

introduction to
numerical relativity

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2026 수치상대론 및 중력과 여름학교

Simulating black holes and neutron stars on computers

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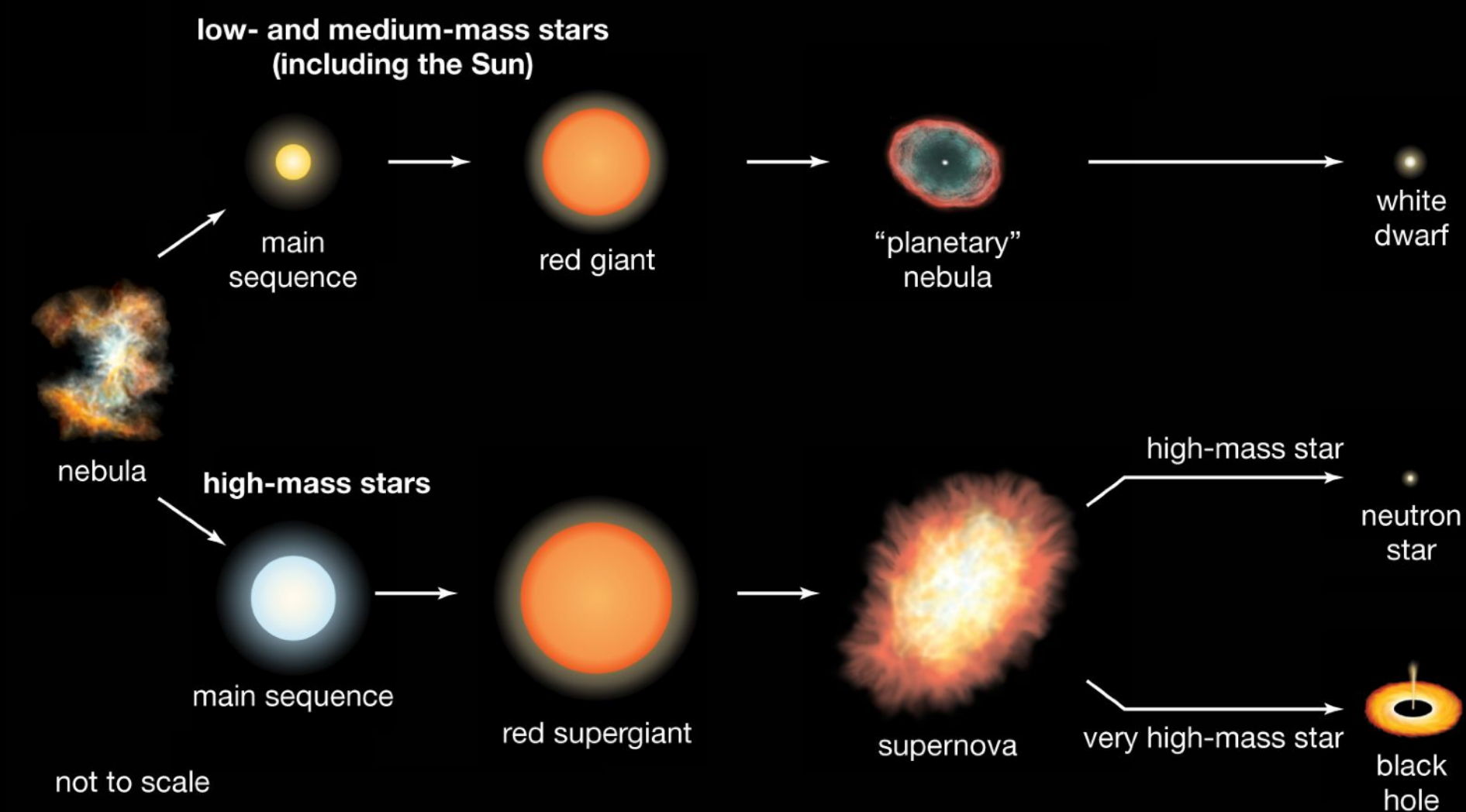
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*remark: massive BHs are not "dense"

strong-gravity compact objects:

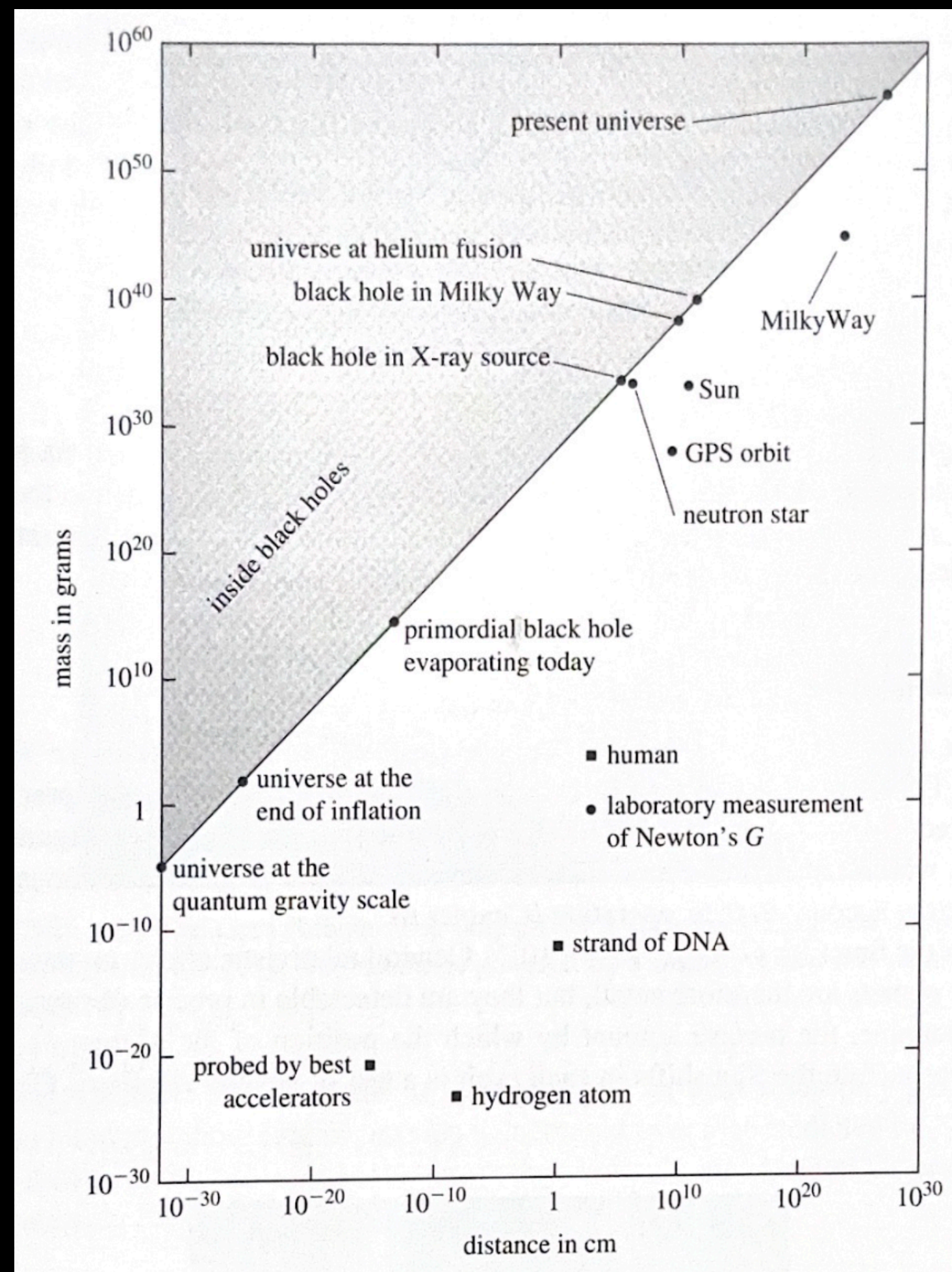
Black holes and Neutron stars

Stellar evolution



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compactness parameter: $\frac{GM}{Rc^2}$

