Gravitational Wave Cosmology

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(2026년 7월 31 일)

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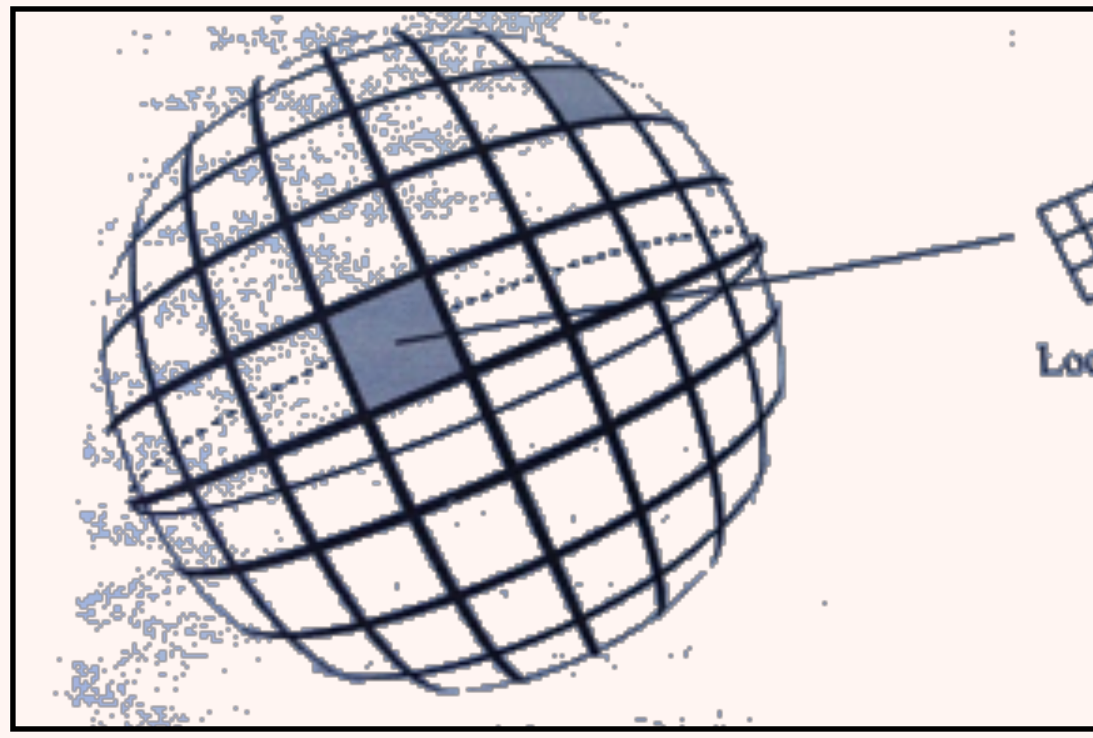
Topics

- A brief overview of GR and Cosmology
- Hubble constant and Hubble Tension
- Measurements of distances to GW sources
- Multi-messenger astronomy for Hubble constant
- Future prospects of dark siren measurements

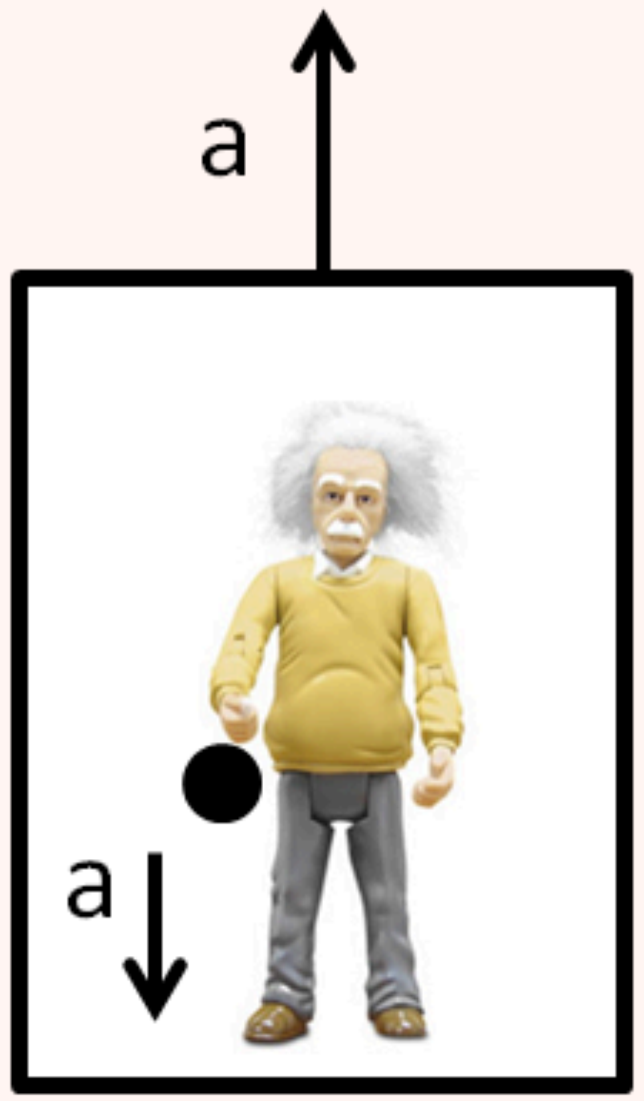
Equivalence Principle and geometrical nature of the gravity

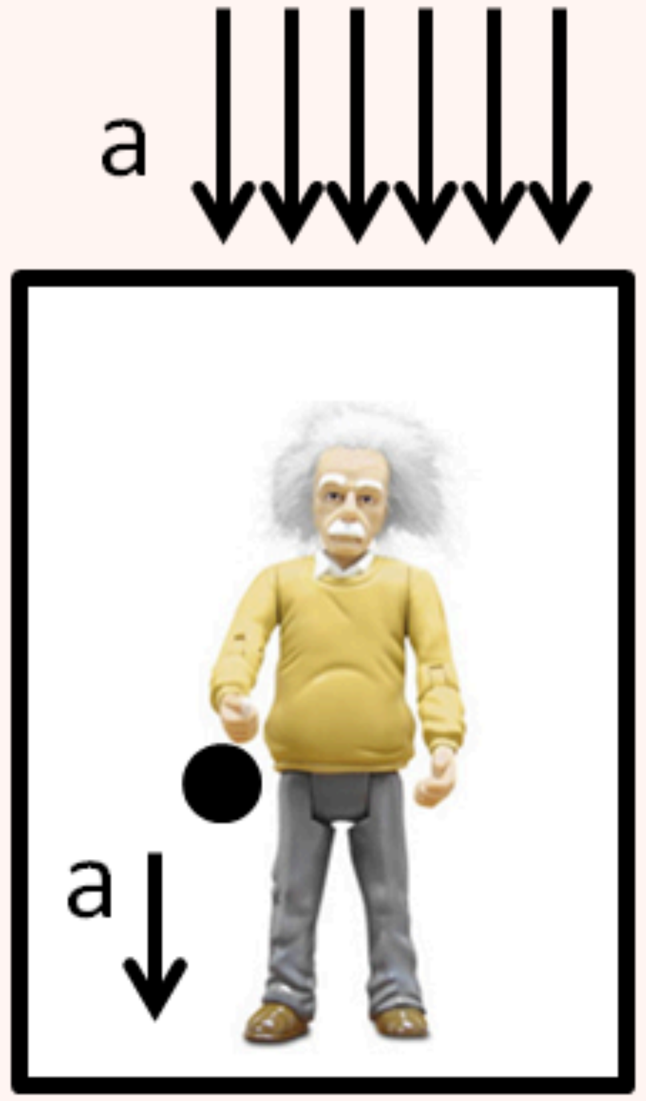


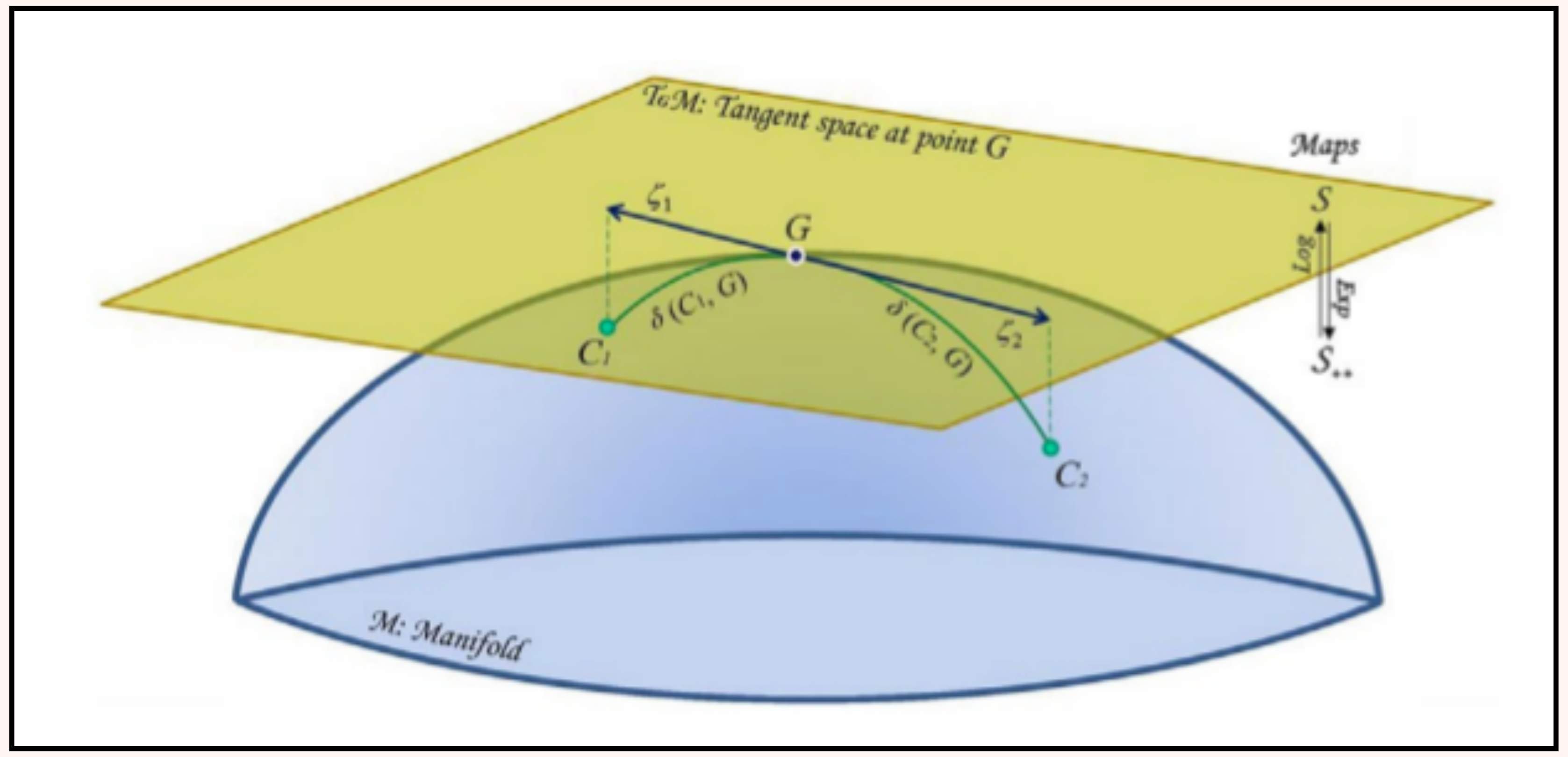
- In Newtonian dynamics:
  - Gravitational force $\mathbf{f}_g = m_g \mathbf{g}$, where m_g is the gravitational mass and $\mathbf{g}$ is gravitational field.
  - Inertia force: $\mathbf{f}_i = m_i \mathbf{a}$ where m_i is the inertial mass.
  - The motion in gravitational field is the same for all masses, of different composition (Galileo's experiment), and therefore gravitational mass and inertial masses are the same
- In freely falling systems, the gravitational force can be cancelled (locally)
  - This is similar to the Riemanian geomery: At any point on a curved surface, we may erect a locally Cartesian coordinate system in which distances obey the law of Pythagoras.

Why sufficiently small region?

- Gravitational field is not uniform
- The differential force becomes important as the region becomes larger, and the presence of the gravitational field cannot be ignored.
- Geometry is the same: the curvature can be ignored only for a 'sufficiently small' region.







Coordinates and Distances

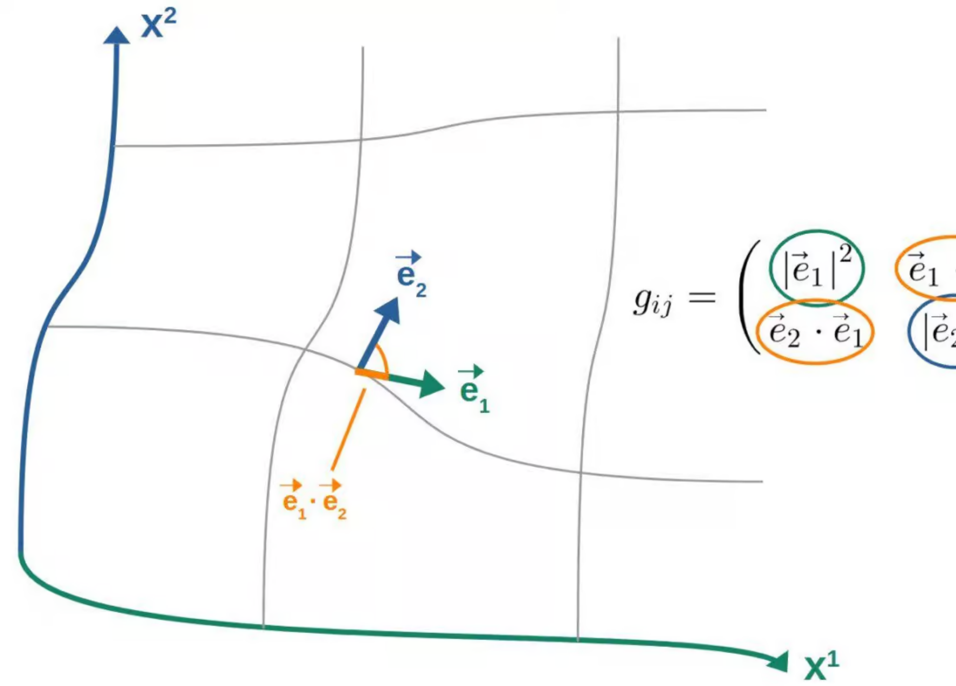
- Manifold of spacetime can be covered with patches or charts on which coordinates can be erected.
- General relativity is formulated in a **covariant** way so that the physical quantities are **invarian**t under changes of coordinates.
- Distance between two points that are sufficiently close together is geometrically invariant.

$ds^2 = g_{\mu\nu}(x^\alpha)dx^\mu dx^\nu$, where $g_{\mu\nu}$ is the metric tensor of spacetime

- In a flat spacetime or Minkowski spacetime, we use the symbol $\eta_{\alpha\beta}$ for the metric, i.e.,

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Metric tensor and its properties



- The coordinate vector $\vec{x} = \sum x^\mu \hat{e}_\mu$, then the length between $\vec{x}$ and $\vec{x} + d\vec{x}$ is

$$ds^2 = d\vec{x} \cdot d\vec{x} = \sum_\mu \sum_\nu \hat{e}_\mu \cdot \hat{e}_\nu dx^\mu dx^\nu = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu, \text{ where } g_{\mu\nu} \equiv \hat{e}_\mu \cdot \hat{e}_\nu$$

- An index appears twice in one term means summation over all possible values of that index.
- Metric tensor has following properties
  - Symmetric –> number of independent components : $\dfrac{n(n+1)}{2}$ independent components
  - Inverse metric: $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$

Metric tensor in other coordinates

- Transform to a new coordinates $x'^{\nu}(x^{\mu})$:

$dx^{\mu} = \dfrac{\partial x^{\mu}}{\partial x'^{\nu}} dx'^{\nu}$, where $x^{\mu}(x'^{\nu})$ is the inverse transformation.

- Since ds^2 is invariant under such transformation,

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial x'^{\kappa}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} dx'^{\kappa} dx'^{\sigma} = g'_{\kappa\sigma} dx'^{\kappa} dx'^{\sigma}$$

where $g'_{\kappa\sigma} = g_{\mu\nu} \dfrac{\partial x^{\mu}}{\partial x'^{\kappa}} \dfrac{\partial x^{\nu}}{\partial x'^{\sigma}}$

- Any physical quantity does not depend on the choice of the coordinate system: freedom of choice of coordinates is **gauge freedom.**

Gravitational Forces and Affine Connection

- According to the principle of equivalence, there is a freely-falling coordinate system ξ^α in which

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0$$

$d\tau^2 = -\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta$ is the proper time.

- Now suppose that we use any other coordinate system x^μ which could be any coordinate system at rest in lab.

$$0 = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right)$$

$$= \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

- Multiply this by $\partial x^\lambda/\partial\xi^\alpha$

$$0 = \frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

where $\Gamma^\lambda_{\mu\nu}$ is the affine connection defined as

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}$$

Note that:

$$d\tau^2 = -\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu$$

$$= g_{\mu\nu}dx^\mu dx^\nu$$

$$g_{\mu\nu} \equiv \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

Covariant and Contravariant Vectors

Let $V = V^{\mu}\hat{e}_{\mu}$

Under the coordinate change, the basis vectors transform as

$$\hat{e}'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\hat{e}_{\nu}$$

- Contravariant vector is a list of numbers that transforms oppositely to a change of basis
- Covariant vector is a list of numbers that transforms in the same way

- Contravariant vector
  - Under the transformation $x^{\mu} \to x'^{\mu}$:

$$V'^{\mu} = V^{\nu}\frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

  - The rules of partial differentiation give

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}dx^{\nu}$$ $\to$ a contravariant vector.

- Covariant vector

$$U'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}U_{\nu}.$$

  - If ϕ is a scalar field, then $\partial\phi/\partial x^{\mu}$ is a covariant vector since

$$\frac{\partial \phi}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial \phi}{\partial x^{\nu}}$$

Tensors

- From the contravariant and covariant vectors we can immediately generalize to tensors:

$$T'^{\mu}{}_{\nu}{}^{\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\kappa}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} T^{\kappa}{}_{\rho}{}^{\sigma}$$

- Inverse of the metric tensor: $g^{\lambda\mu} g_{\mu\nu} = \delta^{\lambda}_{\nu}$

$$g'^{\lambda\mu} g'_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x'^{\mu}}{\partial x^{\sigma}} g^{\rho\sigma} g'_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x'^{\mu}}{\partial x^{\sigma}} g^{\rho\sigma} \frac{\partial x^{\kappa}}{\partial x'^{\mu}} \frac{\partial x^{\eta}}{\partial x'^{\nu}} g_{\kappa\eta}$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} g^{\rho\kappa} \frac{\partial x^{\eta}}{\partial x'^{\nu}} g_{\kappa\eta} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} = \delta^{\lambda}_{\nu}$$

and therefore,

$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x'^{\mu}}{\partial x^{\sigma}} g^{\rho\sigma} = g'^{\lambda\mu}$ as required for a contravariant tensor.

Transformation of Affine Connection

- In a coordinate transformation $x \to x'$,

$$\Gamma^{\lambda}_{\mu\nu} \equiv \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$

$$\begin{aligned}
\Gamma'^{\lambda}_{\mu\nu} &= \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right) \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \left[\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right]
\end{aligned}$$

- Therefore.

