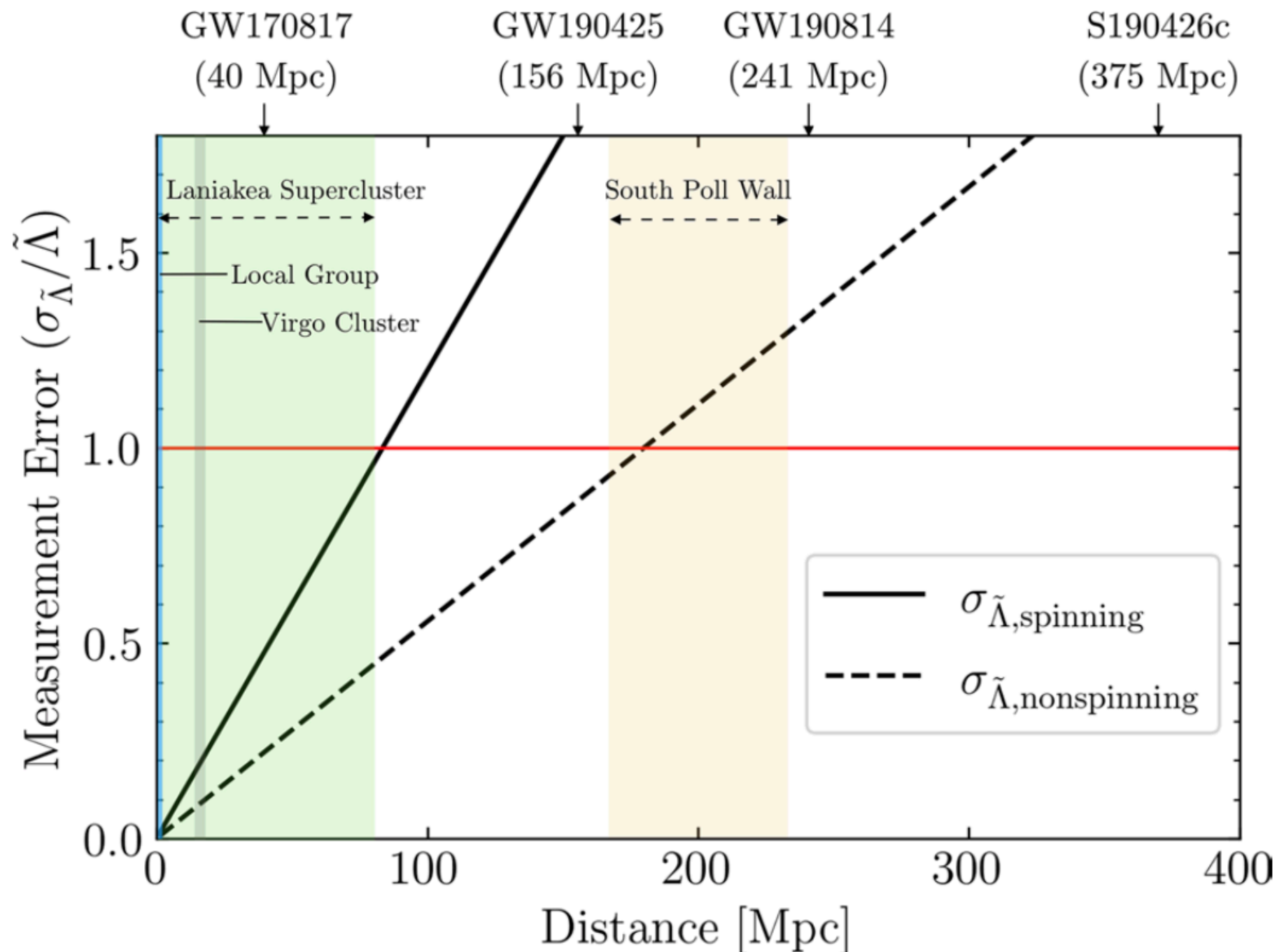
$$c_{ij} = \Sigma_{ij} / \sqrt{\Sigma_{ii} \Sigma_{jj}}$$

$$\begin{aligned} \Psi_T(f) = \varphi_c + 2\pi f t_c + \frac{3}{128\eta v^5} (\Delta\Psi_{3.5\text{PN}}^{\text{pp}} \\ + \Delta\Psi_{3\text{PN}}^{\text{spin}} + \Delta\Psi_{2\text{PN}}^{\text{ecc.}} + \Delta\Psi_{6\text{PN}}^{\text{tidal}} + \Delta\Psi_{6\text{PN}}^{\text{tm}}) \end{aligned}$$

Measurement Errors

$$\sigma_i = \sqrt{\Sigma_{ii}}$$

Measurement Errors of Tidal Deformability



Calculation of Tidal deformability

TOV Eq. vs. Diff. Eq. for Tidal deformability

a spherical symmetric star in hydrostatic equilibrium

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$



$$\begin{aligned} ds_0^2 &= g_{\alpha\beta}^{(0)} dx^\alpha dx^\beta \\ &= -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \end{aligned}$$

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + pg_{\alpha\beta}^{(0)},$$

TOV eq.

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \left(1 + \frac{P}{\rho c^2}\right) \left(1 + \frac{4\pi Pr^3}{Mc^2}\right) \left(1 - \frac{2GM}{rc^2}\right)$$

$$\frac{dM}{dr} = 4\pi r^2 \left(\frac{\epsilon}{c^2}\right)$$



Mass & Radius

TOV Eq. vs. Diff. Eq. for Tidal deformability

static linearized perturbations due to an external tidal field

$$\delta G_{\mu\nu} = 8\pi\delta T_{\mu\nu}$$



$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + h_{\alpha\beta},$$

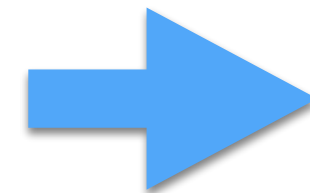
T. Hinderer (2008), K. Thorne and A. Campolattaro (1967)

$$h_{\alpha\beta} =$$

$$\text{diag}[-e^{\nu(r)}H_0(r), e^{\lambda(r)}H_2(r), r^2K(r), r^2\sin^2\theta K(r)]Y_{2m}(\theta, \varphi).$$

$$\delta T_0^0 = -\delta\rho = -(dp/d\rho)^{-1}\delta p \quad \delta T_i^i = \delta p$$

$$H'' + H' \left\{ \frac{2}{r} + e^\lambda \left[\frac{2m(r)}{r^2} + 4\pi r(p - \rho) \right] \right\} + H \left[-\frac{6e^\lambda}{r^2} + 4\pi e^\lambda \left(5\rho + 9p + \frac{\rho + p}{dp/d\rho} \right) - \nu'^2 \right] = 0,$$



k2 or λ

To obtain Tidal Love number, k2 (I)

T. Hinderer, ApJ, 677, 1216 (2008)

$$k_2 = \frac{8C^5}{5} (1 - 2C)^2 [2 + 2C(y - 1) - y] \\ \times \left\{ 2C[6 - 3y + 3C(5y - 8)] + 4C^3[13 - 11y + C(3y - 2) + 2C^2(1 + y)] \right. \\ \left. + 3(1 - 2C)^2[2 - y + 2C(y - 1)] \ln(1 - 2C) \right\}^{-1}$$

$$y = \frac{R\beta(R)}{H(R)} \quad C = \frac{GM}{Rc^2}$$

$$\frac{dH}{dr} = \beta$$

$$\frac{d\beta}{dr} = 2 \left(1 - 2\frac{M}{r} \right)^{-1} \times H \left\{ -2\pi [5\epsilon + 9P + (d\epsilon/dP)(\epsilon + P)] \right. \\ \left. + \frac{3}{r^2} + 2 \left(1 - 2\frac{M}{r} \right)^{-1} \left(\frac{M}{r^2} + 4\pi r P \right)^2 \right\} \\ + \frac{2\beta}{r} \left(1 - 2\frac{M}{r} \right)^{-1} \left\{ -1 + \frac{M}{r} + 2\pi r^2 (\epsilon - P) \right\}$$

To obtain Tidal Love number, k2 (2)

$$\frac{dH}{dr} = \beta \quad \frac{d\beta}{dr} = 2 \left(1 - 2\frac{M}{r}\right)^{-1} \times H \left\{ -2\pi [5\epsilon + 9P + (d\epsilon/dP)(\epsilon + P)] \right. \\ \left. + \frac{3}{r^2} + 2 \left(1 - 2\frac{M}{r}\right)^{-1} \left(\frac{M}{r^2} + 4\pi r P\right)^2 \right\} \\ + \frac{2\beta}{r} \left(1 - 2\frac{M}{r}\right)^{-1} \left\{ -1 + \frac{M}{r} + 2\pi r^2(\epsilon - P) \right\}$$

TOV

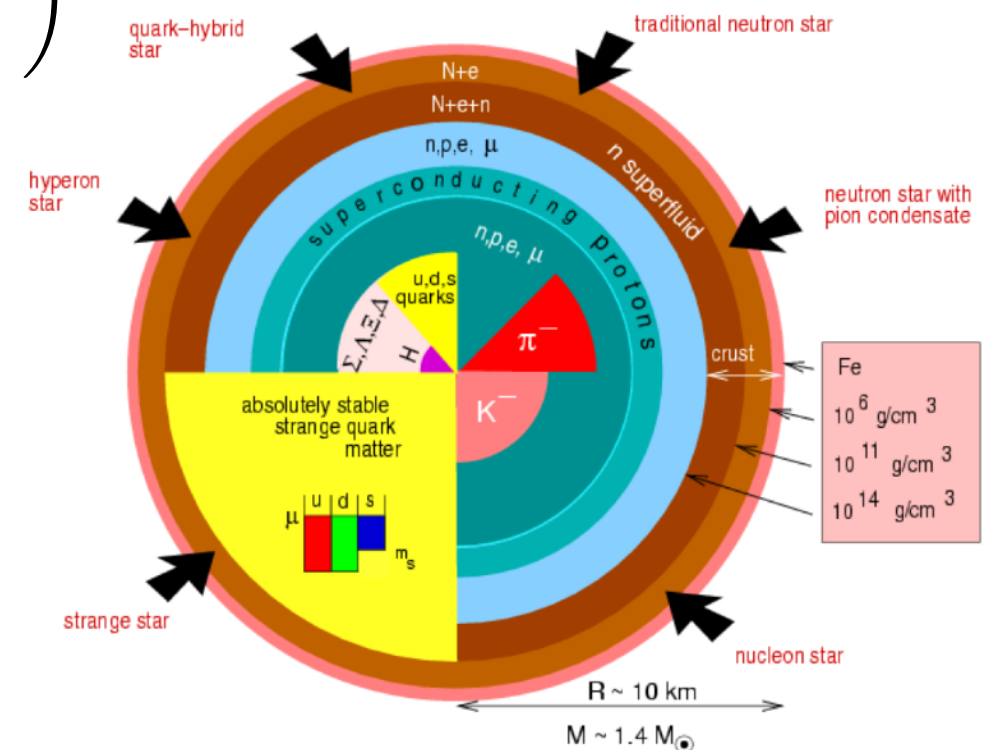
$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \left(1 + \frac{P}{\rho c^2}\right) \left(1 + \frac{4\pi P r^3}{Mc^2}\right) \left(1 - \frac{2GM}{rc^2}\right)$$

$$\frac{dM}{dr} = 4\pi r^2 \left(\frac{\epsilon}{c^2}\right)$$

k2 (λ,Λ) depends on NS EoS !!

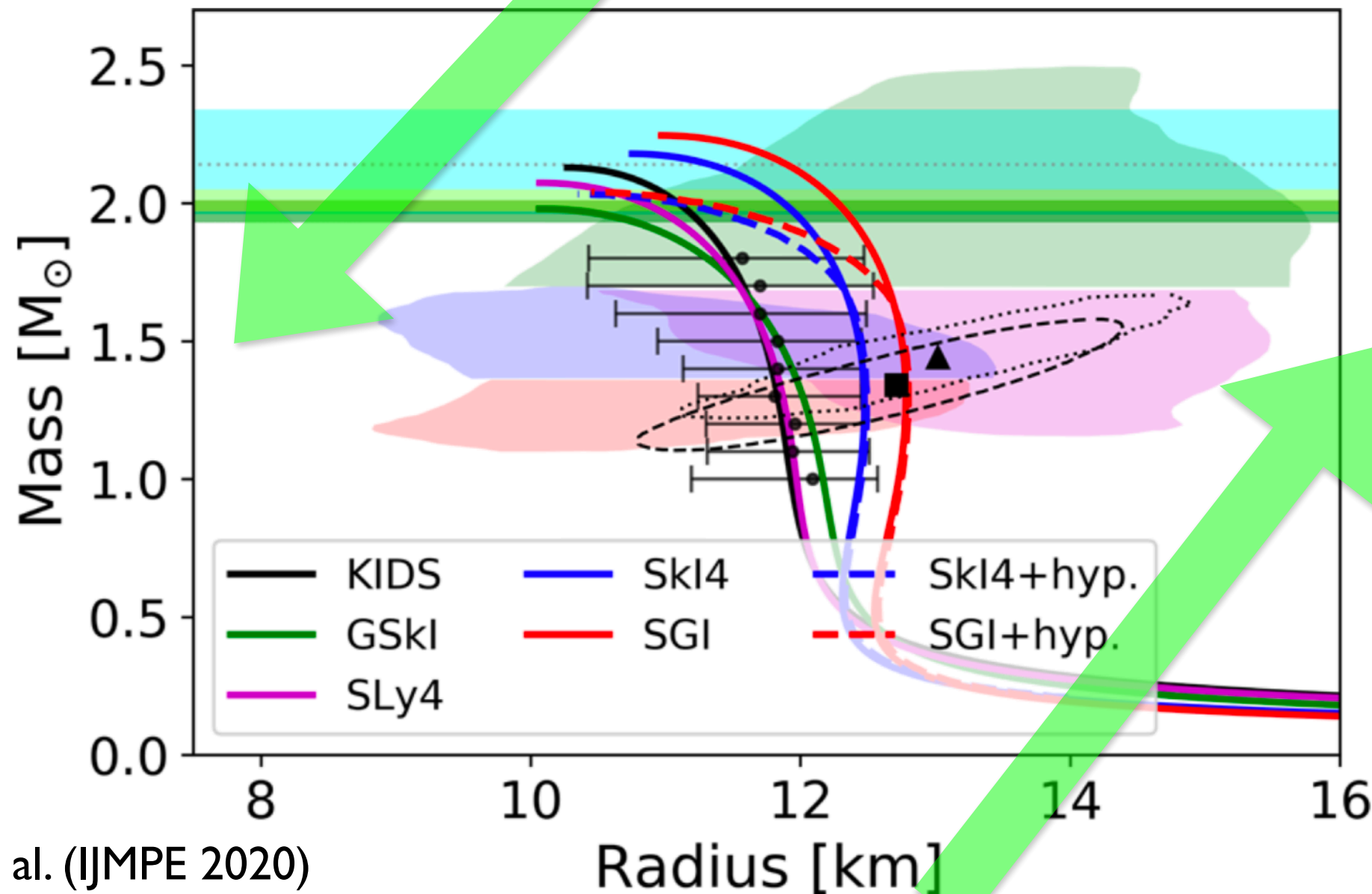
Compositions of a NS

F. Weber 2005



Constraints on Equation of State

prefer soft EoS: GW170817, strangeness



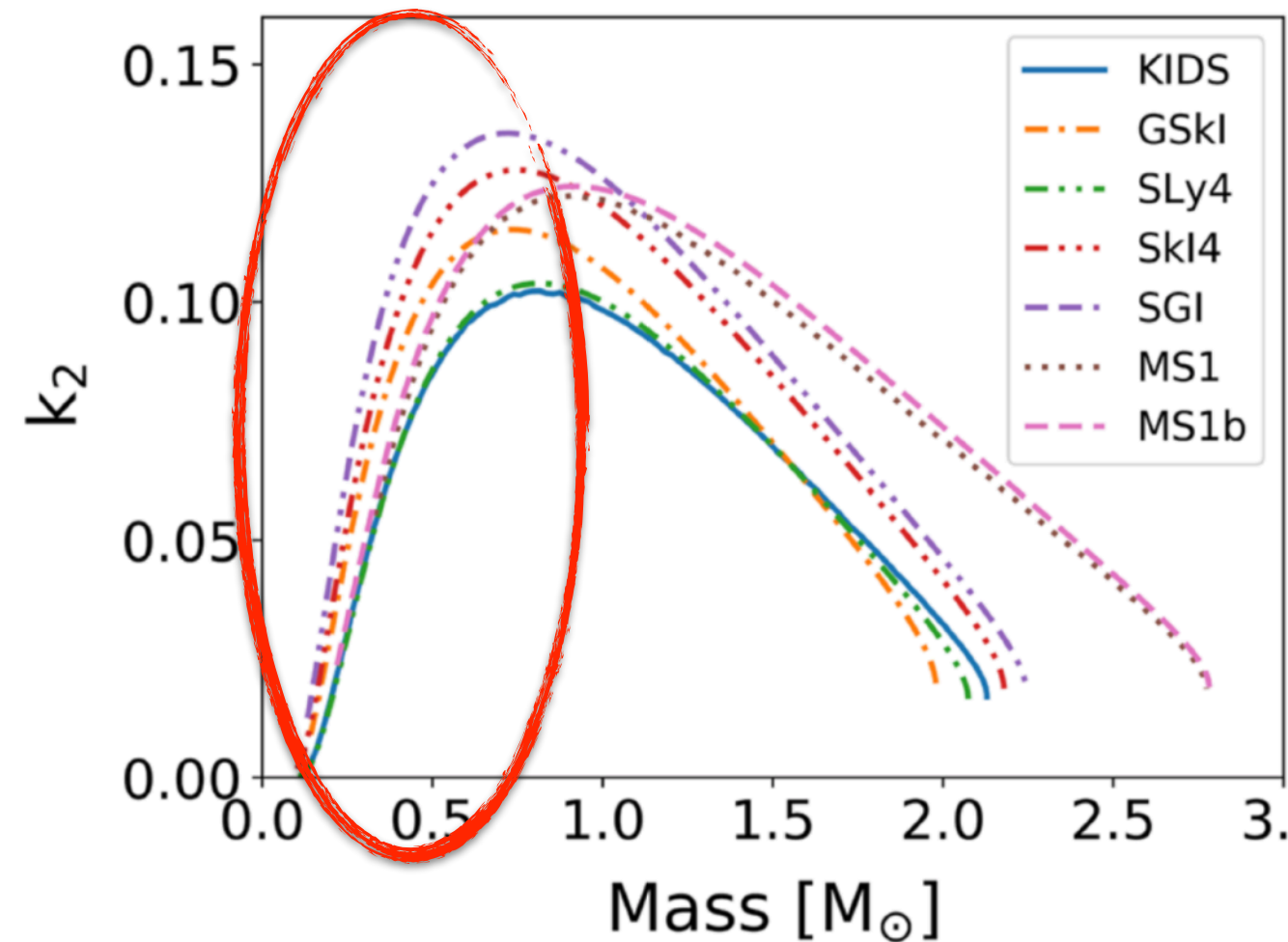
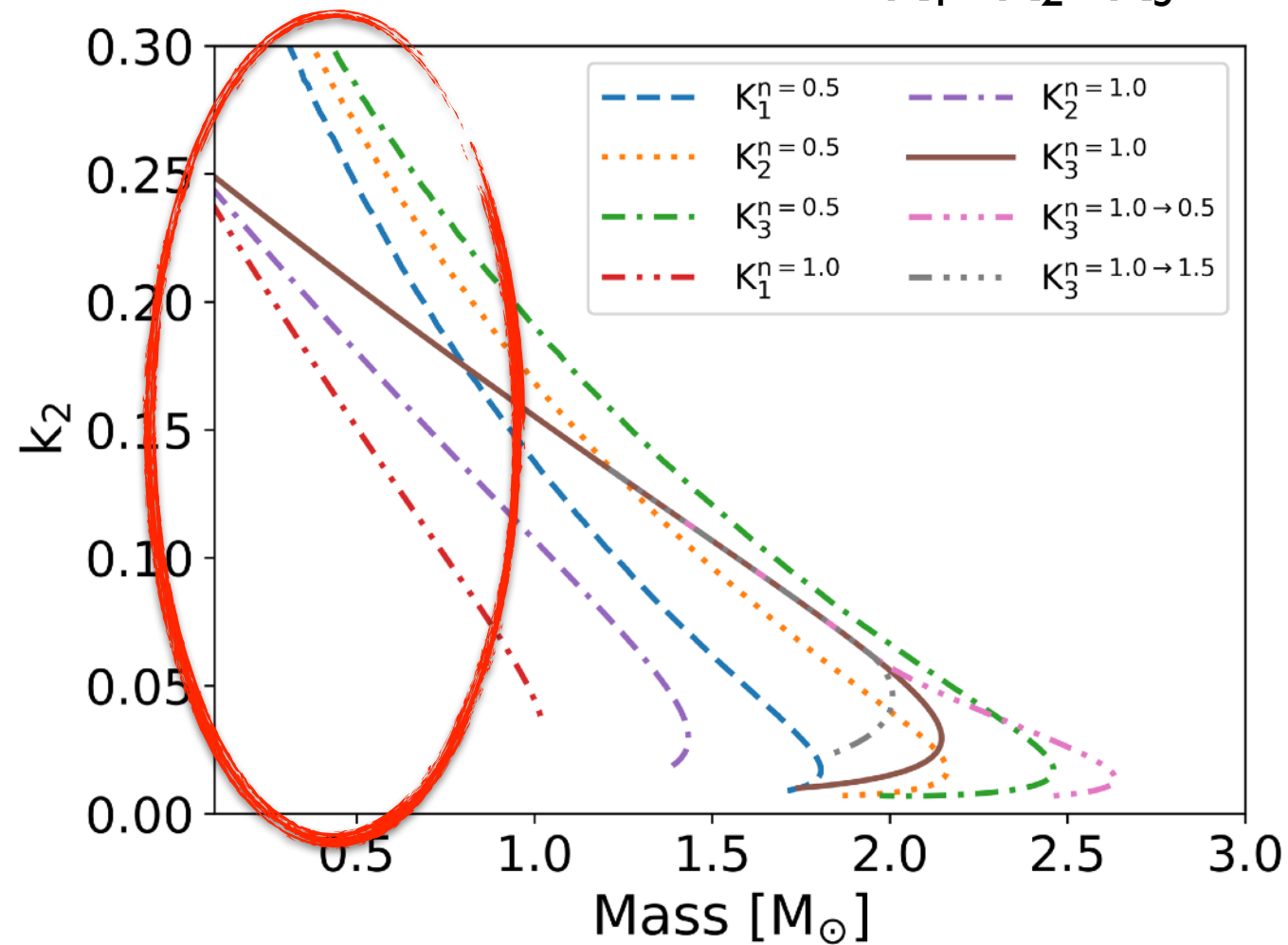
M. Kim et al. (IJMPE 2020)