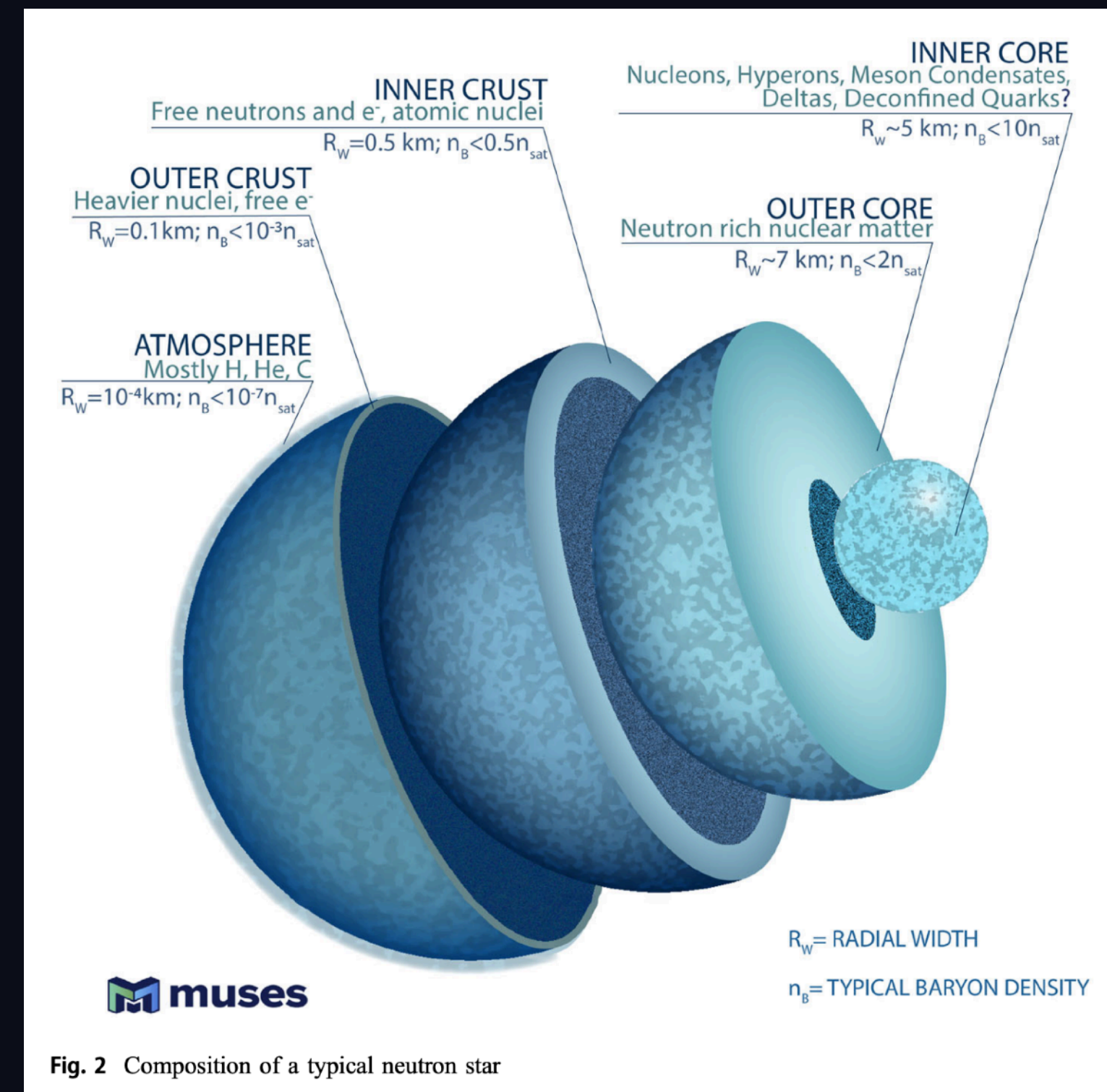


Hartle 2002

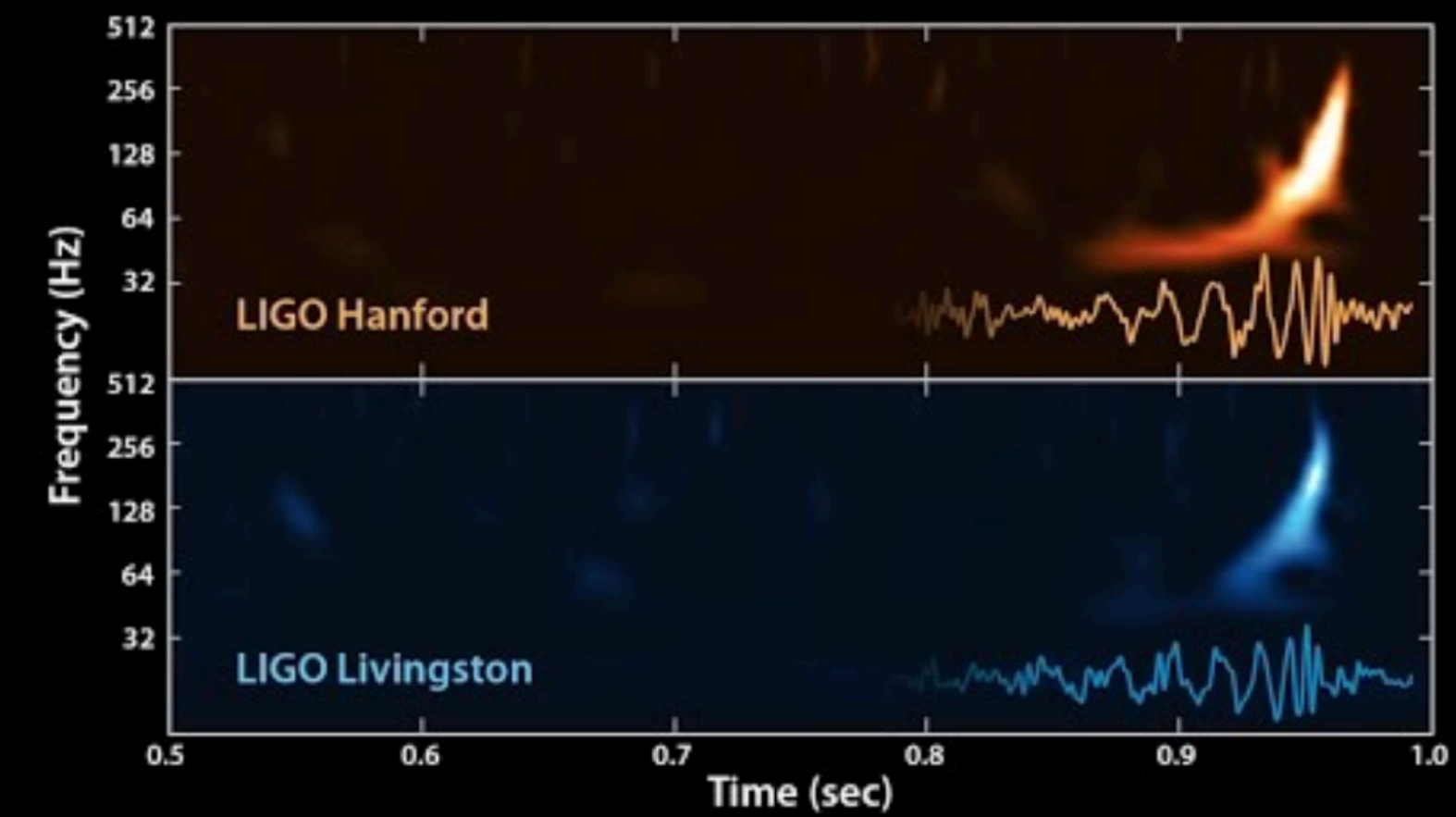
nature's laboratory for physics at extreme



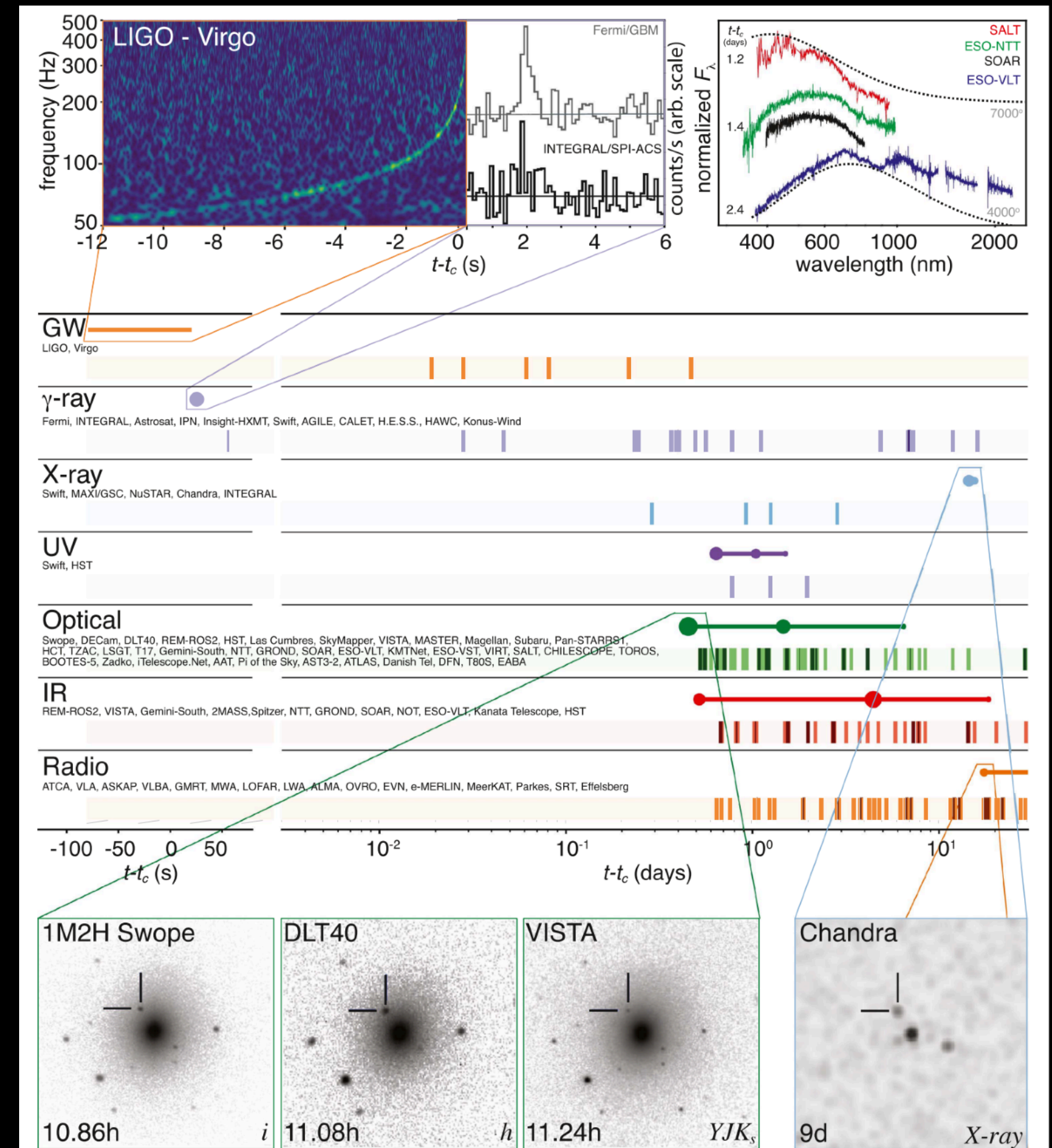
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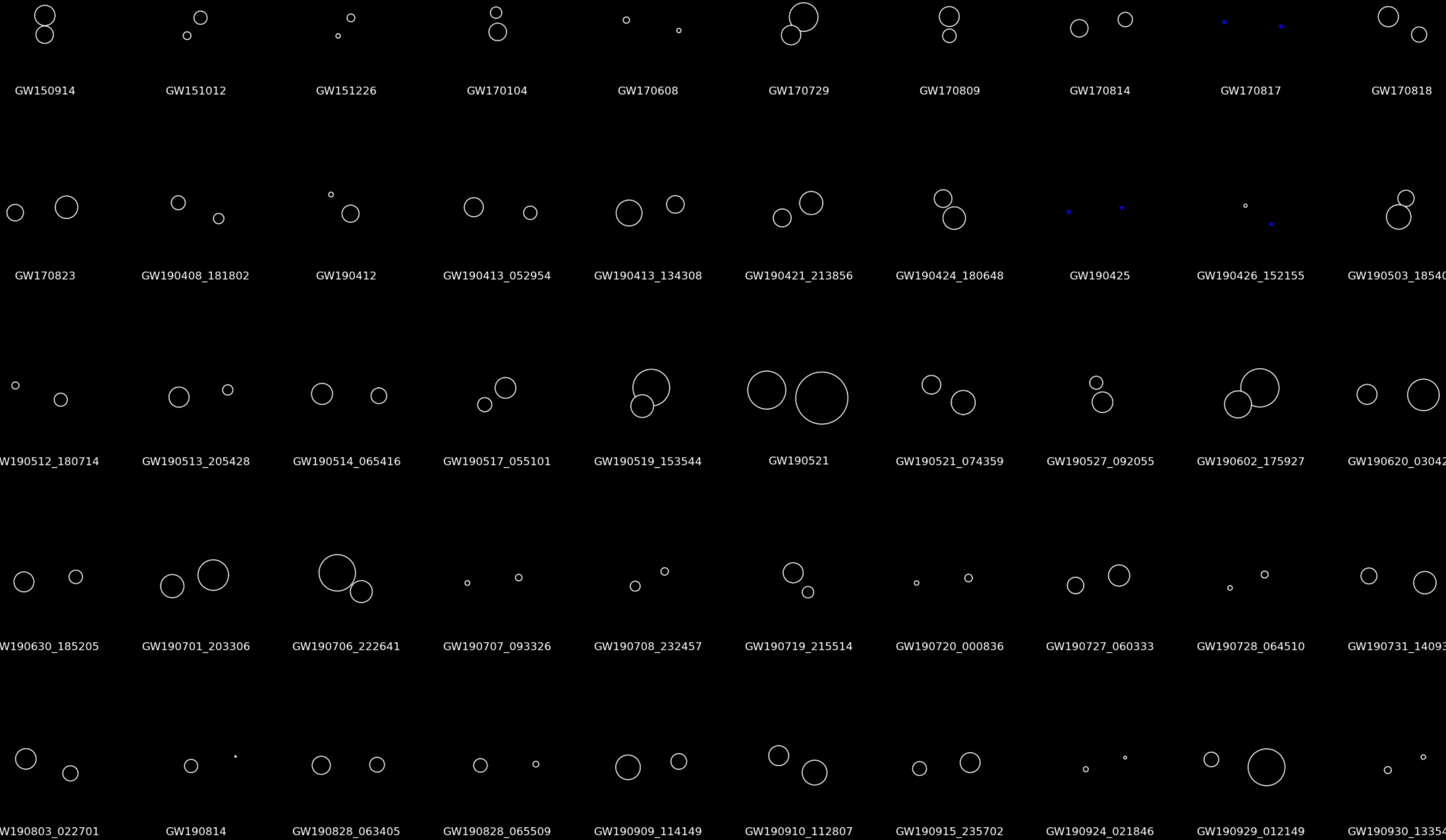
MUSES Collaboration