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} \quad \to \Gamma^{\lambda}_{\mu\nu} \text{ is not a tensor!}$$

General covariance of equation of motion

- Equation of motion in freely falling particle:

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

- In a primed coordinate system,

$$\frac{d^2x'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial x'^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau}$$

whereas

$$\Gamma'^\mu_{\sigma\tau}\frac{dx'^\sigma}{d\tau}\frac{dx'^\tau}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\Gamma^\nu_{\lambda\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} - \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}$$

- According these two equations, we find that

$$\frac{d^2x'^\mu}{d\tau^2} + \Gamma'^\mu_{\nu\lambda}\frac{dx'^\nu}{d\tau}\frac{dx'^\lambda}{d\tau} = \frac{\partial x'^\mu}{\partial x^\kappa}\left(\frac{d^2x'^\mu}{d\tau^2} + \Gamma^\kappa_{\sigma\rho}\frac{dx^\kappa}{d\tau}\frac{dx^\sigma}{d\tau}\right)$$

Covariant Differentiation

- Differentiation of a tensor does not generally yield another tensor. For example, consider a contravariant vector V^{μ}

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}$$

- Differentiation with respect to x^{λ}

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}$$

- We can construct a tense using the affine connection:

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} + \Gamma'^{\mu}_{\lambda\kappa} V'^{\kappa} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left(\frac{\partial V}{\partial x^{\rho}} + \Gamma^{\nu}_{\rho\sigma} V^{\sigma} \right)$$

- Define covariant differentiation

$$V^{\mu}_{;\lambda} \equiv \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa}$$

- Similarly, covariant differentiation of a covariant vector:

$$V_{\mu;\nu} \equiv \frac{\partial V_{\mu}}{\partial x^{\nu}} - \Gamma^{\lambda}_{\mu\nu} V_{\lambda}$$

(Show that $V^{\mu}_{;\nu}$ and $V_{\mu;\nu}$ are tensors)

Curvature

- What tensors can be formed from the metric tensor and its derivatives?
  - If we use only $g_{\mu\nu}$ and its first derivatives, no new tensor can be constructed since we can find a coordinate system in which the first derivatives of $g_{\mu\nu}$ vanish.
  - Thus the simplest possibility is to construct a tensor out of $g_{\mu\nu}$ and its first and second derivatives → curvature tensor.

$$R^{\lambda}{}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}$$

  - Ricci tensor: $R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa}$
  - Curvature scalar: $R = g^{\mu\kappa} R_{\mu\kappa}$

Bianchi identities

- In locally inertial coordinate system in which $\Gamma^{\lambda}_{\mu\nu} = 0$ (but not its derivatives):

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2}\frac{\partial}{\partial x^{\eta}}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right]$$

- By permuting ν, κ, η cyclically, we get Bianchi identities:

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0$$

- Contracted form: multiply by $g^{\lambda\nu}$:

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}_{\mu\kappa\eta;\nu} = 0$$

- Contracting it again (i.e., multiplying $g^{\mu\kappa}$):

$$R_{;\eta} - R^{\mu}{}_{\eta;\mu} - R^{\nu}{}_{\eta;\nu} = 0 \text{ or}$$

$$\left(R^{\mu}{}_{\eta} - \frac{1}{2}\delta^{\mu}{}_{\eta}R\right)_{;\mu} = 0$$

- Equivalent, but familiar form: i.e., multiplying $g^{\eta\nu}$)

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0$$

$$g^{\mu\kappa}R^{\nu}{}_{\mu\kappa\eta;\nu} = R^{\kappa\nu}{}_{\kappa\eta;\nu} = -R^{\nu\kappa}{}_{\kappa\eta;\nu} = -R^{\nu}{}_{\eta;\nu}$$

Energy Momentum Four-Vector

- We define an energy-momentum four vector in special relativity,

$$p^\alpha \equiv m\frac{dx^\alpha}{d\tau}$$

- Then the Newton' second law can be written

$$m\frac{d^2x^\alpha}{d\tau^2} = m\frac{dp^\alpha}{d\tau} = f^\alpha$$

- Recall that $d\tau \equiv (dt^2 - d\mathbf{x}^2)^{1/2} = (1 - \mathbf{v}^2)^{1/2}dt$ where $\mathbf{v} \equiv d\mathbf{x}/dt$, then
  - The space component: $\mathbf{p} = m\gamma\mathbf{v}$
  - Time component: $p^0 = m\gamma = E$

where $\gamma \equiv \dfrac{dt}{d\tau} = (1 - \mathbf{v}^2)^{-1/2}$ is the Lorentz factor

Energy momentum tensor

- In electro-dynamics, the source terms of Maxwell's equations are charge density (ϵ) and current density ($\mathbf{J}$), and they obey the continuity equation: $\nabla \cdot \mathbf{J} = -\dfrac{\partial \epsilon}{\partial t}$

- Similarly we can define the density and current of the energy-momentum four vector:
  - The density: $T^{\alpha 0}(\mathbf{x}, t) \equiv \sum_n p_n^\alpha \delta^3(\mathbf{x} - \mathbf{x}_n(t))$
  - The current:

$$T^{\alpha i}(\mathbf{x}, t) \equiv \sum_n p_n^\alpha \frac{dx_n^i(t)}{dt} \delta^3(\mathbf{x} - \mathbf{x}_n(t))$$

- These two definitions can be united into a single formula

$$T^{\alpha\beta}(\mathbf{x}, t) \equiv \sum_n p_n^\alpha \frac{dx_n^\beta(t)}{dt} \delta^3(\mathbf{x} - \mathbf{x}_n(t)) = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\mathbf{x} - \mathbf{x}_n(t))$$

- Delta function can be extended to four dimension

$$T^{\alpha\beta}(\mathbf{x}, t) \equiv \sum_n \int d\tau \, p_n^\alpha \frac{dx_n^\beta(t)}{dt} \delta^4(\mathbf{x} - \mathbf{x}_n(t))$$

$$\frac{dx_n^\beta}{dt} = \frac{dx_n^\beta}{d\tau}\frac{d\tau}{dt} = \frac{p_n^\beta}{m\gamma} = \frac{p_n^\beta}{E_n}$$

Conservation of Energy-Momentum tensor

- From the definition of the density and current,

$$\frac{\partial}{\partial x^i}T^{\alpha i}(\mathbf{x},t) = -\sum_n p_n^\alpha(t)\frac{dx_n^i(t)}{dt}\frac{\partial}{\partial x_n^i}\delta^3(\mathbf{x}-\mathbf{x}_n(t))$$

$$= -\sum_n p_n^\alpha(t)\frac{\partial}{\partial t}\delta^3(\mathbf{x}-\mathbf{x}_n(t))$$

$$= -\frac{\partial}{\partial t}T^{\alpha 0}(\mathbf{x},t) + \sum_n \frac{dp_n^\alpha(t)}{dt}\delta^3(\mathbf{x}-\mathbf{x}_n(t))$$

and so

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta} = G^\alpha$$

where G^α is the density of the force

$$G^\alpha(\mathbf{x},t) \equiv \sum_n \delta^3(\mathbf{x}-\mathbf{x}_n(t))\frac{dp_n^\alpha(t)}{dt} = \sum_n \delta^3(\mathbf{x}-\mathbf{x}_n(t))\frac{d\tau}{dt}f_n^\alpha(t)$$

- If the particles are free, then p_n^α is a constant and therefore $T^{\alpha\beta}$ is conserved, i.e.,

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta}(x) = 0$$

Generally covariant form of energy-momentum conservation

- Define $T^{\mu\nu}$ and G^{ν} as contravariant tensors that reduce to the special relativistic $T^{\alpha\beta}$ and G^{β} in the absence of gravitation, then generally covariant equation that agrees with $\partial T^{\alpha\beta}/\partial x^{\alpha} = G^{\beta}$ in locally inertial system is

$$T^{\mu\nu}{}_{;\mu} = G^{\nu} \quad \text{or} \quad \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}T^{\mu\nu}) = G^{\nu} - \Gamma^{\nu}{}_{\mu\lambda}T^{\mu\lambda}$$

$$T^{\mu\nu}{}_{;\mu} \equiv \frac{\partial T^{\mu\nu}}{\partial x^{\mu}} + \Gamma^{\mu}{}_{\mu\lambda}T^{\lambda\nu} + \Gamma^{\nu}{}_{\mu\lambda}T^{\mu\lambda} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}T^{\mu\lambda}) + \Gamma^{\nu}{}_{\mu\lambda}T^{\mu\lambda}$$

Derive this

- The energy-momentum tensor of perfect fluid is

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+\rho)U^{\alpha}U^{\beta} \rightarrow \; T^{\mu\nu} = pg^{\mu\nu} + (p+\rho)U^{\mu}U^{\nu}$$

- The conditions of energy-momentum conservation give the hydrodynamic equations:

$$0 = T^{\mu\nu}{}_{;\nu} = \frac{\partial p}{\partial x^{\nu}}g^{\mu\nu} + g^{-1/2}\frac{\partial}{\partial x^{\nu}}g^{1/2}(p+\rho)U^{\mu}U^{\nu} + \Gamma^{\mu}{}_{\nu\lambda}(p+\rho)U^{\nu}U^{\lambda}$$

where the last term represents the gravitational force on the system.

Equation of motion in weak field

- Equation of motion : $\dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\dfrac{dx^\nu}{d\tau}\dfrac{dx^\lambda}{d\tau} = 0$
- In weak and stationary field, $dt/d\tau \gg |d\mathbf{x}/d\tau|$, and the equation of motion can be approximated by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \text{ where } \Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu}$$

since all time derivatives of $g_{\mu\nu}$ vanish.

- Also, we may adopt a nearly Cartesian coordinate system in which $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad |h_{\alpha\beta}| \ll 1$
- Therefore, to first order in $h_{\alpha\beta}$, $\Gamma^\alpha_{00} = -\dfrac{1}{2}\eta^{\alpha\beta}\dfrac{\partial h_{00}}{\partial x^\beta}$
- Using this affine connection in the equation of motion then becomes

$$\frac{d^2\mathbf{x}}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \nabla h_{00} \text{ and } \frac{d^2t}{d\tau^2} = 0 \;\Rightarrow\; \frac{d^2\mathbf{x}}{dt^2} = \frac{1}{2}\nabla h_{00} (= -\nabla\phi \text{ in Newtonian limit})$$

- Now $h_{00} = -2\phi + constant \rightarrow -1$ at ∞, where $\phi = 0$. Therefore, $g_{00} \approx -(1+2\phi)$ in weak field (i.e., $|\phi| \ll 1$.

Einstein's field equations

- Recall that $g_{00} \approx -(1+2\phi)$, in the weak field, and ϕ is determined by the Poisson's equation: $\nabla^2\phi = 4\pi G\rho \approx T_{00}$. By combining them, we have

$$\nabla^2 g_{00} = -8\pi G T_{00}$$

- This equation holds for weak static field generated by non-relativistic matter. However, we may guess that the weak-field equations for a general distribution of $T_{\alpha\beta}$ take the form

$G_{\alpha\beta} = -8\pi G T_{\alpha\beta}$ in the weak field, which can be generalized to $G_{\mu\nu} = -8\pi G T_{\mu\nu}$

- $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ correctly describes the equation of motion in weak field.

- Therefore, the Enstein's field equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}$$

- Since $g_{\mu\nu;\lambda} = 0$, adding an extra term proportional to $g_{\mu\nu}$ to the field does not violate the energy conservation.

- $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$ with an arbirtary constant Λ. –> Cosmological constant

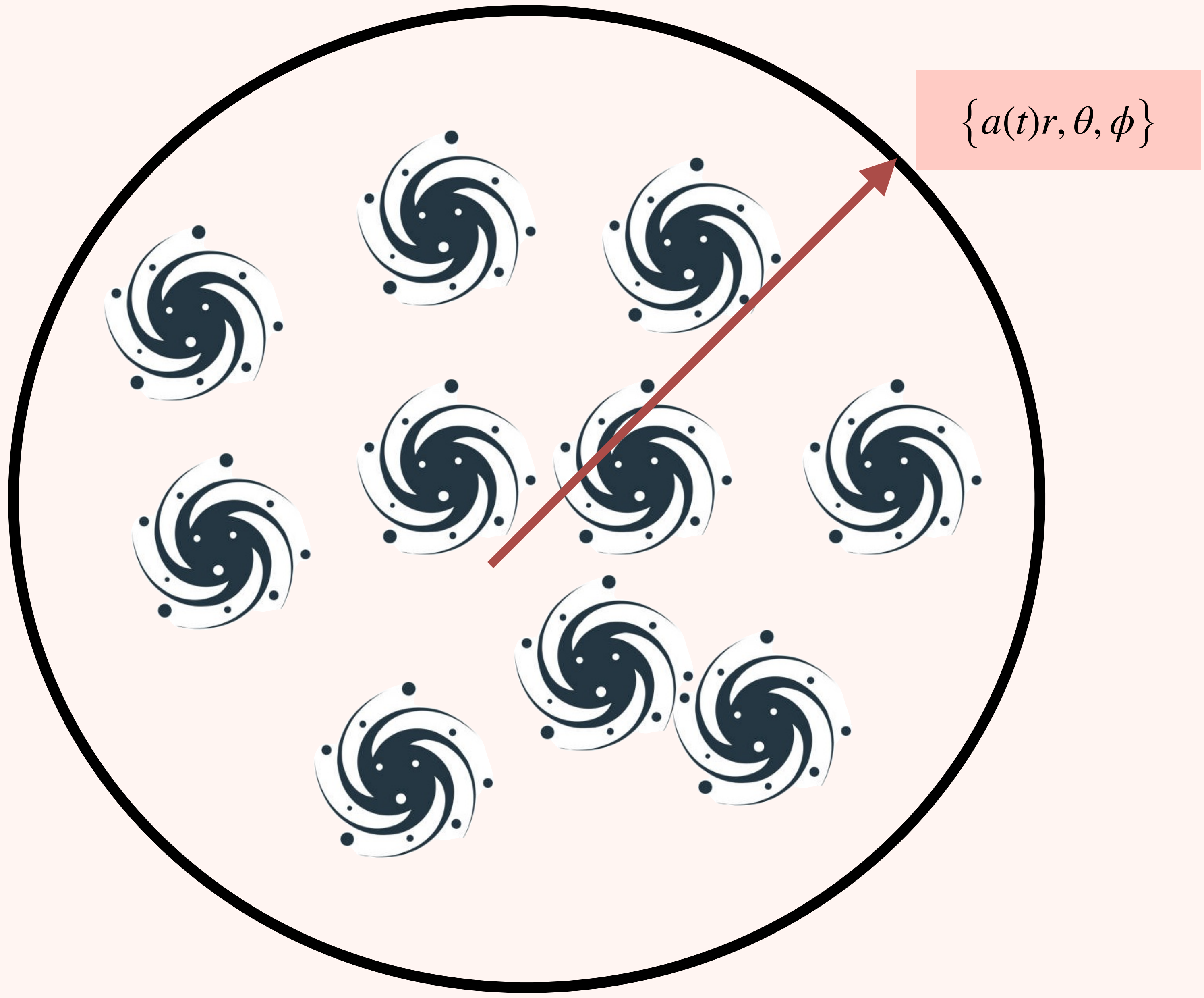
Cosmological Principle and Maximally Symmetric Space

- Cosmological Principle
  - Universe is homogenous and isotropic
  - Maximally symmetric space (=constant spatial curvature)
- Friedman-Robertson-Walker metric for homogenous and isotropic universe (or the only maximally symmetric space)

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right\}; \quad k = +1,\ 0,\ \text{or}\ -1$$

where $(r, \overrightarrow{\Omega})$ is the comoving coordinates and $a(t)$ is the scale factor.

The above metric is called Friedmann-Lemaitre-Robertson-Walker Metric (FRLW)



Redshift

- Assume that we are at $r = 0$. Then equation of motion of a photon along radial direction
$$0 = d\tau^2 = dt^2 - a^2(t)\frac{dr^2}{1 - kr^2}$$
- If the wave crest leaves (r_1, θ_1, ϕ_1) at t_1 and arrives at t_0, then
$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = f(r_1)$$
- Next crest leaves at $t_1 + \delta t_1$ will arrive at $t_1 + \delta t_1$, and since f depends on only r_1,
$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = f(r_1)$$
- Therefore, we have
$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$
- Since the frequencies are related by
$$\frac{\nu_0}{\nu_1} = \frac{\delta t_0}{\delta t_1} = \frac{a(t_1)}{a(t_0}$$
- The redshift is $z = \dfrac{\lambda_0 - \lambda_1}{\lambda_1}$, we find
$$z = \frac{a(t_0)}{a(t_1)} - 1$$

Simplification of Einstein's equation in FRLW metric

- The energy-momentum tensor $T_{\mu\nu}$
  - $T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)U_\mu U_\nu$, U: velocity four vector of the Hubble flow.
  - $U^\mu = (c,0,0,0)$, $U_\nu = (-c,0,0,0)$ [$c = 1$ is used]
- The time–time component of Einstein's equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

- The spatial components:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right),$$

- Energy conservation:

$$\frac{d}{da}(\rho a^3) = -3pa^2$$

p : Pressure, ρ : energy density

Constituents of the Universe

- Friedmann's equations have three unknowns $a(t), \rho(t), p(t)$, but only two independent equations
- An additional constraint comes from equation of state: $p = p(\rho)$, which depends on the constituents of the universe.
  - Matter (non-relativistic): $p = 0$
  - Radiation (or relativistic particles: $p = \dfrac{1}{3}\rho$
- Cosmological constant as a form of perfect fluid:

$$T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} = \left(\rho^{\Lambda} + \frac{p^{\Lambda}}{c^2}\right) u_\mu u_\nu + \frac{p^{\lambda}}{c^2} g_{\mu\nu}$$

  - In order the above quantaty to be proportional to $g_{\mu\nu}$, $\rho_\Lambda + \dfrac{p^{\Lambda}}{c^2}$ has to vanish.
  - Therefore, $p^{\Lambda} = -\rho^{\Lambda}$, i.e., negative pressure. ==> $\dfrac{\ddot{a}}{a} = \dfrac{8\pi G \rho^{\Lambda}}{3} > 0$: acceleration

Expansion of the Universe for different components

- Matter only

$$p = 0\,,\ \frac{d}{da}(\rho a^3) = 0\ \rightarrow \rho \propto a^{-3}.\ a(t) \propto t^{2/3}$$

- Radiation only

$$p = (1/3)\rho,\ \frac{d}{da}(\rho a^3) = -3pa^2 = -\rho a^2$$

  - divide both sides by ρa^3: $\frac{1}{\rho a^3}\frac{d}{da}(\rho a^3) = \frac{d\ln \rho a^3}{da} = -\frac{1}{a}$
  - Integral over a: $\ln(\rho a^3) = -\ln a + C,\ \rho \propto a^{-4};\ a(t) \propto t^{1/2}$

- Cosmological Constant Only:

$$p = -\rho;\ \frac{d}{da}(\rho a^3) = -3pa^2 = 3\rho a^2 \text{->} \rho = \rho_0,\ a(t) = A\exp(Ht),\ \text{where } H = \sqrt{\frac{8\pi G}{3}\rho}$$

Static Universe with matter and Cosmological Constant

- Friedmann's equation:
- $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right)$
- Let $\rho = \rho_m + \rho_\Lambda$; $p = p_m + p_\Lambda = p_\Lambda = -\rho_\Lambda$
- Static Universe: $\ddot{a} = 0$, $\rho_m + \rho_\Lambda - 3\rho_\Lambda = 0$; therefore $\rho_\Lambda = \frac{1}{2}\rho_m$.
- However, the static universe is unstable against small perturbations.
  - Let $a = a_0(1 + \epsilon(t))$, then $\frac{\ddot{a}}{a} = \ddot{\epsilon}(t)$, $\rho_m = \rho_{m,0}\left(\frac{a}{a_0}\right)^{-3} = \rho_{m,0}(1 - 3\epsilon(t))$
  - $\ddot{\epsilon}(t) = 4\pi G\rho_{m,0}\epsilon(t) \text{ –> } \epsilon(t) \propto \exp\left(\pm\sqrt{4\pi G\rho_{m,0}}t\right)$; what is the time scale?

Generalizaion of Cosmological Constant: Dark Energy

- Cosmological constant causes accelerating universe
- Consider the following equation of state:

$$p = -w\rho$$

- Plugging this into $\dfrac{\ddot{a}}{a} = -\dfrac{4\pi G}{3}\left(\rho + 3p\right)$, and we find the following properties
  - If $w > \dfrac{1}{3}$, $\ddot{a} > 0$: acceleration
  - If $w < 1/3$, $\ddot{a} < 0$: deceleration

Density Parameter

- Hubble parameter: $H = \dot{a}/a$

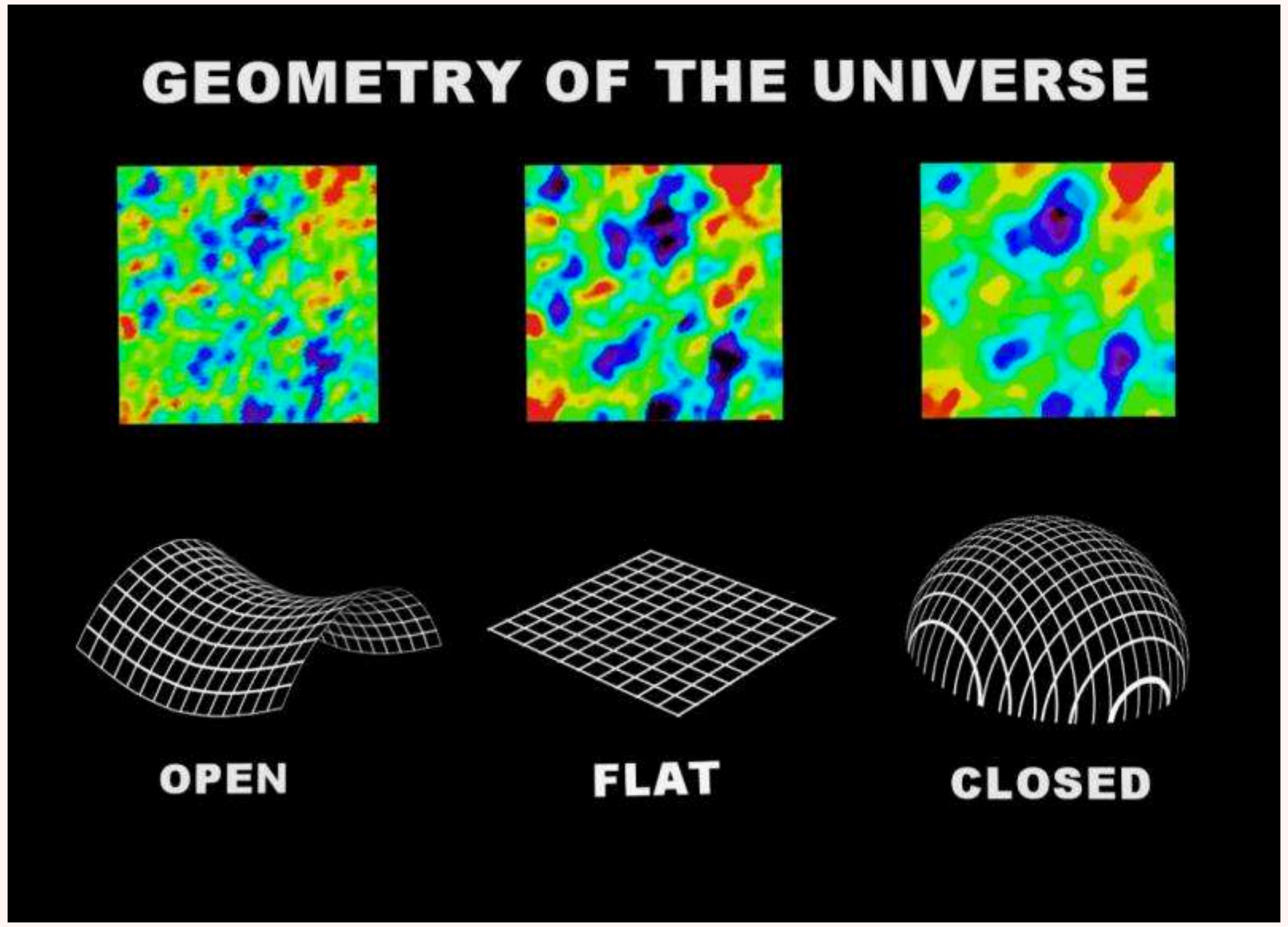
$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{kc^2}{a^2}$$