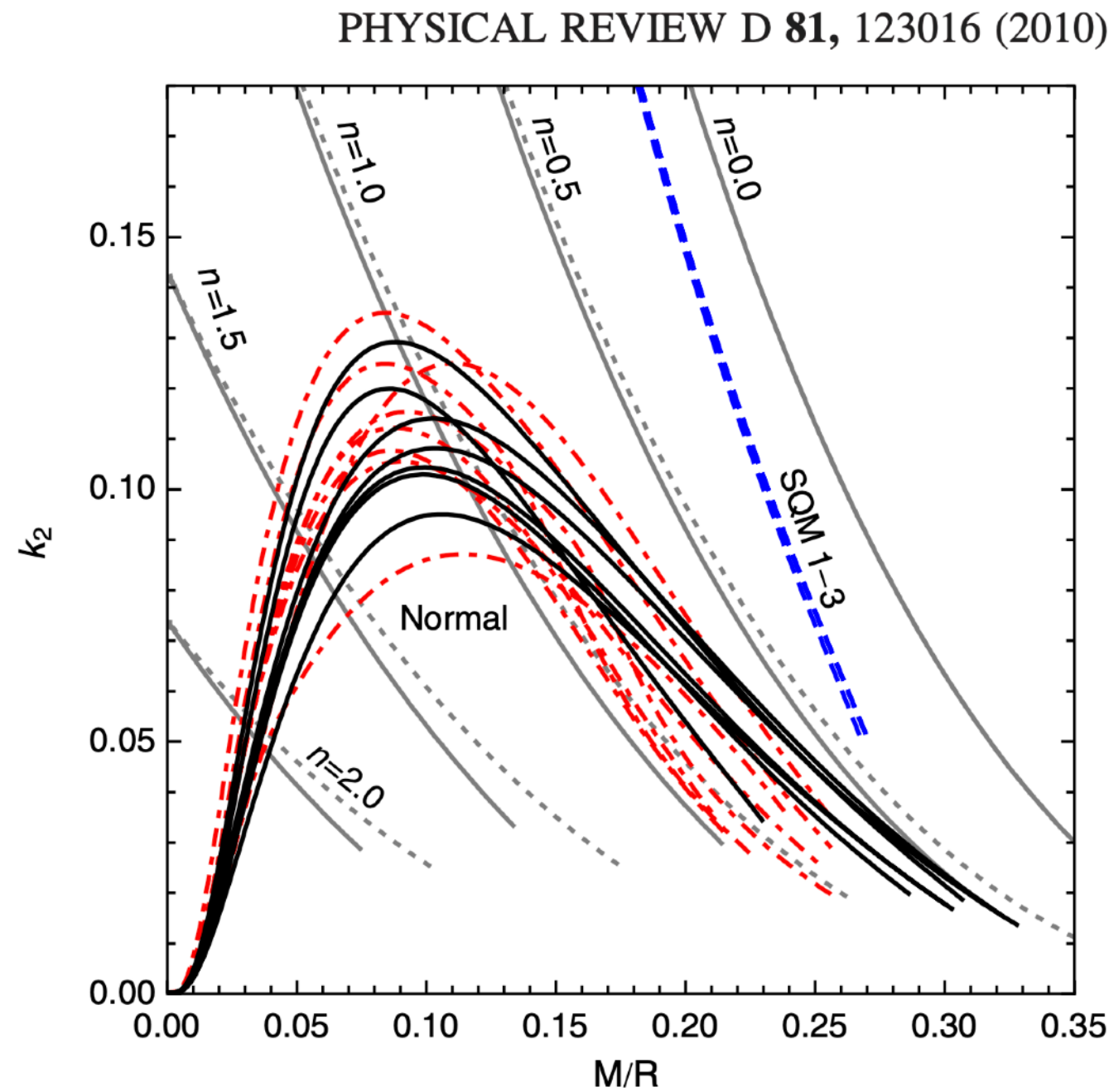
prefer hard EoS: M_{max} , NICER

Polytropic EoSs vs. Realistic EoSs (2)

$$\Lambda = \lambda/M^5 \rightarrow G \left(\frac{c^2}{GM} \right)^5 \quad \lambda = \frac{2}{3} \left(\frac{Rc^2}{GM} \right)^5 k_2$$

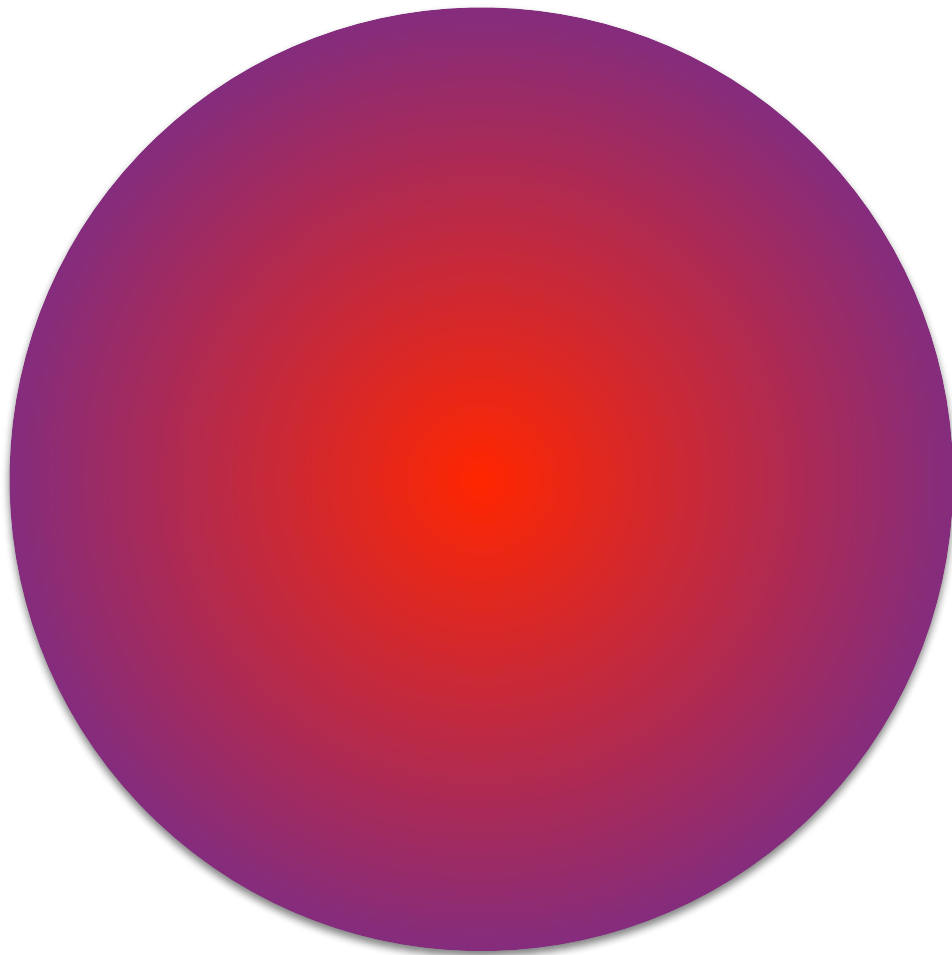
$K_1 < K_2 < K_3$



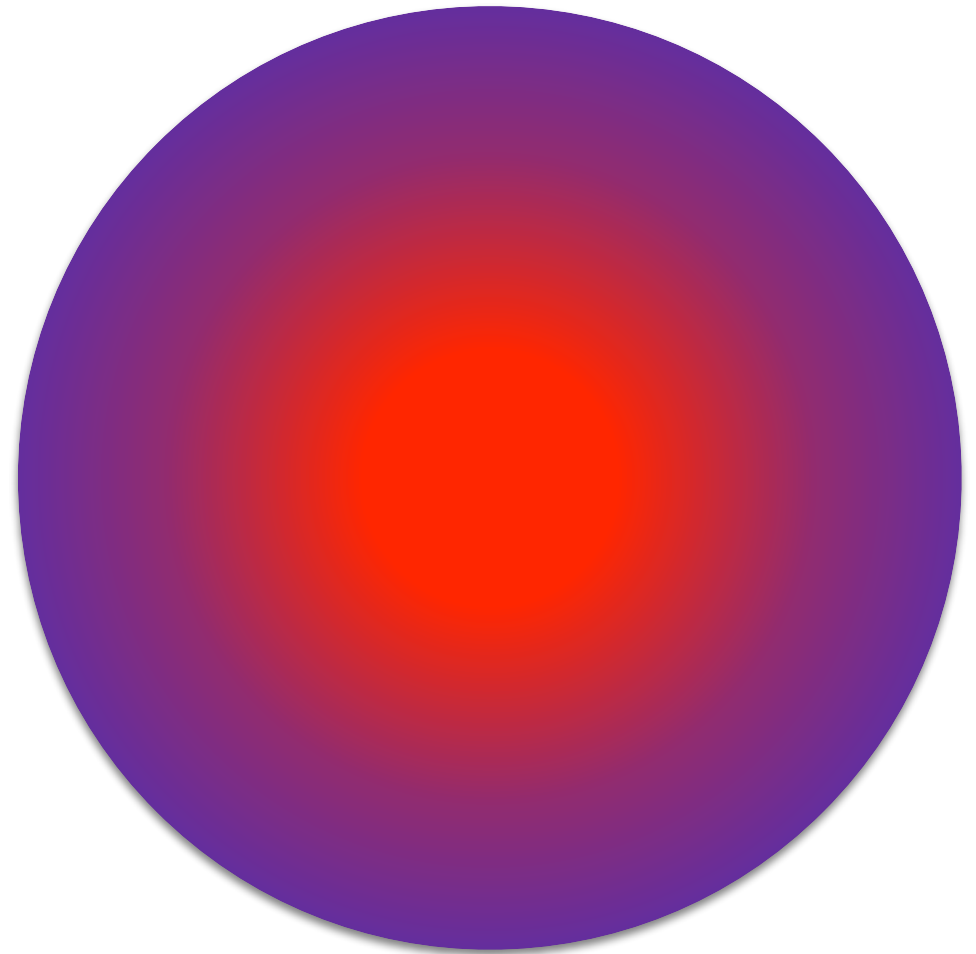


Tidal Love number, k_2

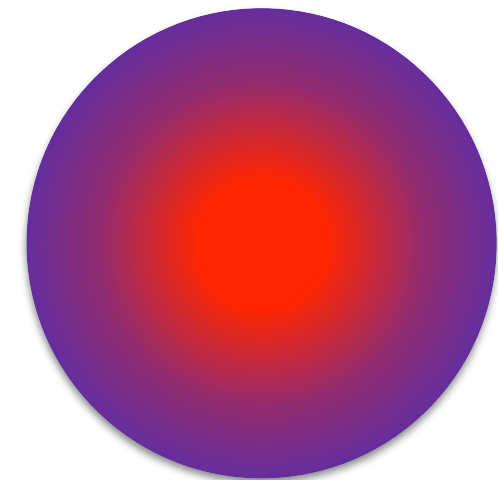
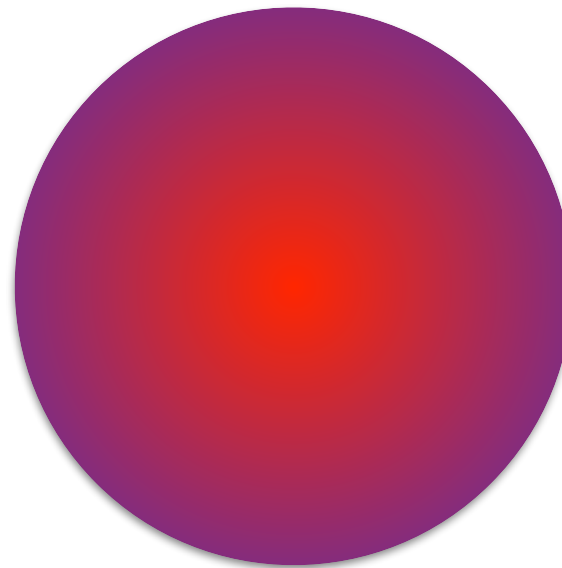
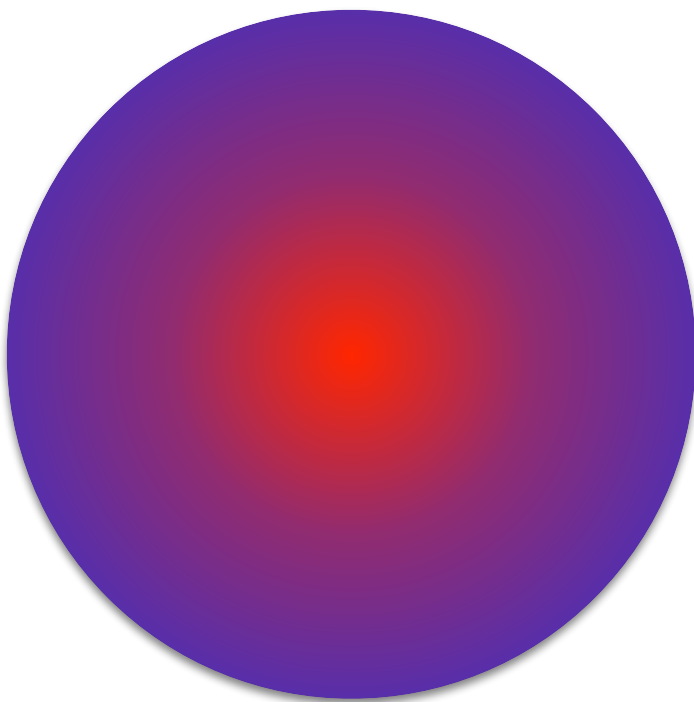
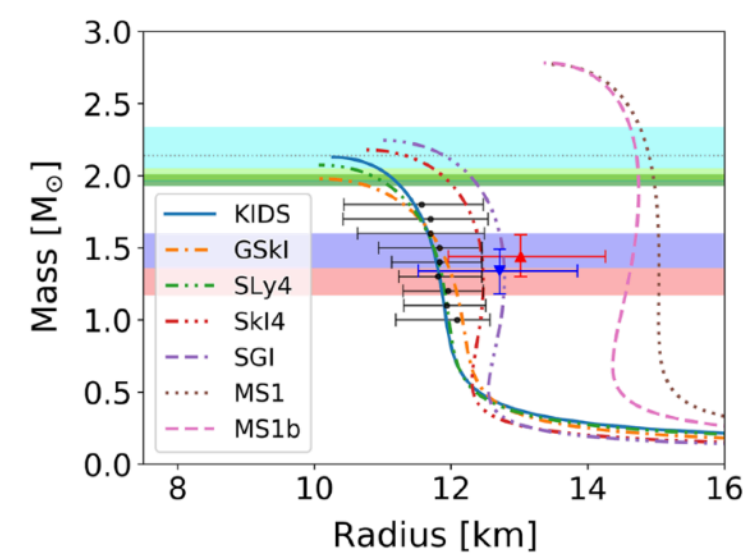
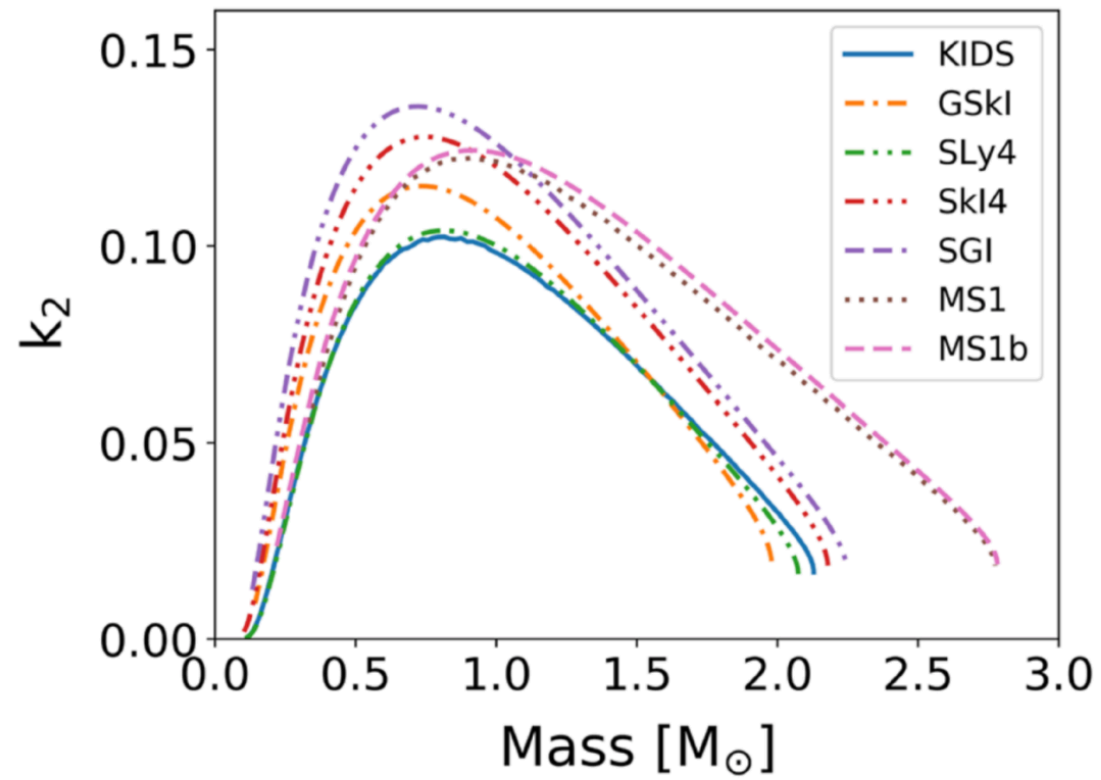
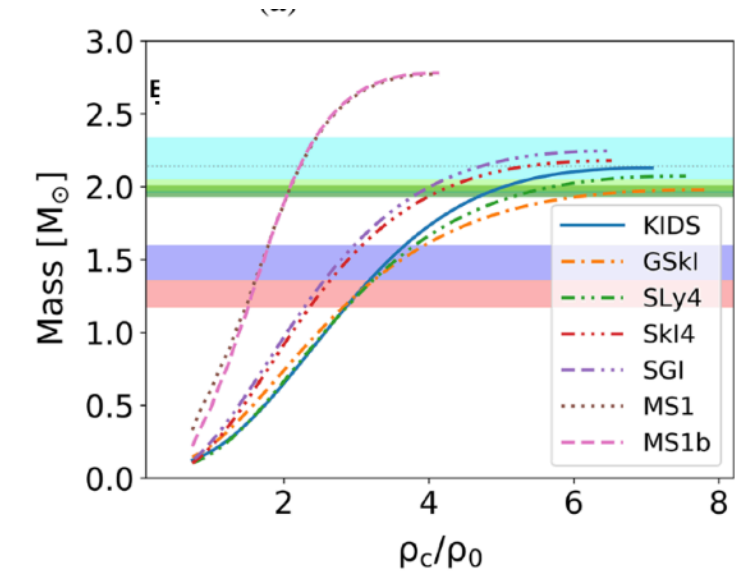
High k_2