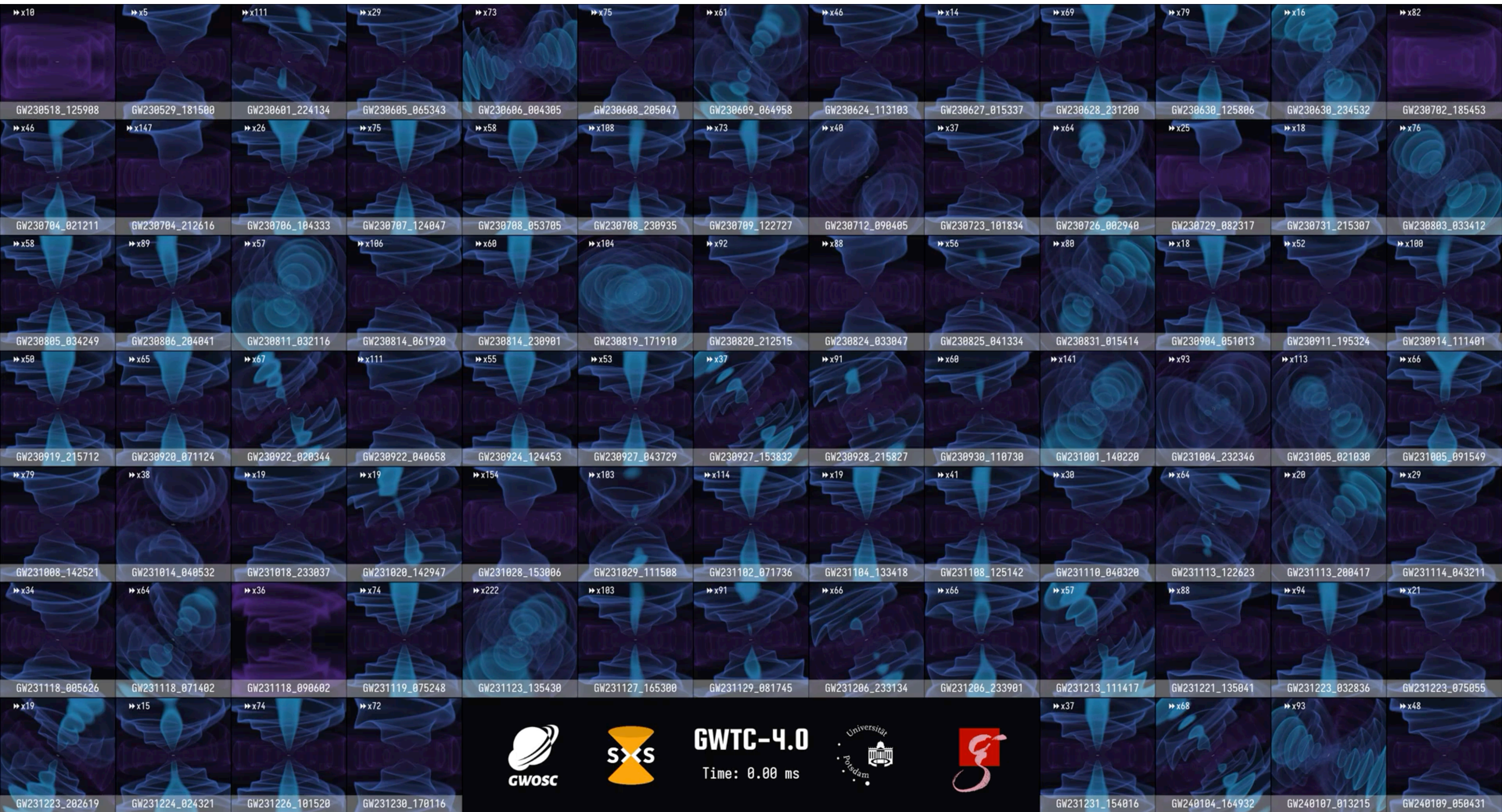


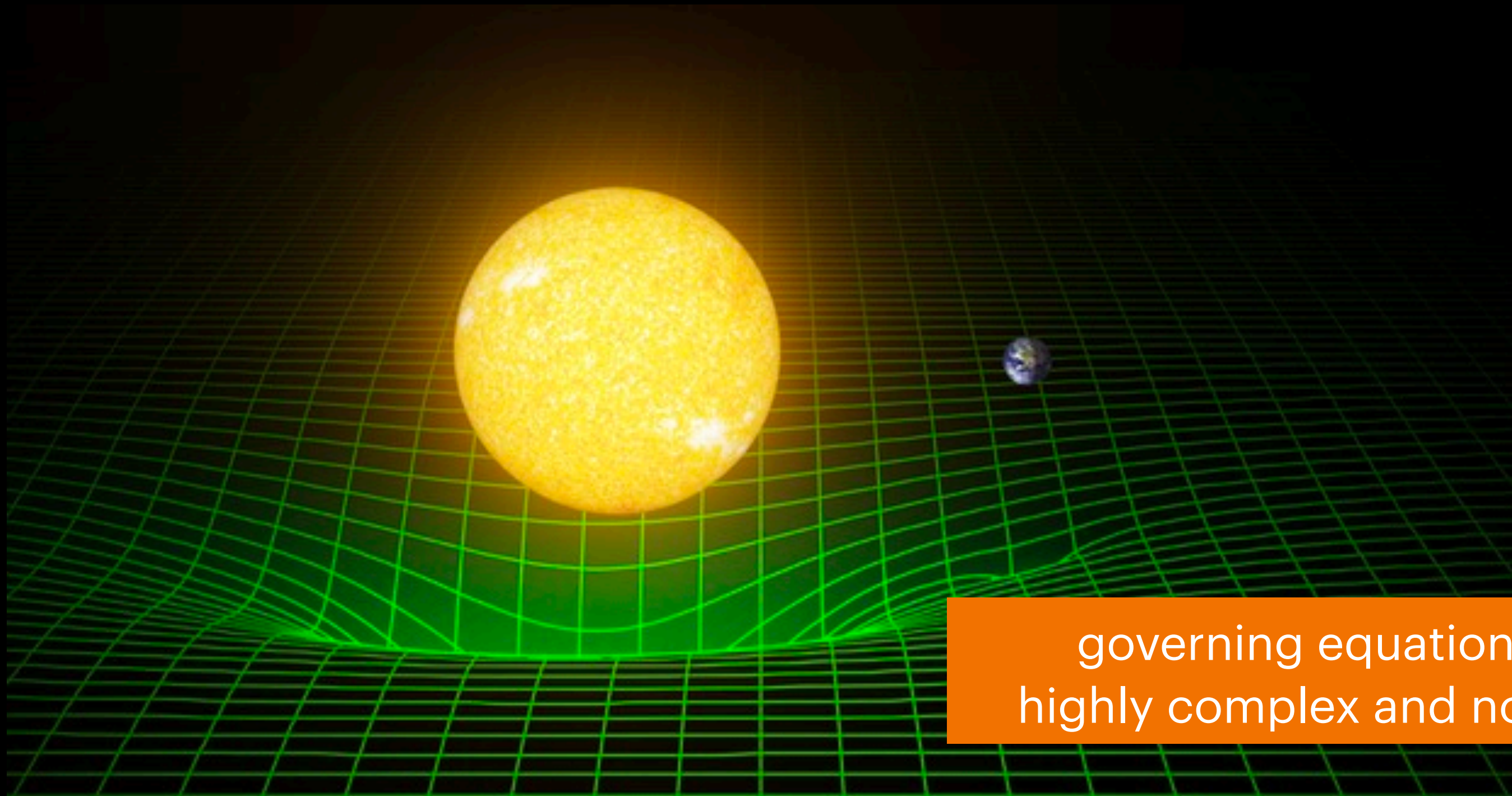
LIGO Lab



Abbott+2017







governing equations are
highly complex and nonlinear

- *Matter tells spacetime how to curve* $G_{\mu\nu} = 8\pi T_{\mu\nu}$
- *Spacetime tells matter how to move* $\nabla_{\mu} T^{\mu\nu} = 0$

you can try this by hands, but...