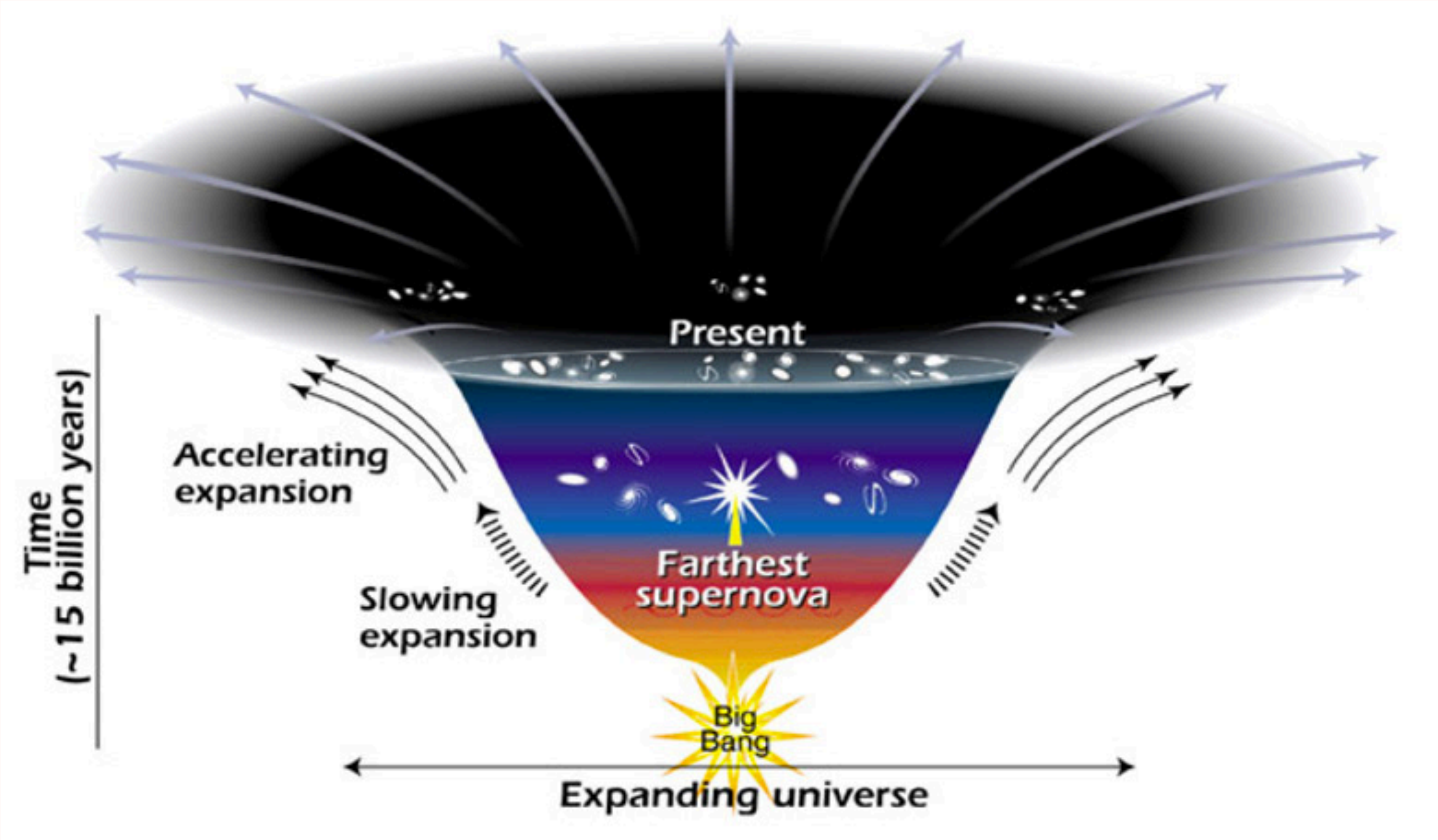
$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$$

- Current universe, $H = H_0$, $(\rho + \rho_\Lambda) = \rho_0$, then $H_0^2 = \dfrac{8\pi G}{3}\rho_0 - \dfrac{kc^2}{a_0^2}$
- The critical density, ρ_{crit}: hypothetical density today that makes the universe flat, i.e., $k = 0$.
- $\rho_{crit} \equiv \dfrac{3H_0^2}{8\pi G}$
- Density parameter is defined as $\Omega \equiv \dfrac{\rho}{\rho_{crit}}$;

Cosmological Parameters

- Hubble parameter
  - $H(t) = \frac{\dot{a}}{a}$, $H_0 = H(t_0)$: Hubble constant
- Current density parameters: $\Omega_0 = \frac{\rho_0}{\rho_{crit}}$
  - $\Omega_0 < 1$: Open,
  - $\Omega_0 > 1$: Closed
  - $\Omega_0 = 1$: Flat
- $\Omega = \Omega_m + \Omega_{Rad} + \Omega_\Lambda$
- $\Omega_m = \Omega_b + \Omega_{DM}$ (Baryon, Dark matter)

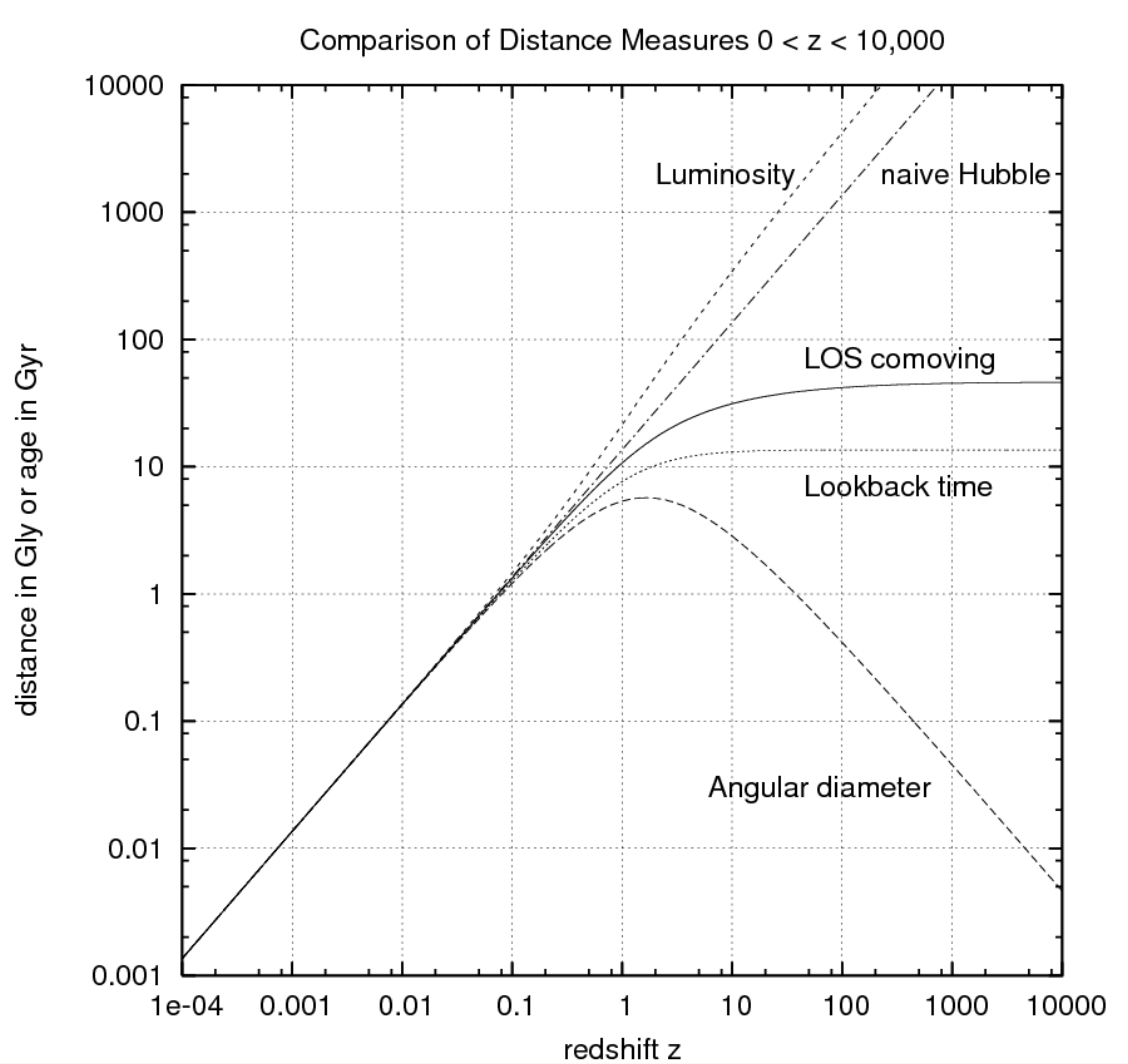




Accelerating universe: $\Omega_\Lambda > \Omega_m$

Cosmography

- Complete desctiption of the expansion history of the universe
  - $a(t)$, or $a(z)$
- However, $a(z)$ cannot be observed directly.
- Instead,, we can construct $a(z)$ by measuring distances of objects as a function of redshift.
- What is the distance?
  - Luminosity distace: $F_{obs} = \frac{L}{4\pi d_L^2}$.
    - We need standard candle
  - Angular diameter distance: $d_A = \frac{R}{d_A}$
    - We need standard ruler
  - Relation between the two: $d_L = d_A(1+z)^2$

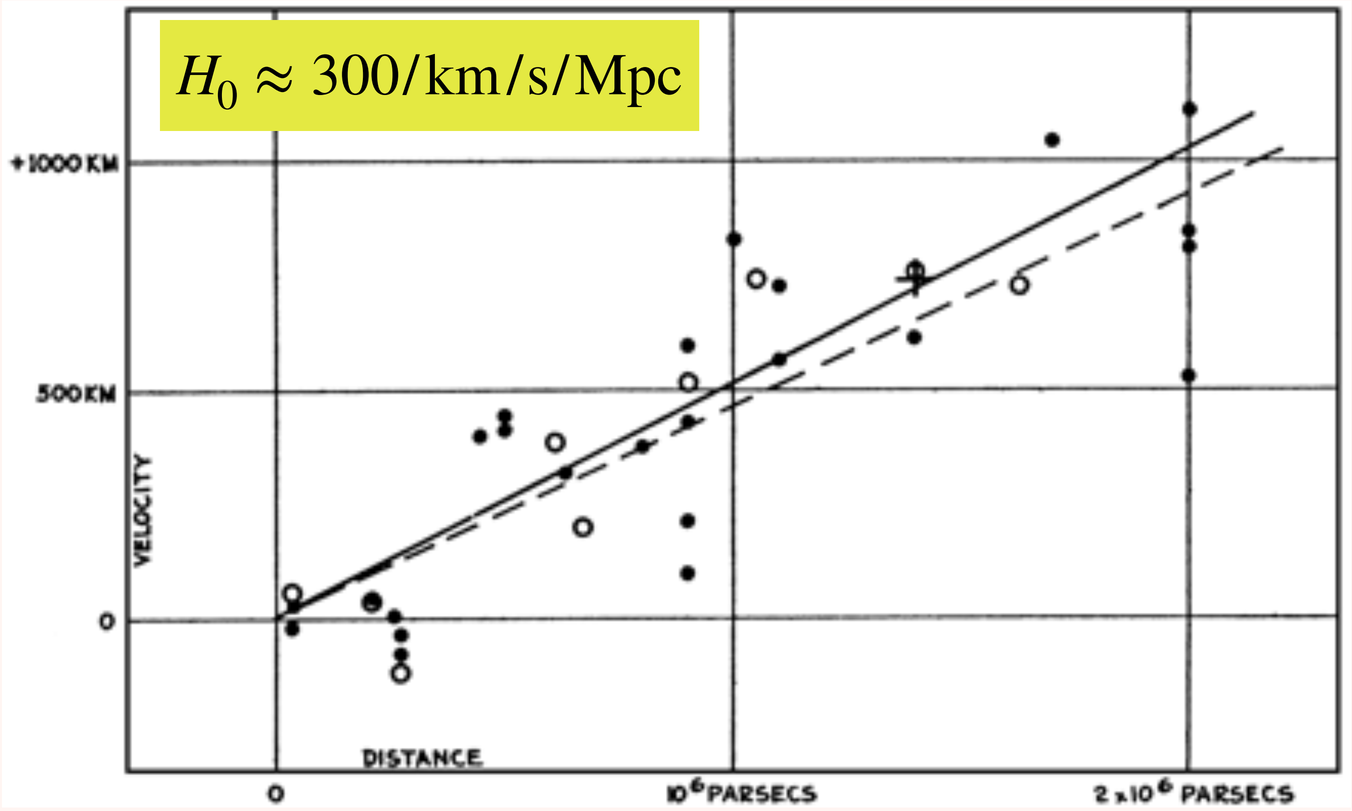


How the Hubble parameter is measured?

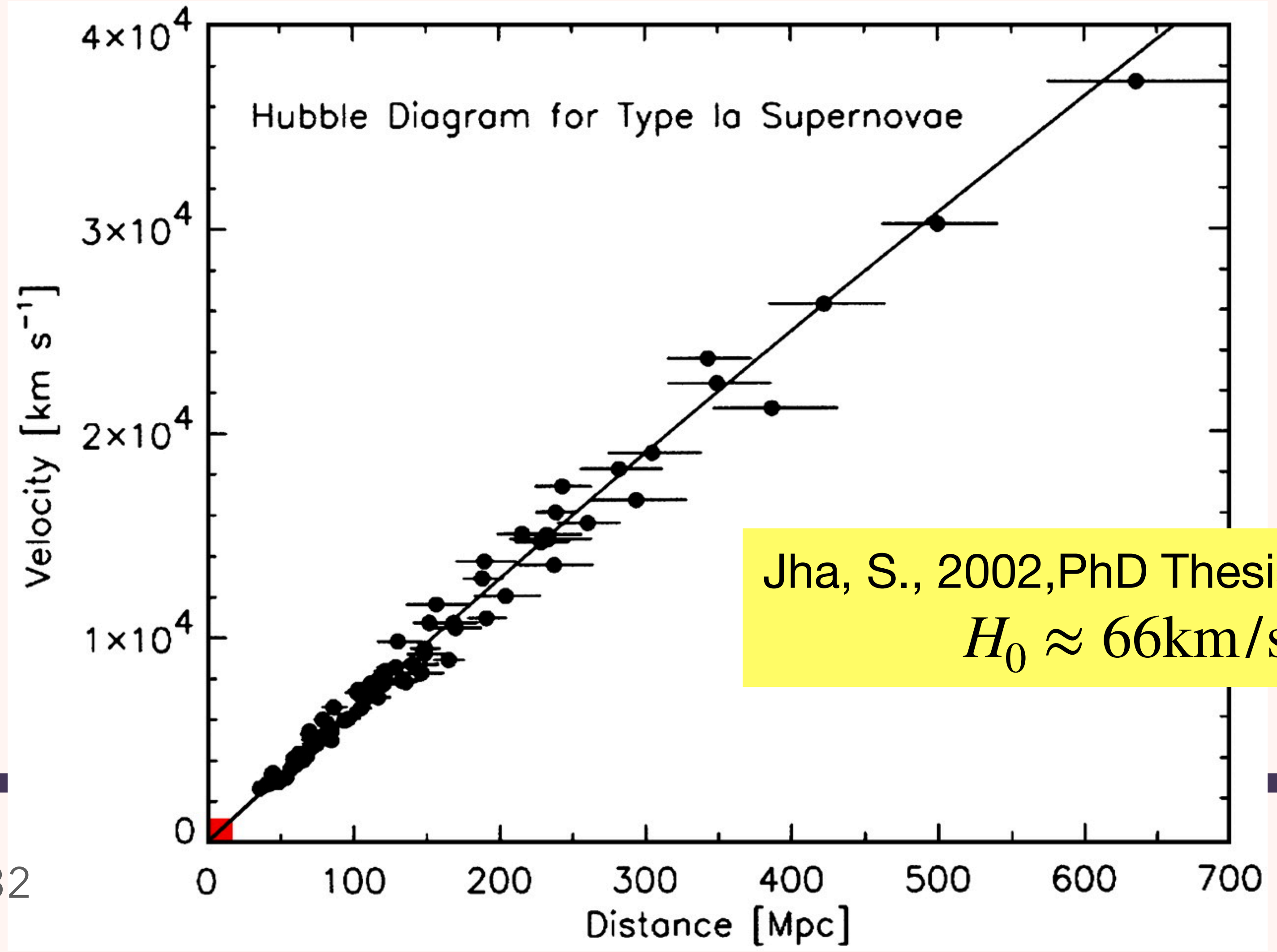
- Hubble's law;

$$v_r = H_0 d$$

- We need to measure radial velocity (v_r, or redshift z) and distance (d) simultaneously for large number of galaxies.
- Galaxies should be not too close (because of the peculiar motion) and not too far (because we are looking at th past).
- Distance measurements are difficult, especially for distant galaxies

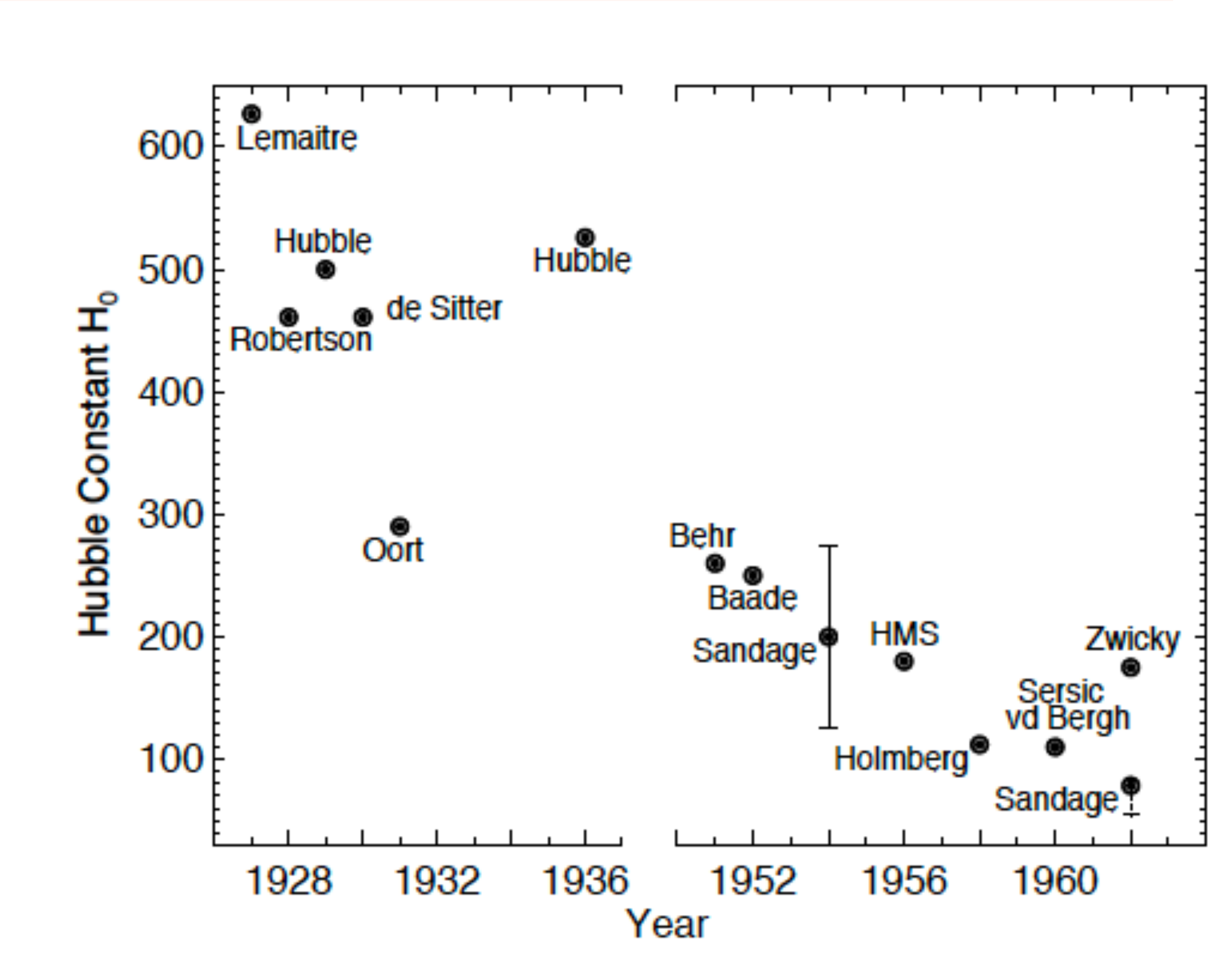


Hubble, E. P., 1929, Prof. Nat. Acad. Sci., 15, 168

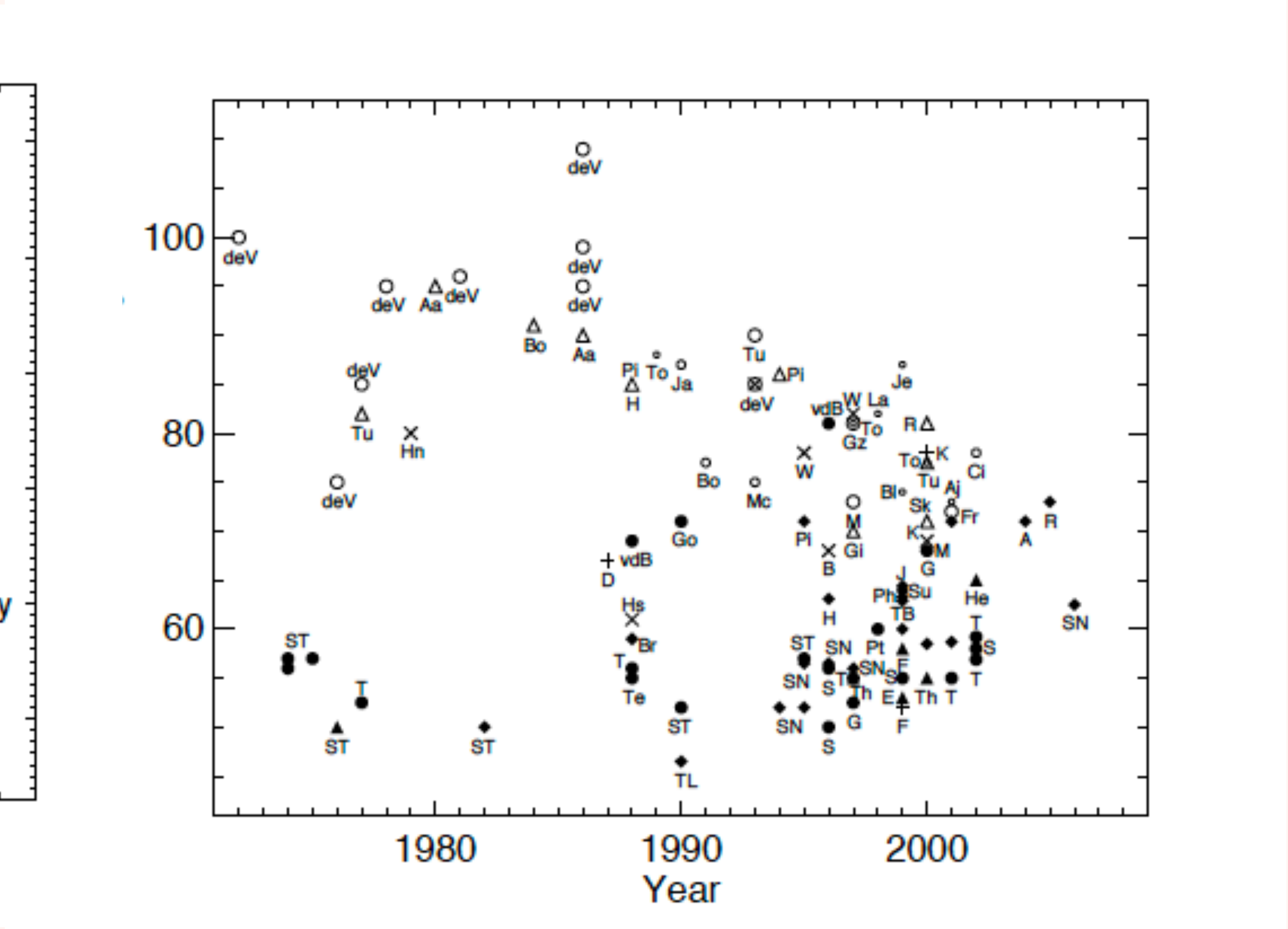


Jha, S., 2002,PhD Thesis, Harvard Univ.
$H_0 \approx 66$km/s/Mpc

H_0 has changed over time



From 1927 to 1962
(Tamman 2005)