low k_2

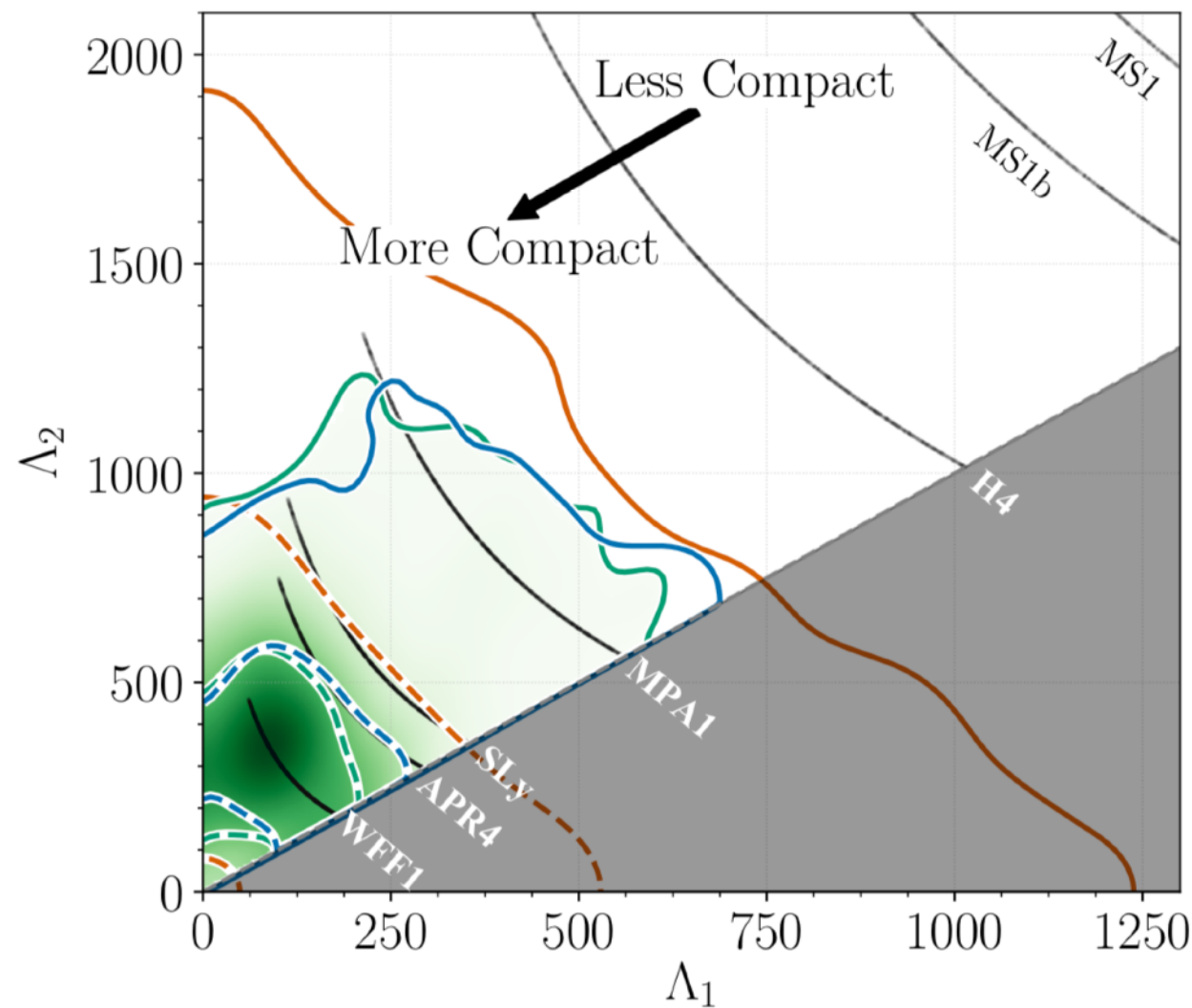


Tidal Love number, k_2



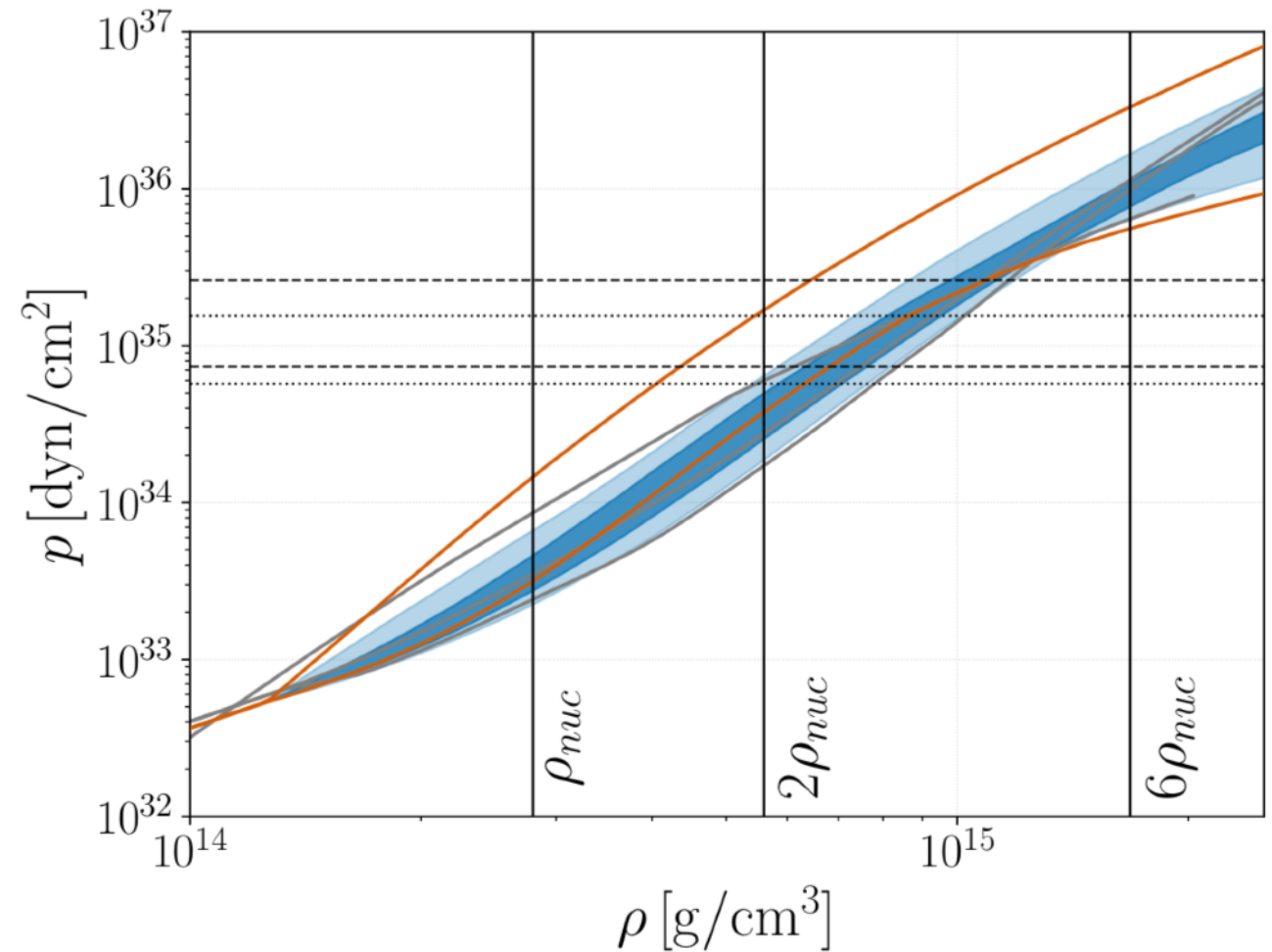
Measurements of GW170817

$$\Lambda(1.4M_{\odot}) = 190^{+390}_{-120}$$



$$P(2 \rho_{\text{nuc}}) = 3.5^{+2.7}_{-1.7} \times 10^{34} \text{ dyne/cm}^2$$

$$P(6 \rho_{\text{nuc}}) = 9.0^{+7.9}_{-2.6} \times 10^{35} \text{ dyne/cm}^2$$



Abbott et al. (LSC and Virgo), arxiv:1805.11581 (PhysRevLett.121.161101)

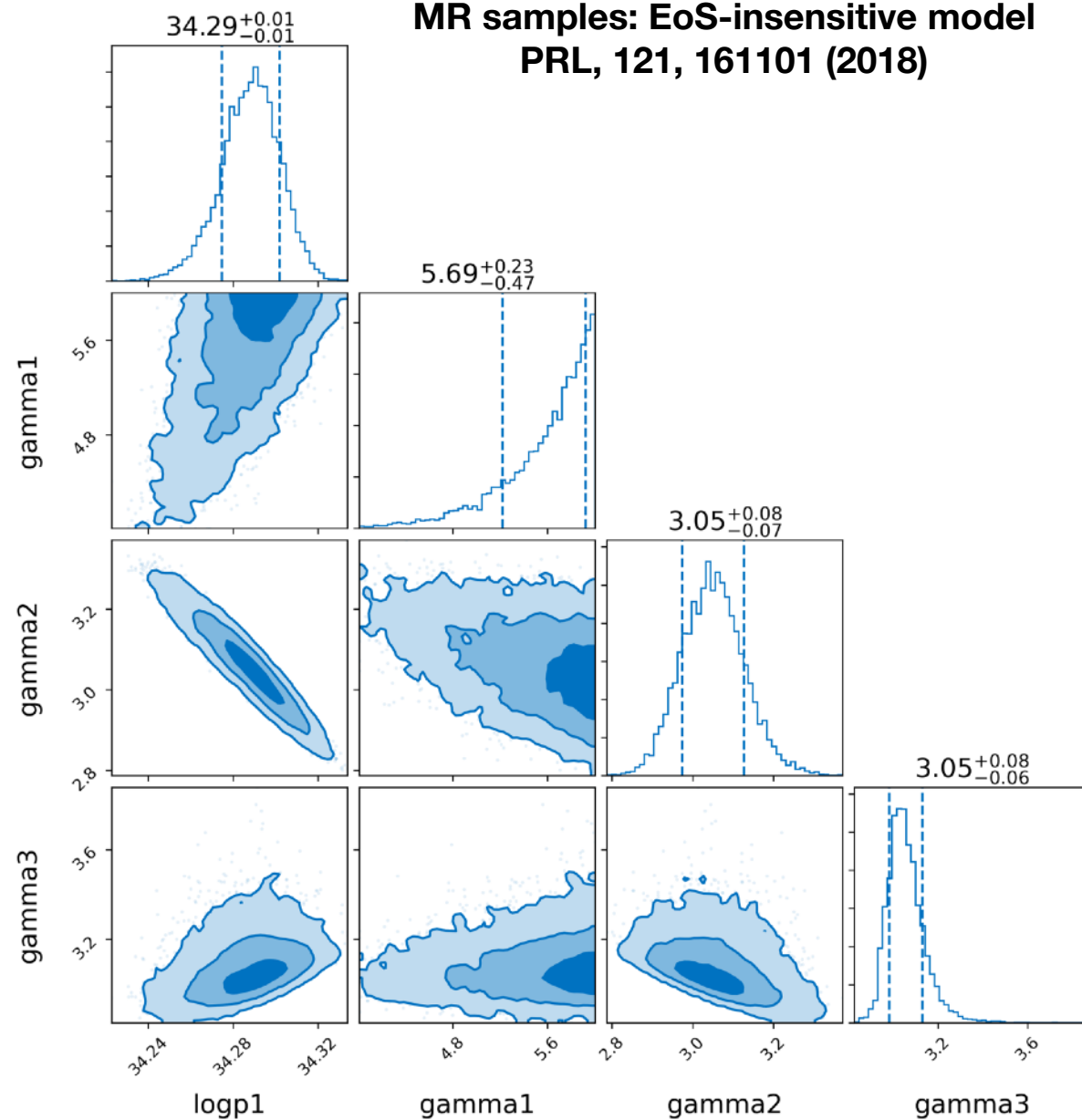
$$\rho_{\text{nuc}} = 2.8 \times 10^{14} \text{ g/cm}^3$$

Posteriors w/ Piece-wise Polytrropic EoSs

Reference for Piece-wise Polytrropic EoSs : Read et al. PRD 79, 124032 (2009)

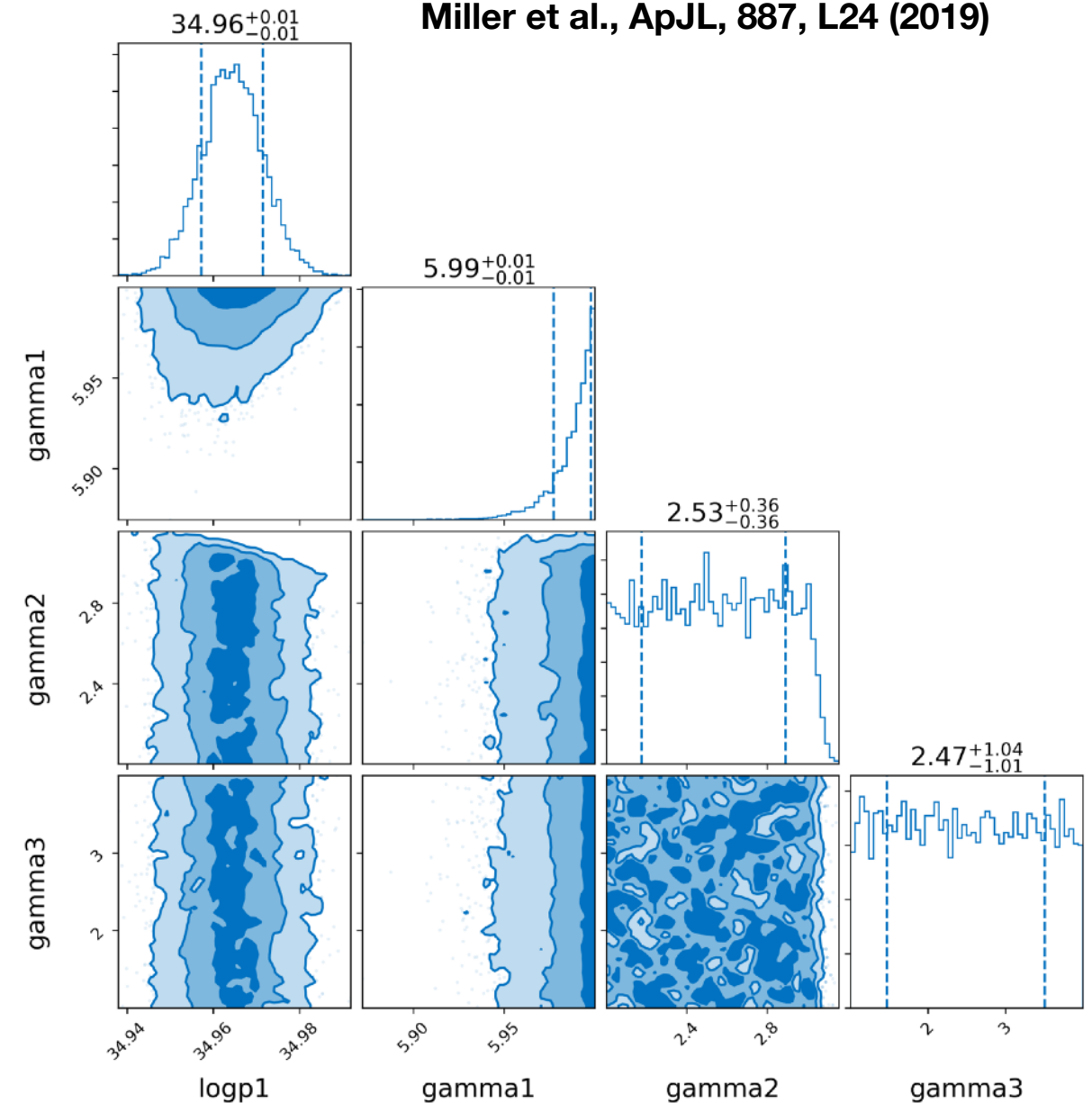
GW170817

MR samples: EoS-insensitive model
PRL, 121, 161101 (2018)



PSR J0030+0451

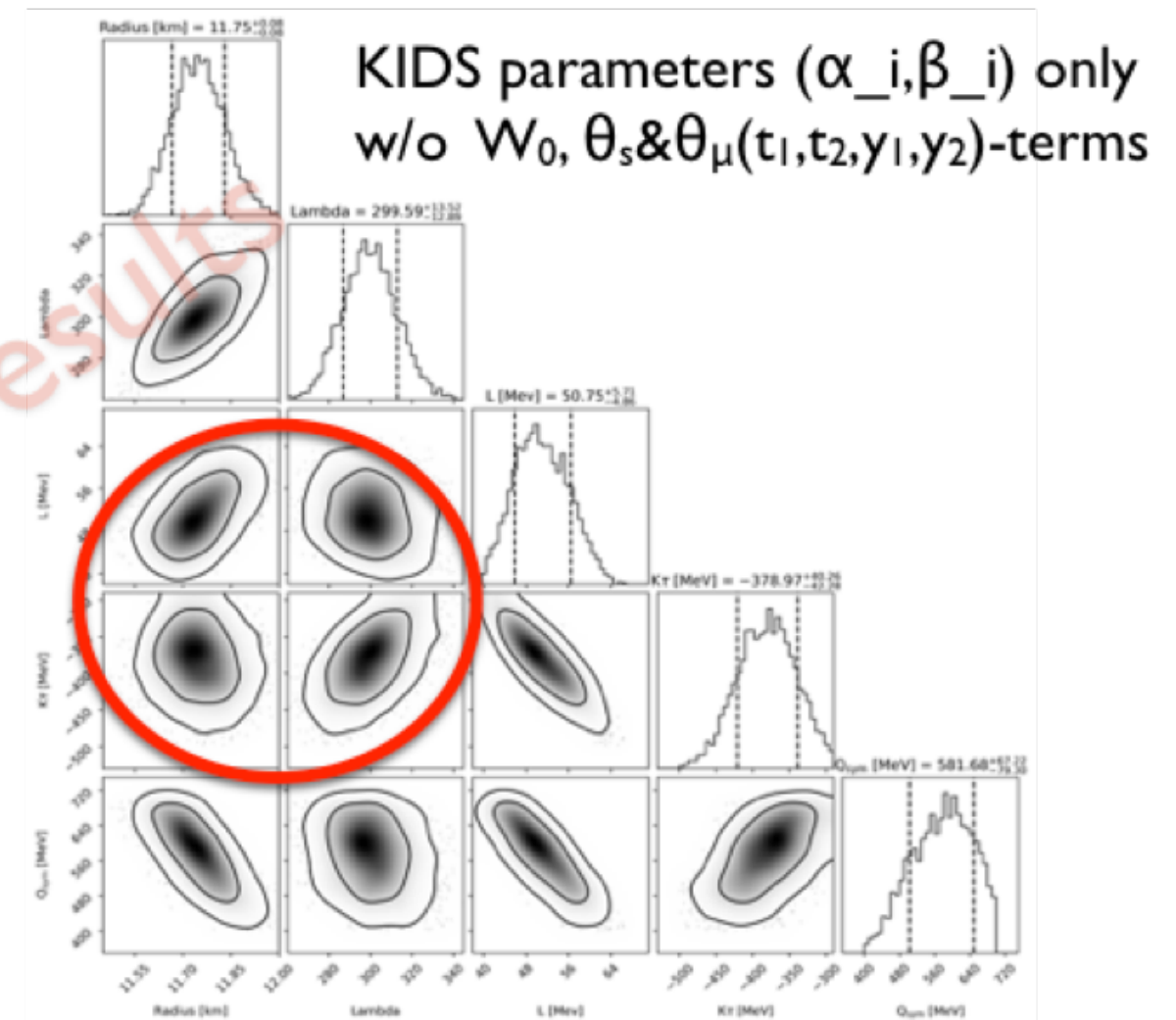
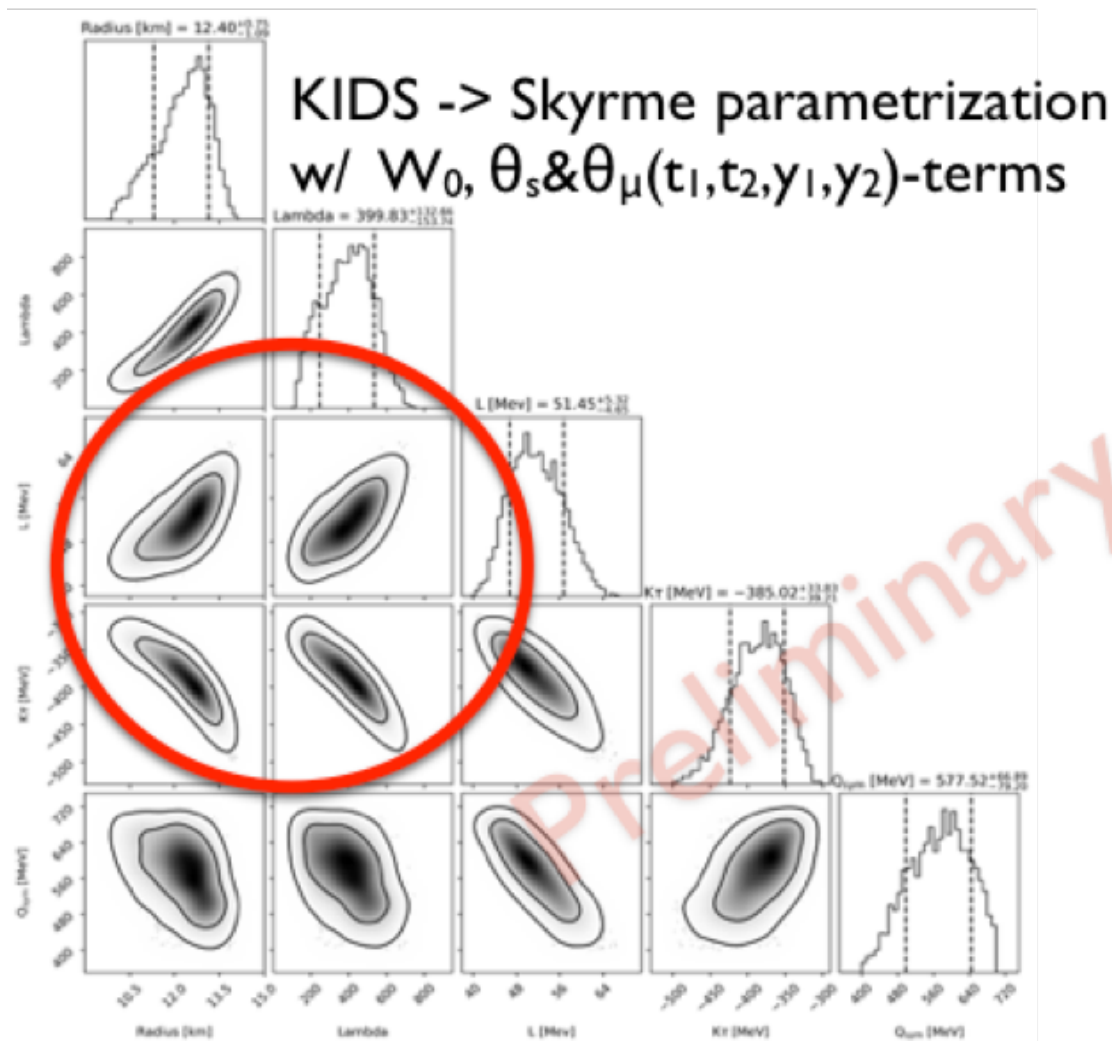
MR samples: 2 spot model
Miller et al., ApJL, 887, L24 (2019)