$$\begin{aligned}
\mathbf{a}_1 = & -\frac{Gm_2}{r_{12}^2} \mathbf{n}_{12} \\
& + \frac{1}{c^2} \left\{ \left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2} \left(\frac{3}{2}(n_{12}v_2)^2 - v_1^2 + 4(v_1v_2) - 2v_2^2 \right) \right] \mathbf{n}_{12} \right. \\
& \quad \left. + \frac{Gm_2}{r_{12}^2} (4(n_{12}v_1) - 3(n_{12}v_2)) \mathbf{v}_{12} \right\} \\
& + \frac{1}{c^4} \left\{ \left[-\frac{57G^3m_1^2m_2}{4r_{12}^4} - \frac{69G^3m_1m_2^2}{2r_{12}^4} - \frac{9G^3m_2^3}{r_{12}^4} \right. \right. \\
& \quad + \frac{Gm_2}{r_{12}^2} \left(-\frac{15}{8}(n_{12}v_2)^4 + \frac{3}{2}(n_{12}v_2)^2v_1^2 - 6(n_{12}v_2)^2(v_1v_2) - 2(v_1v_2)^2 \right. \\
& \quad \left. \left. + \frac{9}{2}(n_{12}v_2)^2v_2^2 + 4(v_1v_2)v_2^2 - 2v_2^4 \right) \right. \\
& \quad + \frac{G^2m_1m_2}{r_{12}^3} \left(\frac{39}{2}(n_{12}v_1)^2 - 39(n_{12}v_1)(n_{12}v_2) + \frac{17}{2}(n_{12}v_2)^2 - \frac{15}{4}v_1^2 - \frac{5}{2}(v_1v_2) + \frac{5}{4}v_2^2 \right) \\
& \quad + \frac{G^2m_2^2}{r_{12}^3} (2(n_{12}v_1)^2 - 4(n_{12}v_1)(n_{12}v_2) - 6(n_{12}v_2)^2 - 8(v_1v_2) + 4v_2^2) \Big] \mathbf{n}_{12} \\
& \quad + \left[\frac{G^2m_2^2}{r_{12}^3} (-2(n_{12}v_1) - 2(n_{12}v_2)) + \frac{G^2m_1m_2}{r_{12}^3} \left(-\frac{63}{4}(n_{12}v_1) + \frac{55}{4}(n_{12}v_2) \right) \right. \\
& \quad + \frac{Gm_2}{r_{12}^2} \left(-6(n_{12}v_1)(n_{12}v_2)^2 + \frac{9}{2}(n_{12}v_2)^3 + (n_{12}v_2)v_1^2 - 4(n_{12}v_1)(v_1v_2) \right. \\
& \quad \left. \left. + 4(n_{12}v_2)(v_1v_2) + 4(n_{12}v_1)v_2^2 - 5(n_{12}v_2)v_2^2 \right) \right] \mathbf{v}_{12} \Big\} \\
& + \frac{1}{c^5} \left\{ \left[\frac{208G^3m_1m_2^2}{15r_{12}^4}(n_{12}v_{12}) - \frac{24G^3m_1^2m_2}{5r_{12}^4}(n_{12}v_{12}) + \frac{12G^2m_1m_2}{5r_{12}^3}(n_{12}v_{12})v_{12}^2 \right] \mathbf{n}_{12} \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + \left[\frac{8G^3m_1^2m_2}{5r_{12}^4} - \frac{32G^3m_1m_2^2}{5r_{12}^4} - \frac{4G^2m_1m_2}{5r_{12}^3}v_{12}^2 \right] \mathbf{v}_{12} \right\} \\
& + \frac{1}{c^6} \left\{ \left[\frac{Gm_2}{r_{12}^2} \left(\frac{35}{16}(n_{12}v_2)^6 - \frac{15}{8}(n_{12}v_2)^4v_1^2 + \frac{15}{2}(n_{12}v_2)^4(v_1v_2) + 3(n_{12}v_2)^2(v_1v_2)^2 \right. \right. \right. \\
& \quad - \frac{15}{2}(n_{12}v_2)^4v_2^2 + \frac{3}{2}(n_{12}v_2)^2v_1^2v_2^2 - 12(n_{12}v_2)^2(v_1v_2)v_2^2 - 2(v_1v_2)^2v_2^2 \\
& \quad \left. \left. + \frac{15}{2}(n_{12}v_2)^2v_2^4 + 4(v_1v_2)v_2^4 - 2v_2^6 \right) \right. \\
& \quad + \frac{G^2m_1m_2}{r_{12}^3} \left(-\frac{171}{8}(n_{12}v_1)^4 + \frac{171}{2}(n_{12}v_1)^3(n_{12}v_2) - \frac{723}{4}(n_{12}v_1)^2(n_{12}v_2)^2 \right. \\
& \quad + \frac{383}{2}(n_{12}v_1)(n_{12}v_2)^3 - \frac{455}{8}(n_{12}v_2)^4 + \frac{229}{4}(n_{12}v_1)^2v_1^2 \\
& \quad - \frac{205}{2}(n_{12}v_1)(n_{12}v_2)v_1^2 + \frac{191}{4}(n_{12}v_2)^2v_1^2 - \frac{91}{8}v_1^4 - \frac{229}{2}(n_{12}v_1)^2(v_1v_2) \\
& \quad + 244(n_{12}v_1)(n_{12}v_2)(v_1v_2) - \frac{225}{2}(n_{12}v_2)^2(v_1v_2) + \frac{91}{2}v_1^2(v_1v_2) \\
& \quad - \frac{177}{4}(v_1v_2)^2 + \frac{229}{4}(n_{12}v_1)^2v_2^2 - \frac{283}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 \\
& \quad \left. \left. + \frac{259}{4}(n_{12}v_2)^2v_2^2 - \frac{91}{4}v_1^2v_2^2 + 43(v_1v_2)v_2^2 - \frac{81}{8}v_2^4 \right) \right. \\
& \quad + \frac{G^2m_2^2}{r_{12}^3} \left(-6(n_{12}v_1)^2(n_{12}v_2)^2 + 12(n_{12}v_1)(n_{12}v_2)^3 + 6(n_{12}v_2)^4 \right. \\
& \quad + 4(n_{12}v_1)(n_{12}v_2)(v_1v_2) + 12(n_{12}v_2)^2(v_1v_2) + 4(v_1v_2)^2 \\
& \quad \left. \left. - 4(n_{12}v_1)(n_{12}v_2)v_2^2 - 12(n_{12}v_2)^2v_2^2 - 8(v_1v_2)v_2^2 + 4v_2^4 \right) \right. \\
& \quad + \frac{G^3m_2^3}{r_{12}^4} \left(-(n_{12}v_1)^2 + 2(n_{12}v_1)(n_{12}v_2) + \frac{43}{2}(n_{12}v_2)^2 + 18(v_1v_2) - 9v_2^2 \right) \\
& \quad + \frac{G^3m_1m_2^2}{r_{12}^4} \left(\frac{415}{8}(n_{12}v_1)^2 - \frac{375}{4}(n_{12}v_1)(n_{12}v_2) + \frac{1113}{8}(n_{12}v_2)^2 - \frac{615}{64}(n_{12}v_{12})^2\pi^2 \right. \\
& \quad \left. + 18v_1^2 + \frac{123}{64}\pi^2v_{12}^2 + 33(v_1v_2) - \frac{33}{2}v_2^2 \right) \\
& \quad + \frac{G^3m_1^2m_2}{r_{12}^4} \left(-\frac{45887}{168}(n_{12}v_1)^2 + \frac{24025}{42}(n_{12}v_1)(n_{12}v_2) - \frac{10469}{42}(n_{12}v_2)^2 + \frac{48197}{840}v_1^2 \right. \\
& \quad \left. - \frac{36227}{420}(v_1v_2) + \frac{36227}{840}v_2^2 + 110(n_{12}v_{12})^2 \ln \left(\frac{r_{12}}{r'_0} \right) - 22v_{12}^2 \ln \left(\frac{r_{12}}{r'_0} \right) \right) \\
& \quad + \frac{16G^4m_2^4}{r_{12}^5} + \frac{G^4m_1^2m_2^2}{r_{12}^5} \left(175 - \frac{41}{16}\pi^2 \right) + \frac{G^4m_1^3m_2}{r_{12}^5} \left(-\frac{3187}{1260} + \frac{44}{3} \ln \left(\frac{r_{12}}{r'_0} \right) \right) \\
& \quad \left. + \frac{G^4m_1m_2^3}{r_{12}^5} \left(\frac{110741}{630} - \frac{41}{16}\pi^2 - \frac{44}{3} \ln \left(\frac{r_{12}}{r'_0} \right) \right) \right] \mathbf{n}_{12}
\end{aligned}$$

...

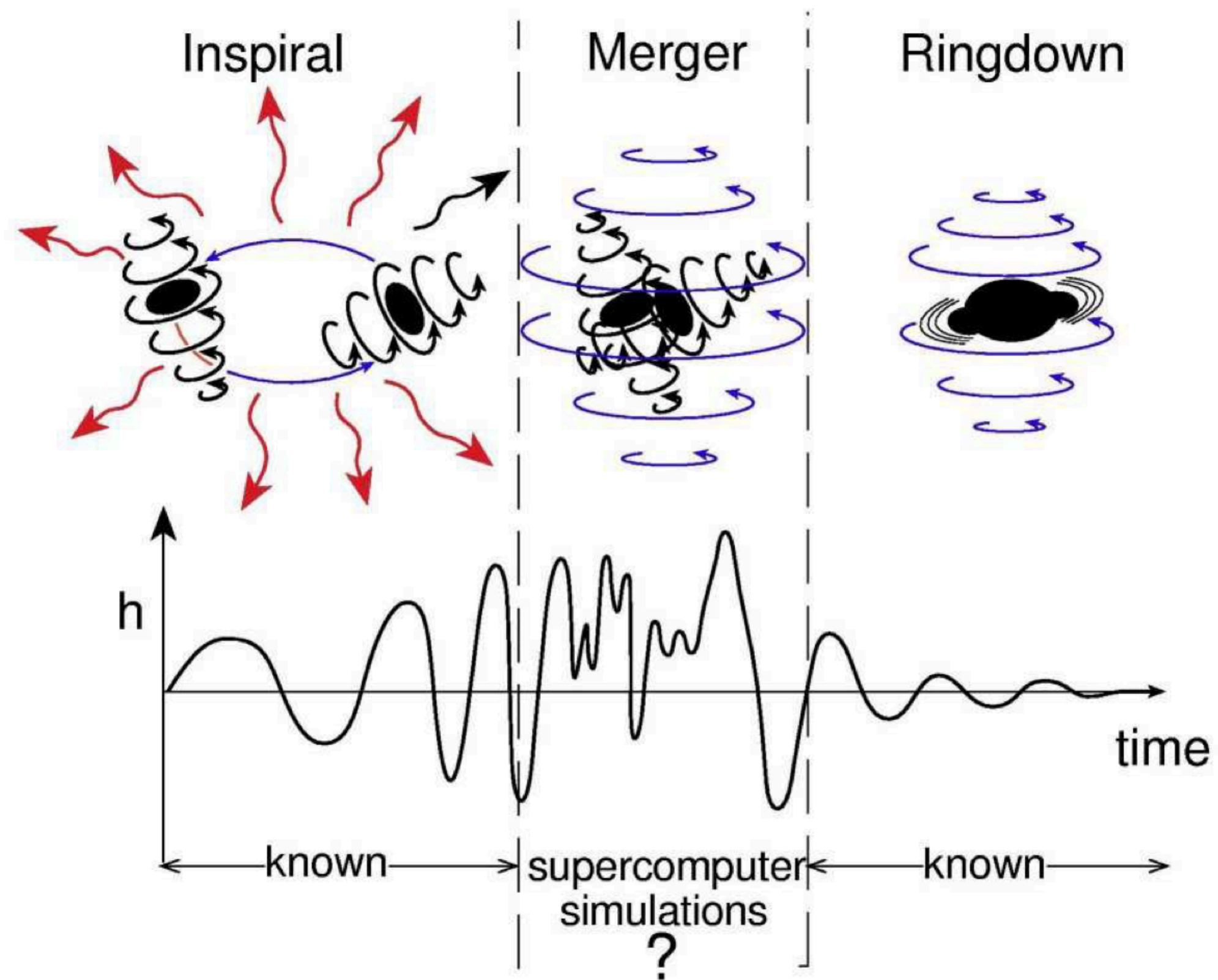
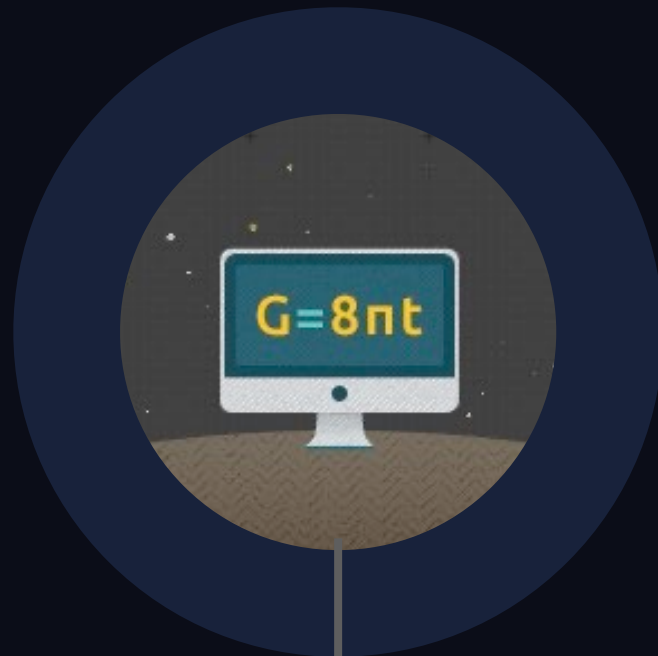