From 1975
(Tamman 2005)

Summary of local measurement: Hubble Key Project

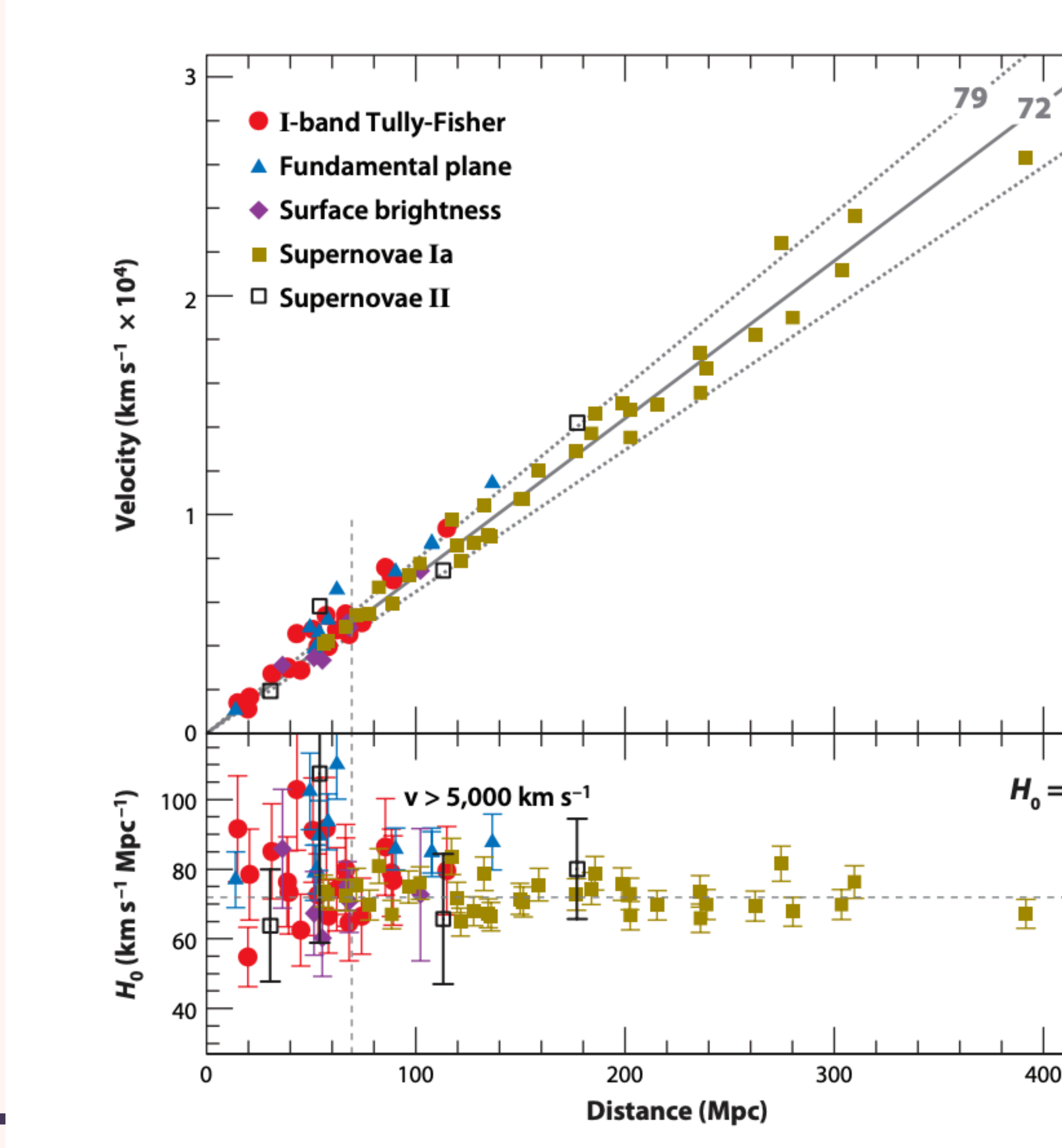
(Freedman et al., 2001, ApJ, 553, 47)

- Cepheid Hubble Diagram (up to 25 Mc): $H_0 = 75 \pm 10(r)\mathrm{km/sec/Mpc}$
- Relative distance methods (60 - 400 Mpc)
  - Type Ia SN: $H_0 = 71 \pm 2(r) \pm 6(s)\mathrm{km/sec/Mpc}$
  - Tully-Fisher Relation $H_0 = 71 \pm 3(r) \pm 7(s)\mathrm{km/sec/Mpc}$
  - Fundamental Plane: $H_0 = 70 \pm 5(r) \pm 6(s)\mathrm{km/sec/Mpc}$
  - Surface Brightness Fluctuation $H_0 = 82 \pm 6(r) \pm 9(s)\mathrm{km/sec/Mpc}$
  - Type II SN $H_0 = 72 \pm 9(r) \pm 7(s)\mathrm{km/sec/Mpc}$
- Combination:

$$H_0 = 72 \pm 8\mathrm{km/sec/Mpc}$$

HST Key Project

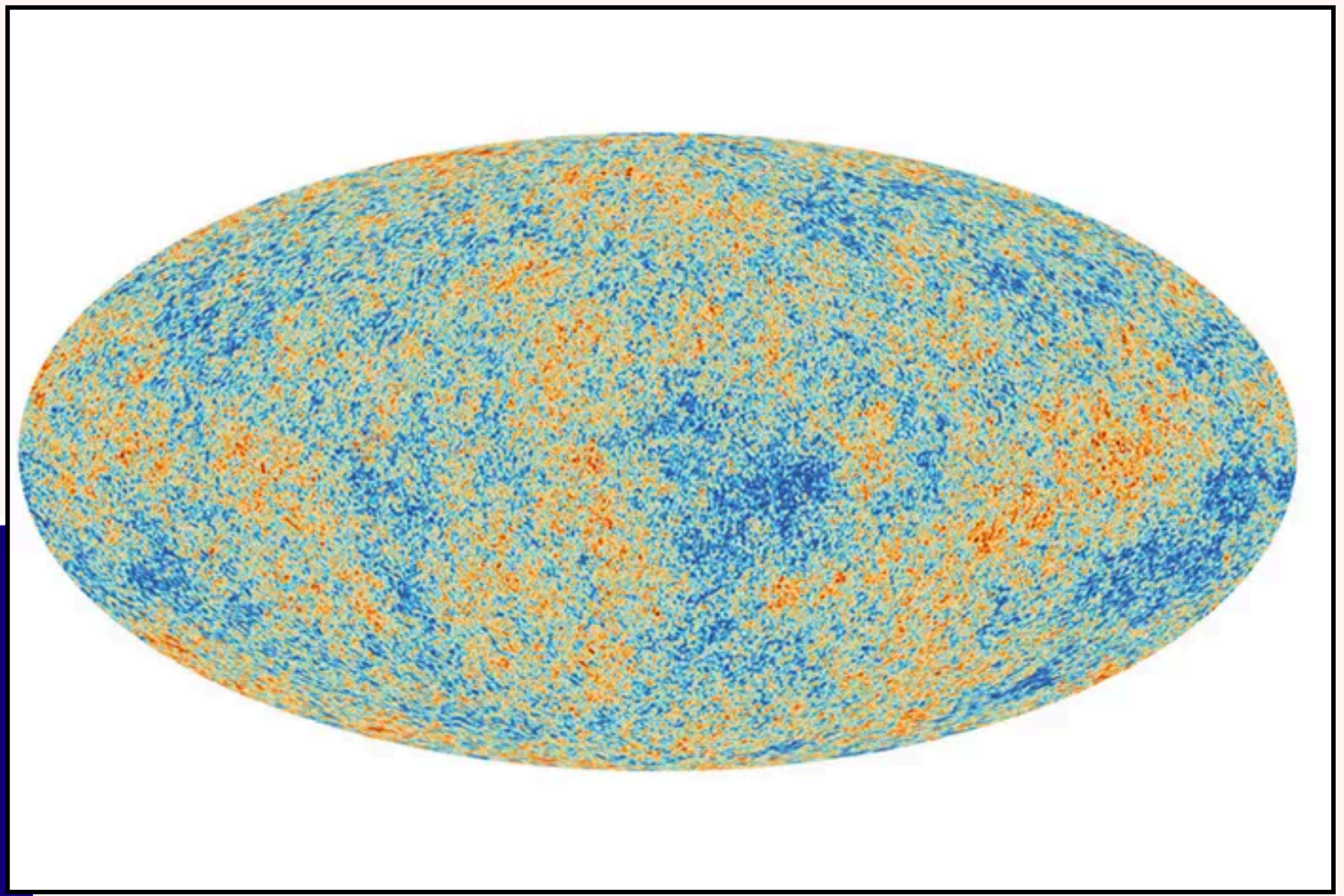
- Hubble Space Telescope (HST) Kery Project aimed to measure H_0 to an accuracy of $\pm 10\%$ by measuring the distances to 18 galaxies containing Cepheid variables
- Result: (Freedman et al. 2001, later some updates)
  - $H_0 = 72 \pm 3\ \text{(statistic)} \pm 4\ \text{(systematic)}\ \text{km/sec/Mpc}$
  - Issue was resolved?

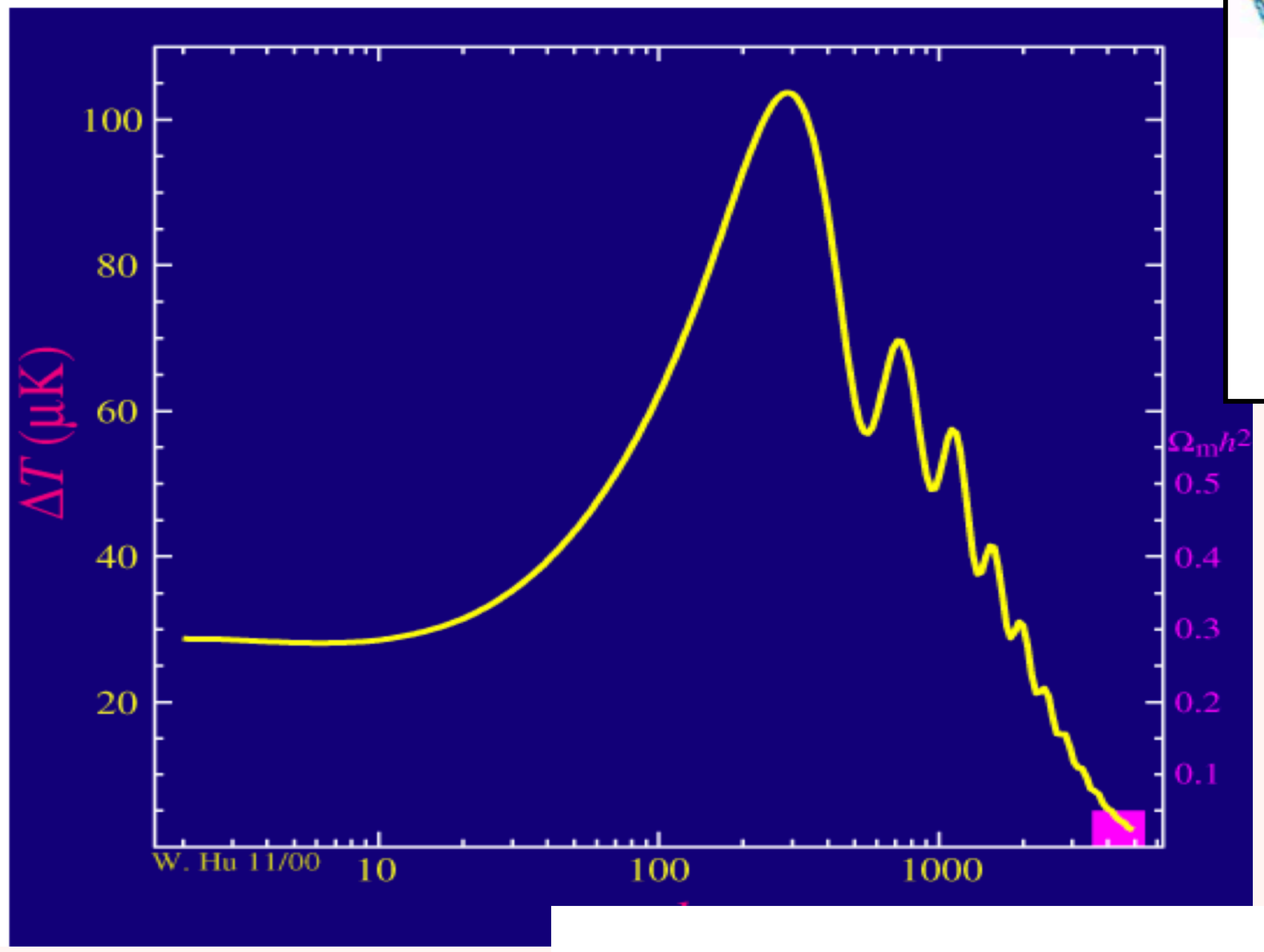


Cosmic Microwave Background contains Rich Information on the Universe

- CMB power spectrum depends on various parameters: Ω_0, Ω_b, Ω_m and H_0.
- For flat universe,
  - Peak positions: $\sim \Omega_m h^3$
  - Height of the second peak $\sim \Omega_b h^2$
  - Peak Ratios $\sim \Omega_m h^2$

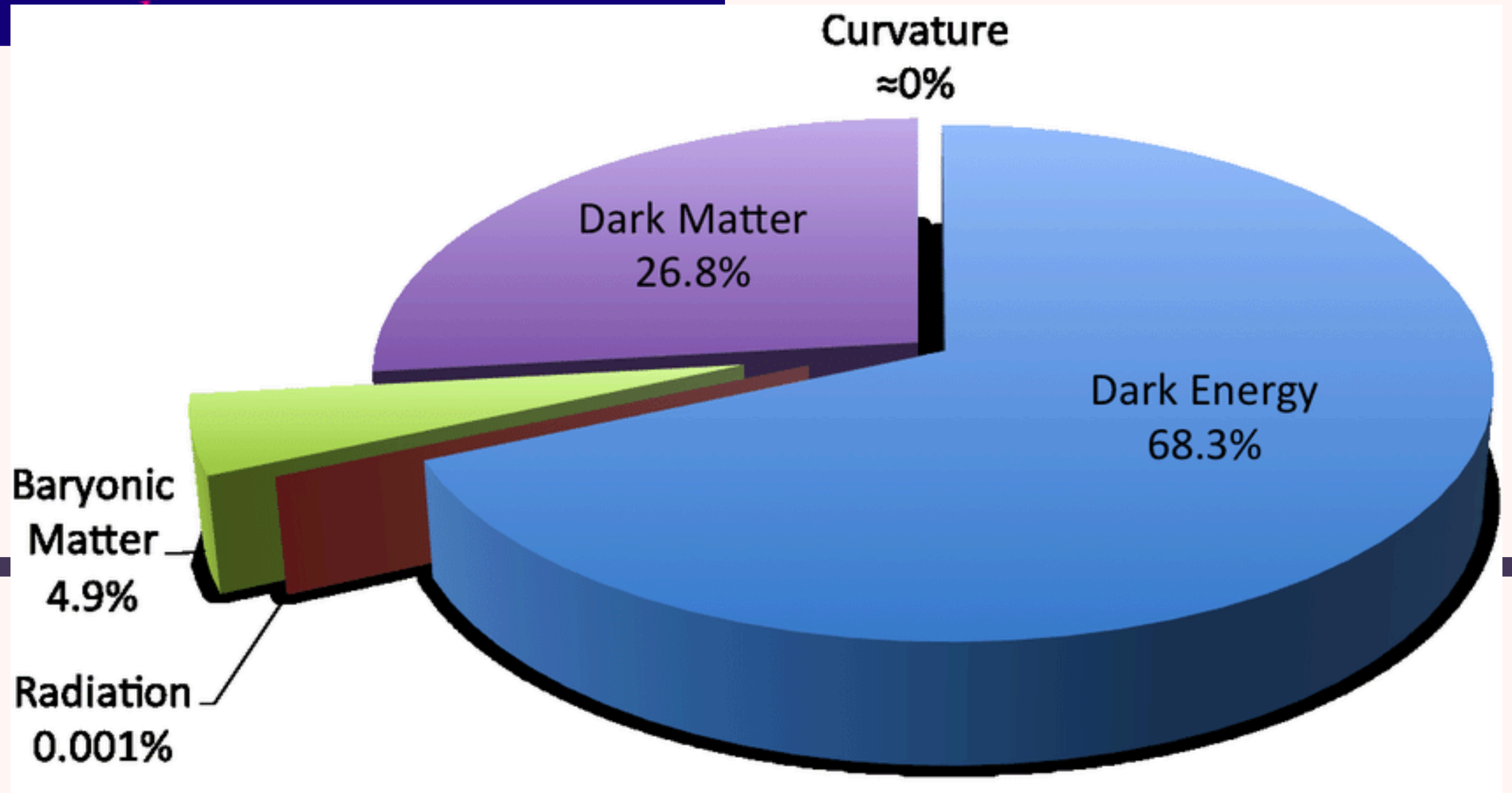
$$h = H_0/100 \text{ km/s}$$





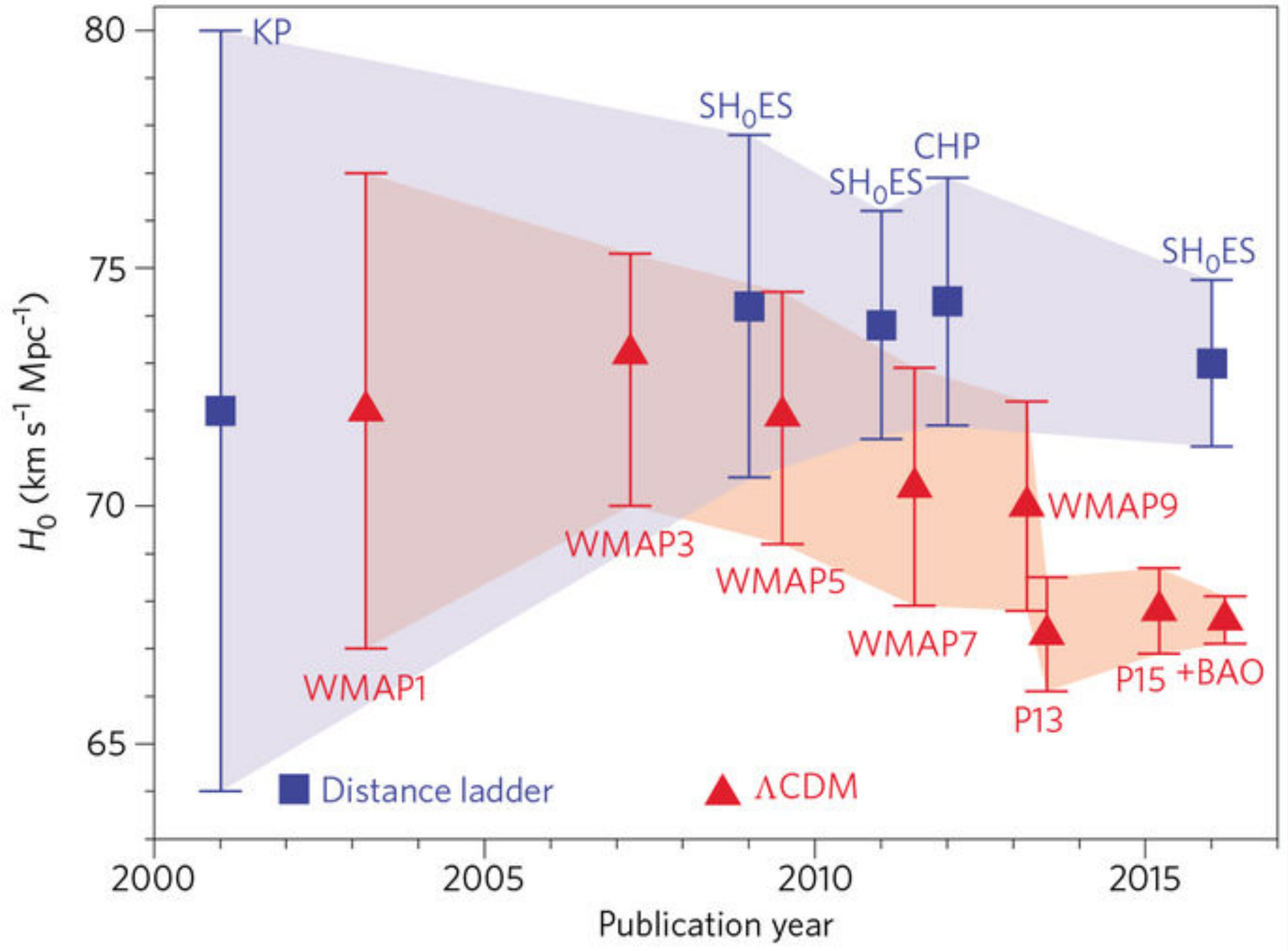
Results from Planck (2018)