Bayesian analysis on EoS parameters

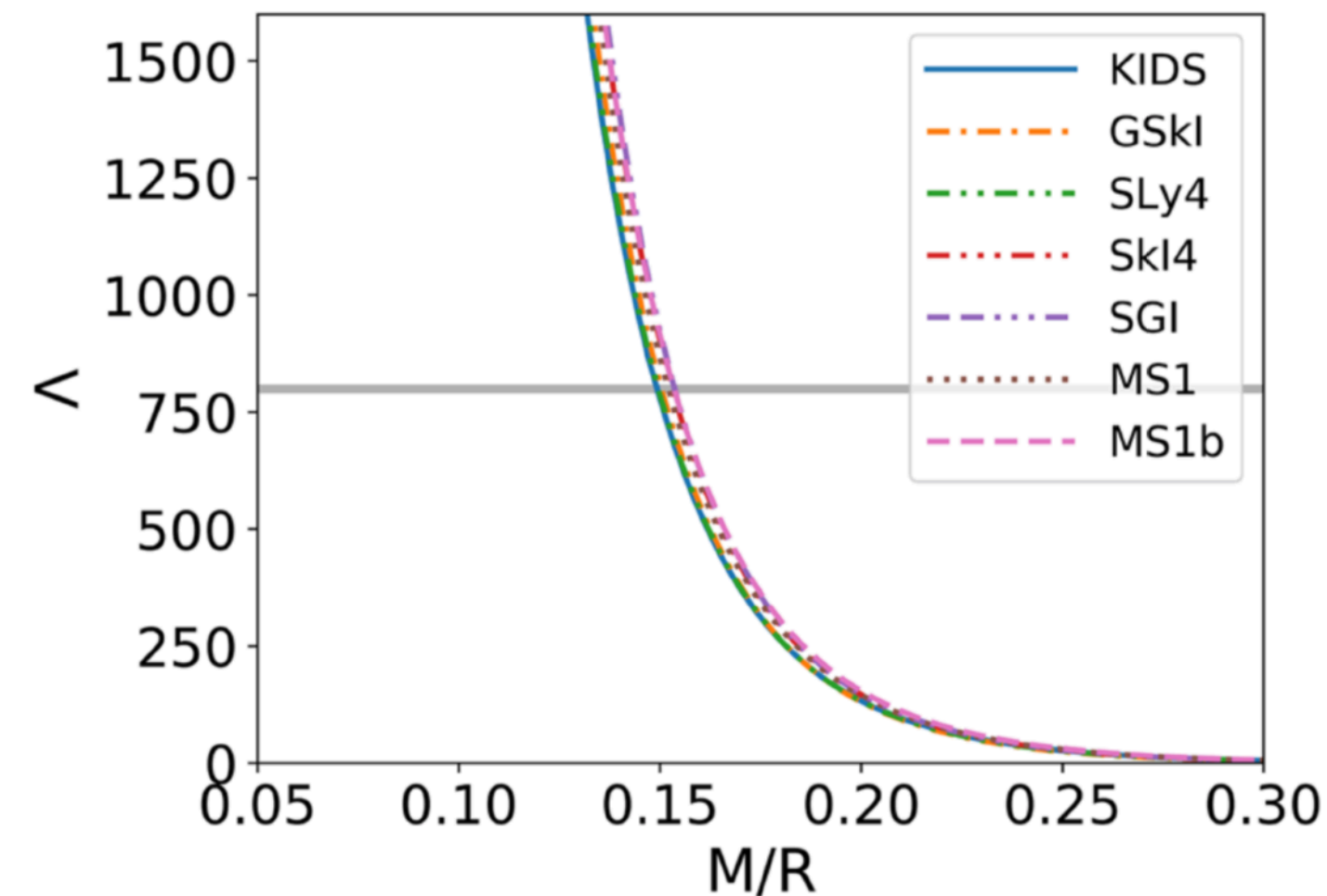
$$P(\{E_0, K_0, Q_0, J, L, K_\tau, Q_{\text{sym}}\}|\text{data})$$

Draft in preparation



Precise measurements of R, Λ
=> constraints of EoS & Nuclei properties

Eos-insensitive relation



Insensitive to EoS

K.Yagi and N.Yunes, Phys. Rep. 681 (2017) 1

$$C = a_0 + a_1 (\ln \Lambda) + a_2 (\ln \Lambda)^2$$

$$a_0=0.360, a_1= - 0.0355, a_2= 0.000705$$

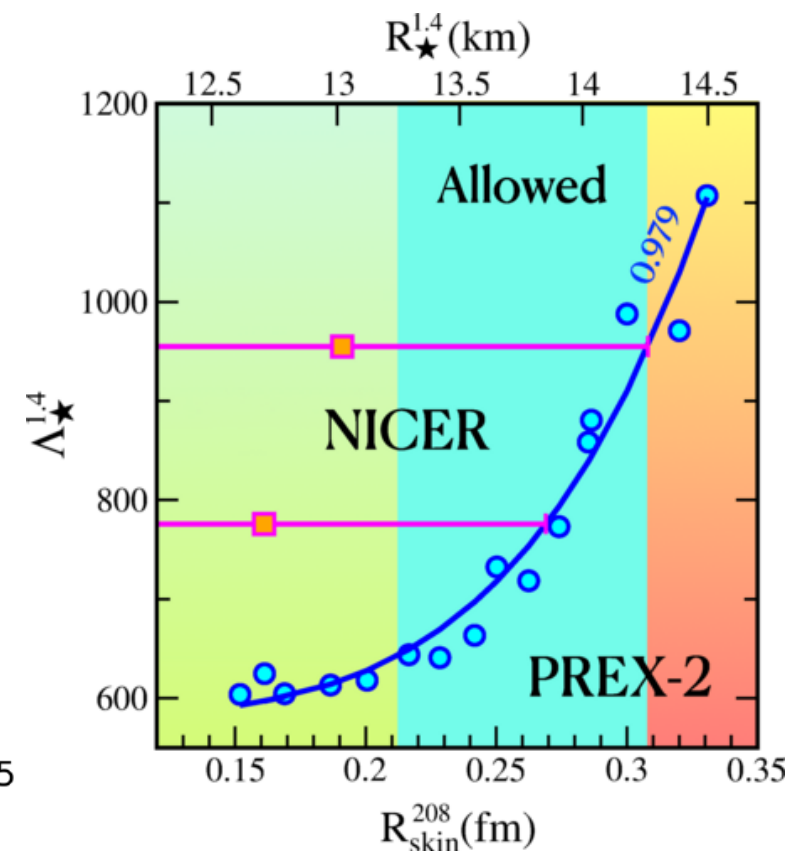
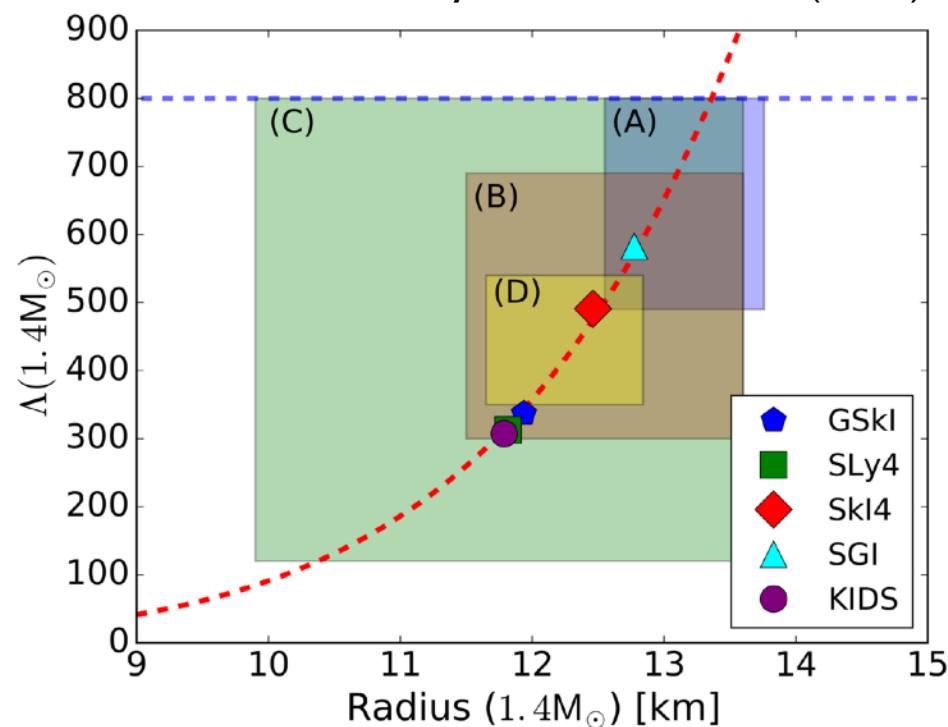
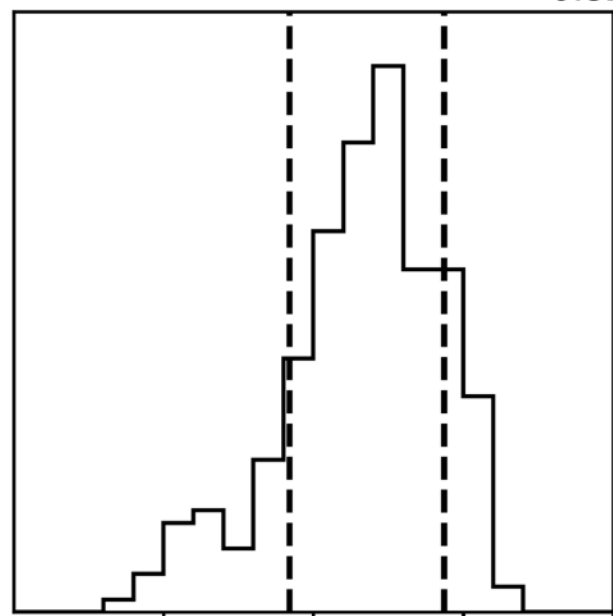
$$C=GM/Rc^2$$

Lambda-Radius relation

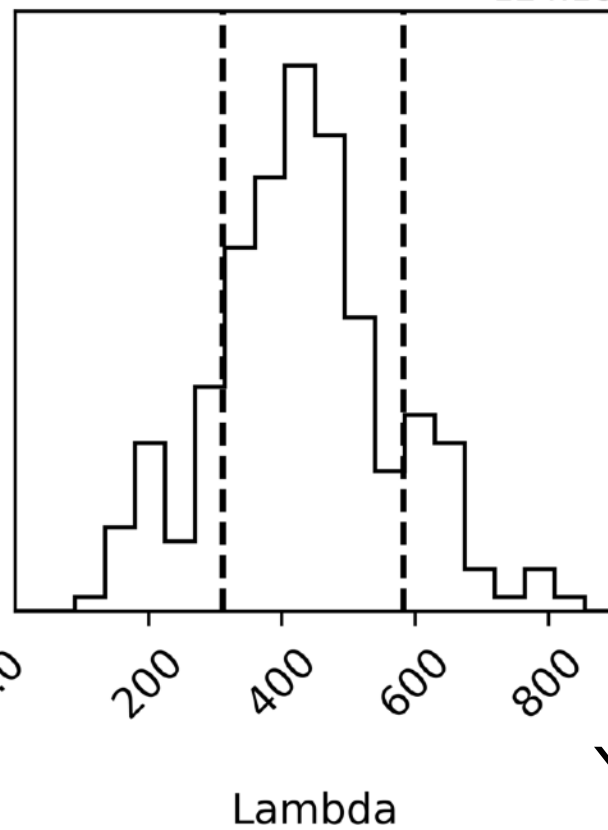
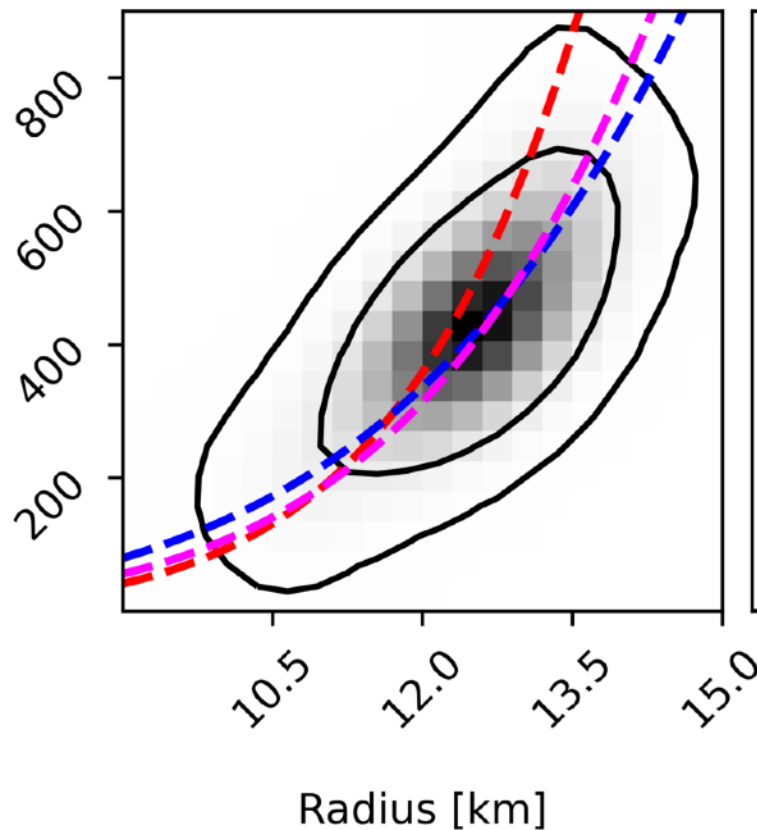
Implication of PREX 2 -
PhysRevLett.126.172503 (2021)

Kim et al., PhysRevC.98.065805 (2018)

Radius [km] = $12.58^{+0.72}_{-0.82}$



Lambda = $435.74^{+147.15}_{-124.18}$



Red line: $\Lambda(1.4M_{\odot}) = 2.88 * 10^{-6} (R/\text{km})^{7.5}$

[C] PhysRevLett.120.172703.pdf

Blue line: $\Lambda(1.4M_{\odot}) = 1.35 * 10^{-3} (R/\text{km})^{5.0}$

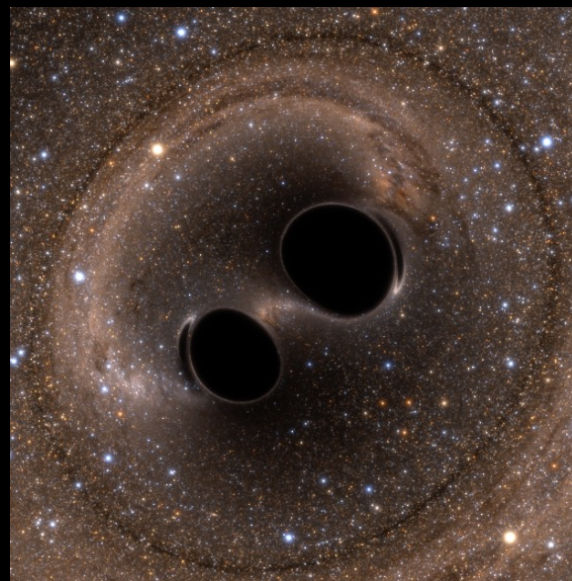
Implication of PREX 2 - PhysRevLett.126.172503 (2021)

=> $\Lambda \sim R^{4.8}$

Magenta line: $\Lambda(1.4M_{\odot}) = 1.05 * 10^{-4} (R/\text{km})^{6.0}$

Y.-M. Kim, in progress

Astrophysical GW Sources - NS



Credit: Bohn, Hébert, Throwe, SXS

Coalescing Binary Systems

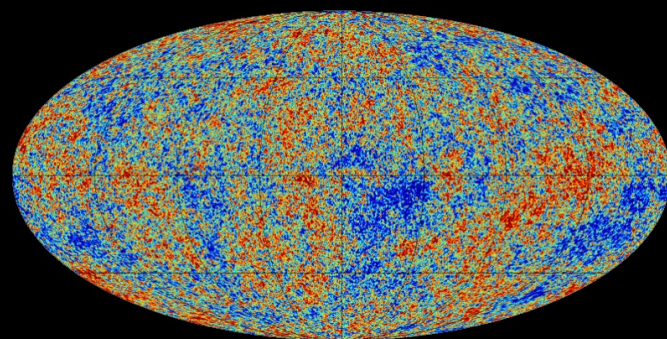
- Black hole – black hole
 - Black hole – neutron star
 - Neutron star – neutron star
- (modeled waveform)



Credit: Chandra X-ray Observatory

Transient ‘Burst’ Sources

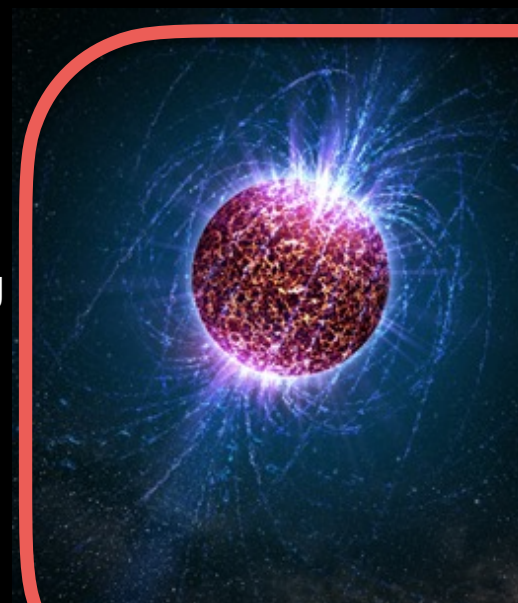
- asymmetric core collapse supernovae
 - cosmic strings
 - ???
- (Unmodeled waveform)



Credit: Planck Collaboration

Stochastic Background

- residue of the Big Bang
 - incoherent sum of unresolved ‘point’ sources
- (stochastic, incoherent noise background)



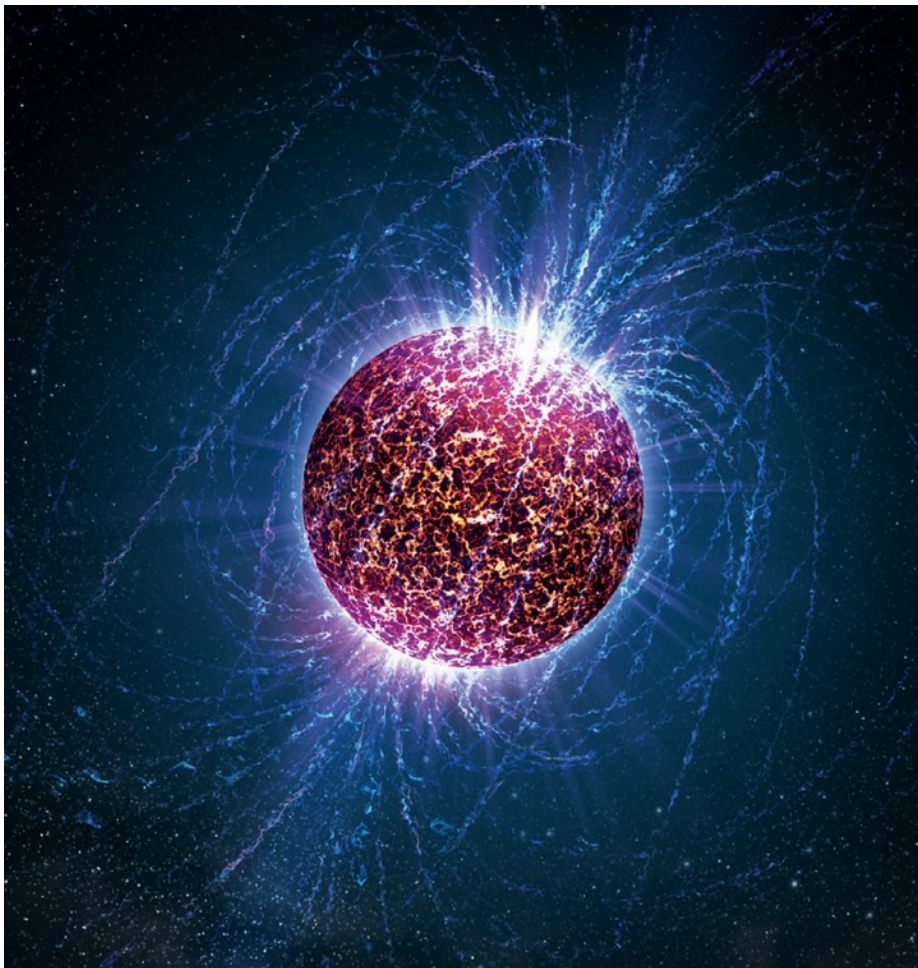
Credit: Casey Reed, Penn State

Continuous Sources

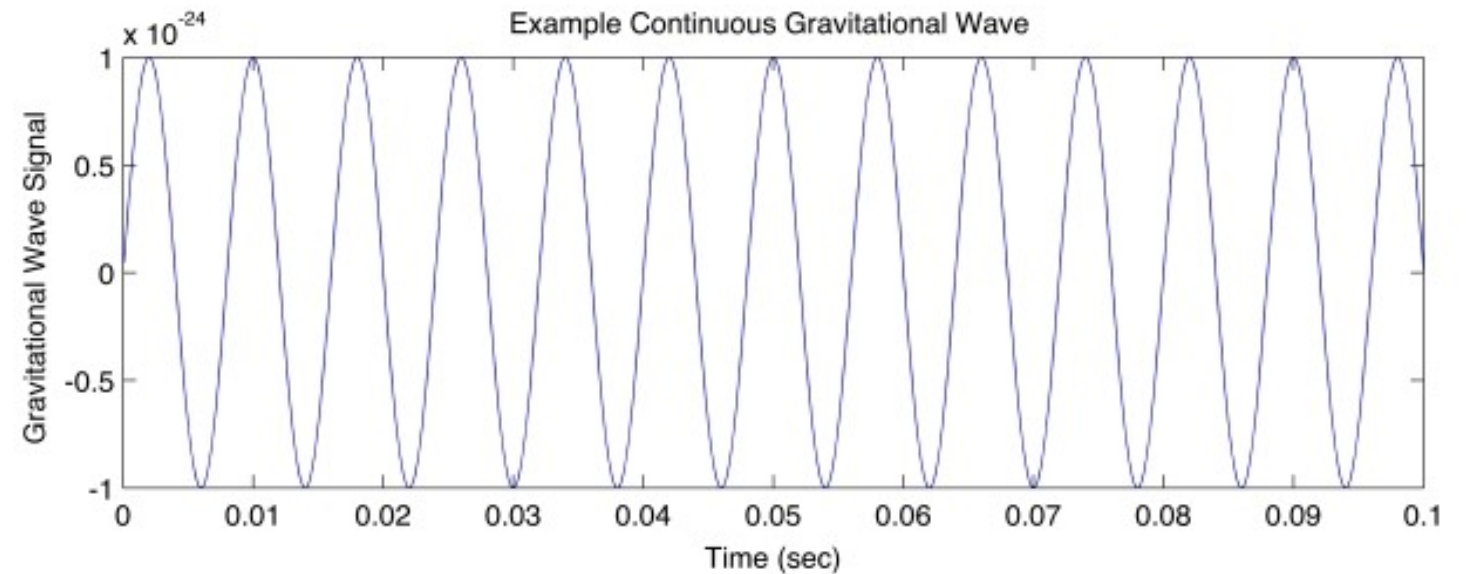
- Spinning neutron stars
- (monotone waveform)