Fig. 1. The three phases of black hole merger (courtesy Kip Thorne).

physics in strong gravity, tackled with supercomputers

numerical relativity



solving Einstein's equations on computers

- modeling GW sources
- astrophysical phenomena with strong gravity & dynamic spacetime

• *Einstein's GR (1915)*

• *ADM / 3+1 formulation (circa 1960)*

• **Breakthroughs in 2005**
(annus mirabilis)

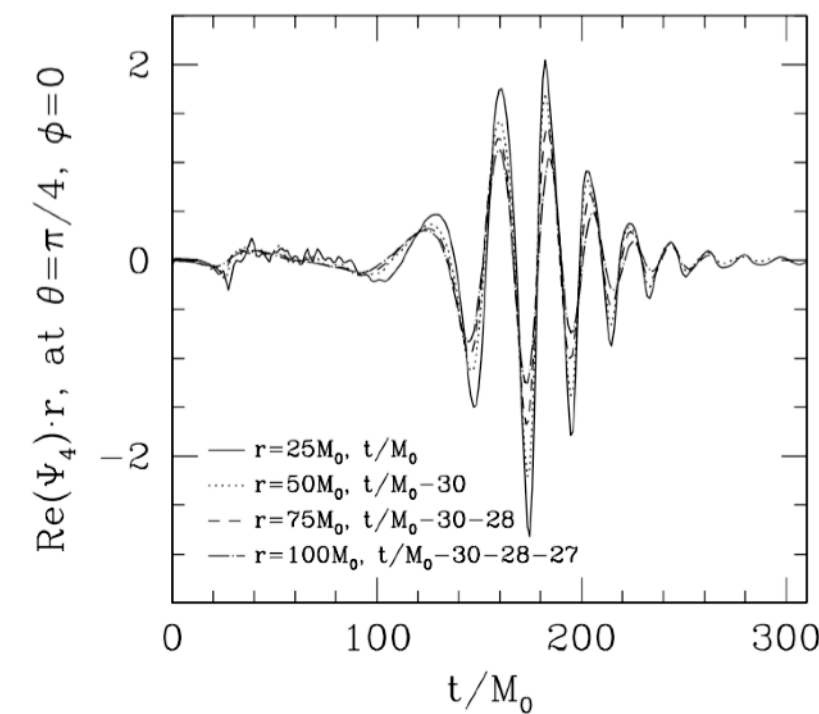
Pretorius 2005

Campanelli+ 2006

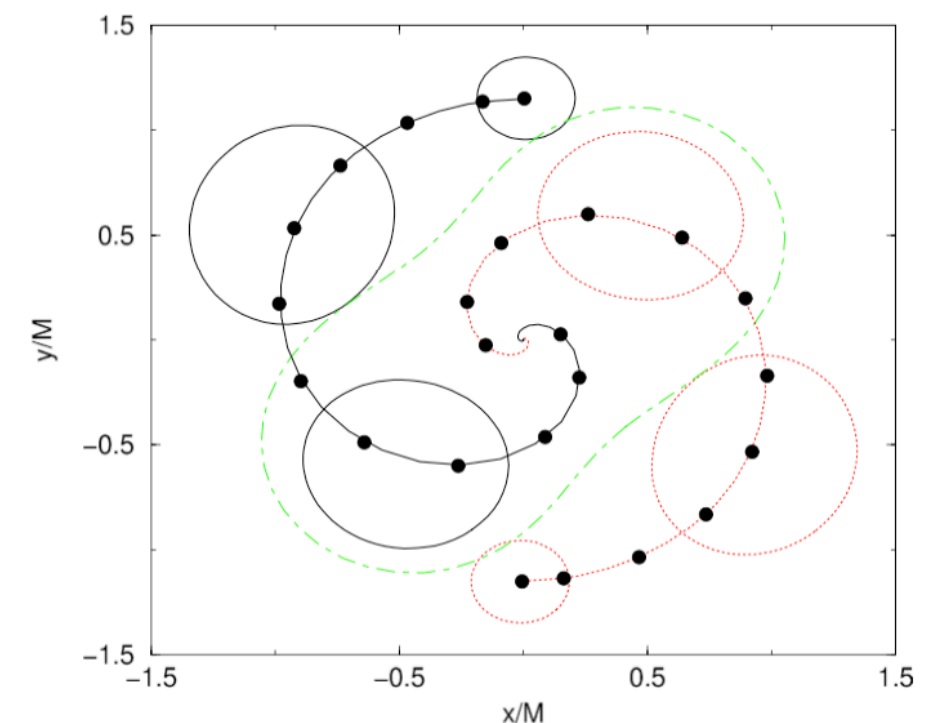
Baker+ 2006

• *LIGO's first detection of GW (2015)*

• *seminal kilonova event (2017)*

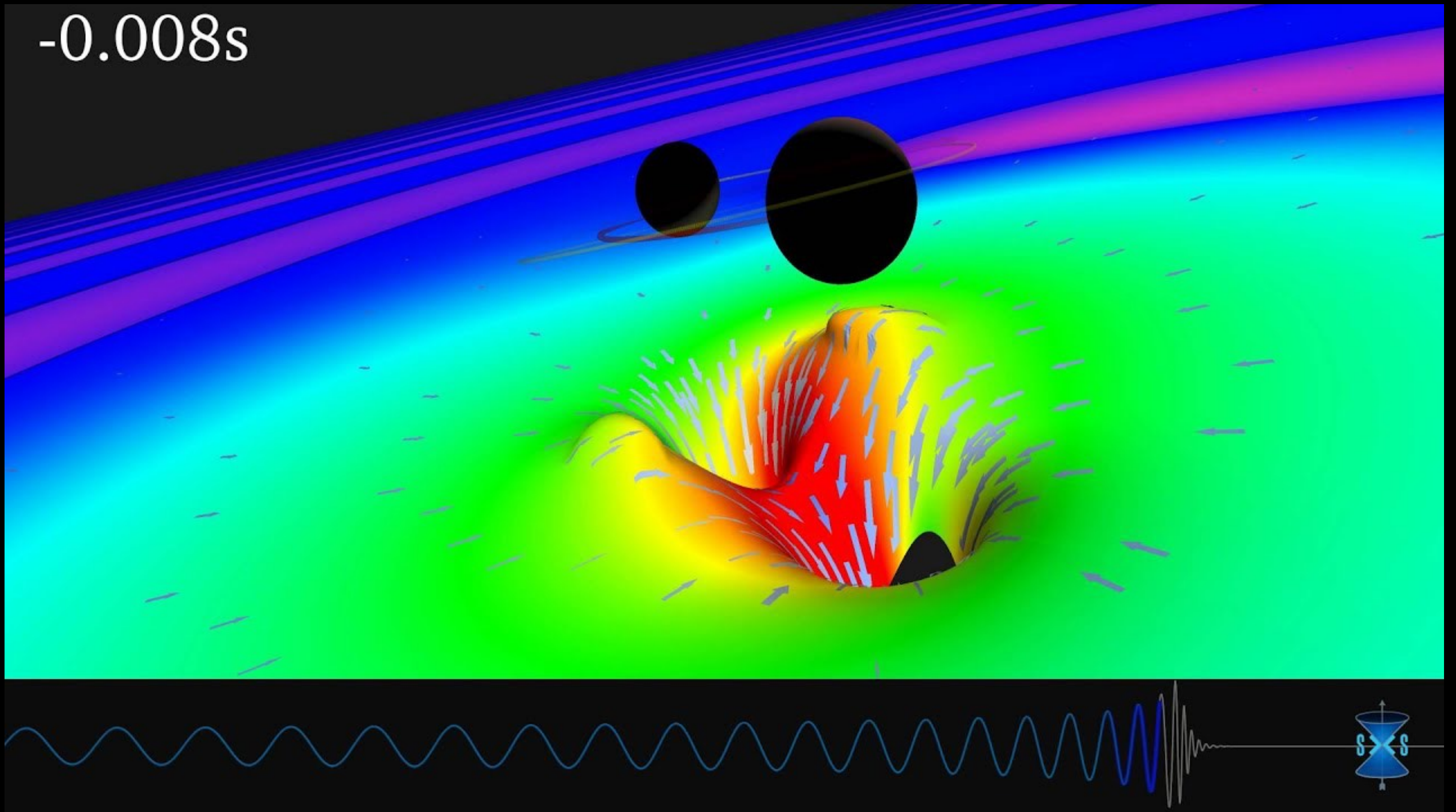


Pretorius 2005

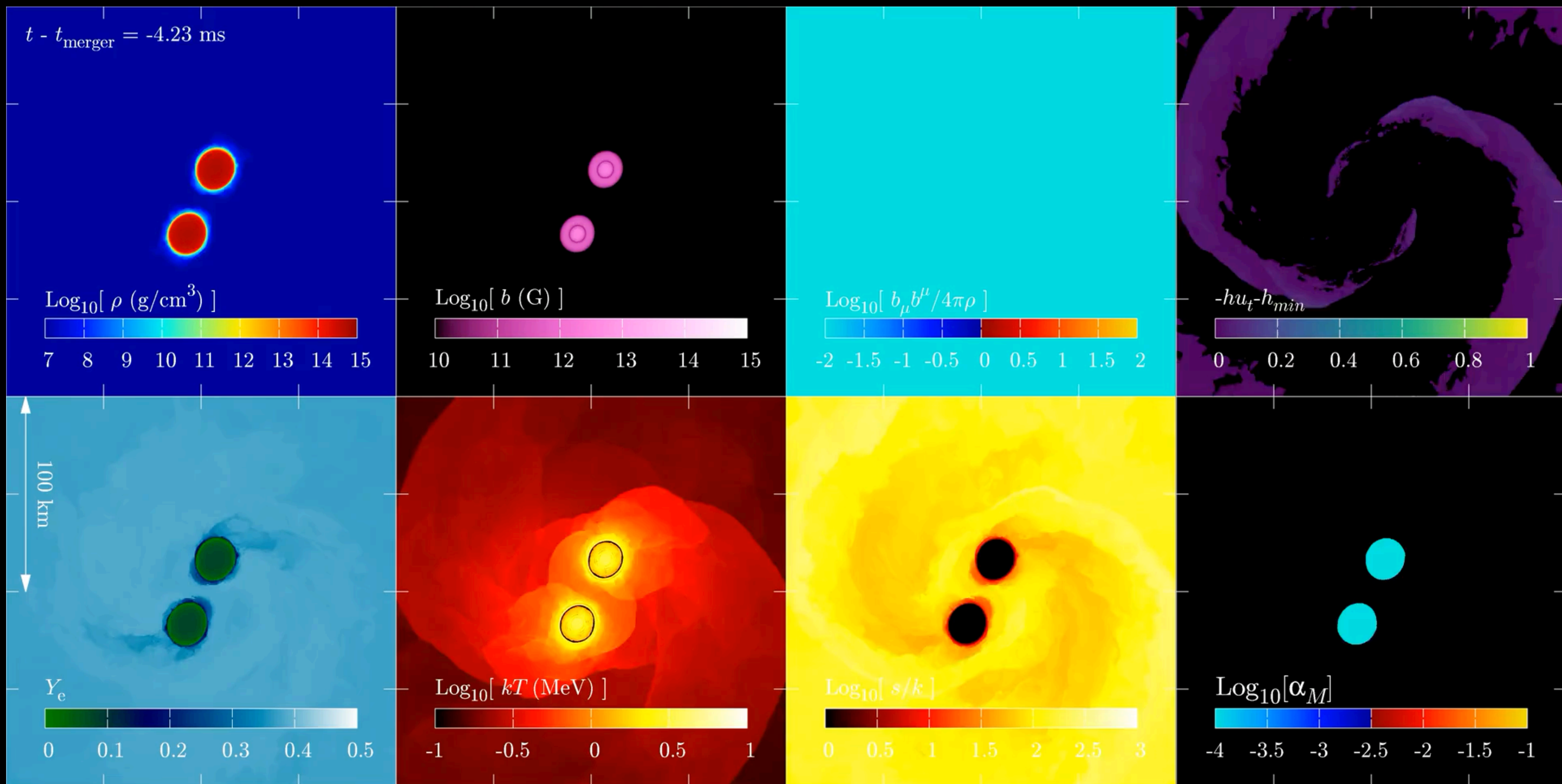


Campanelli+ 2006

-0.008s



Video credit : SXS collaboration (<https://www.youtube.com/watch?v=c-2XluNFgD0>)



Contents

- Why NR?
- 1. Numerical techniques
- 2. Formulations
- 3. Matter physics
- 4. Applications
- Remarks

Part I

Solving PDEs on computers

methods / codes / numerics / ...

Partial differential equations (PDEs)

e.g. Schrodinger equation in 1D

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