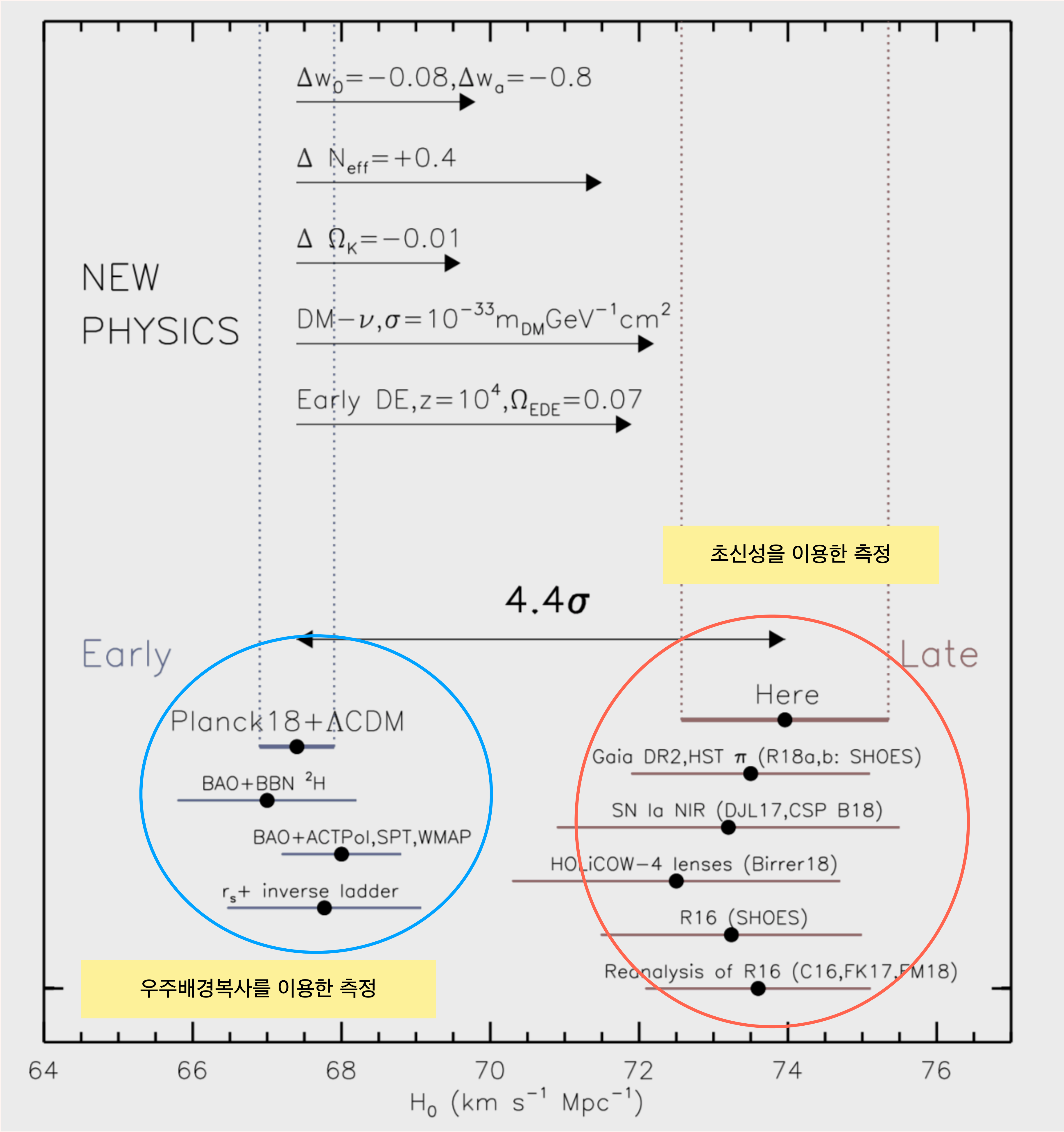
$$H_0 = 67.4 \pm 0.5 \text{ km/sec/Mpc}$$



Hubble tension

- SH_0ES: SNe, H0
- WMAP: Wilkinson Microwave Probe
- P: Planck
- BAO: Baryonic Acoustic Oscillation





Riess et al. (2019) ApJ, 876, 85

W. Freedman, 2017, Nature Astronomy, 1, 0121

Cosmological Research with GWs

- Reconstruction of expansion history of the universe through direct measurements of distances
  - GWs from compact binaries are 'standard sirens' as the distances can be estimated from the detected waveforms and amplitudes
    - Bright sirens [optical counterparts] vs. Dark sirens [no optical counterparts]
  - Need to identify the host galaxies in order to measure the redshifts in order to construct Hubble diagram

Estimation of Parameters from GW Observations

- What we measure is the waveforms form for certain duration

$$h(t) = F_+(\alpha, \delta, \psi, t_c)h_+(t) + F_\times(\alpha, \delta, \psi, t_c)h_\times(t)$$

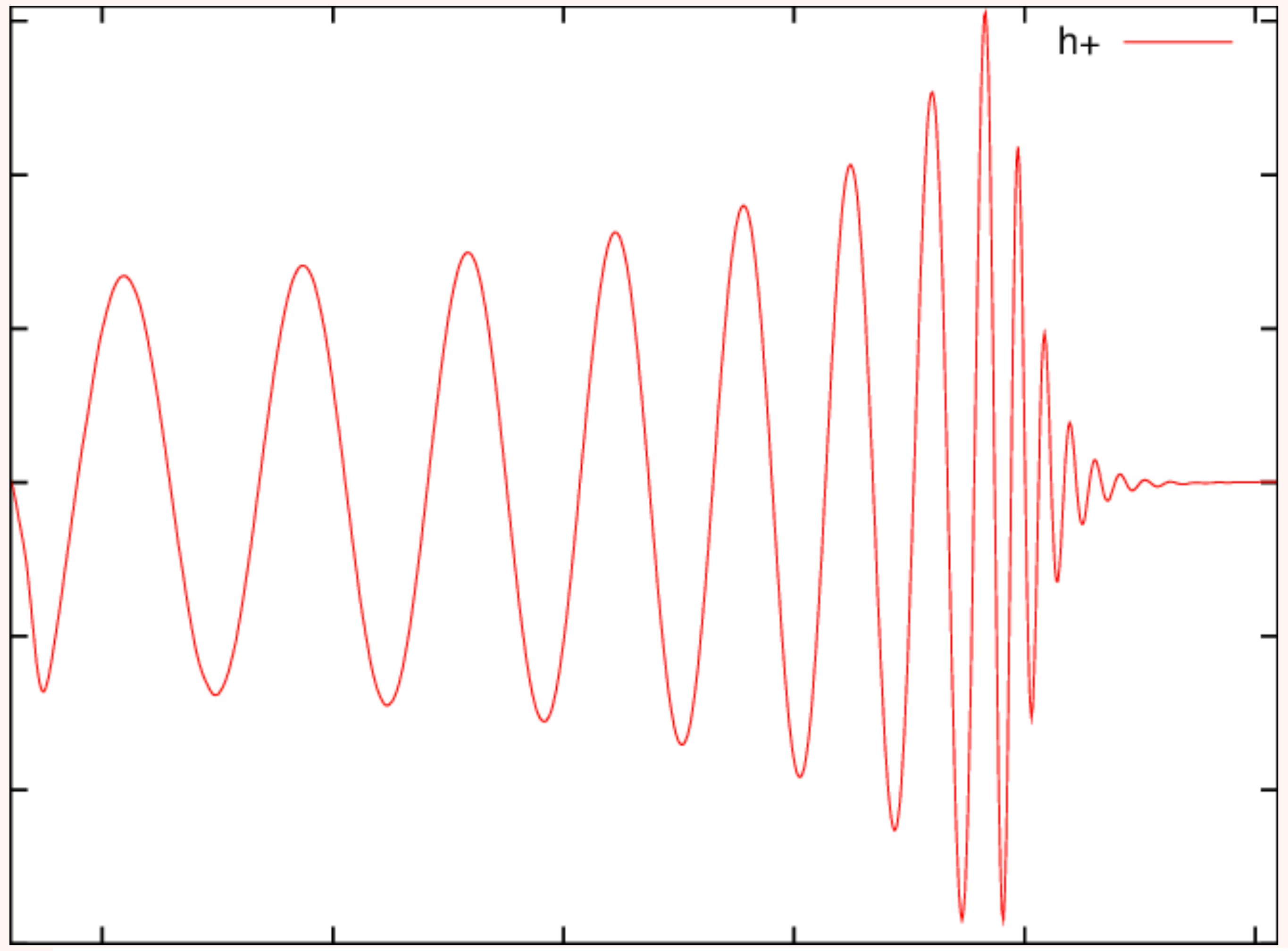
- For circular binaries, 15 parameters are imprinted on the waveforms
  - Intrinsic: $m_1, m_2, \mathbf{s_1}, \mathbf{s_2}$ (8)
  - Extrinsic: $\mathbf{\Omega} = (\alpha, \delta), \iota, d, \psi, t_c, \phi_c$ (7)
  - If we ignore spins, the number of parameters reduces to 9.

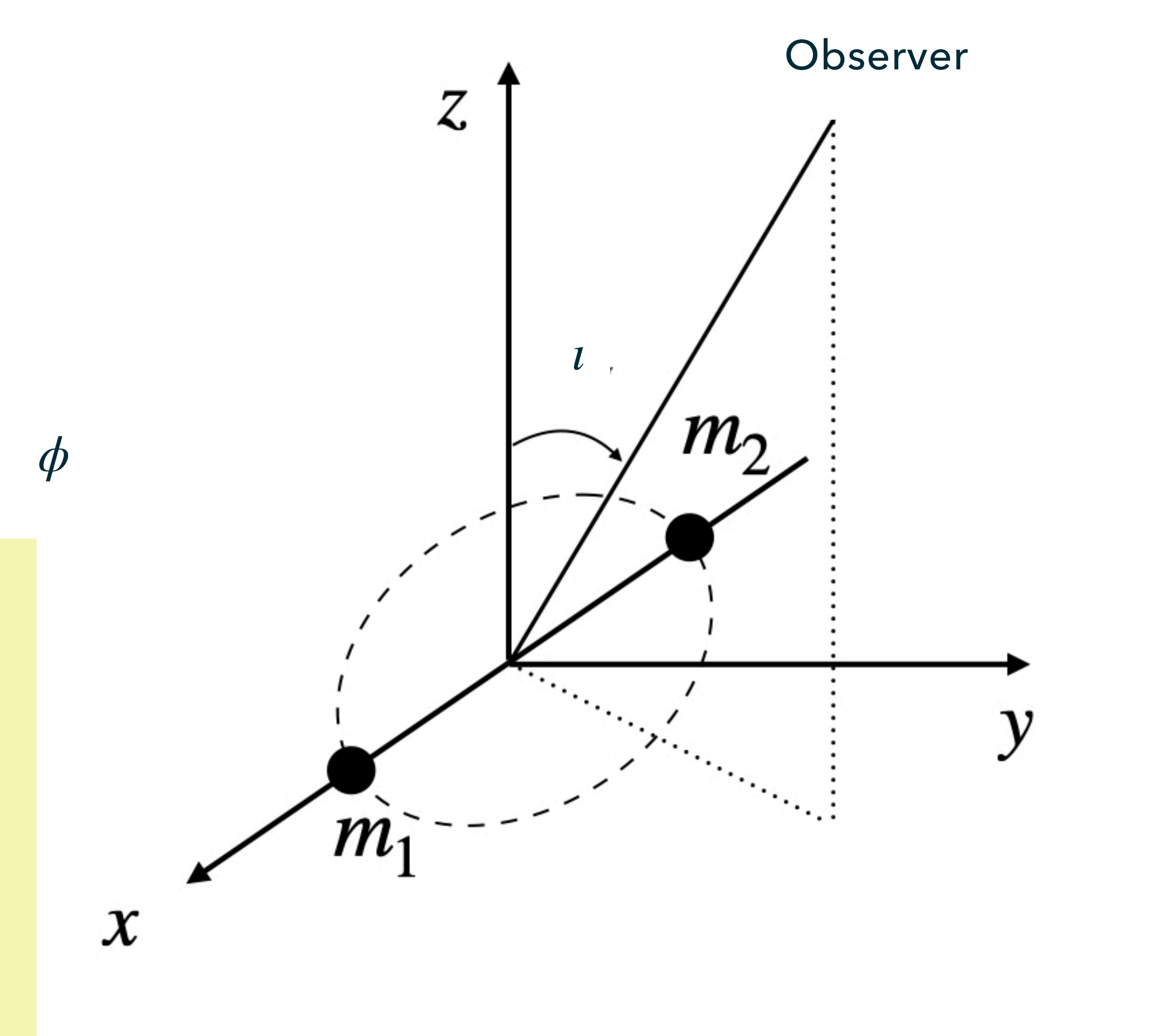
Polarization angle

Coleascence Phase

- Duration of the observations from frequency f until merger $(f_{merge} \gg f)$

$$T = \frac{f}{\dot{f}} = \frac{5}{96}\pi^{-8/3}\frac{c^5}{\eta(GM)^{5/3}}f^{-8/3}$$





For ground based detectors, $f_0 \sim 30$ Hz
- $T < 1$ second for BBH
- $T \sim$ minute for BNS

For lower frequency detectors, T could become very long (weeks to years)

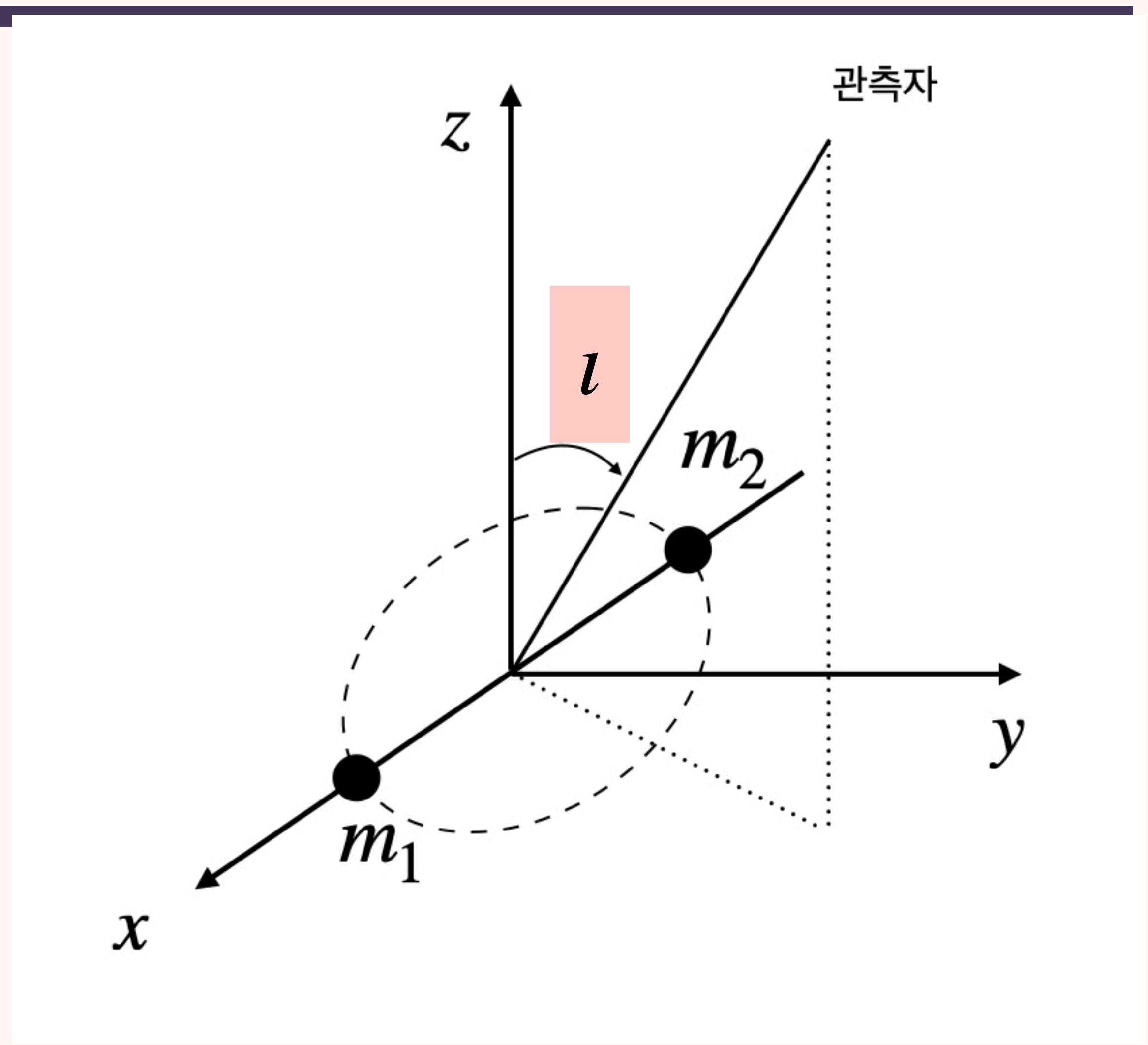
How GWs can help?

- The astronomical measurements use the 'standard candles such as Cepheids and Type Ia Supernovae
  - However, the absolute luminosities are not well known.
- For GWs from coalescing compact binaries, the shapes (Waveforms) and amplitudes can be computed from GR.
  - The waveforms does depend only on intrinsic parameters such as masses & spins
- Luminosity distances can be estimated because the amplitude of the GW signal is **inversely proportional** to the distance.

$$\boxed{\begin{aligned} d_L &= \frac{5}{96\pi^2}\frac{c}{h}\frac{\dot{f}_{GW}}{f_{GW}^3} \\ &= 512\frac{1}{h_{21}}\left(\frac{0.01s}{\tau}\right)\left(\frac{100\text{Hz}}{f_{GW}}\right)^2 \text{ Mpc} \end{aligned}}$$

Schutz 1982, Nature

Uncertainty in Distances



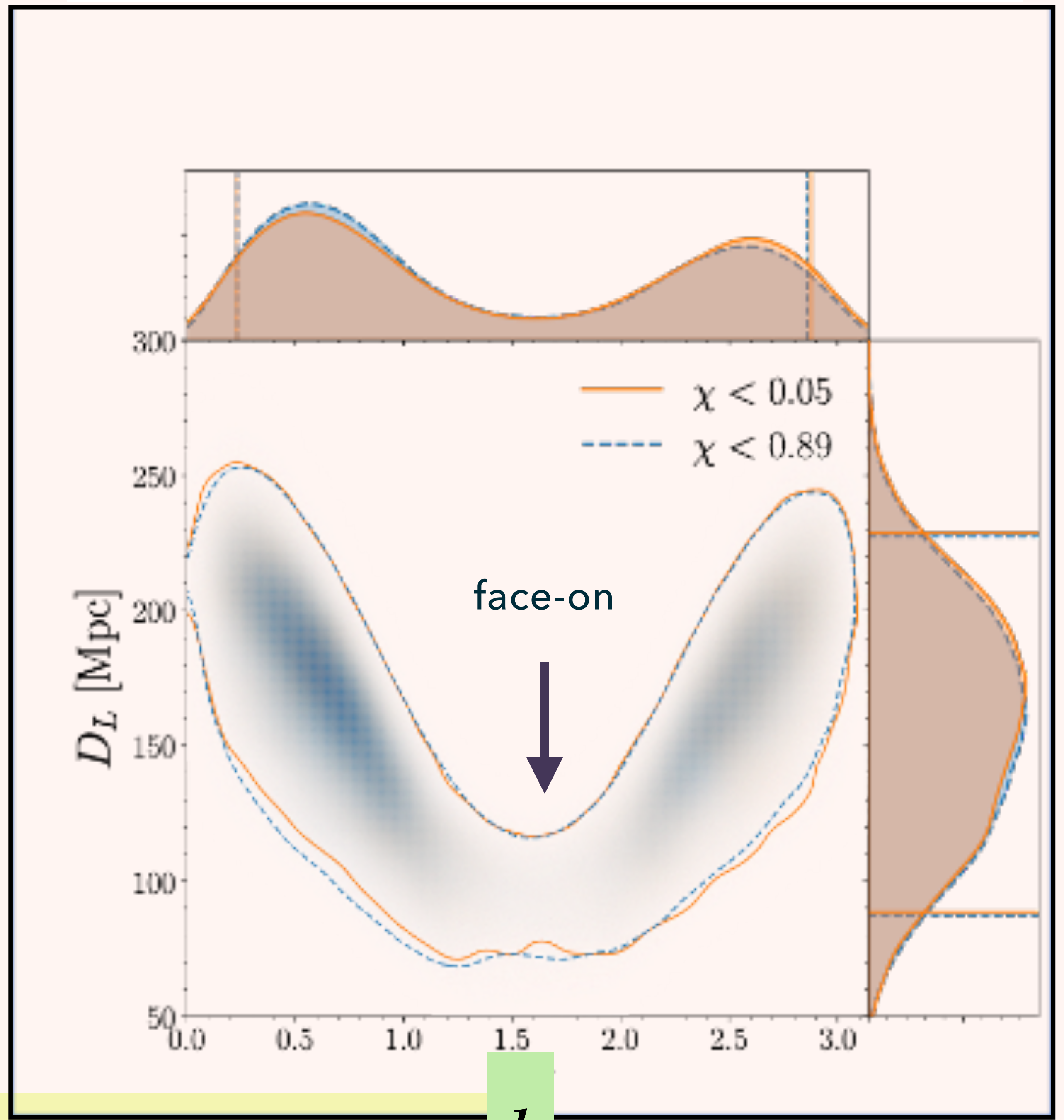
- The distance estimation from GWs does not suffer from **systematic uncertainty**
- However, individual GW distance estimation is subject to large **statistical uncertainty** due to the lack of information on the viewing angle (ι)

$$h_+ = \frac{h_c}{d}\frac{1}{2}(1+\cos^2\iota),\ h_c \equiv 2\mu(M\Omega^{2/3})\cos(2[\Omega t - \phi_0])$$

$$h_\times = \frac{h_s}{d}\cos\iota,\ h_s \equiv 2\mu(M\Omega^{2/3})\sin(2[\Omega t - \phi_0])$$