Continuous Gravitational Waves (CW)

non-Transient sources



Credit: Casey Reed, Penn State

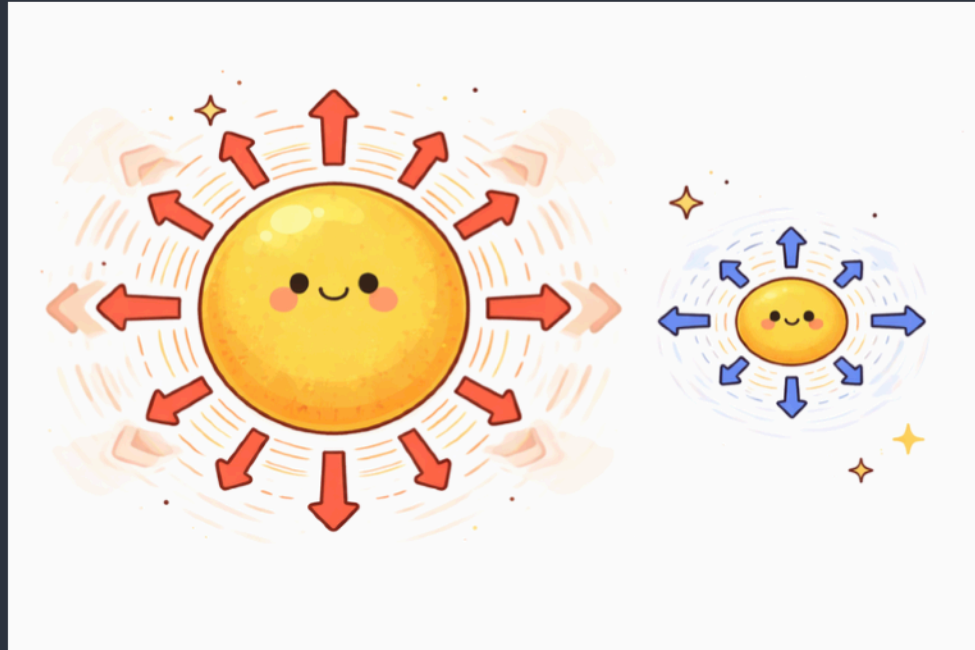


credit: A. Stuver/ LIGO

Continuous Sources

NS Oscillations

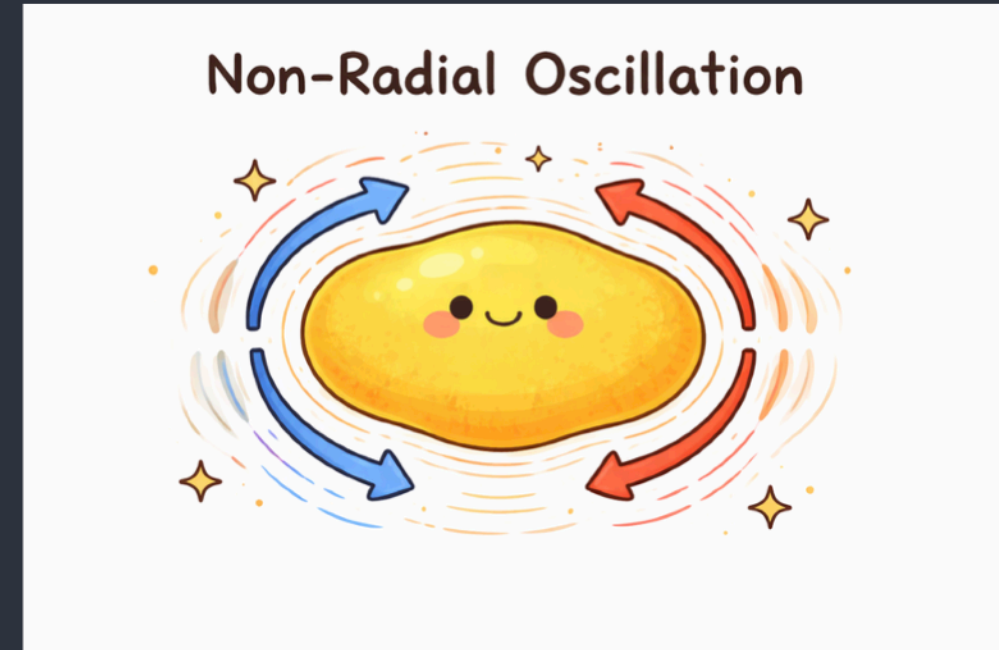
2026 겨울학교 슬라이드 - 김진호/박경빈



★ 방사형 진동 (Radial)

- 별이 **중심을 기준으로 안팎으로만** 팽창·수축
- 진동 중에도 **구형** 대칭 유지
- 모든 지점이 **같은 위상**으로 움직임

별이 숨 쉬듯이 커졌다 작아졌다 하는 진동



🌊 비방사형 진동 (Non-radial)

- 별의 표면이 **위·아래, 옆** 방향으로도 움직임
- **구형 대칭이 깨짐**
- 위치에 따라 진동 위상이 다름

별 표면에 물결무늬 패턴이 생기는 진동

Oscillation Modes

- **p-modes** (pressure modes)

- 복원력: 압력
- 주로 태양형 진동

- **g-modes** (gravity modes)

- 복원력: 부력(buoyancy)
- 백색왜성

- **f-mode**

- fundamental mode ($n=0$)
- Cepheid variable



- 별이 중
- 진동 중
- 모든 지

별이 숨 쉬듯

Non-Radial Oscillation



비방사형 진동 (Non-radial)

- 별의 표면이 위·아래, 옆 방향으로도 움직임
- 구형 대칭이 깨짐
- 위치에 따라 진동 위상이 다름

별 표면에 물결무늬 패턴이 생기는 진동

Equilibrium Structure of Star

Fluid equation (Euler equation).

1. Continuity Equation

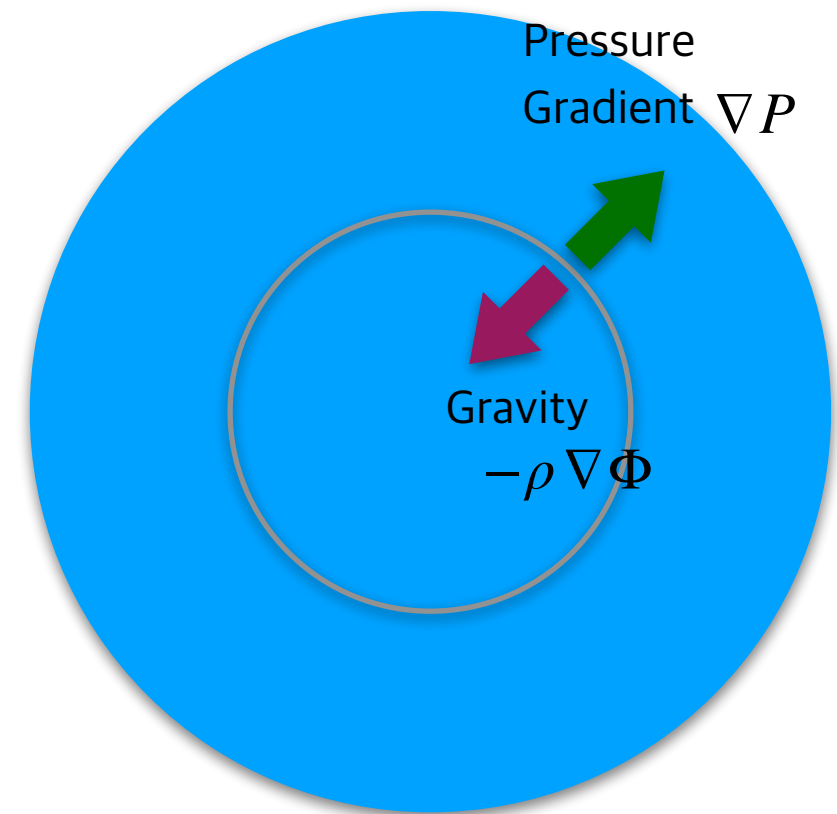
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

2. Momentum Equation

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} + \nabla P = -\rho \nabla \Phi$$

3. Energy Equation

$$\rho \frac{\partial e}{\partial t} + \vec{v} \cdot \nabla e + \frac{P}{\rho} \nabla \cdot \vec{v} = 0$$



For spherical star (non-rotating, non-magnetized), LHS of 1. Continuity and 3. Energy equation is 0. The remaining equation is 2. Momentum equation and can be rewritten in spherical coordinates as

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \quad \text{where } M_r = 4\pi \int_0^r \rho(r)r^2 dr \quad \rightarrow \quad \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G\rho$$

Linear Stability Analysis

Let us revisit hydrodynamic (Euler) equation in spherical coordinates

1. Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = 0,$$

2. Momentum Equation

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial r} + \frac{\partial P}{\partial r} = -\rho \frac{\partial \Phi}{\partial r},$$

3. Equation of State

$$P = K \rho^\Gamma,$$

4. Poisson Equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) = 4\pi G \rho.$$

Equilibrium (background) solution

$$\frac{\partial P_0}{\partial r} = -\rho_0 \frac{\partial \Phi_0}{\partial r}.$$

Assume,

$$\rho = \rho_0 + \delta \rho,$$

$$P = P_0 + \delta P,$$

$$v = 0 + \delta v,$$

$$\Phi = \Phi_0 + \delta \Phi,$$

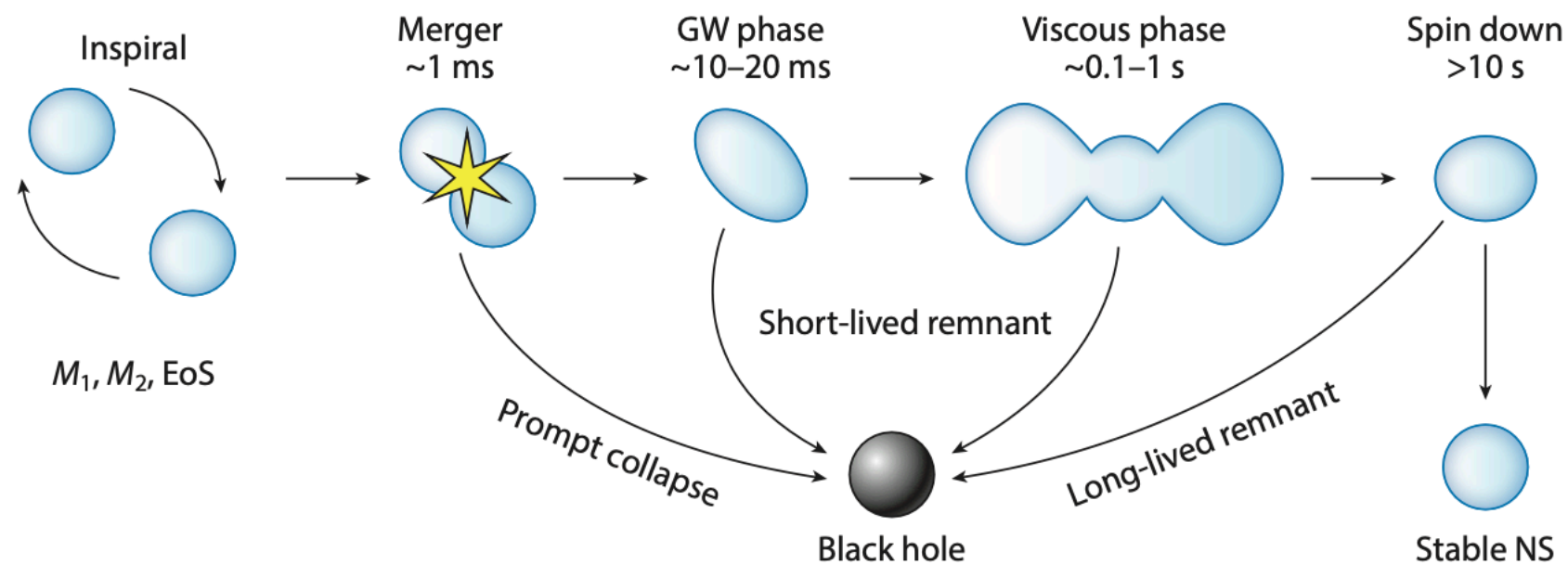
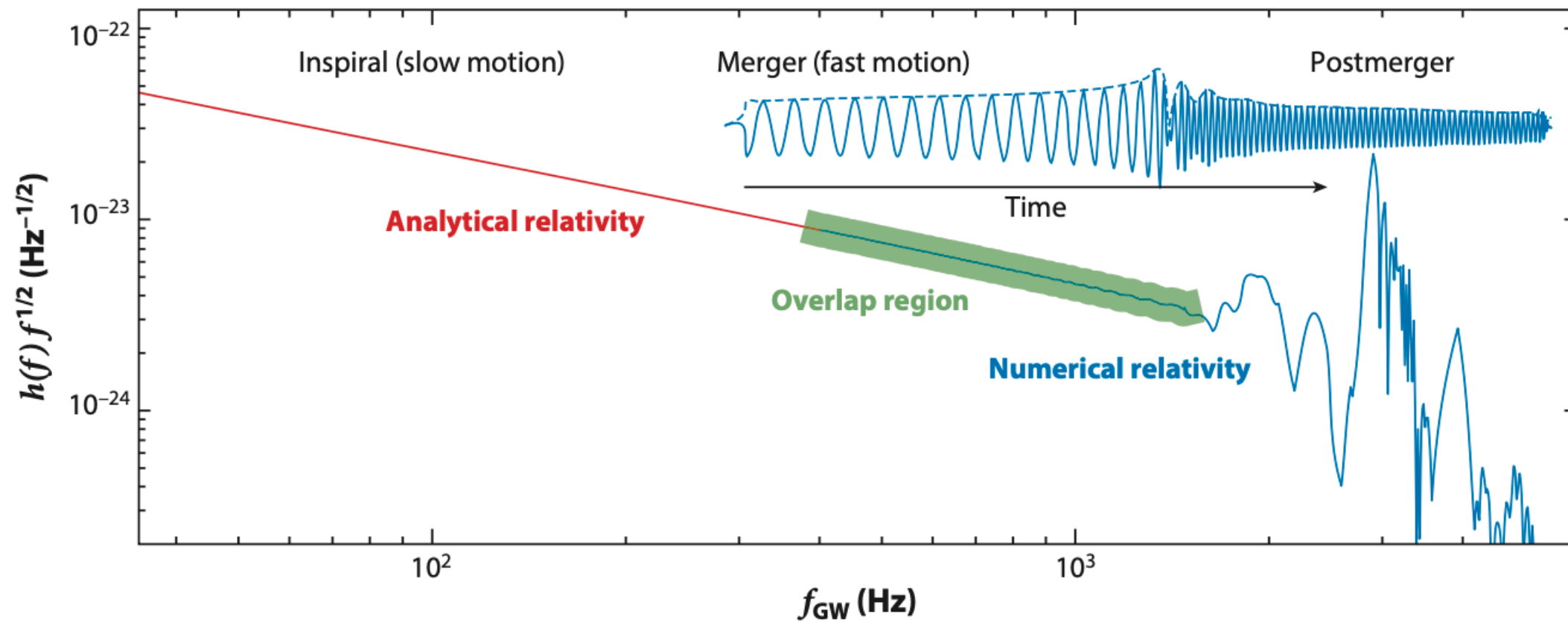
where subscript 0 denotes background (equilibrium) solution while δ -ed variables represents Eulerian perturbations.

We can define radial displacement, ζ , then the perturbation of velocity can be written as

$$\delta v = \frac{\partial \zeta}{\partial t}.$$

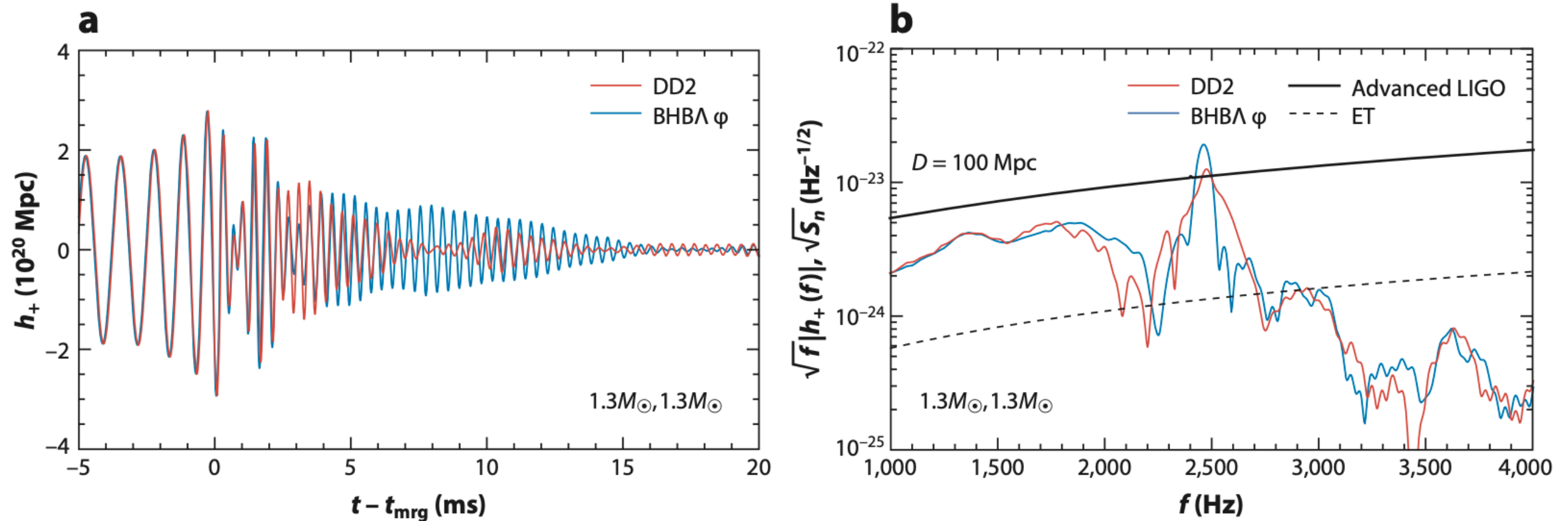
Radice et al., Annu. Rev. Nucl. Part. Sci. 70, 95–119 (2020)

“The Dynamics of Binary Neutron Star Mergers and GW170817”



Radice et al., Annu. Rev. Nucl. Part. Sci. 70, 95–119 (2020)

“The Dynamics of Binary Neutron Star Mergers and GW170817”



“Gravitational waves and non-axisymmetric oscillation modes in mergers of compact object binaries ”