PDE, 2 independent vars, 1 dependent var $\Psi(x, t)$

e.g. Lotka-Volterra equations (predator-prey model)

$$\frac{dx}{dt} = Ax - Bxy$$

$$\frac{dy}{dt} = Cxy - Dy$$

system of 2 ODEs (not PDE)

e.g. Einstein equations in vacuum

$$G_{\mu\nu} = 0$$

system of 10 PDEs,

4 independent vars, 10 dependent vars $g_{\mu\nu}(x^a)$

PDEs may permit analytic (exact) solutions

e.g. Wave equation (3D)	$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$	→	Plane wave Spherical wave ...
e.g. vacuum Einstein equations	$G_{\mu\nu} = 0$	→	Flat (Minkowski) space Schwarzschild geometry ...

however, in general situations, they need to be solved numerically

Example — heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- $x \in [0, L]$
- $u(x, 0) = u_0(x)$
- $u(0) = u(L) = 0$

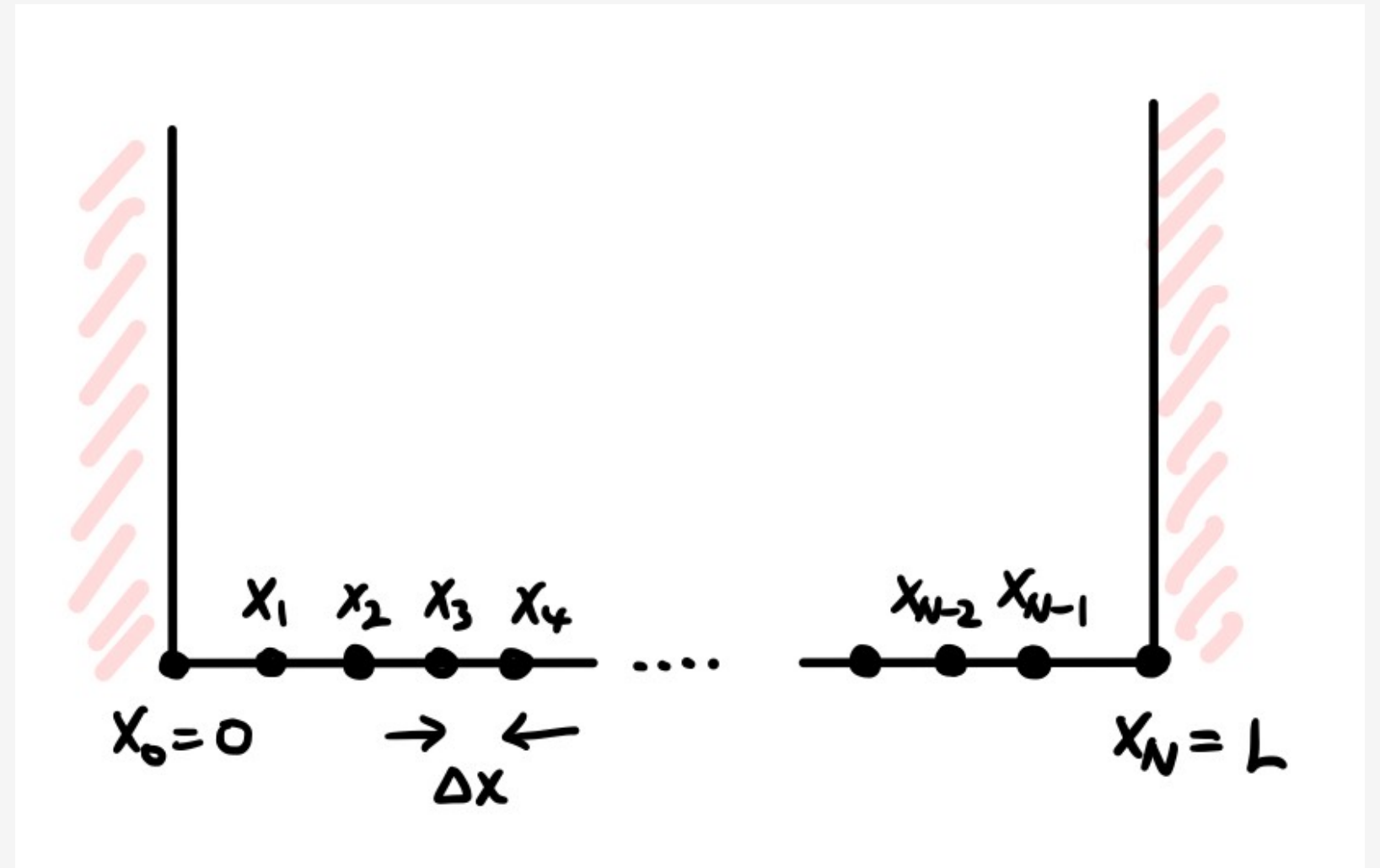
Example — heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- $x \in [0, L]$
- $u(x, 0) = u_0(x)$
- $u(0) = u(L) = 0$

Common practice :

- discretize computational domain



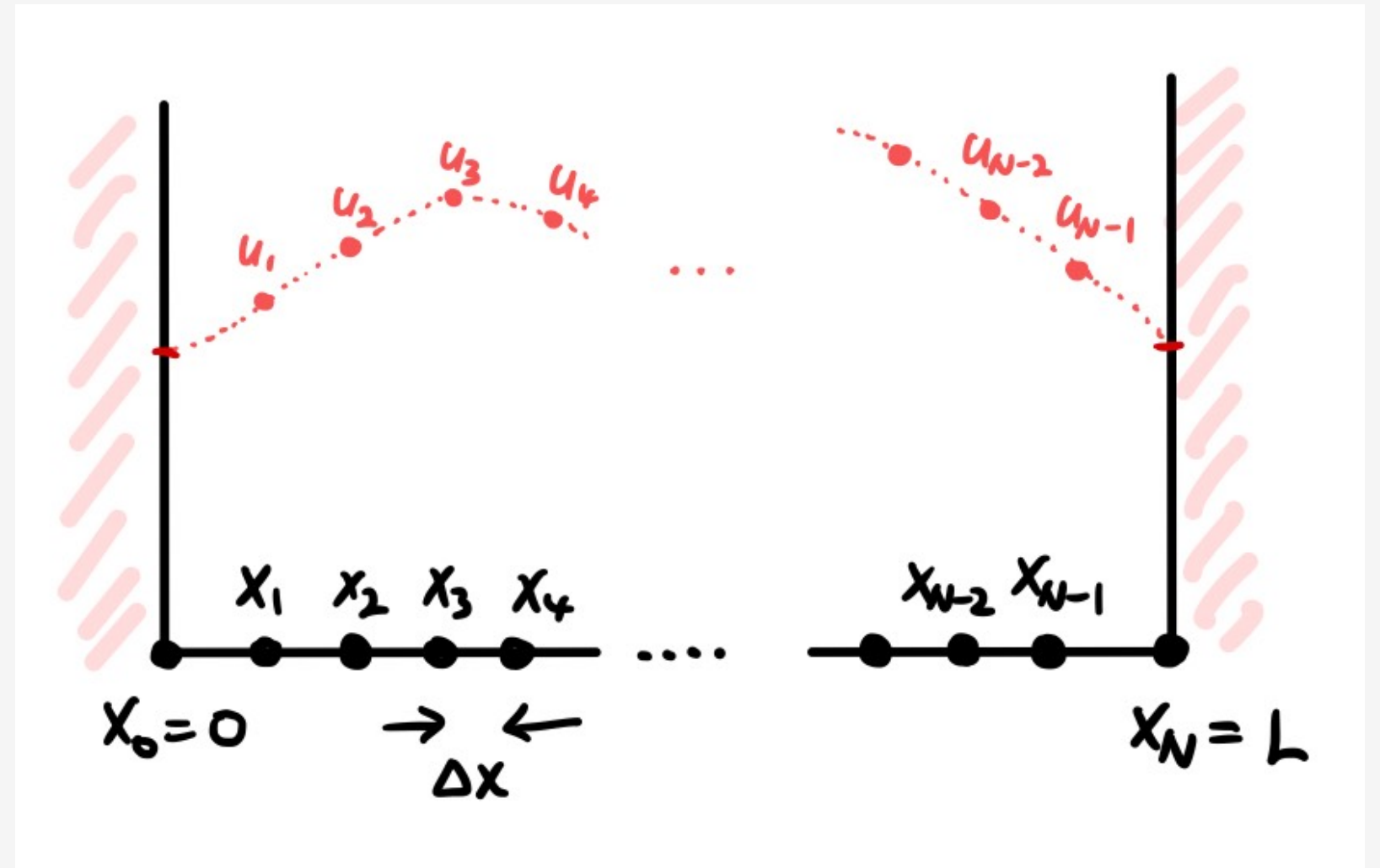
Example — heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- $x \in [0, L]$
- $u(x, 0) = u_0(x)$
- $u(0) = u(L) = 0$