Ω : Orbital frequency

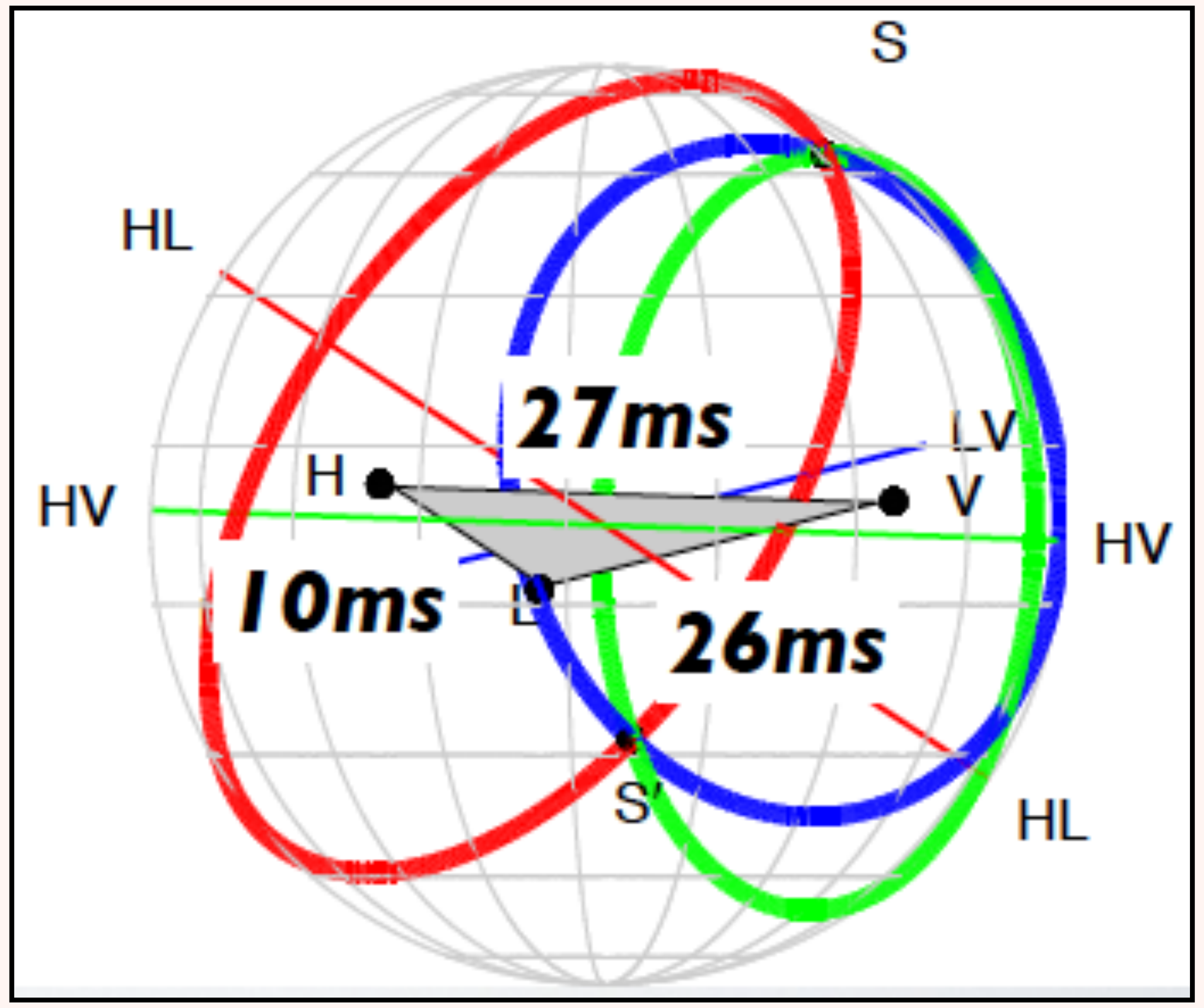
- Also, we need to determine redshift separately: identification of the host galaxy

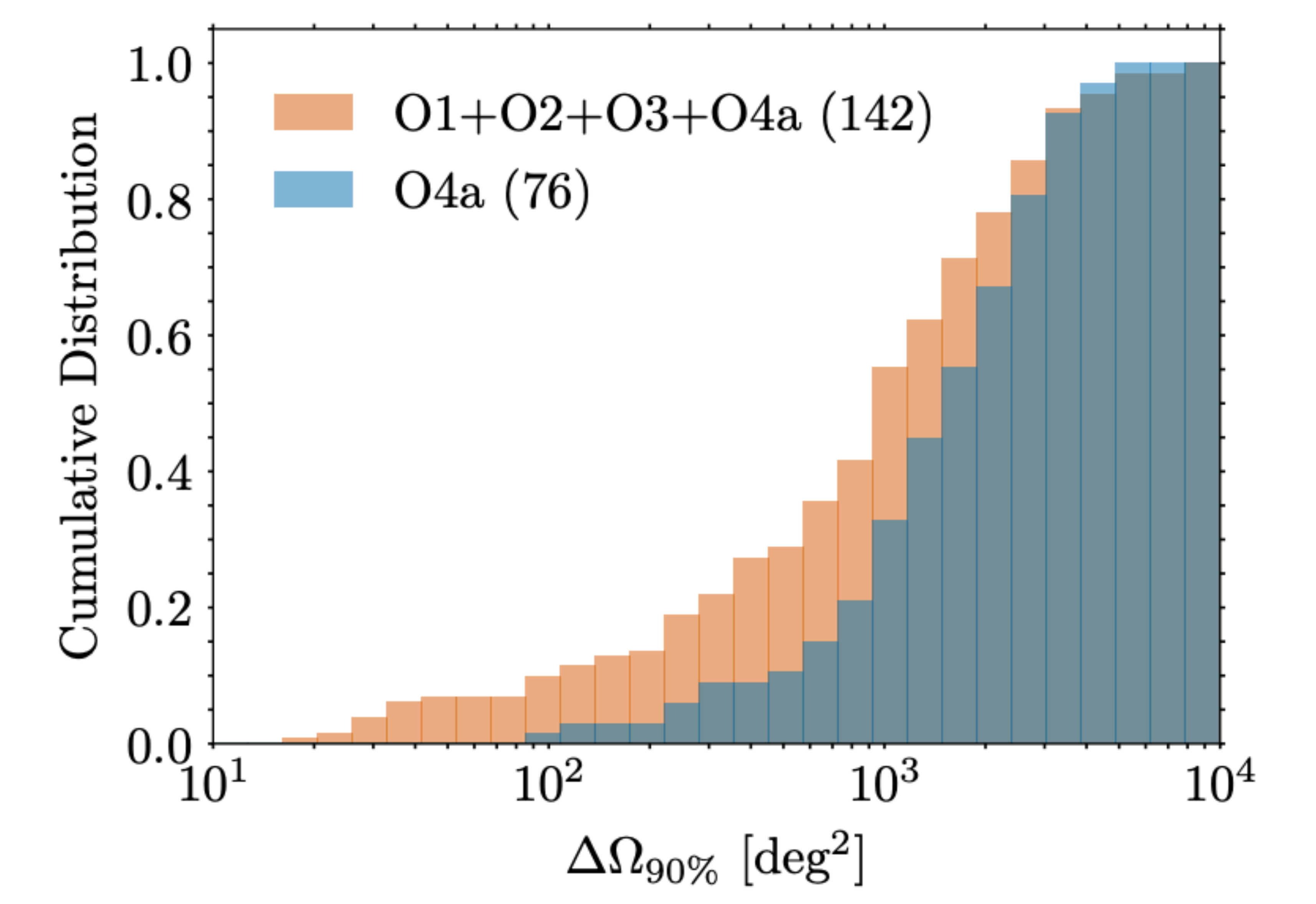


BNS GW190425(Abbott et al. 2020, ApJL, 892, L3)

Bright Sirens, Dark sirens

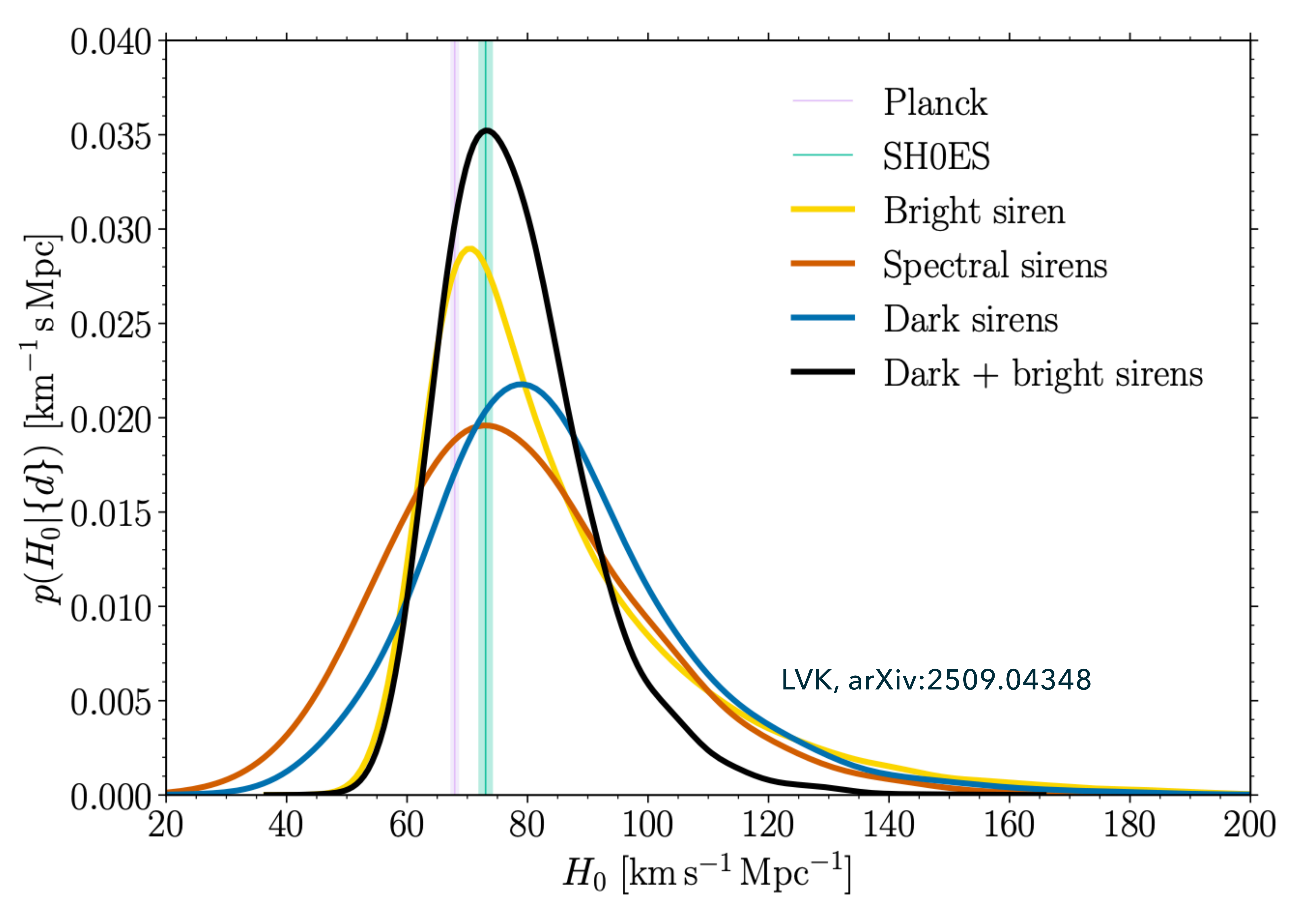
- Identification of host galaxies to determine redshift is difficult
  - **Bright Sirens**: BNS or BH-NS with optical counterparts (so far only GW170817)
  - **Dark Sirens**: BBH or BNS/NS-BH with no observed optical counterparts.
    - One can still use dark sirens to constrain cosmological parameters statistically.
  - Currently, the accuracy in sky localization is rather poor





LVK, arXiv:2509.04348

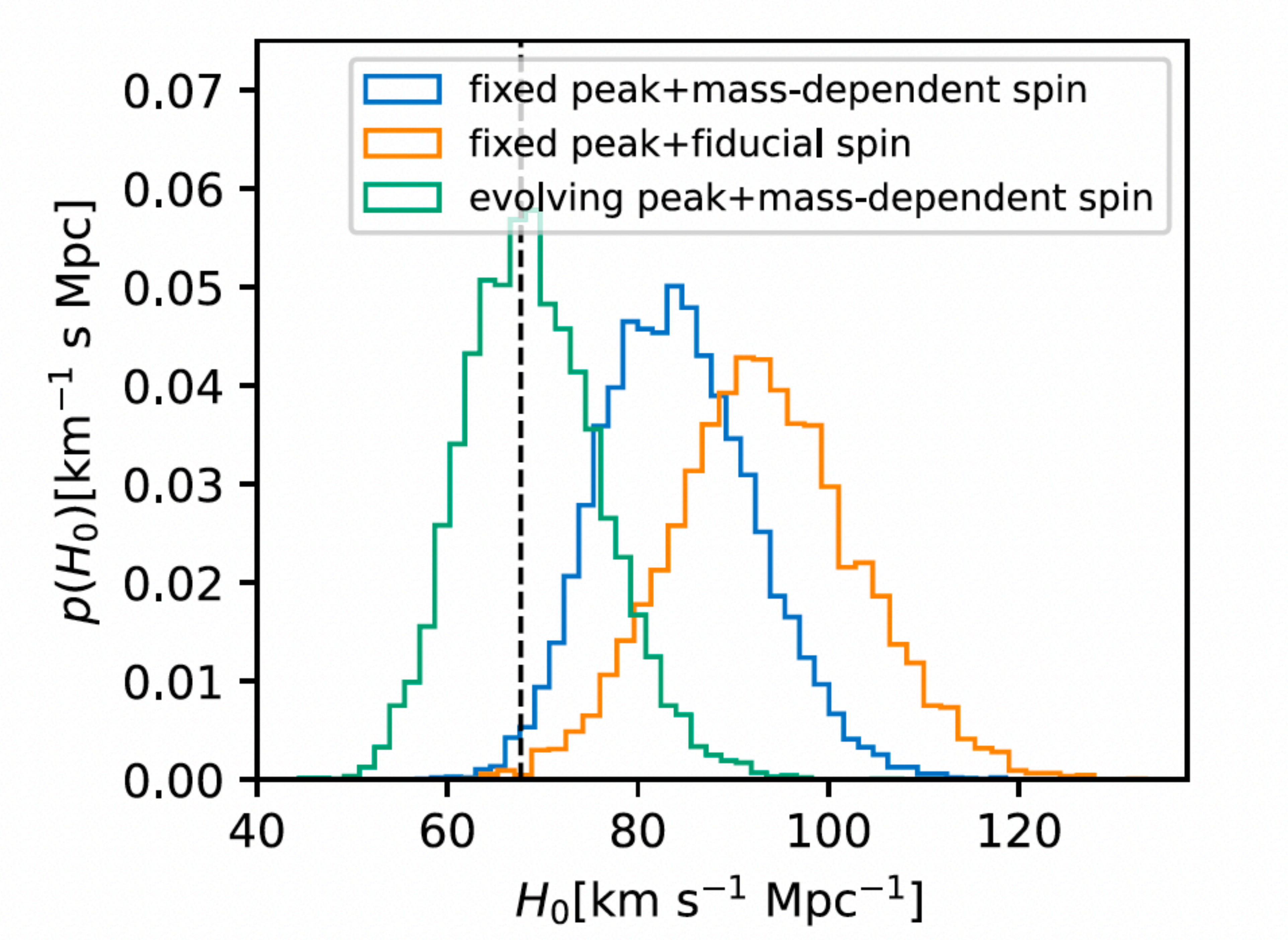
Measurements of H_0



- Accuracy of Hubble constant can be improved if we have many multi-messenger BNS events.
  - Dark sirens: use the galaxies within the localization area
  - Spectral Sirens: next page

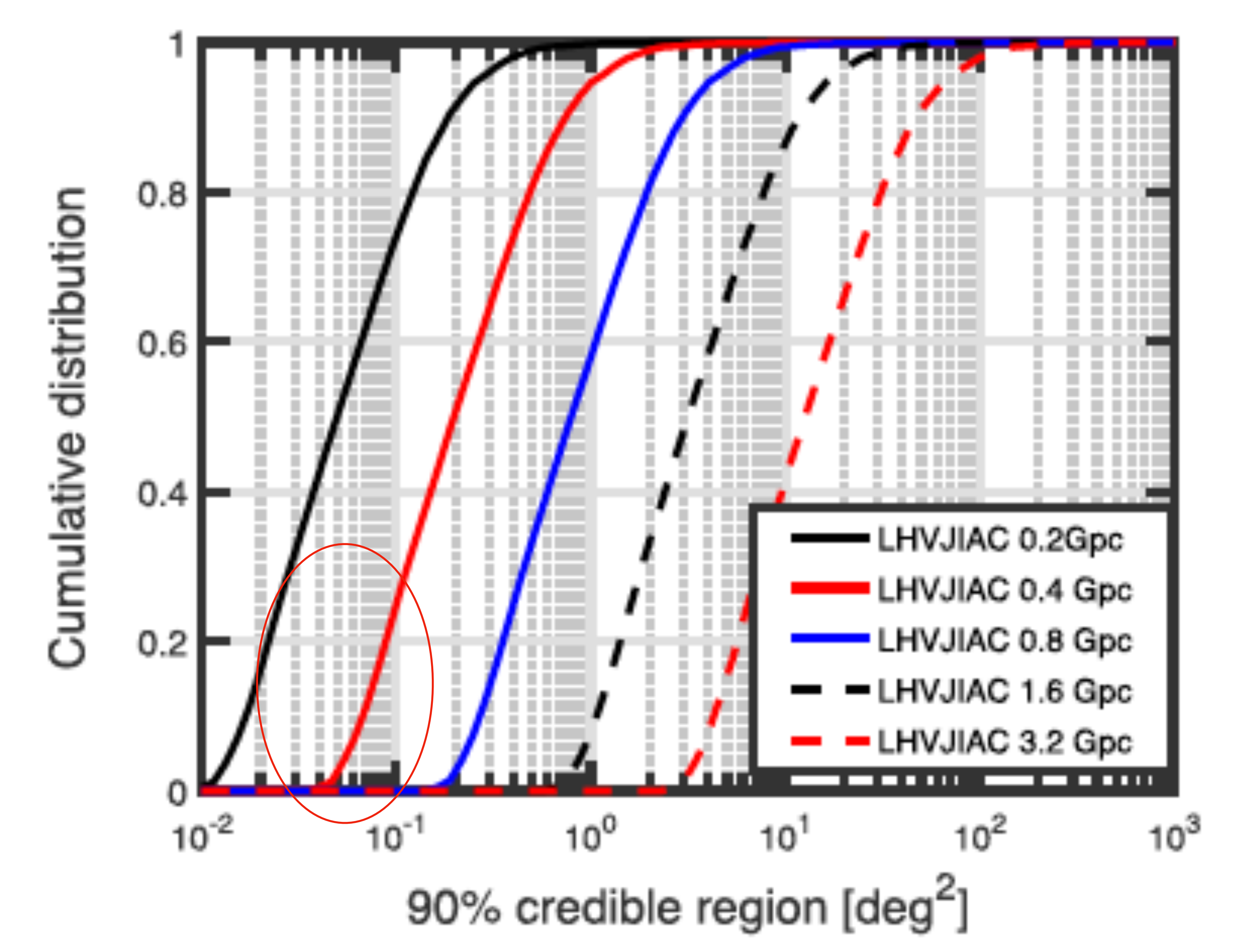
Spectral siren

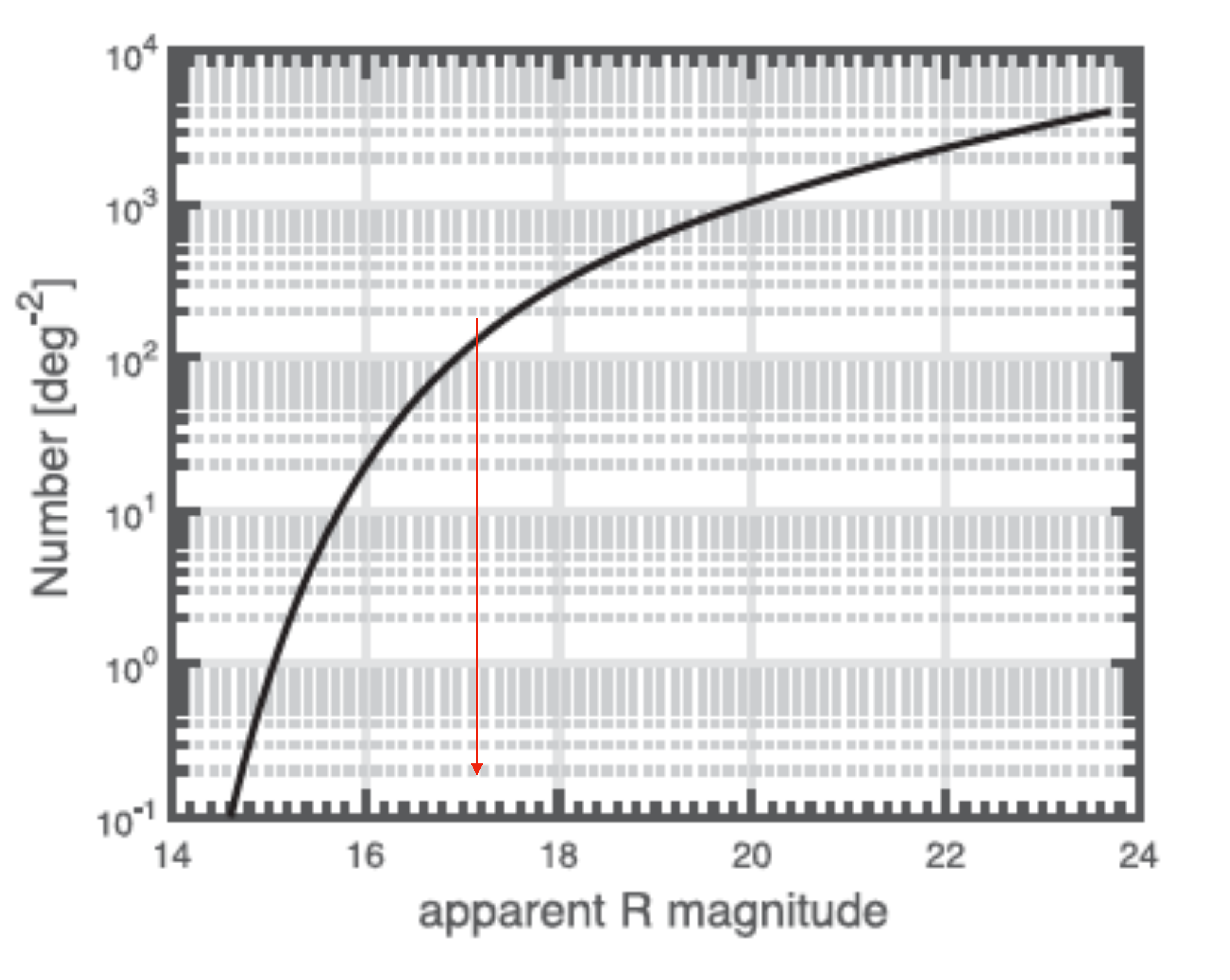
Explicitly factoring the population model in terms of detector-frame masses, luminosity distances, and spins, multiple GW source parameters carry information about cosmology.