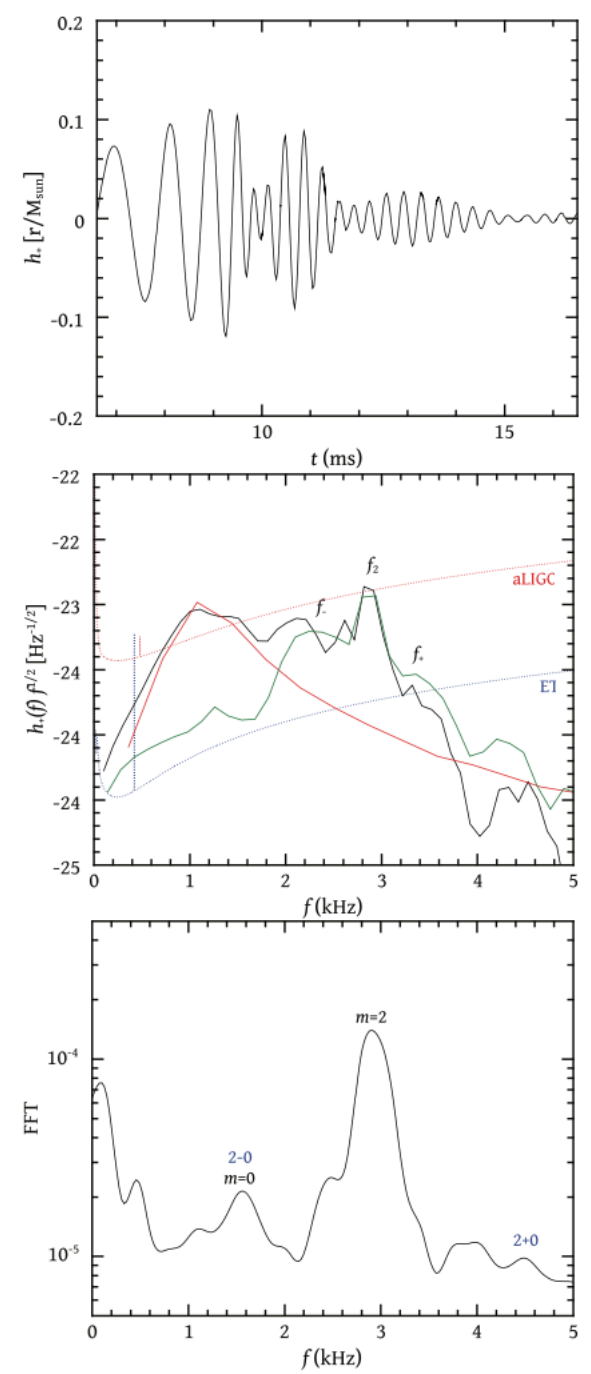
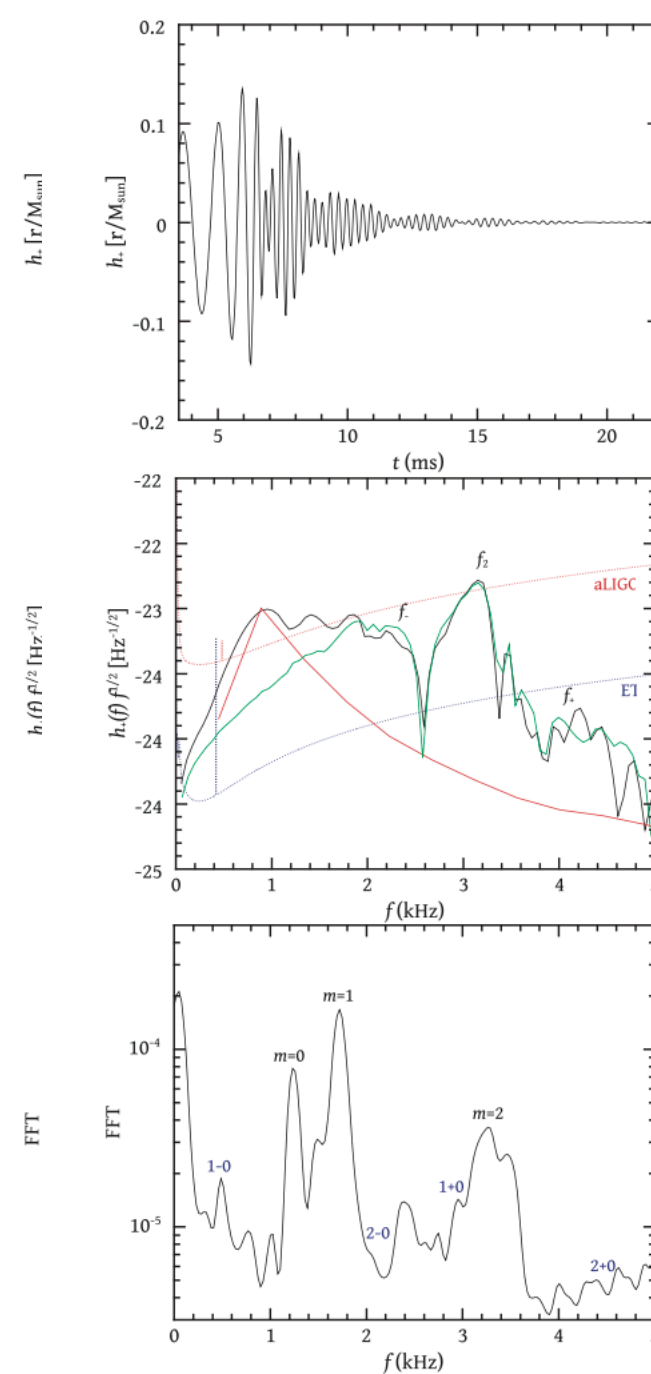
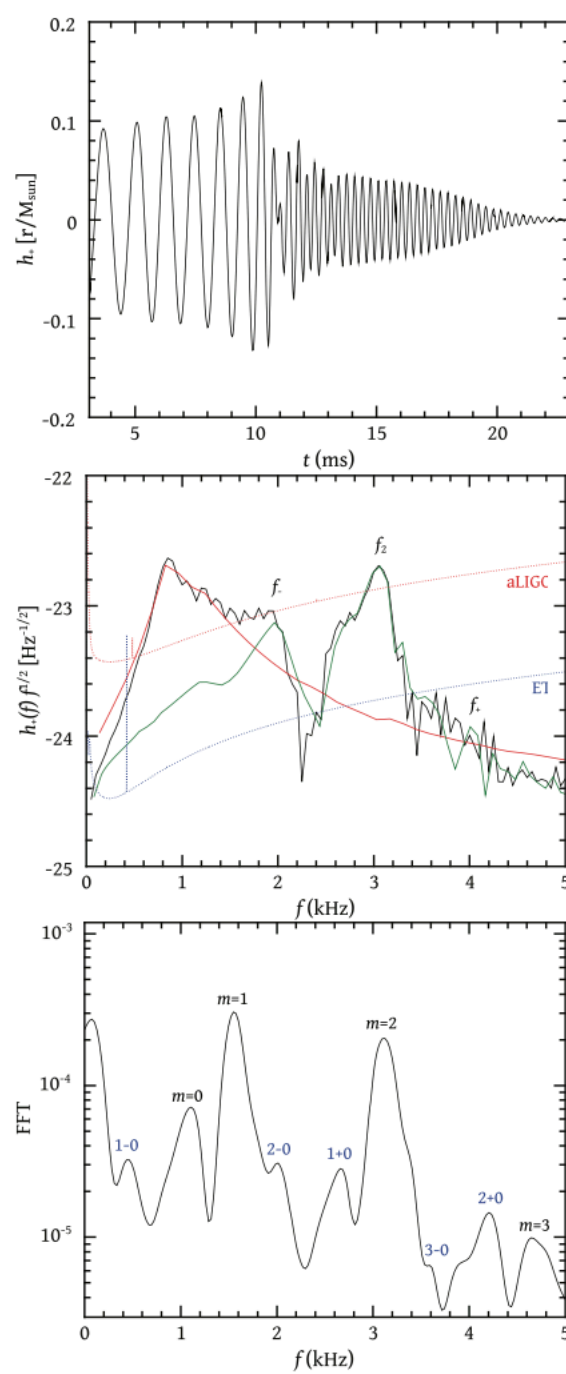
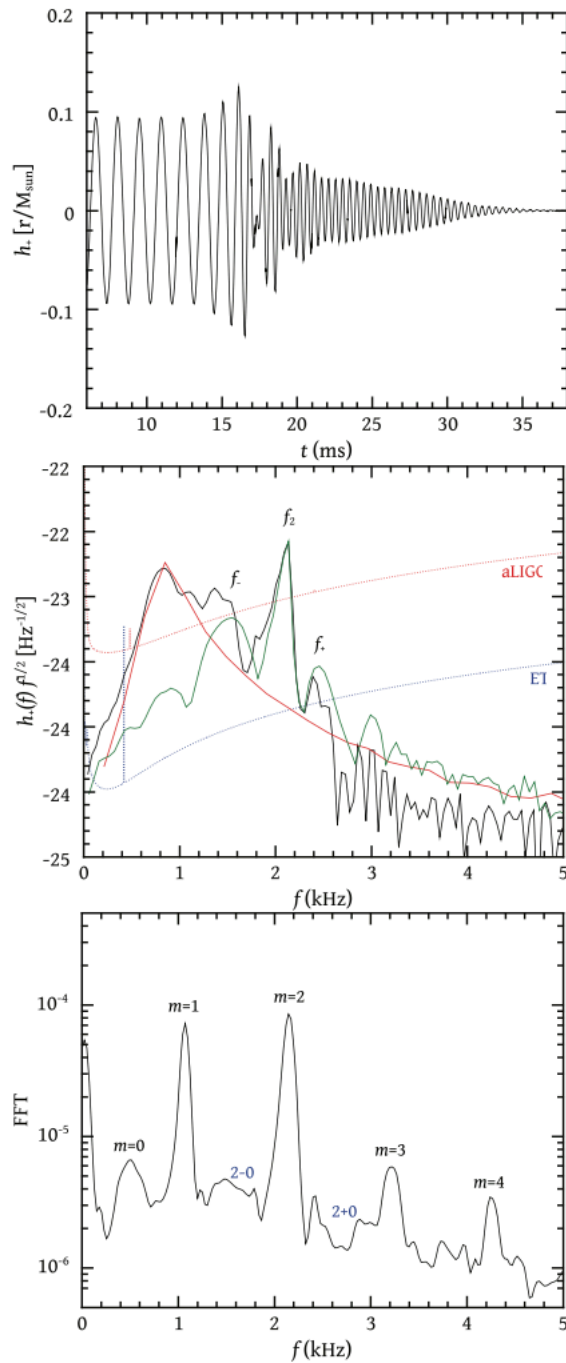


Figure 3. Model Shen 12135. Top panel: time evolution of the GW amplitude h_+ . Middle panel: total (black), pre-merger (red) and post-merger (green) scaled power spectral density, compared to the Advanced LIGO and Einstein Telescope noise curves. Bottom panel: Fast Fourier Transform (FFT) of the signal.

Figure 5. Same as Fig. 3, but for model LS 12135.

Figure 8. Same as Fig. 3, but for model MIT60 12135.

Figure 7. Same as Fig. 3, but for model MIT60 1111.

“Gravitational waves and non-axisymmetric oscillation modes in mergers of compact object binaries ”

Table 1. Extracted mode frequencies.

Model	$f_{m=0}$ (kHz)	$f_{m=1}$ (kHz)	$f_{m=2}$ (kHz)	$f_{m=3}$ (kHz)	$f_{m=4}$ (kHz)
Shen 12135	0.50	1.07	2.14	3.22	4.26
Shen 135135	0.46	—	2.24	—	4.11
LS 12135	1.10	1.55	3.12	4.66	—
LS 135135	0.98	—	3.30	—	—
MIT60 1111	1.56	—	2.92	—	5.94
MIT60 12135	1.24	1.72	3.26	—	—

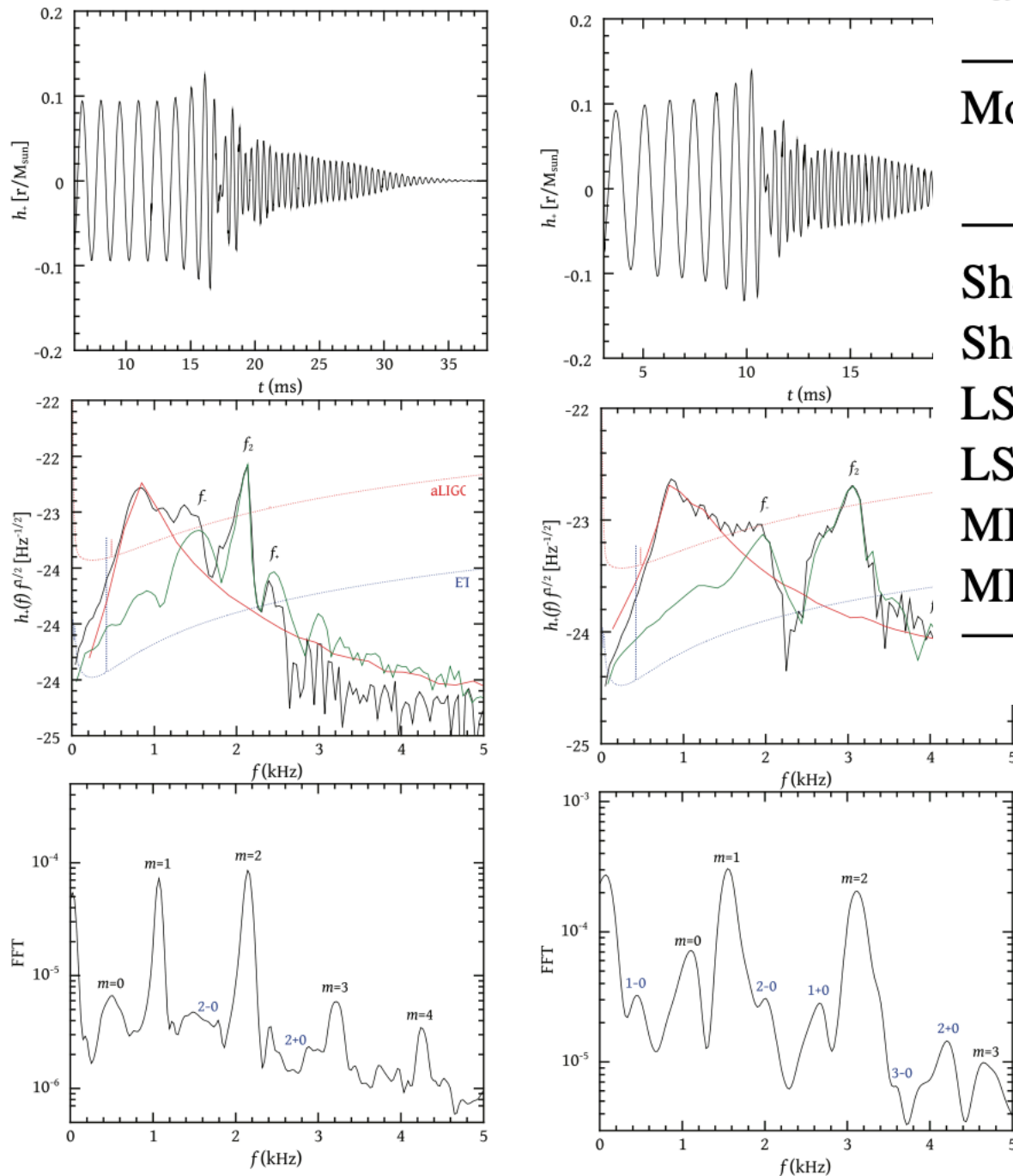


Figure 5. Same as Fig. 3, but for model LS 12135.

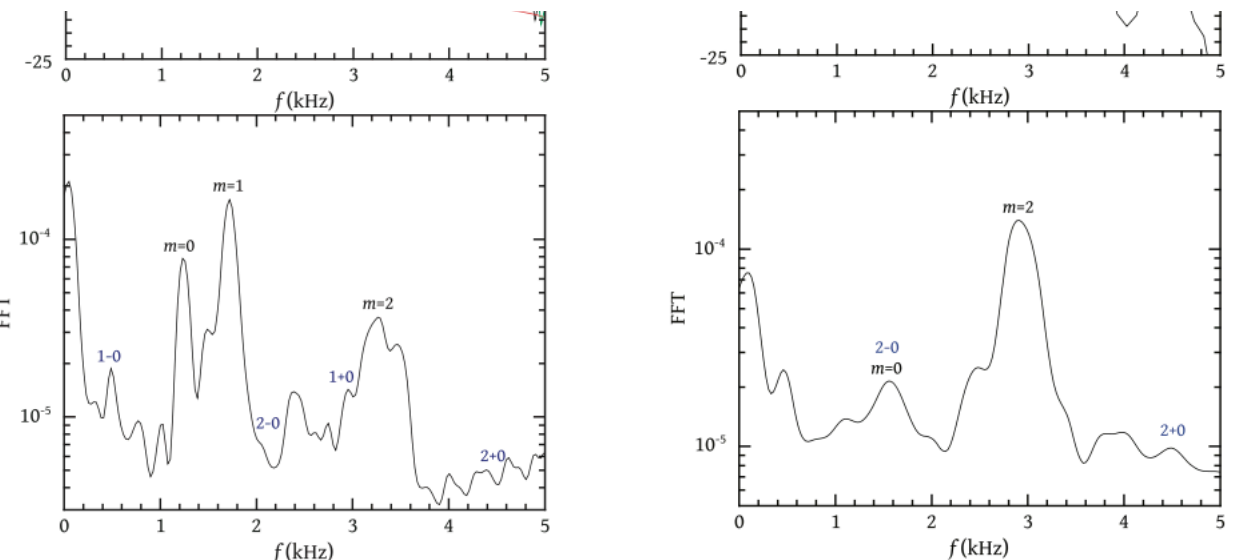


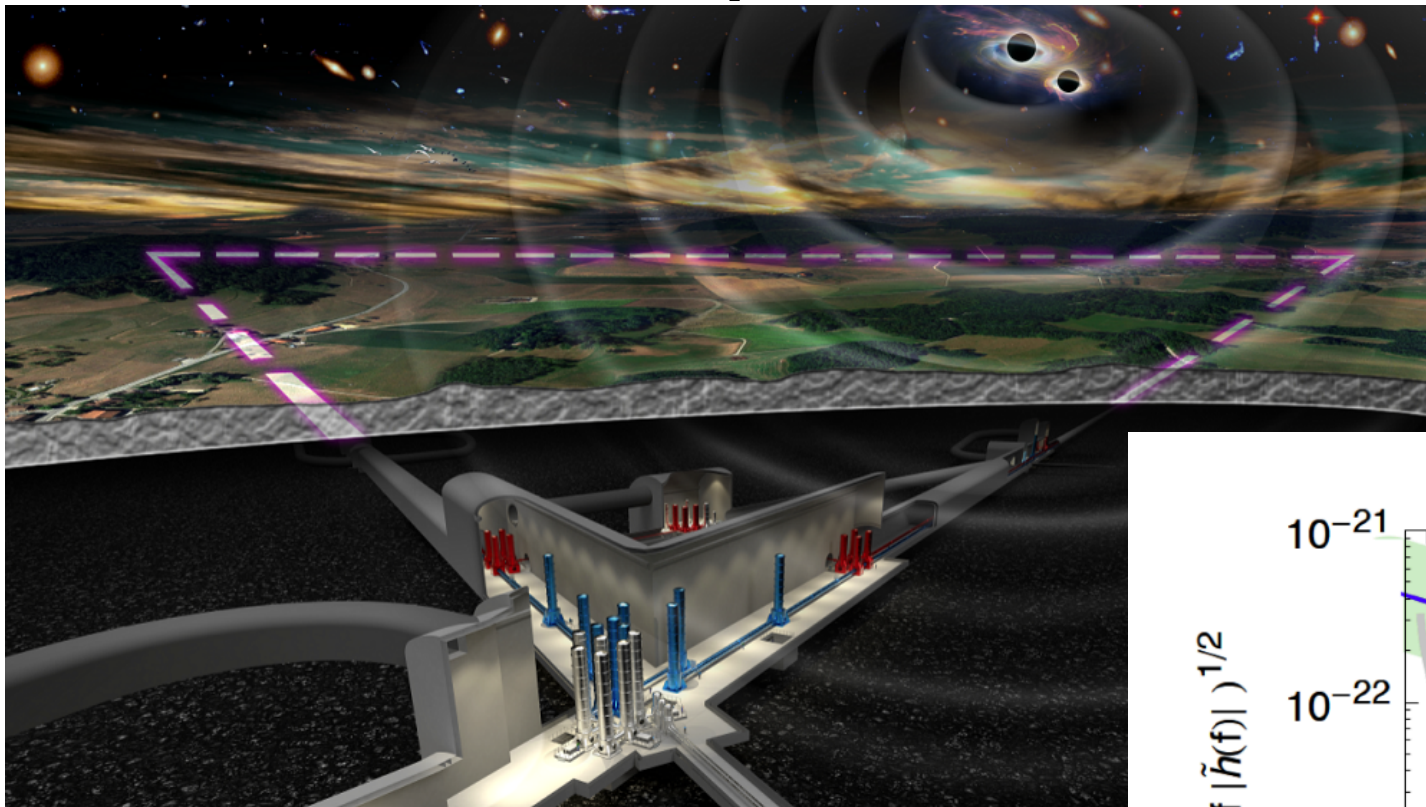
Figure 8. Same as Fig. 3, but for model MIT60 12135.

Figure 7. Same as Fig. 3, but for model MIT60 1111.

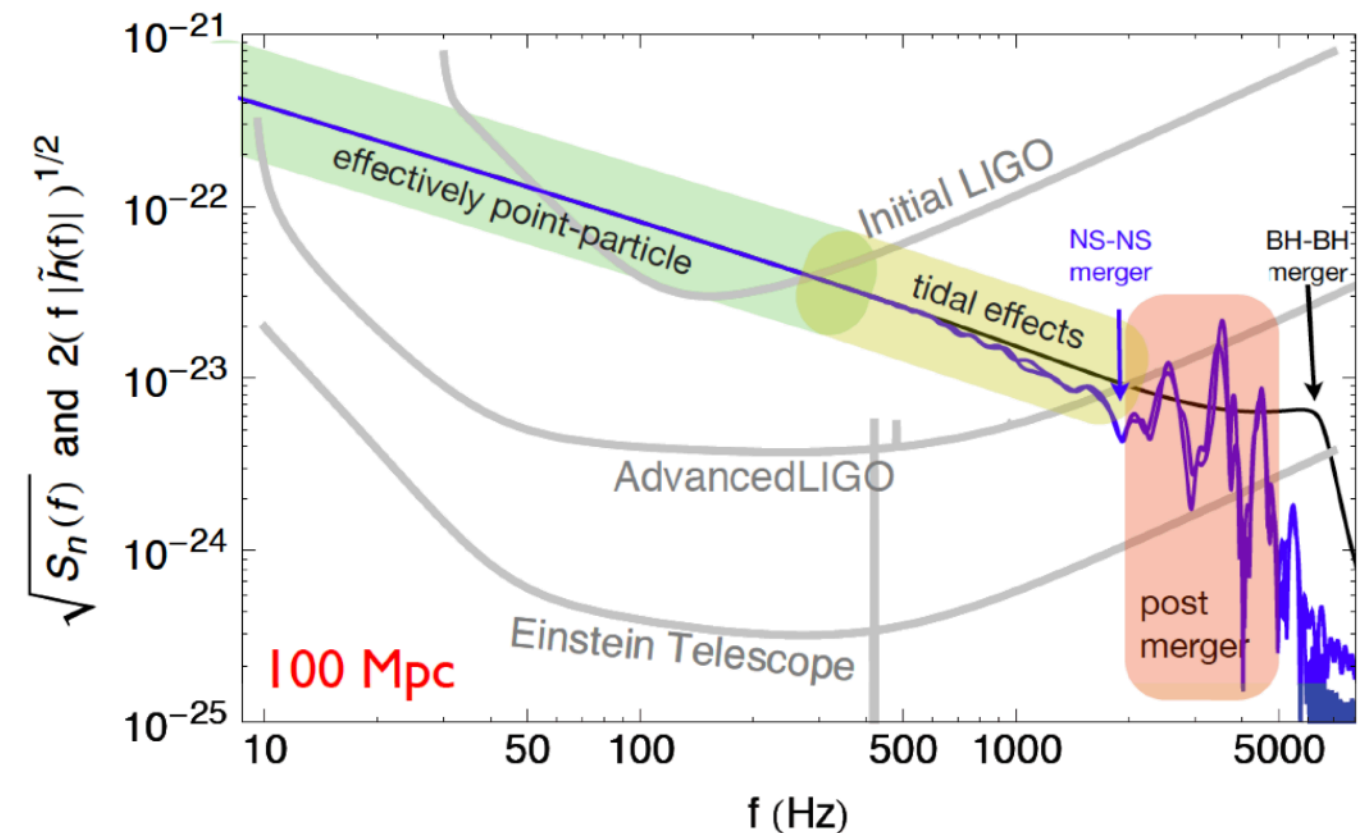
Figure 3. Model Shen 12135. Top panel: time evolution of the GW amplitude h_+ . Middle panel: total (black), pre-merger (red) and post-merger (green) scaled power spectral density, compared to the Advanced LIGO and ET noise curves. Bottom panel: FFT of the post-merger signal showing various modes ($m=0, 1, 2, 3, 4$).