Common practice :

- discretize computational domain
- approximate the PDE into discrete version
- apply initial condition



$$\frac{du_i}{dt} = \alpha \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} \right)$$

$$u_i = u_0(x_i)$$

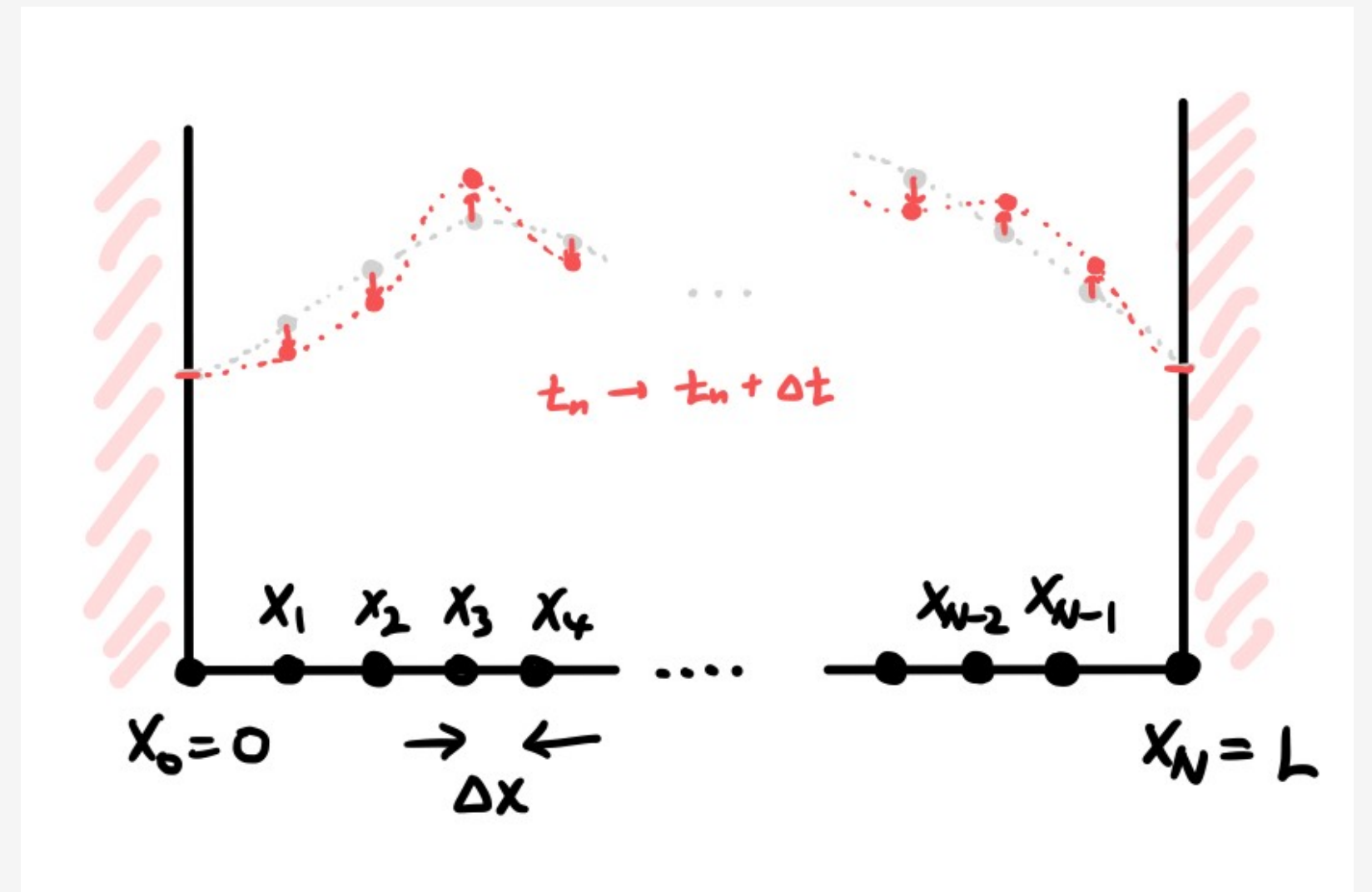
Example — heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- $x \in [0, L]$
- $u(x, 0) = u_0(x)$
- $u(0) = u(L) = 0$

Common practice :

- discretize computational domain
- approximate the PDE into discrete version
- apply initial condition
- evolve with time
- complete simulation



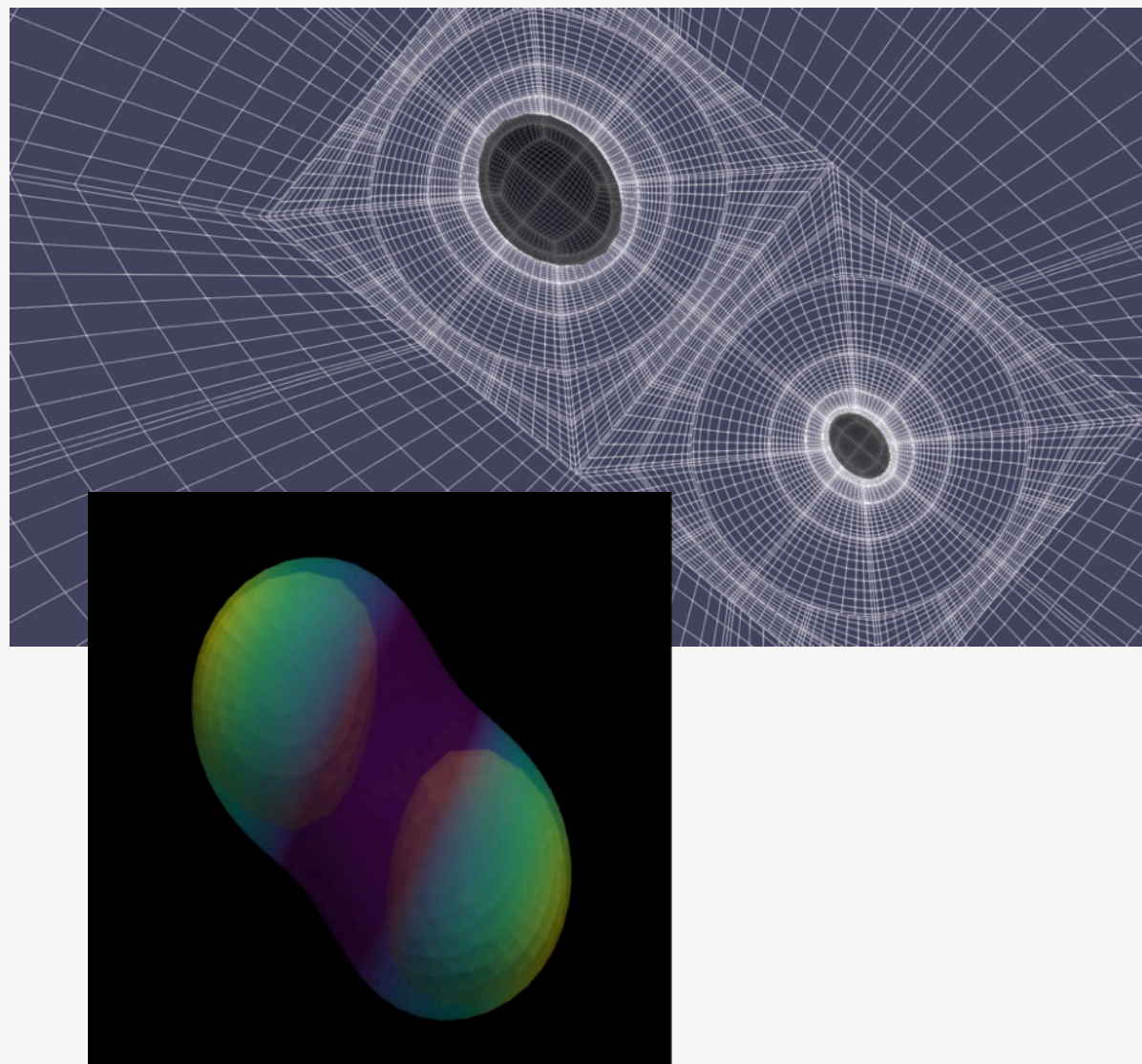
$$\frac{du_i}{dt} = \alpha \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} \right) \equiv (*)$$