Tong, Fishbach & Thrane 2025, ApJ

Localization accuracy increases with multiple detectors





Milky Way Galaxy at 0.4 Gpc

Howell, … Lee, et al. 2018, MNRAS

- With network of 5 advanced detectors and two additional detectors with better sensitivity, ~20% of the BBHs at 400 Mpc can be localized within 0.1 sq. deg.
- There could be 10-20 milky-way like galaxies within the 0.1 sq. deg.
- **BBH host identification by ground-based detectors is very challenging!**

In-band Duration could be important

- Duration of the observations from frequency f until merger ($f_{merge} \gg f$)

$$T = \frac{f}{\dot{f}} = \frac{5}{96}\pi^{-8/3}\frac{c^5}{\eta(GM)^{5/3}}f^{-8/3}$$

For ground based detectors, $f_0 \sim 30$ Hz

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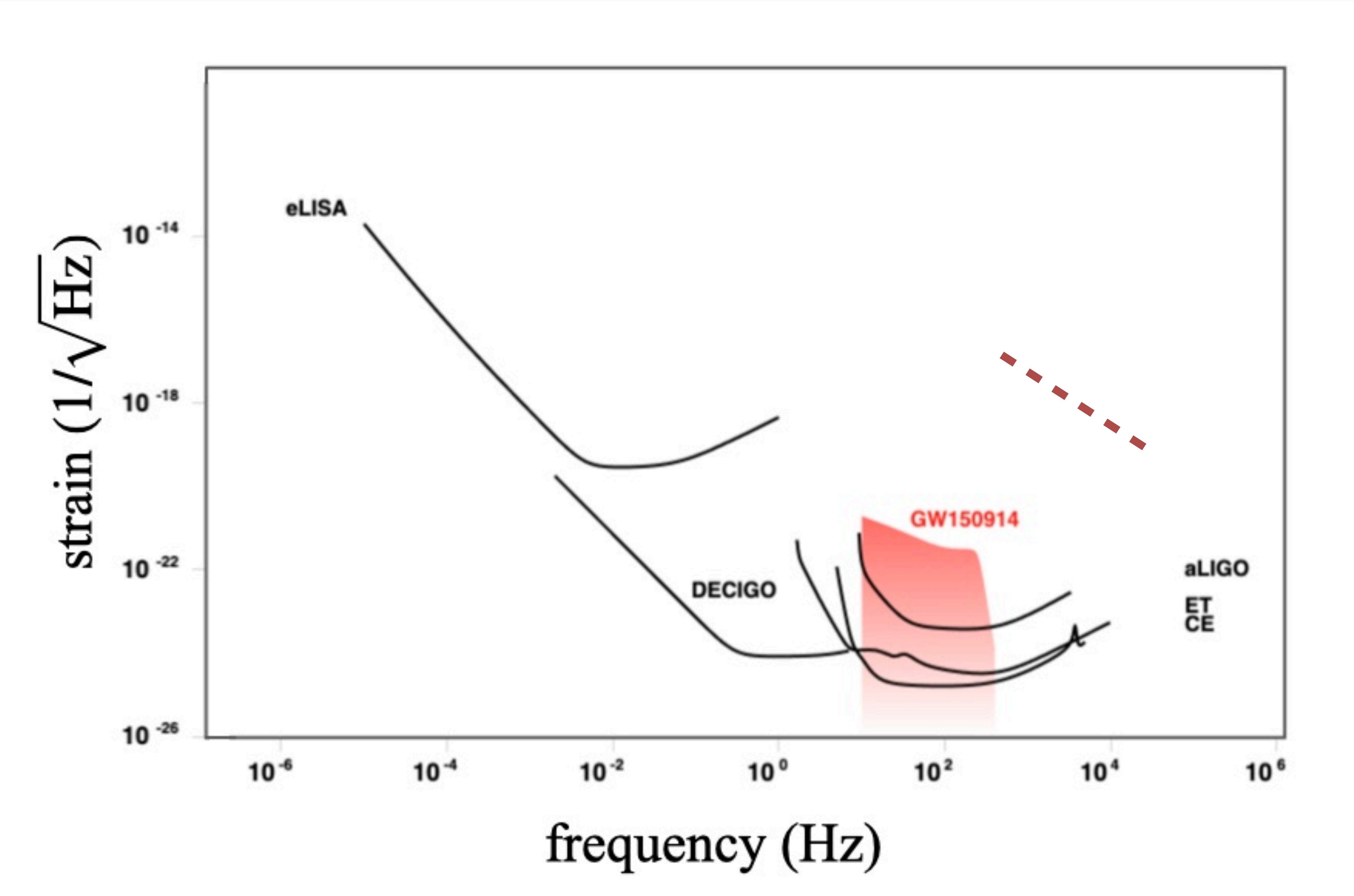
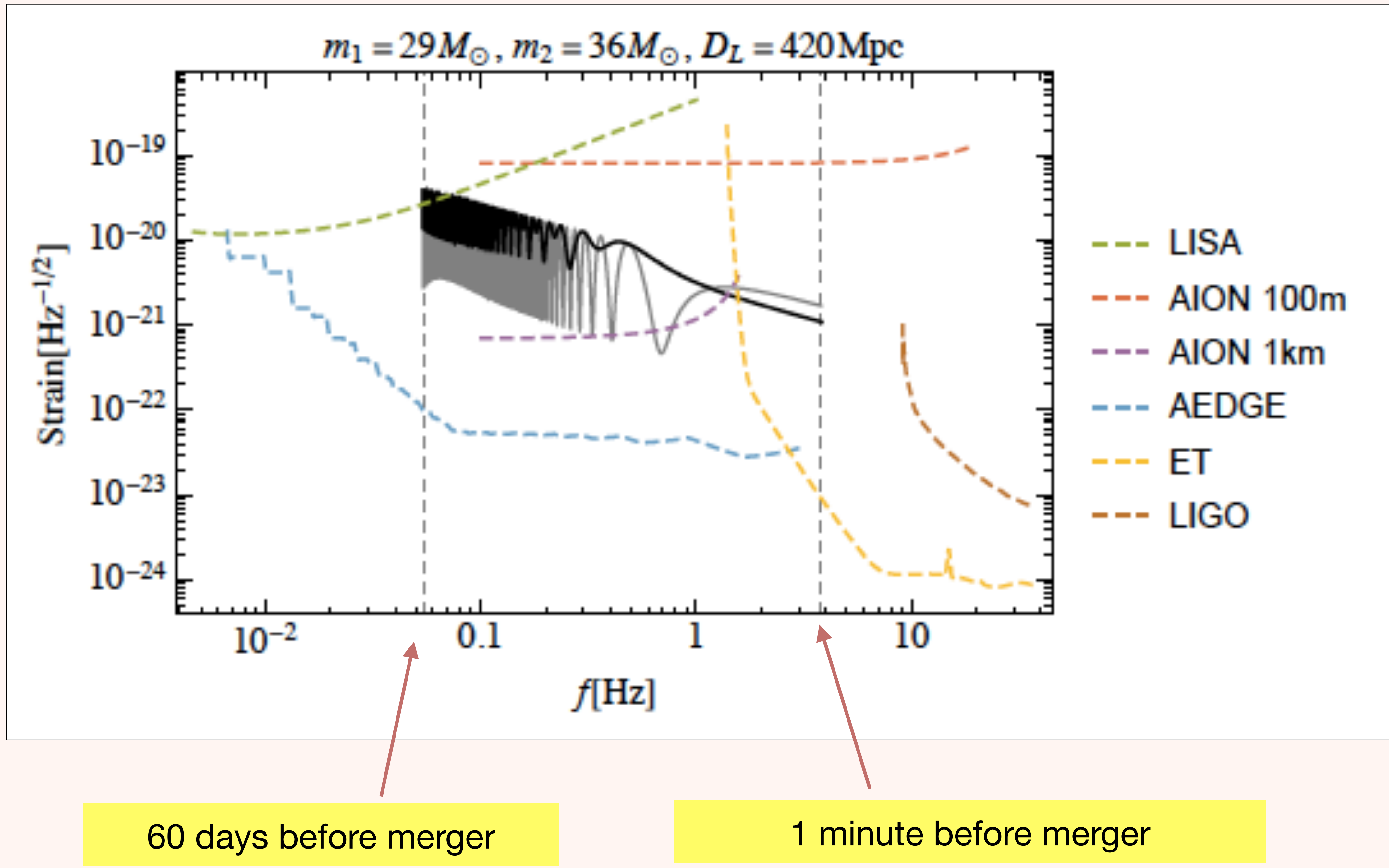


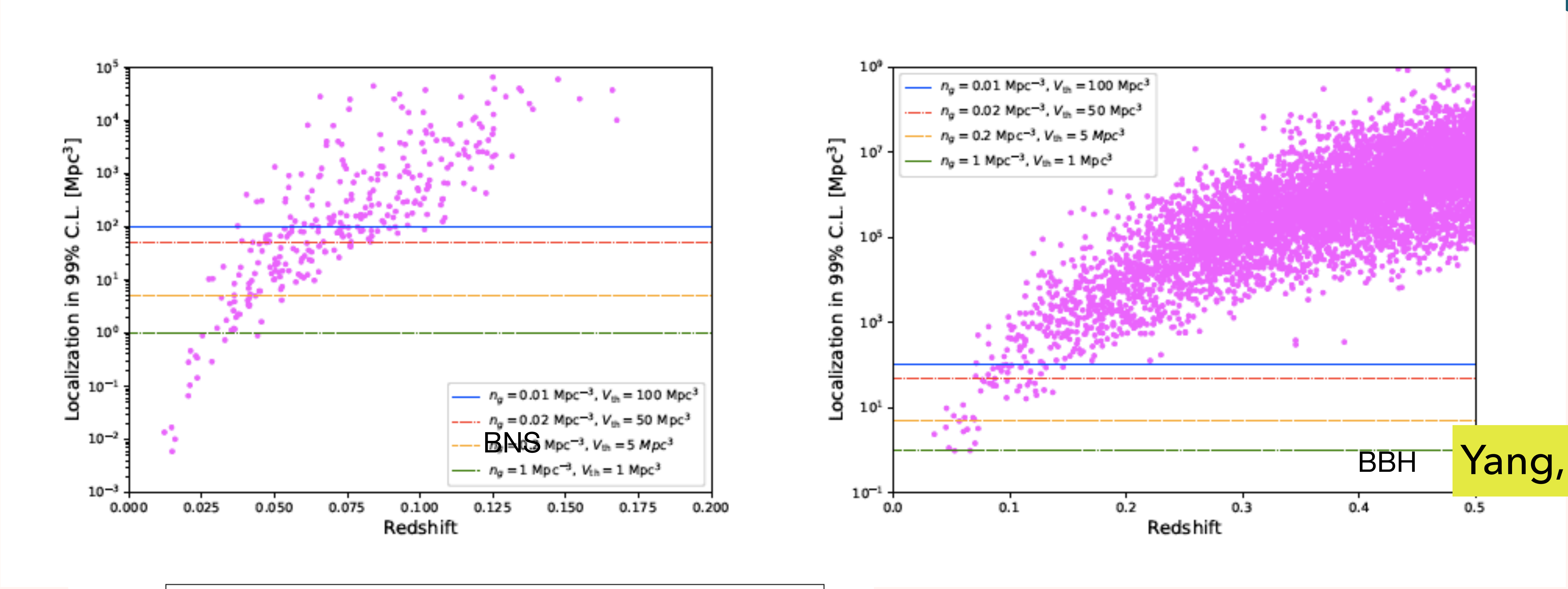
Dark Siren Observations with mid-frequency detectors

- The source position and inclination angle are encoded in the measured signal through
  - Relative amplitudes and phases of the two polarization components,
  - **Periodic Doppler shift** imposed on the signal by the detector’s motion around the Sun,
  - Further modulation of the signal caused by the detector’s **time-varying orientation**.
- Accuracies of Ω and d_L can be significantly improved

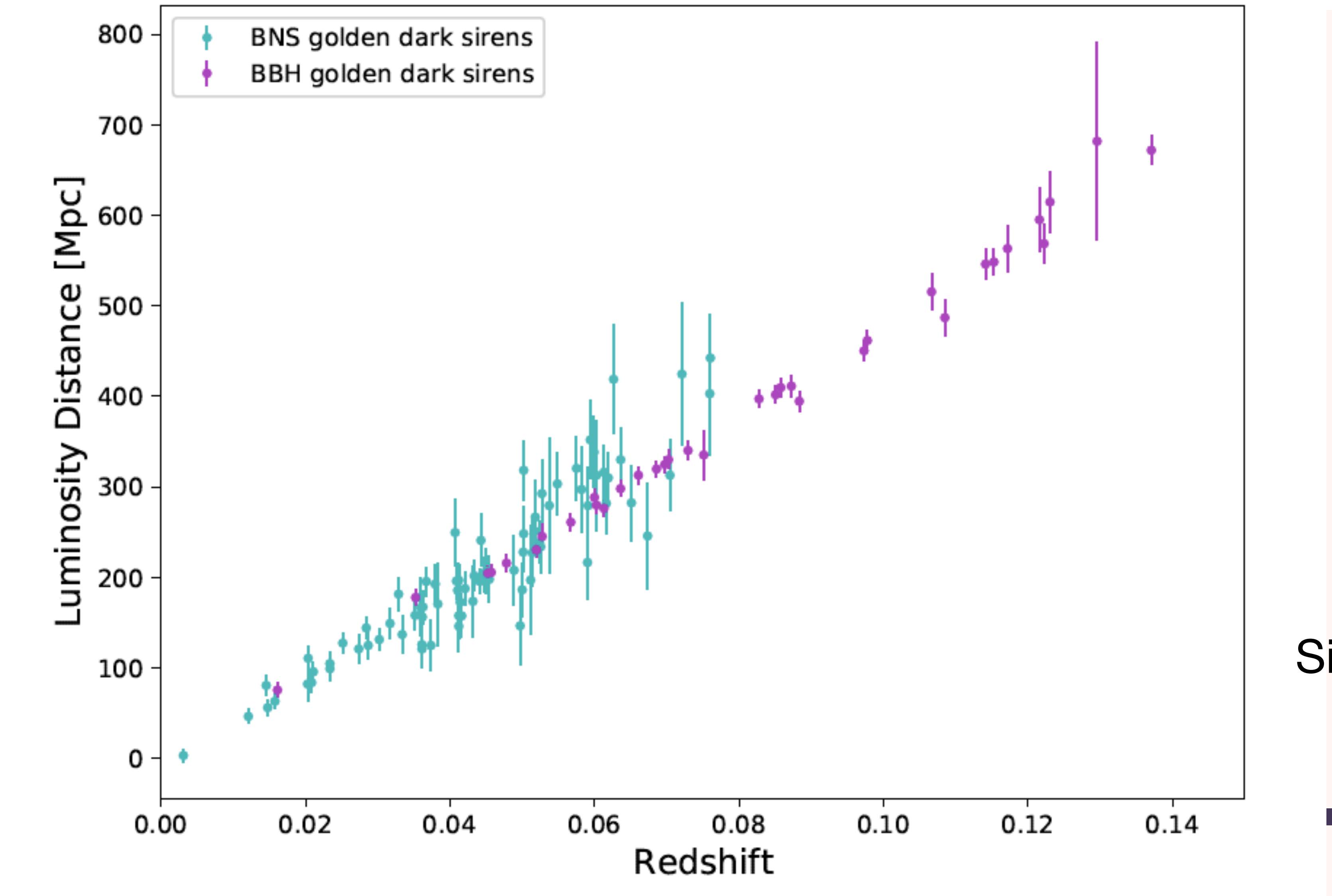


Yang, Lee et al, 2022, JCAP

Localization volume and Hubble Diagram



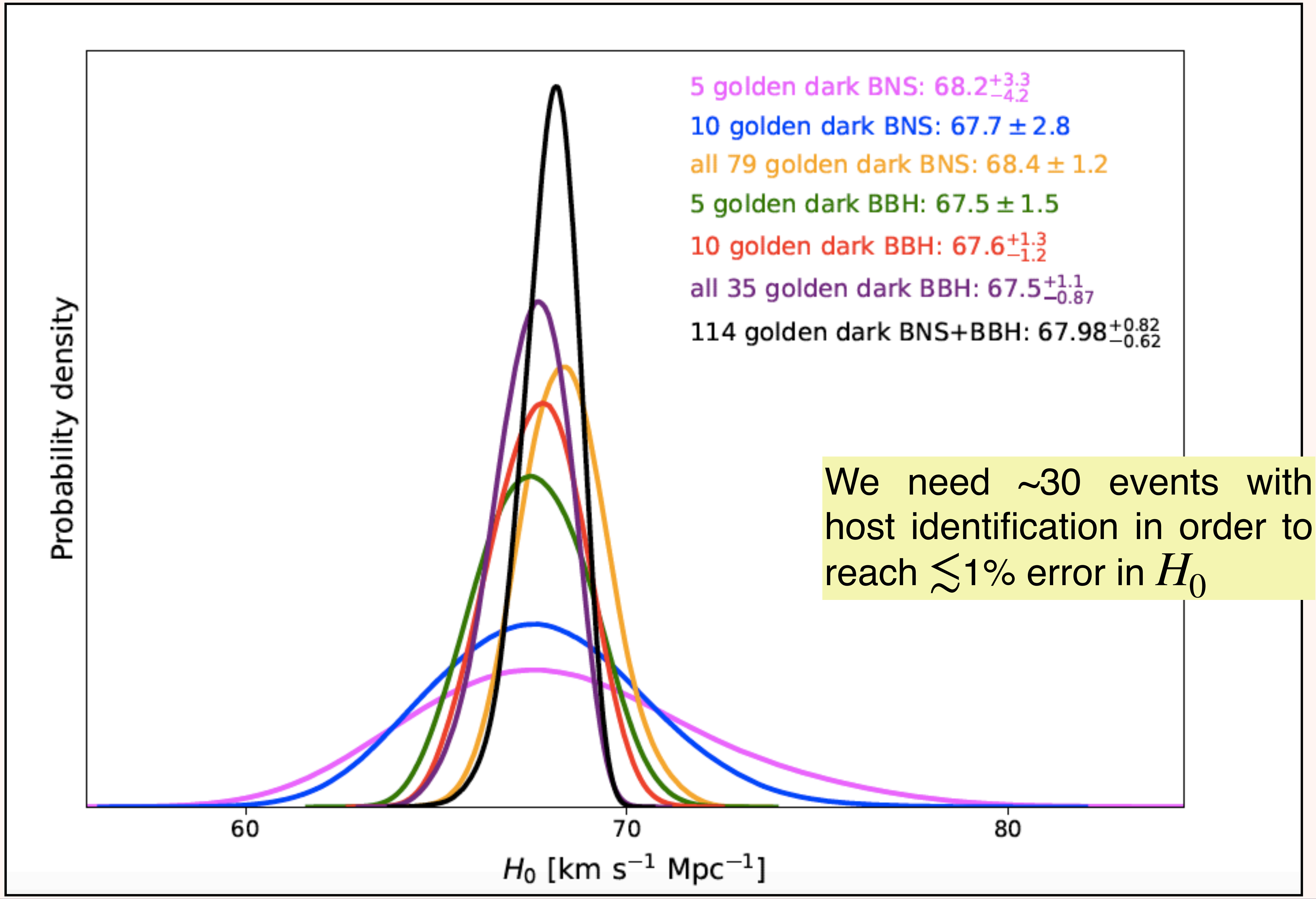
Yang, Lee et al, 2022, JCAP



Various cuts are assumed galaxy number densities: below these lines, we can uniquely identify host galaxies within 5 year observation