Next Generation GW detectors

Einstein Telescope



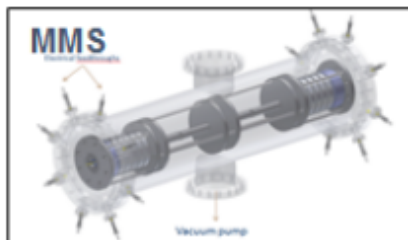
ET-bluebook



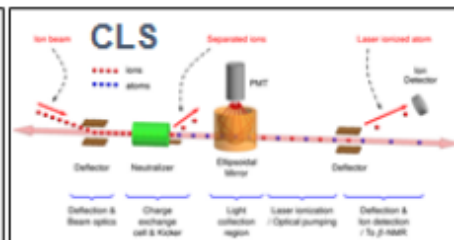
Astrophysics w/ nuclear physics

Experimental Systems

• 동위원소 질량 측정

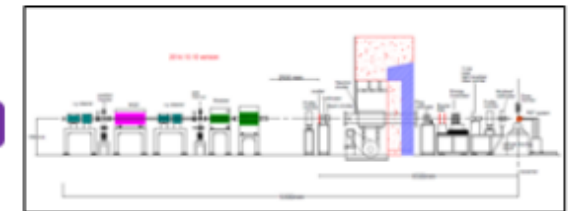


• Charge Radius 측정



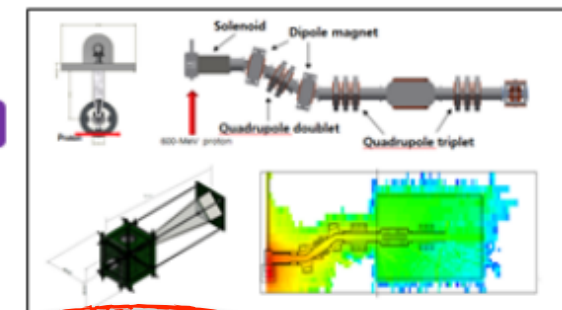
MMS/CLS

BIS



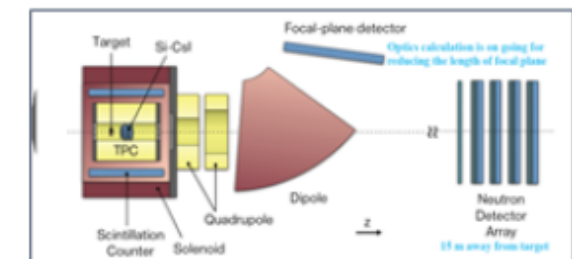
- 암 치료 연구
- 새로운 육종 연구

• 물성 연구



LAMPS

- 핵 물질 연구
- 중성자별 연구

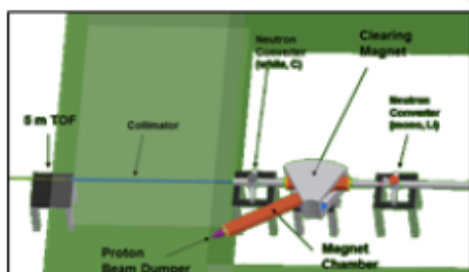
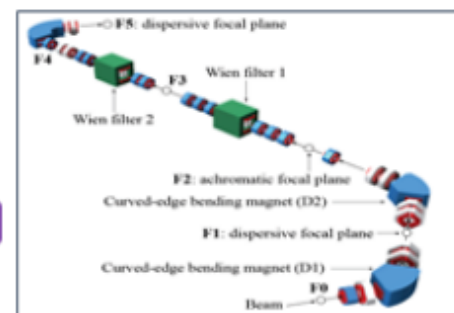


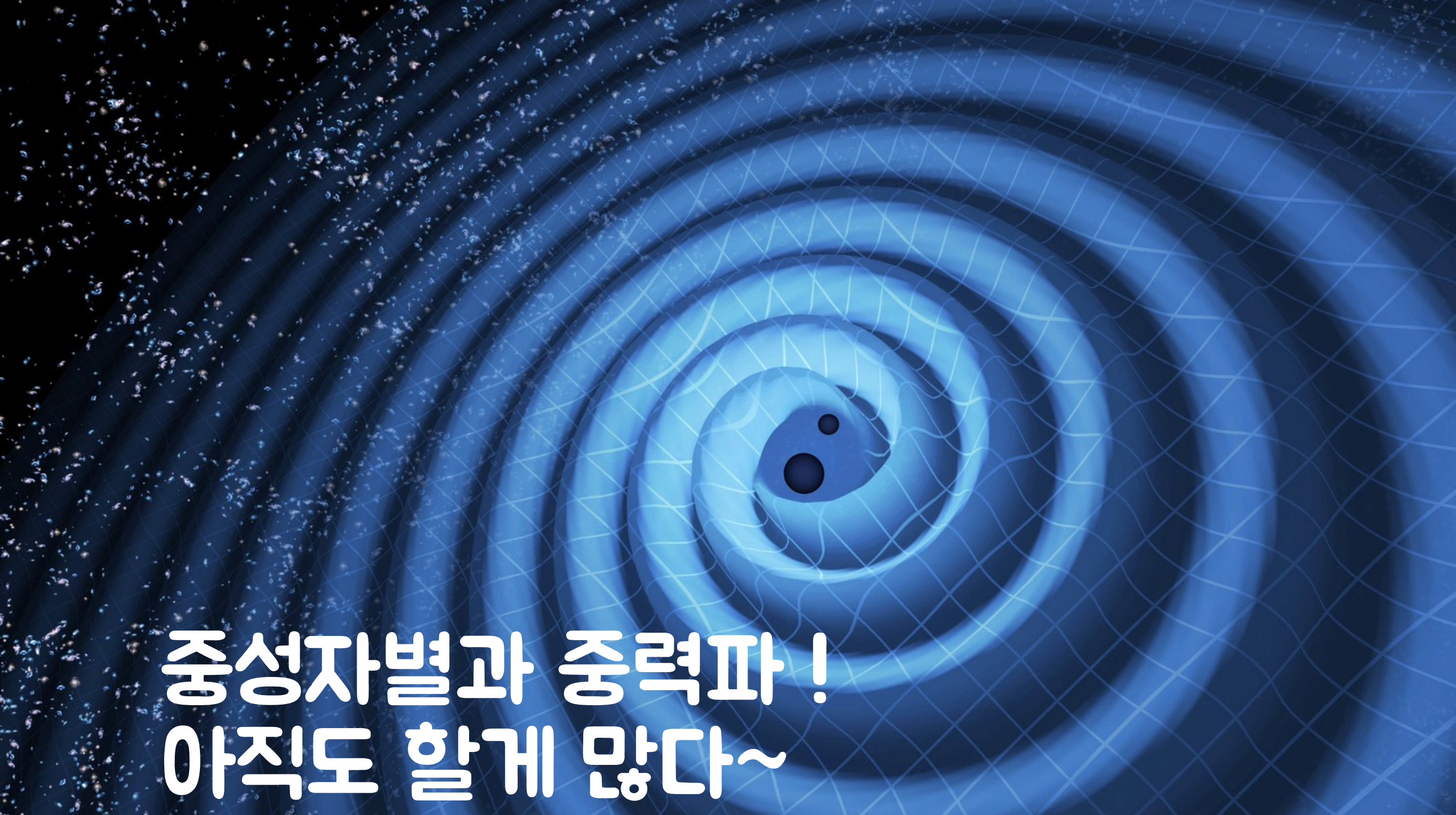
NDPS

- 핵자료 연구

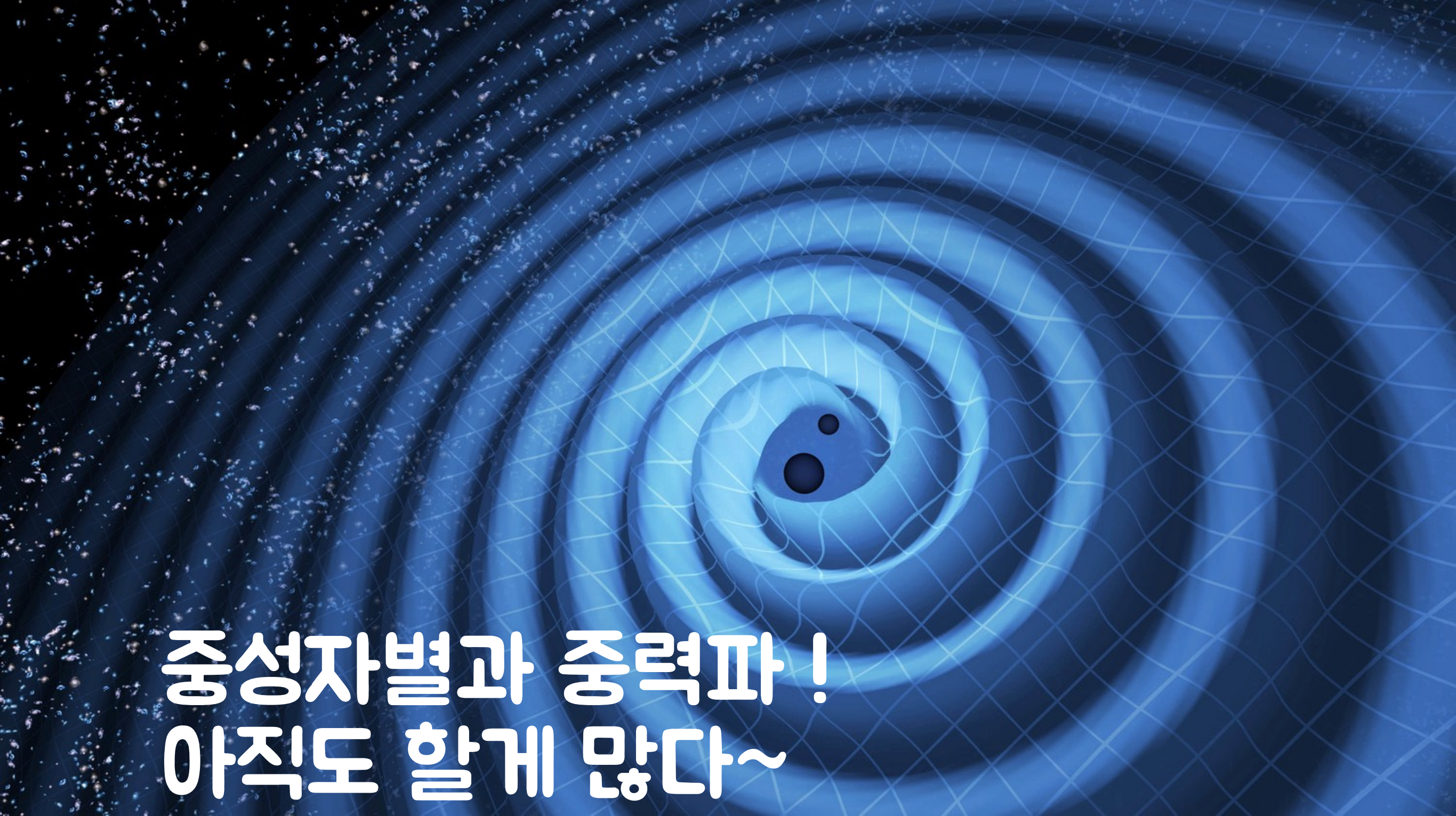
KOBRA

- Astrophysics
- 핵 구조 연구





중성자별과 중력파!
아직도 할게 많다~

A blue-toned image of a spiral galaxy, viewed from an angle, creating a sense of depth. A white grid pattern is overlaid on the galaxy's structure. The text is in white, bold Korean characters.

**중성자별과 중력파 !
아직도 할게 많다~**

궁금한 것은 이메일로 물어보세요~ : ymkim@kasi.re.kr

Derive 2nd order Diff. eq of H
for tidal deformability

Selected references

- **A.E.H. Love** (1909) - The Yielding of the Earth to Disturbing Forces
- **K.S. Thorne** & A. Campolattaro (1967) - non-radial pulsation of NS
- J.B. Hartle & **K.S. Thorne** (1969) - stability of rotating NS
-
- **K.S. Thorne** (1998) - Tidal stabilization of rigid rotating, fully relativistic neutron star
-
- **T. Hinderer** (2008) - Tidal Love numbers of neutron stars
- **T. Damour and A. Nagar** (2009) - Relativistic tidal properties of neutron stars

Love number

The Yielding of the Earth to Disturbing Forces.

By A. E. H. LOVE, F.R.S., Sedleian Professor of Natural Philosophy in
the University of Oxford.

(Received November 28, 1908,—Read January 14, 1909.)

1. Any estimate of the rigidity of the Earth must be based partly on some observations from which a deformation of the Earth's surface can be inferred, and partly on some hypothesis as to the internal constitution of the Earth. The observations may be concerned with tides of long period, variations of the vertical, variations of latitude, and so on. The hypothesis must relate to the arrangement of the matter as regards density in different parts, and to the state of the parts in respect of solidity, compressibility, and so on.

Love number

terms which are the products of the spherical solid harmonic W_n and functions of r . In the most important cases $n = 2$, and we may write

$$U = H(r) \frac{W_2}{g}, \quad \Delta = f(r) \frac{W_2}{g}, \quad V = V_0 + K(r) W_2, \quad (5)$$

where $H(r)$, $f(r)$, $K(r)$ are functions of r , and the constant $1/g$ has been inserted in the expression for U for the sake of convenience. If, in fact, W_2 is a periodic term of the tide-generating potential, W_2/g is the “true equilibrium height” of the corresponding tide, that is to say, it is the height of the harmonic inequality which the forces answering to W_2 would produce in an ocean covering a rigid spherical nucleus, of the same size and mass as the Earth, if the depth and density of the ocean were negligible. From the definition of $K(r)$ we easily find the formula

$$K(a) = \frac{3}{5\rho a^6} \int_0^a \rho_0 \left[\frac{d}{dr} \{r^6 H(r)\} - r^6 f(r) \right] dr. \quad (6)$$

In what follows we shall write h for $H(a)$ and k for $K(a)$, so that the equation of the disturbed surface is

$$r = a + hW_2/g,$$

Tidal Love number and Tidal deformability

T. Hinderer, ApJ, 677, 1216 (2008)

$$\frac{(1 - g_{tt})}{2} = -\frac{M}{r} - \frac{3Q_{ij}}{2r^3} \left(n^i n^j - \frac{1}{3} \delta^{ij} \right) + O\left(\frac{1}{r^3}\right) + \frac{1}{2} \mathcal{E}_{ij} x^i x^j + O(r^3), \quad (1)$$

$$Q_{ij} = -\lambda \mathcal{E}_{ij}$$

$$\lambda = \frac{2}{3} \frac{R^5}{G} k_2$$

λ : Tidal deformability

Q_{ij} : Quadrupole moment of NS

$\mathcal{E}_{ij}$: External quadrupole tidal field

k_2 : $l = 2$ Tidal Love number

$$\Lambda = \lambda / M^5 \rightarrow G \left(\frac{c^2}{GM} \right)^5 \lambda = \frac{2}{3} \left(\frac{Rc^2}{GM} \right)^5 k_2$$

Tidal coefficients (deformability)

Multipole moments induced by a external tidal field

electric-type tidal coefficient (even parity)

Mass moments $M_L^A = \mu_\ell^A G_L^A,$ electric-type tidal field

Spin moments $S_L^A = \sigma_\ell^A H_L^A.$ magnetic-type tidal field

magnetic-type tidal coefficient (odd parity)

Metric in linearized perturbation

even parity

$$h_{\mu\nu} = \begin{vmatrix} (1-2m^*/r)H_0(T,r)Y_L^M & H_1(T,r)Y_L^M & h_0(T,r)(\partial/\partial\theta)Y_L^M & h_0(T,r)(\partial/\partial\varphi)Y_L^M \\ H_1(T,r)Y_L^M & (1-2m^*/r)^{-1}H_2(T,r)Y_L^M & h_1(T,r)(\partial/\partial\theta)Y_L^M & h_1(T,r)(\partial/\partial\varphi)Y_L^M \\ \text{Sym} & \text{Sym} & r^2[K(T,r) + G(T,r)(\partial^2/\partial\theta^2)]Y_L^M & \text{Sym} \\ \text{Sym} & \text{Sym} & r^2G(T,r)(\partial^2/\partial\theta\partial\varphi - \cos\theta\partial/\sin\theta\partial\varphi)Y_L^M & r^2[K(T,r)\sin^2\theta + G(T,r)(\partial^2/\partial\varphi\partial\varphi + \sin\theta\cos\theta\partial/\partial\theta)]Y_L^M \end{vmatrix}. \quad (13)$$

odd parity

$$h_{\mu\nu} = \begin{vmatrix} 0 & 0 & -h_0(T,r)(\partial/\sin\theta\partial\varphi)Y_L^M & h_0(T,r)(\sin\theta\partial/\partial\theta)Y_L^M \\ 0 & 0 & -h_1(T,r)(\partial/\sin\theta\partial\varphi)Y_L^M & h_1(T,r)(\sin\theta\partial/\partial\theta)Y_L^M \\ \text{Sym} & \text{Sym} & h_2(T,r)(\partial^2/\sin\theta\partial\theta\partial\varphi - \cos\theta\partial/\sin^2\theta\partial\varphi)Y_L^M & \text{Sym} \\ \text{Sym} & \text{Sym} & \frac{1}{2}h_2(T,r)(\partial^2/\sin\theta\partial\varphi\partial\varphi + \cos\theta\partial/\partial\theta - \sin\theta\partial^2/\partial\theta\partial\theta)Y_L^M & -h_2(T,r)(\sin\theta\partial^2/\partial\theta\partial\varphi - \cos\theta\partial/\partial\varphi)Y_L^M \end{vmatrix}. \quad (12)$$

Regge-Wheeler gauge and M=0

even parity $h_{\mu\nu} = \exp(-ikT) P_L(\cos\theta)$

$$\times \begin{vmatrix} H_0(1-2m^*/r) & H_1 & 0 & 0 \\ H_1 & H_2(1-2m^*/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 K \sin^2\theta \end{vmatrix} \quad (20)$$

odd parity

$$h_{\mu\nu} = \begin{vmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ \text{Sym} & \text{Sym} & 0 & 0 \end{vmatrix}$$

$$\times \exp(-ikT) (\sin\theta \partial/\partial\theta) P_L(\cos\theta). \quad (18)$$

Regge-Wheeler gauge and M=0

even parity $h_{\mu\nu} = \exp(-ikT) P_L(\cos\theta)$

$$H=H_0=H_2, K$$

$$\times \begin{vmatrix} H_0(1-2m^*/r) & \cancel{H_1}^0 & 0 & 0 \\ \cancel{H_1}^0 & H_2(1-2m^*/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 K \sin^2\theta \end{vmatrix} \quad (20)$$

$k=0$, stationary case

odd parity

$$h_{\mu\nu} = \begin{vmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & \cancel{h_1(r)}^0 \\ 0 & 0 & 0 & 0 \\ \text{Sym} & \cancel{\text{Sym}}^0 & 0 & 0 \end{vmatrix}$$

$$\Psi = h_0$$

$$\times \exp(-ikT) (\sin\theta \partial / \partial \theta) P_L(\cos\theta). \quad (18)$$

Even parity (Electric-type) metric Fn. H

Regge & Wheeler (1957), Thorne & Campolattaro (1967)

$$\delta G_{\mu\nu} = 8\pi\delta T_{\mu\nu}$$



$$H'' + C_1 H' + C_0 H = 0.$$

$$h_{\alpha\beta} =$$

$$\text{diag}[-e^{\nu(r)}H_0(r), e^{\lambda(r)}H_2(r), r^2K(r), r^2 \sin^2\theta K(r)] Y_{2m}(\theta, \varphi).$$

$$\delta T_0^0 = -\delta\rho = -(dp/d\rho)^{-1}\delta p \quad \delta T_i^i = \delta p$$

$$C_1 = \frac{2}{r} + \frac{1}{2}(\nu' - \lambda') = \frac{2}{r} + e^\lambda \left[\frac{2m}{r^2} + 4\pi r(p - e) \right], \quad (28)$$

$$\begin{aligned} C_0 &= e^\lambda \left[-\frac{\ell(\ell+1)}{r^2} + 4\pi(e+p)\frac{de}{dp} + 4\pi(e+p) \right] \\ &\quad + \nu'' + (\nu')^2 + \frac{1}{2r}(2 - r\nu')(3\nu' + \lambda') \\ &= e^\lambda \left[-\frac{\ell(\ell+1)}{r^2} + 4\pi(e+p)\frac{de}{dp} + 4\pi(5e+9p) \right] \\ &\quad - (\nu')^2, \end{aligned} \quad (29)$$

Electric-type tidal coefficients (I)

General solution

$$H = a_P \hat{P}_{\ell 2}(x) + a_Q \hat{Q}_{\ell 2}(x), \quad x \equiv r/M - 1,$$

$$c \equiv M/R$$

$$y^{\text{ext}} \equiv rH'/H$$

$$a_\ell \equiv a_Q/a_P$$

$$y_\ell^{\text{ext}}(x) = (1+x) \frac{\hat{P}'_{\ell 2}(x) + a_\ell \hat{Q}'_{\ell 2}(x)}{\hat{P}_{\ell 2}(x) + a_\ell \hat{Q}_{\ell 2}(x)}.$$

$$a_\ell = - \left. \frac{\hat{P}'_{\ell 2}(x) - cy_\ell \hat{P}_{\ell 2}(x)}{\hat{Q}'_{\ell 2}(x) - cy_\ell \hat{Q}_{\ell 2}(x)} \right|_{x=1/c-1}.$$

Electric-type tidal coefficients (2)

Damour and Nagar , PRD 80, 084035 (2009)

$$\begin{aligned} -(\delta H_{00}e^{-\nu})^{\text{growing}} &= H^{\text{growing}}(r) = a_P \hat{P}_{\ell 2}(x) Y_{\ell m} \\ &\simeq a_P \left(\frac{r}{M}\right)^\ell Y_{\ell m}, \end{aligned} \quad (40)$$

$$\begin{aligned} -(\delta H_{00}e^{-\nu})^{\text{decreasing}} &= H^{\text{decreasing}}(r) = a_Q \hat{Q}_{\ell 2}(x) Y_{\ell m} \\ &\simeq a_Q \left(\frac{r}{M}\right)^{-(\ell+1)} Y_{\ell m}, \end{aligned} \quad (41)$$

$$\partial_L r^{-1} = (-)^\ell (2\ell - 1)!! \hat{n}^L r^{-(\ell+1)},$$

$$M_L^A = \mu_\ell^A G_L^A,$$

$$(2\ell - 1)!! G \mu_\ell = \frac{a_Q}{a_P} \left(\frac{GM}{c_0^2}\right)^{2\ell+1} = a_\ell \left(\frac{GM}{c_0^2}\right)^{2\ell+1}.$$

$$\bar{W} = \frac{1}{\ell!} \hat{X}^L G_L^A = \frac{1}{\ell!} r^\ell \hat{n}^L G_L^A,$$

$$W^+ = G \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{M_L^A}{r} \right),$$

$$G_{00}^A(X) = -\exp(-2W^A/c^2),$$

$$G_{0a}^A(X) = -\frac{4}{c^3} W_a^A,$$

$$E_a^A(X) = \partial_a W^A + \frac{4}{c^2} \partial_T W_a^A,$$

$$G_L^A(T) \equiv [\partial_{\langle L-1} \bar{E}_{a_\ell}^A(T, \mathbf{X})]_{X^a \rightarrow 0}$$

Electric-type tidal coefficients (3)