Simulated Hubble Diagram

Simulation of Hubble Constant Estimation with Dark Sirens

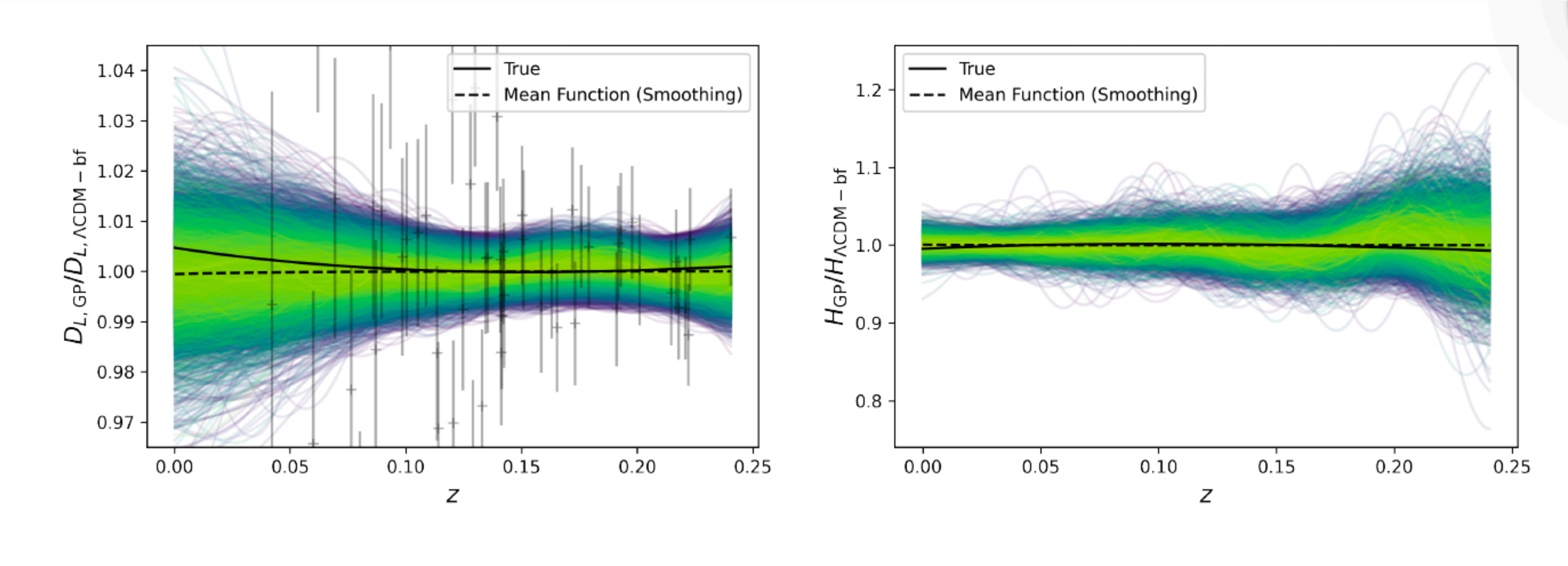


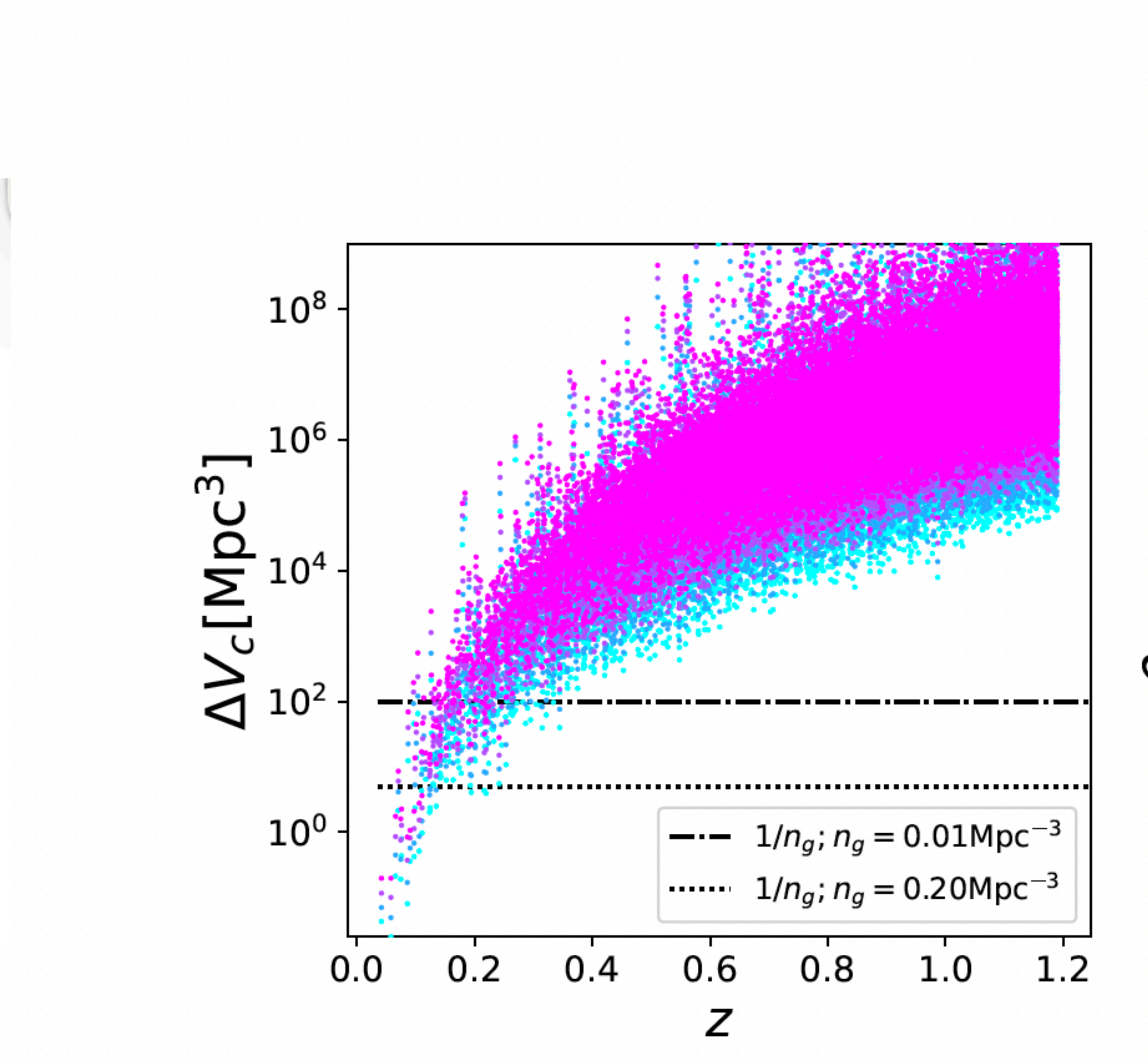
We need ~30 events with host identification in order to reach $\lesssim$1% error in H_0

Reconstruction of the Expansion History

- With ~50 'golden events' expansion history can be accurately reconstructed without parameterization (CPL, LCDM...)

Chatier et al., in preparation

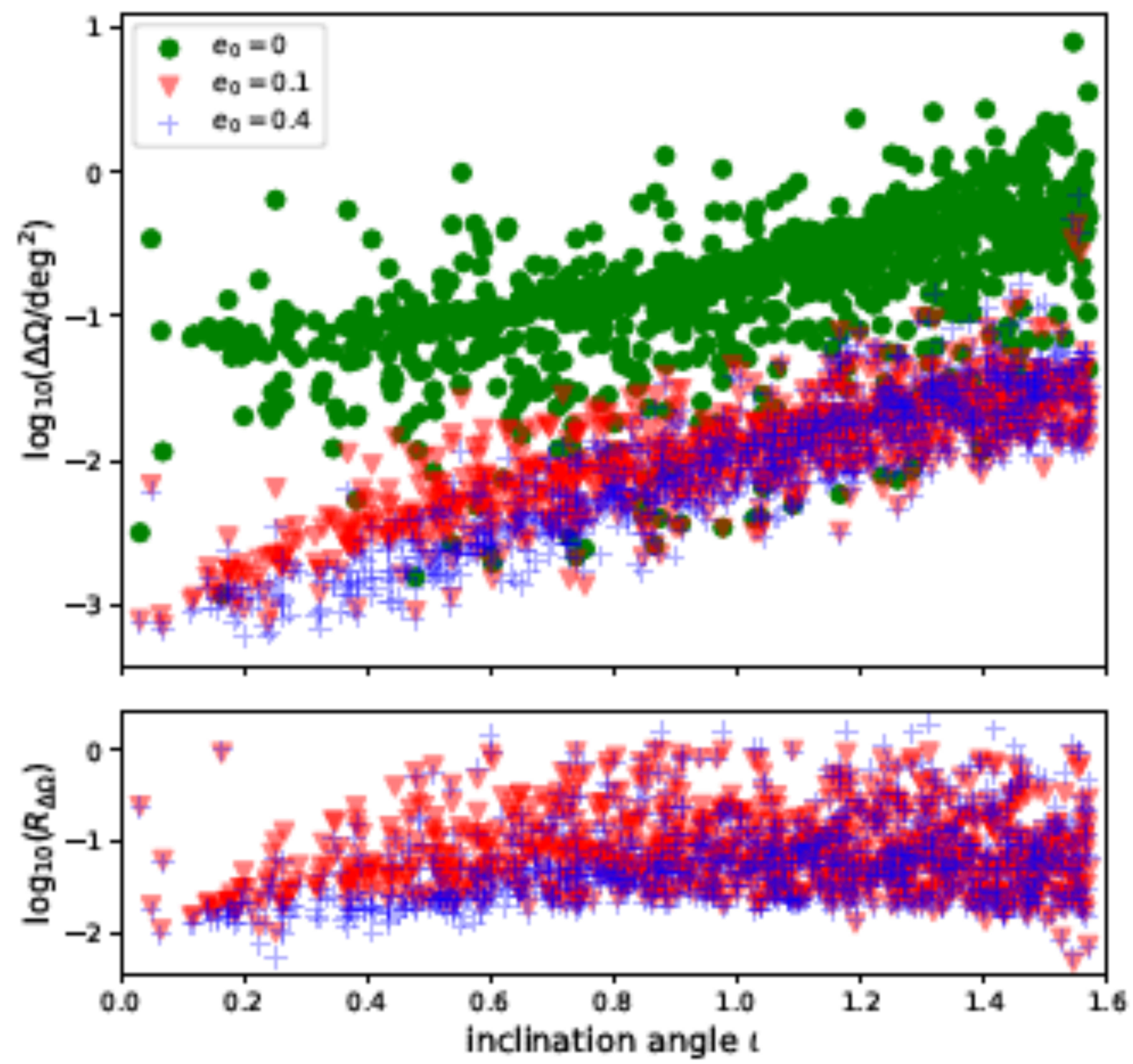




Further improvements of estimated parameters for eccentric binaries

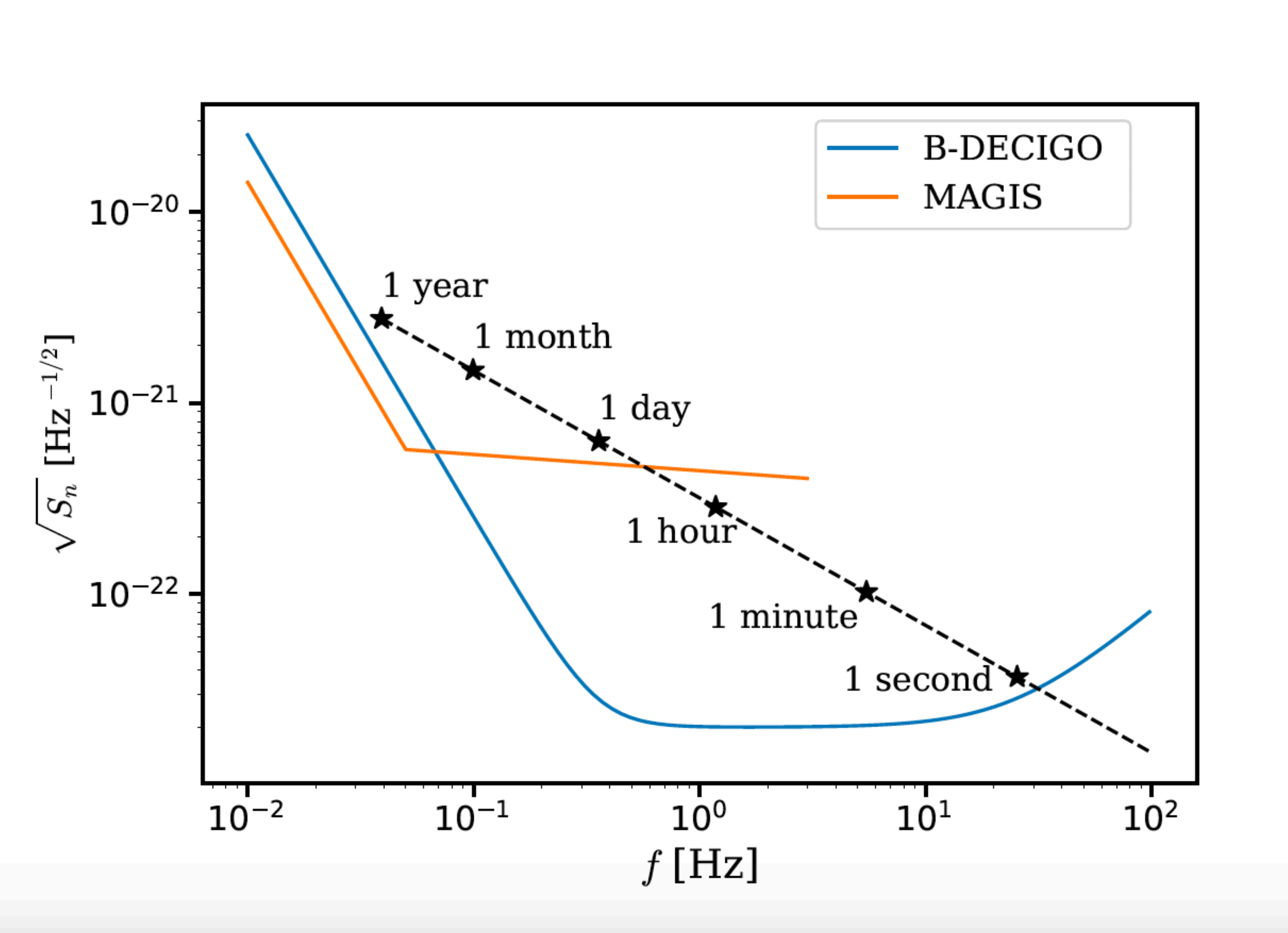
Yang, Cai. Cao & Lee, 2022, PRL

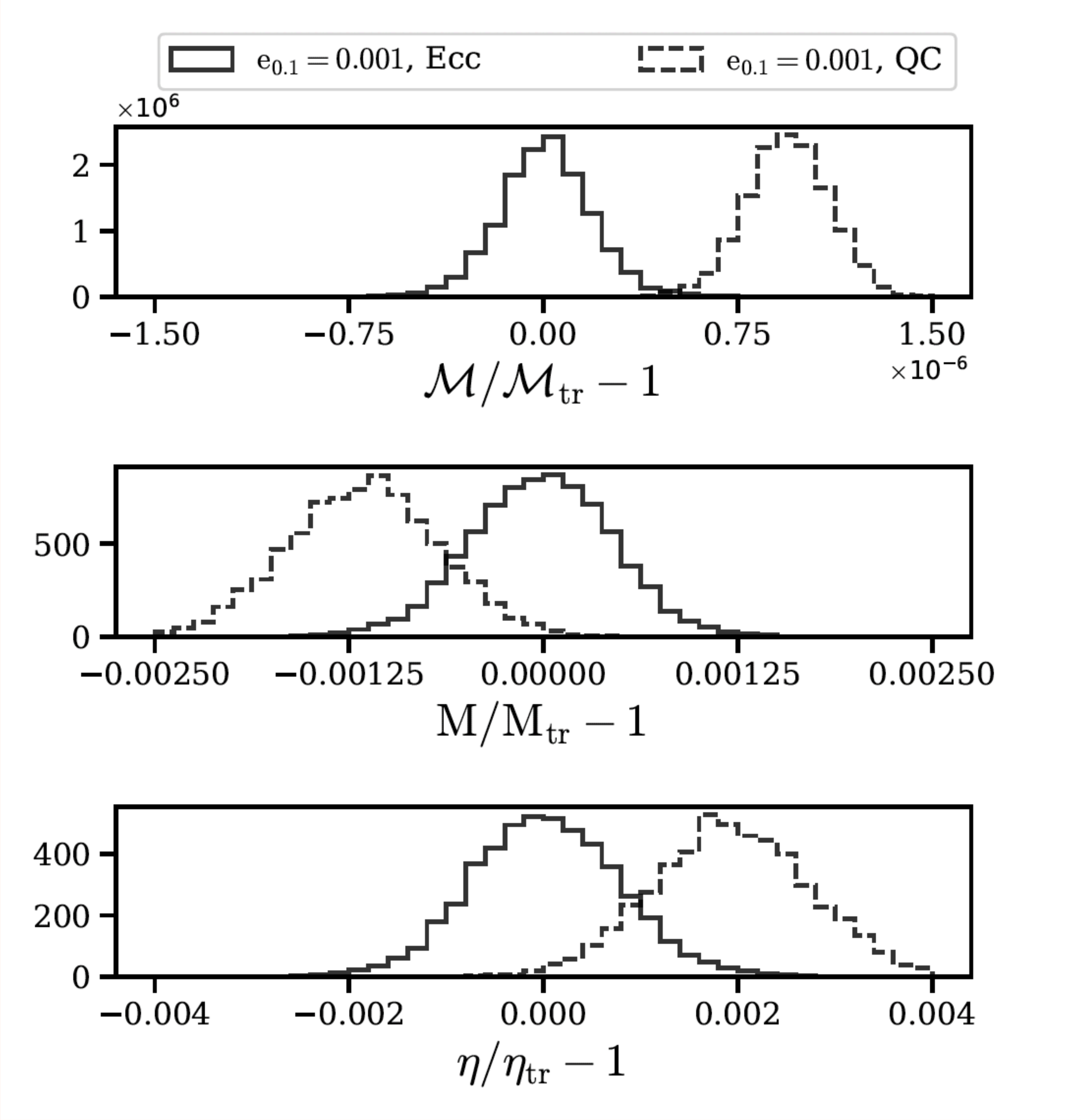
- In mid-frequency band, some binaries may have significant eccentricity (i.e., $e > 0.1$)
- The eccentric waveforms have more features than circular ones, and thus enable us to break some of the degeneracies during the inspiral phase → more accurate parameters can be inferred
- A case study with B-DECIGO:
  - $\Delta d_L / d_L$ can be improved near $\iota = 0$.
  - $(\Delta\Omega)_{e=0.1} \lesssim (\Delta\Omega)_{e=0}$
  - More improvement for larger e.



Systematic Errors due to uncertain waveforms (Choi, Yang & Lee, 2024, PRD

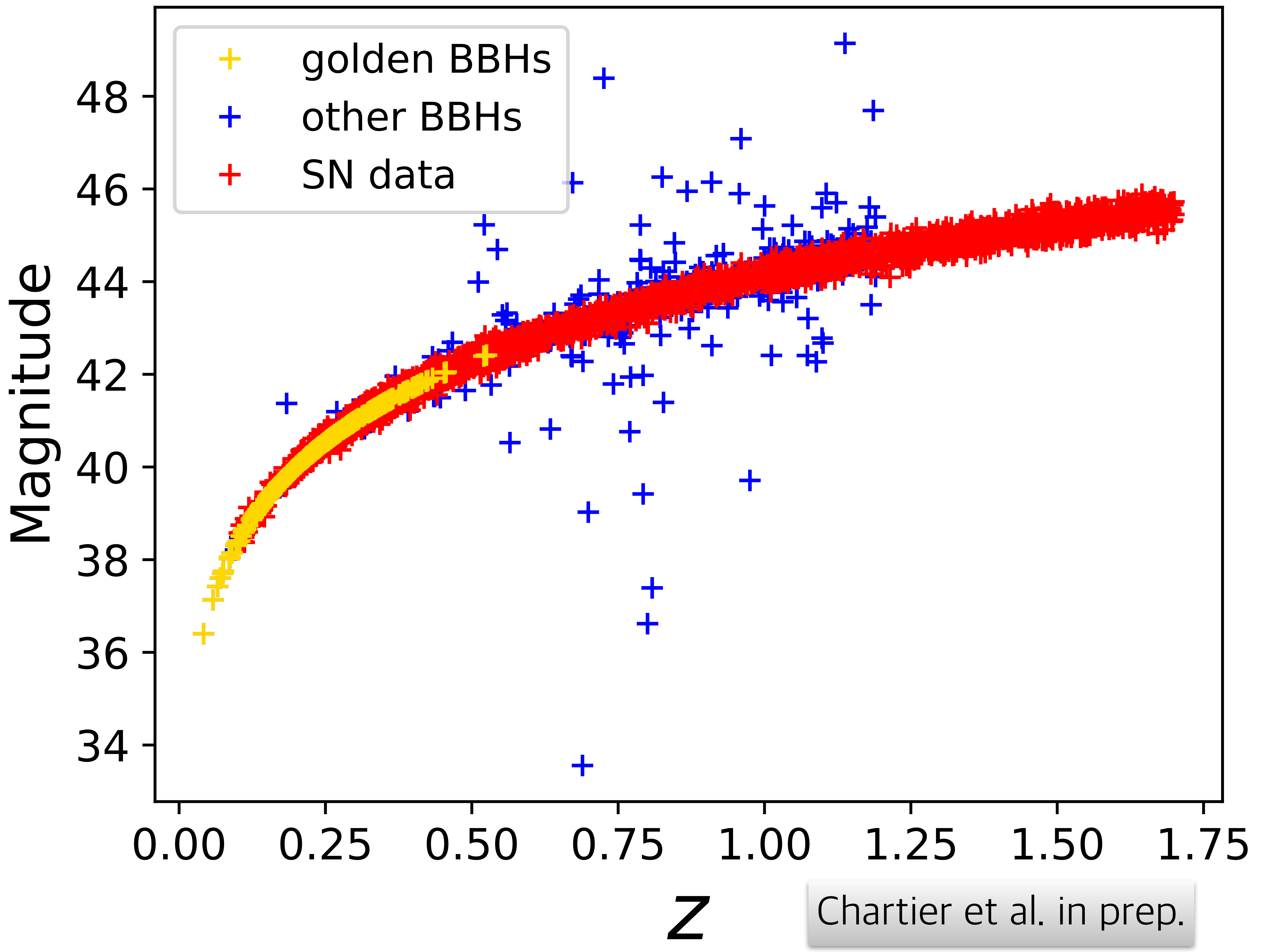
- We made detailed parameter estimation for assumed B-DECIGO and MAGIS Detectors for GW150914-like BBH
- The injected waveform is eccentric model (TaylorF2ECC) but analyzed with Quasi-circular model
- Systematic Errors in mass estimations (thus eventually distance) greater than statistical errors arise.





Hubble Diagram with GW Observations

- Sensitive GW detectors allow us to accurately constrain $z - d_L$ relation within z<0.5



Chartier et al. in prep.

Summary

- Hubble tension could be due to our lack of understanding of the universe or systematic error in local measurements
- GW sources can be used to make completely independent measurements of the Hubble constant.
- However, individual measurement of distances with GW sources have large random error due to the lack of information on inclination angle of the coalescing binaries.
- BBHs do not emit EM radiation. The pointing accuracy of the ground-based detectors (including the future ones) is very poor for host identification.
- Localization can be significantly improved with future detectors in miid-band (0.1 ~ 1 Hz).
  - If some binaries are eccentric, accuracies of directions and distances can be further improved.
- Cosmological parameters could be precisely constrained with dark sirens alone with mid-band detectors.