Damour and Nagar , PRD 80, 084035 (2009)

$$(2\ell - 1)!! G\mu_\ell = \frac{a_Q}{a_P} \left(\frac{GM}{c_0^2} \right)^{2\ell+1} = a_\ell \left(\frac{GM}{c_0^2} \right)^{2\ell+1}. \quad (45)$$

$$G\mu_\ell = \frac{a_\ell}{(2\ell - 1)!!} \left(\frac{GM}{c_0^2} \right)^{2\ell+1} = \frac{2k_\ell}{(2\ell - 1)!!} R^{2\ell+1}. \quad (48)$$

$$\begin{aligned} k_\ell &= \frac{1}{2} c^{2\ell+1} a_\ell \\ &= -\frac{1}{2} c^{2\ell+1} \frac{\hat{P}'_{\ell 2}(x) - cy_\ell \hat{P}_{\ell 2}(x)}{\hat{Q}'_{\ell 2}(x) - cy_\ell \hat{Q}_{\ell 2}(x)} \Big|_{x=1/c-1}. \end{aligned} \quad (49)$$

Electric-type tidal coefficients (4)

Damour and Nagar , PRD 80, 084035 (2009)

Electric-type tidal coefficients

$$C_1 = \frac{2}{r} + \frac{1}{2}(\nu' - \lambda') = \frac{2}{r} + e^\lambda \left[\frac{2m}{r^2} + 4\pi r(p - e) \right], \quad (28)$$

$$G\mu_\ell = \frac{a_\ell}{(2\ell - 1)!!} \left(\frac{GM}{c_0^2} \right)^{2\ell+1} = \frac{2k_\ell}{(2\ell - 1)!!} R^{2\ell+1}. \quad (48)$$

$$H'' + C_1 H' + C_0 H = 0.$$

$$\begin{aligned} C_0 &= e^\lambda \left[-\frac{\ell(\ell + 1)}{r^2} + 4\pi(e + p) \frac{de}{dp} + 4\pi(e + p) \right] \\ &\quad + \nu'' + (\nu')^2 + \frac{1}{2r} (2 - r\nu')(3\nu' + \lambda') \\ &= e^\lambda \left[-\frac{\ell(\ell + 1)}{r^2} + 4\pi(e + p) \frac{de}{dp} + 4\pi(5e + 9p) \right] \\ &\quad - (\nu')^2, \end{aligned} \quad (29)$$

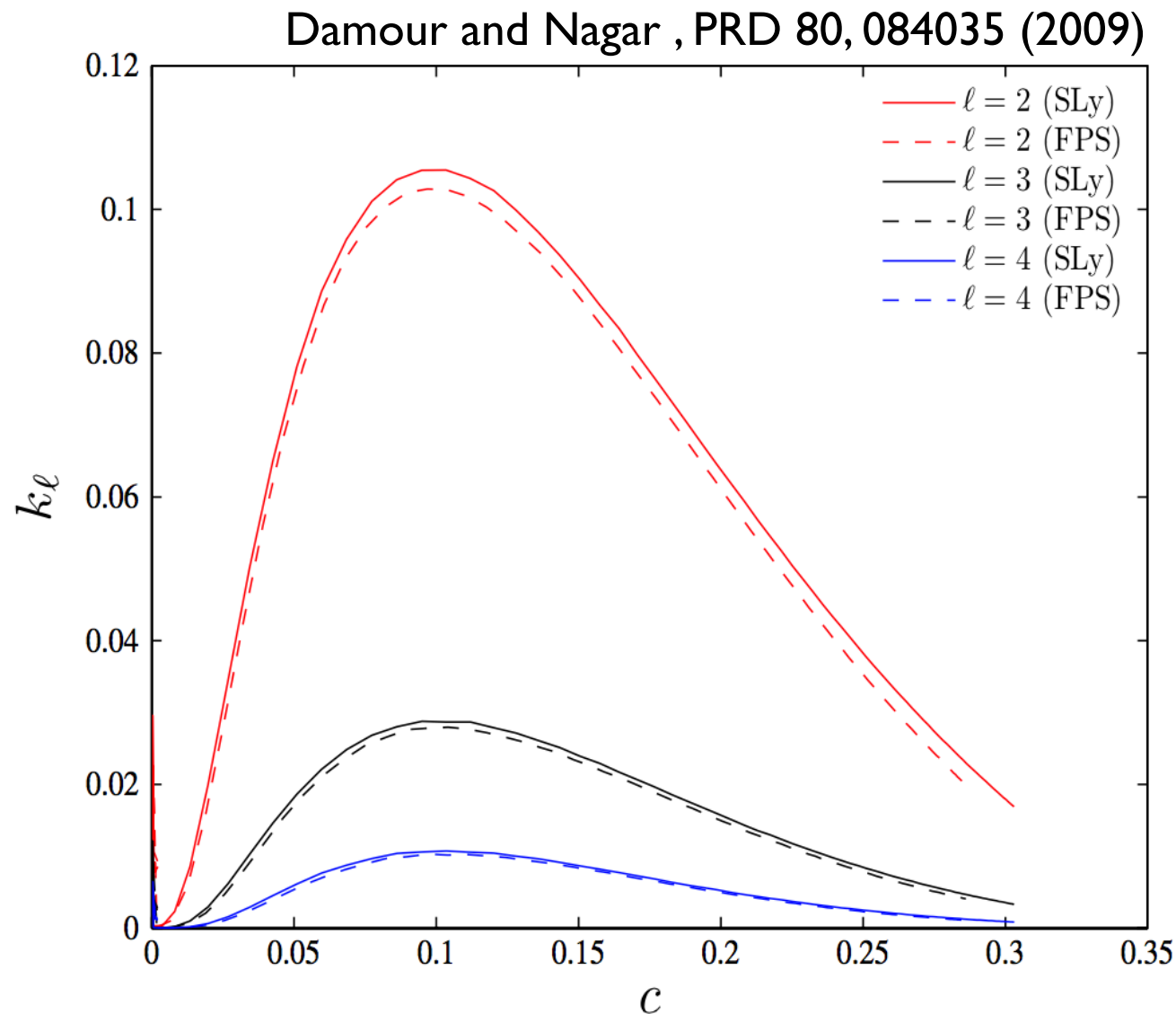
we calculated the case, $\ell=2$

$$\begin{aligned} k_2 &= \frac{8}{5} (1 - 2c)^2 c^5 [2c(y - 1) - y + 2] \left\{ 2c(4(y + 1)c^4 + (6y - 4)c^3 + (26 - 22y)c^2 + 3(5y - 8)c - 3y + 6) \right. \\ &\quad \left. - 3(1 - 2c)^2 (2c(y - 1) - y + 2) \log\left(\frac{1}{1 - 2c}\right) \right\}^{-1}, \end{aligned} \quad (50)$$

$$y_R = \frac{rH'(r)}{H(r)} \Big|_{r=R}$$

$$\begin{aligned} k_3 &= \frac{8}{7} (1 - 2c)^2 c^7 [2(y - 1)c^2 - 3(y - 2)c + y - 3] \left\{ 2c[4(y + 1)c^5 + 2(9y - 2)c^4 - 20(7y - 9)c^3 + 5(37y - 72)c^2 \right. \\ &\quad \left. - 45(2y - 5)c + 15(y - 3)] - 15(1 - 2c)^2 (2(y - 1)c^2 - 3(y - 2)c + y - 3) \log\left(\frac{1}{1 - 2c}\right) \right\}^{-1}, \end{aligned} \quad (51)$$

Higher Tidal Love Numbers



$$x \equiv (M\omega)^{2/3}$$

$$\Delta\Psi_2^{tidal} \sim \lambda_2 x^{5/2}$$

$$\Delta\Psi_3^{tidal} \sim \lambda_3 x^{9/2}$$

$$|\Delta\Psi_3^{tidal} / \Delta\Psi_2^{tidal}| \sim \mathcal{O}(10^{-3})$$

We hardly expect to observe higher terms of tidal deformability in the waveform

Magnetic-type tidal coefficients

Damour and Nagar , PRD 80, 084035 (2009)

Likewise,

Magnetic-type tidal coefficients

$$S_L^A = \sigma_\ell^A H_L^A.$$

$$G\sigma_\ell = \frac{\ell-1}{4(\ell+2)} \frac{b_\ell}{(2\ell-1)!!} \left(\frac{GM}{c_0^2}\right)^{2\ell+1} \\ = \frac{\ell-1}{4(\ell+2)} \frac{j_\ell}{(2\ell-1)!!} R^{2\ell+1},$$

$$G\sigma_2 = \frac{1}{48} j_2 R^5 = \frac{1}{48} b_2 \left(\frac{GM}{c_0^2}\right)^5,$$

$$j_2 = \frac{96c^5(2c-1)(y-3)}{5(2c(12(y+1)c^4 + 2(y-3)c^3 + 2(y-3)c^2 + 3(y-3)c - 3y + 9) + 3(2c-1)(y-3)\log(1-2c))}. \quad (73)$$

$$\psi'' + \frac{e^\lambda}{r^2} [2m + 4\pi r^3(p - e)] \psi' \\ - e^\lambda \left[\frac{\ell(\ell+1)}{r^2} - \frac{6m}{r^3} + 4\pi(e - p) \right] \psi = 0. \quad (31)$$

$$j_\ell \equiv c^{2\ell+1} b_\ell = -c^{2\ell+1} \frac{\psi'_P(\hat{r}) - cy_{\text{odd}} \psi_P(\hat{r})}{\psi'_Q(\hat{r}) - cy_{\text{odd}} \psi_Q(\hat{r})} \Big|_{\hat{r}=1/c}.$$

Magnetic-type tidal coefficients

Damour and Nagar , PRD 80, 084035 (2009)

Likewise,

Magnetic-type tidal coefficients

$$S_L^A = \sigma_\ell^A H_L^A.$$

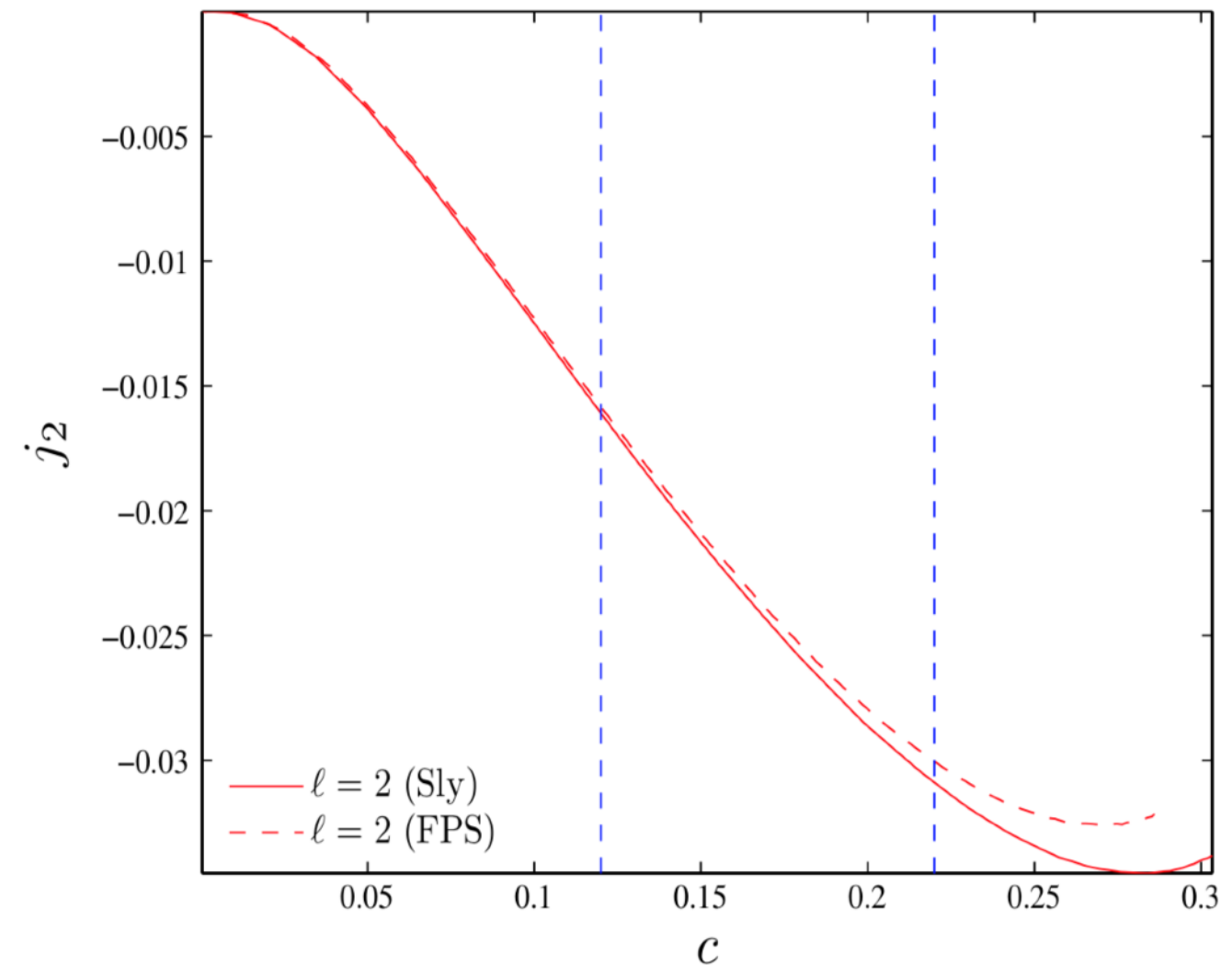
$$\begin{aligned} G\sigma_\ell &= \frac{\ell-1}{4(\ell+2)} \frac{b_\ell}{(2\ell-1)!!} \left(\frac{GM}{c_0^2}\right)^{2\ell+1} \\ &= \frac{\ell-1}{4(\ell+2)} \frac{j_\ell}{(2\ell-1)!!} R^{2\ell+1}, \end{aligned}$$

$$G\sigma_2 = \frac{1}{48} j_2 R^5 = \frac{1}{48} b_2 \left(\frac{GM}{c_0^2}\right)^5,$$

$$j_2 = \frac{96c^5(2c-1)(y-3)}{5(2c(12(y+1)c^4 + 2(y-3)c^3 + 2(y-3)c^2 + 3(y-3)c - 3y + 9) + 3(2c-1)(y-3)\log(1-2c))}. \quad (73)$$

$$\psi'' + \frac{\epsilon}{r}$$

$$j_\ell \equiv$$



Tidal Love number and Tidal deformability

$$\frac{(1 - g_{tt})}{2} = -\frac{M}{r} - \frac{3Q_{ij}}{2r^3} \left(n^i n^j - \frac{1}{3} \delta^{ij} \right) + O\left(\frac{1}{r^3}\right) + \frac{1}{2} \mathcal{E}_{ij} x^i x^j + O(r^3), \quad (1)$$

T. Hinderer, ApJ, 677, 1216 (2008)

$$Q_{ij} = -\lambda \mathcal{E}_{ij}$$

$$\lambda = \frac{2}{3} \frac{R^5}{G} k_2$$

λ : Tidal deformability

Q_{ij} : Quadrupole moment of NS

$\mathcal{E}_{ij}$: External quadrupole tidal field

k_2 : $l = 2$ Tidal Love number

$$k_2 = \frac{8C^5}{5} (1 - 2C)^2 [2 + 2C(y - 1) - y] \times \left\{ 2C[6 - 3y + 3C(5y - 8)] + 4C^3[13 - 11y + C(3y - 2) + 2C^2(1 + y)] + 3(1 - 2C)^2[2 - y + 2C(y - 1)] \ln(1 - 2C) \right\}^{-1}$$

$$y = \frac{R\beta(R)}{H(R)} \quad C = \frac{GM}{Rc^2}$$

Tidal term in GW waveform

$$\tilde{h}_T(f) = \mathcal{A} f^{-7/6} e^{i\Psi_T(f)}$$

M. Favata, PRL.112.101101 (2014)

$$\Psi_T(f) = \varphi_c + 2\pi f t_c + \frac{3}{128\eta v^5} (\Delta\Psi_{3.5\text{PN}}^{\text{pp}} + \Delta\Psi_{3\text{PN}}^{\text{spin}} + \Delta\Psi_{2\text{PN}}^{\text{ecc.}} + \Delta\Psi_{6\text{PN}}^{\text{tidal}} + \Delta\Psi_{6\text{PN}}^{\text{tm}}), \quad (1)$$

$$\Delta\Psi_{6\text{PN}}^{\text{tidal}} = -\frac{39}{2} \tilde{\Lambda} v^{10} + v^{12} \left(\frac{6595}{364} \delta\tilde{\Lambda} - \frac{3115}{64} \tilde{\Lambda} \right), \quad (4)$$

$$\tilde{\Lambda} = \frac{16}{13} \frac{(m_1 + 12m_2)m_1^4\Lambda_1 + (m_2 + 12m_1)m_2^4\Lambda_2}{(m_1 + m_2)^5}$$

$$\Lambda = \lambda/M^5 \rightarrow G \left(\frac{c^2}{GM} \right)^5 \lambda = \frac{2}{3} \left(\frac{Rc^2}{GM} \right)^5 k_2$$

Higher order terms in GW phase (I)

Post-Newtonian spin-tidal couplings for compact binaries -
arxiv:1805.01487

$$\psi(x) = \frac{3}{128\nu x^{5/2}} \left\{ 1 + \left(\frac{3715}{756} + \frac{55}{9}\nu \right) x + \left(\frac{113}{3} \times \right. \right. \\ \left. \left. \times (\eta_1\chi_1 + \eta_2\chi_2) - \frac{38}{3}\nu(\chi_1 + \chi_2) \right) x^{1.5} + O(x^2) \right. \\ \left. + \Lambda x^5 + (\delta\Lambda + \Sigma)x^6 + (\tilde{\Lambda} + \tilde{\Sigma} + \tilde{\Gamma})x^{6.5} + O(x^7) \right\}, \quad (15)$$

PN order	λ_2	σ_2	$\lambda_{23,32}, \sigma_{23,32}$	λ_3	σ_3
5	LO $\propto \Lambda$				
6	NLO $\propto \delta\Lambda$	LO $\propto \Sigma$			
6.5	NNLO $\propto \tilde{\Lambda}$	NLO $\propto \tilde{\Sigma}$	LO $\propto \tilde{\Gamma}$		
7	...	...	...	LO	
8	...	...	...	...	LO

$$\Lambda = \left(264 - \frac{288}{\eta_1} \right) \frac{c^{10}\lambda_2^{(1)}}{M^5} + (1 \leftrightarrow 2), \quad (16)$$

$$\delta\Lambda = \left(\frac{4595}{28} - \frac{15895}{28\eta_1} + \frac{5715\eta_1}{14} - \frac{325\eta_1^2}{7} \right) \frac{c^{10}\lambda_2^{(1)}}{M^5} \\ + (1 \leftrightarrow 2), \quad (17)$$

$$\Sigma = \left(\frac{6920}{7} - \frac{20740}{21\eta_1} \right) \frac{c^8\sigma_2^{(1)}}{M^5} + (1 \leftrightarrow 2).$$

$$\tilde{\Lambda} = \left[\left(\frac{593}{4} - \frac{1105}{8\eta_1} + \frac{567\eta_1}{8} - 81\eta_1^2 \right) \chi_2 \right. \\ \left. + \left(-\frac{6607}{8} + \frac{6639\eta_1}{8} - 81\eta_1^2 \right) \chi_1 \right] \frac{c^{10}\lambda_2^{(1)}}{M^5} \\ + (1 \leftrightarrow 2), \quad (19)$$

$$\tilde{\Sigma} = \left[\left(-\frac{9865}{3} + \frac{4933}{3\eta_1} + 1644\eta_1 \right) \chi_2 - \chi_1 \right] \frac{c^8\sigma_2^{(1)}}{M^5} \\ + (1 \leftrightarrow 2), \quad (20)$$

$$\tilde{\Gamma} = \frac{c^{10}\chi_1}{M^4} \left[(856\eta_1 - 816\eta_1^2) \lambda_{23}^{(1)} \right. \\ \left. - \left(\frac{4993\eta_1}{18} - \frac{2497\eta_1^2}{9} \right) \sigma_{23}^{(1)} \right. \\ \left. - \nu \left(272\lambda_{32}^{(1)} - 204\sigma_{32}^{(1)} \right) \right] + (1 \leftrightarrow 2). \quad (21)$$

Higher order terms in GW phase (2)

Rotational-tidal phasing of the binary neutron star waveform

- arxiv:1805.01882

$$\Psi = \frac{3M}{128\mu} x^{-2.5} \left[1 - \frac{39}{2} \tilde{\Lambda} x^5 + \tilde{\Sigma} x^6 - \tilde{X} x^{6.5} - \tilde{\Lambda}_3 x^7 + \tilde{\Sigma}_3 x^8 \right], \quad (8)$$

$$\tilde{X} = \frac{1}{21M^6} c^{12} \left\{ \chi^{(1)} \left[36(35 + 614q) \hat{\lambda}_2^{(1)} - (7 - 4751q) \hat{\sigma}_2^{(1)} - 2316q \hat{\lambda}_3^{(1)} - 3474q \hat{\sigma}_3^{(1)} \right] + \chi^{(2)} \left[36(35 + 614/q) \hat{\lambda}_2^{(2)} - (7 - 4751/q) \hat{\sigma}_2^{(2)} - 2316 \hat{\lambda}_3^{(2)} / q - 3474 \hat{\sigma}_3^{(2)} / q \right] \right\},$$

$$\tilde{\Lambda}_3 = \frac{4000}{9M^7} c^{14} (q \lambda_3^{(1)} + \lambda_3^{(2)} / q),$$
$$\tilde{\Sigma}_3 = \frac{29925}{11M^7} c^{14} (q \sigma_3^{(1)} + \sigma_3^{(2)} / q).$$

Dynamic tide

$$k_{\ell}^{\text{eff}} = k_{\ell} \left[a_{\ell} + \frac{b_{\ell}}{2} \left(\frac{Q_{m=\ell}^{\text{DT}}}{Q_{m=\ell}^{\text{AT}}} + \frac{Q_{m=-\ell}^{\text{DT}}}{Q_{m=-\ell}^{\text{AT}}} \right) \right],$$

$$\begin{aligned} \frac{Q_m^{\text{DT}}}{Q_m^{\text{AT}}} \approx & \frac{\omega_f^2}{\omega_f^2 - (m\Omega)^2} + \frac{\omega_f^2}{2(m\Omega)^2 \epsilon_f \Omega'_f (\phi - \phi_f)} \\ & \pm \frac{i\omega_f^2}{(m\Omega)^2 \sqrt{\epsilon_f}} e^{\pm i\Omega'_f \epsilon_f (\phi - \phi_f)^2} \int_{-\infty}^{\sqrt{\epsilon_f}(\phi - \phi_f)} e^{\mp i\Omega'_f s^2} ds, \end{aligned} \quad (2)$$

PRL.116.181101 (arxiv:1602.00599)