$$u_i = u_0(x_i)$$

$$u_i(t_{n+1}) = u_i(t_n) + (*) \times \Delta t$$

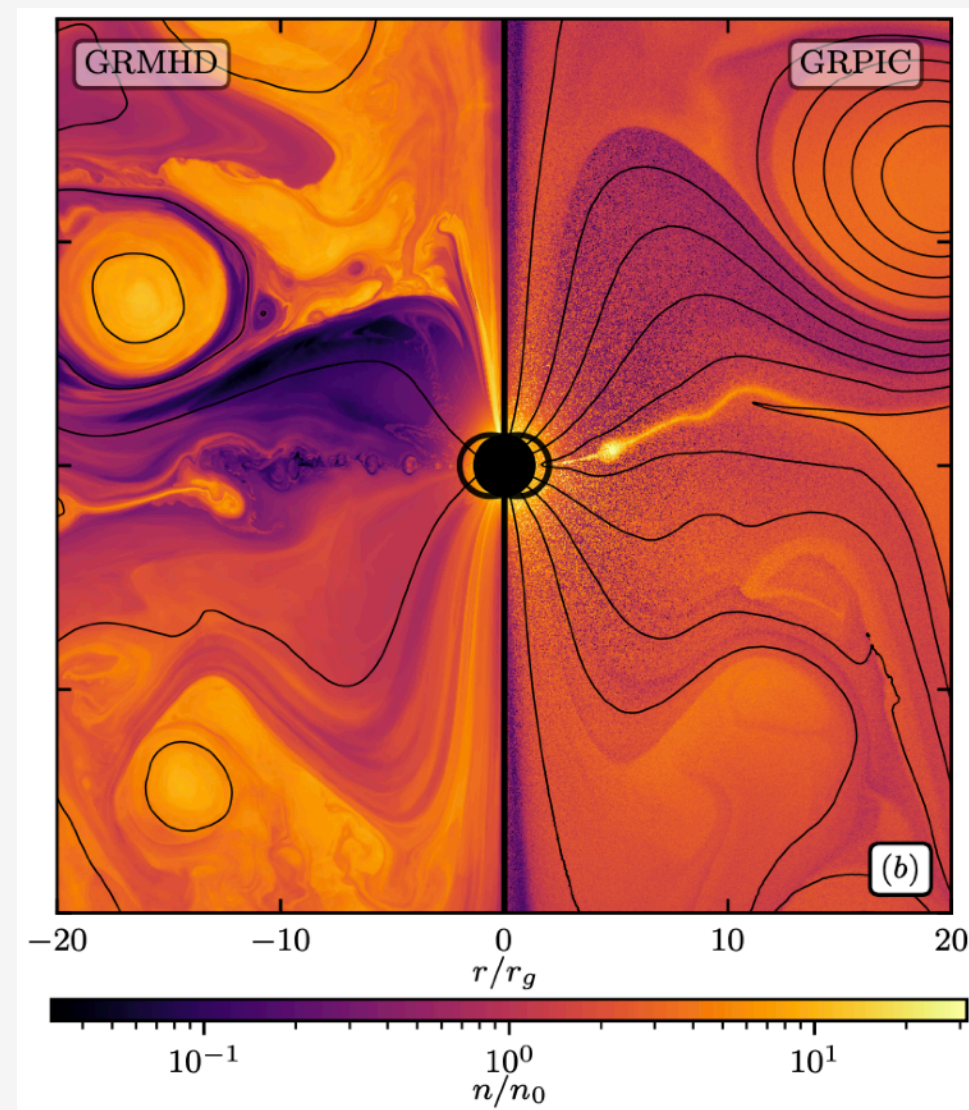
Numerical simulations in astrophysics

Grid-based (Eulerian)



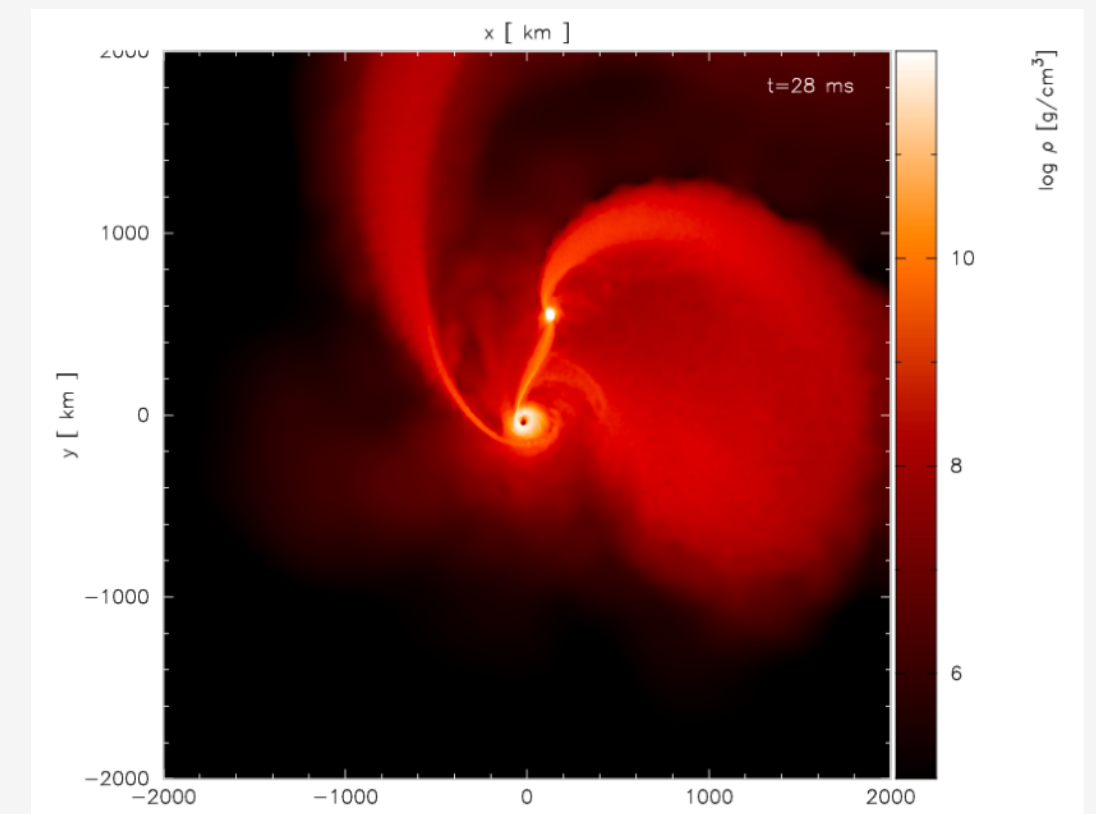
SXS

Particle (kinetic) simulations



Galishnikova+2023

Lagrangian (e.g. SPH)



Rosswog 2015

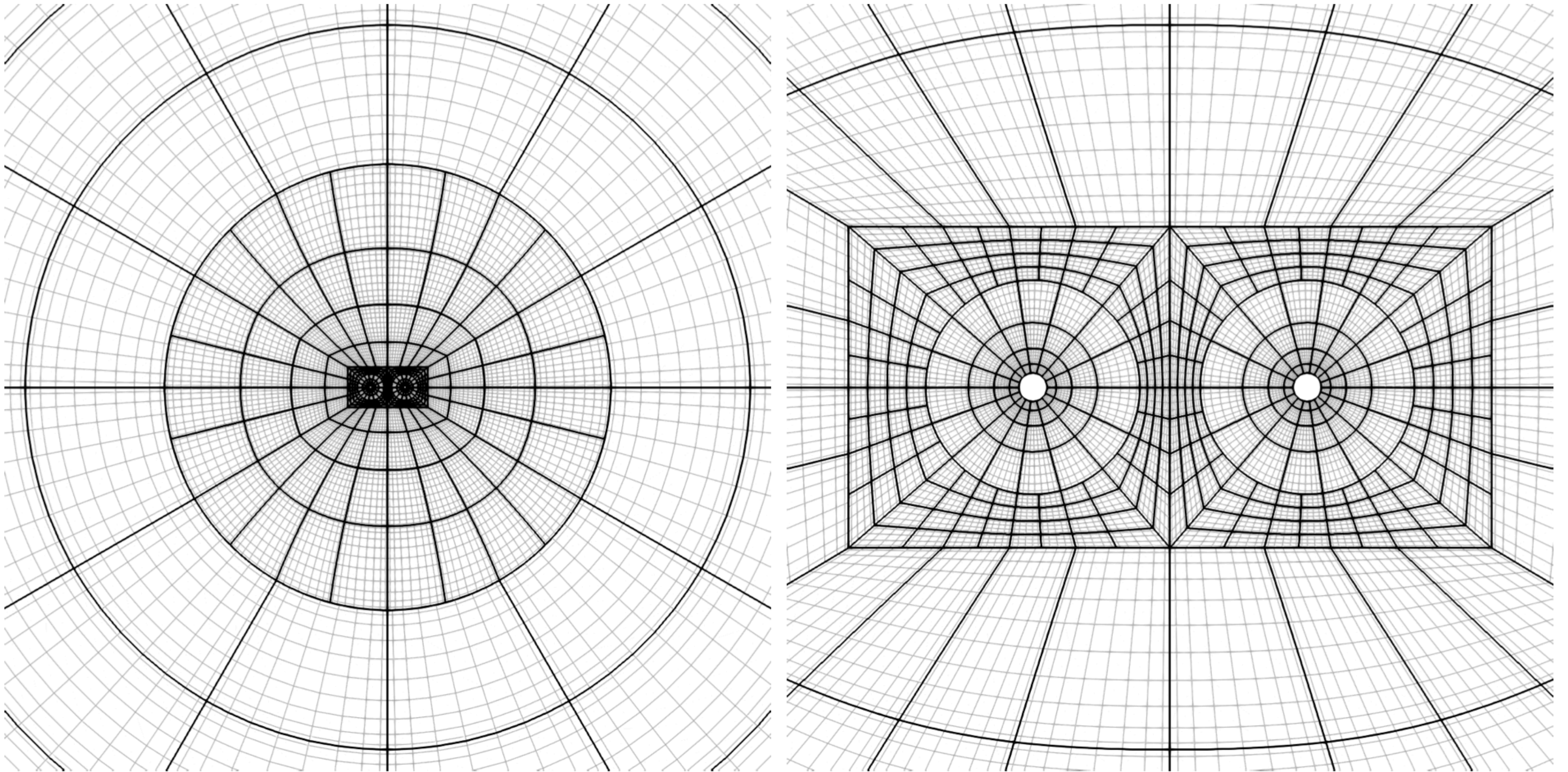


Figure 8. An illustration of the computational grid used during the inspiral. We make use of two excision regions, each region lying inside a black hole’s apparent

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LEARN C++ IN ONE VIDEO



Code	Open Source	Catalog	Formulation	Hydro	Beyond GR
AMSS-NCKU [43–46]	Yes	No	BSSN/Z4c	No	Yes
BAM [47–49]	No	[18]	BSSN/Z4c	Yes	No
BAMPS [50, 51]	No	No	GHG	Yes	No
COFFEE[52, 53]	Yes	No	GCFE	No	Yes
Dendro-GR [54–56]	Yes	No	BSSN/CCZ4	No	Yes
Einstein Toolkit [57, 58]	Yes	No	BSSN/Z4c	Yes	Yes
*Canuda [59–62]	Yes	No	BSSN	No	Yes
*IllinoisGRMHD [63]	Yes	No	BSSN	Yes	No
*LazEv [37, 64]	No	[65–68]	BSSN+CCZ4	No	No
*Lean [69, 70]	Partially	No	BSSN	No	Yes
*MAYA [71]	No	[71]	BSSN	No	Yes
*NRPy+ [72]	Yes	No	BSSN	Yes	No
*SphericalNR [73, 74]	No	No	spherical BSSN	Yes	No
*THC [75–77]	Yes	[18]	BSSN/Z4c	Yes	No
ExaHyPE [78]	Yes	No	CCZ4	Yes	No
FIL[79]	No	No	BSSN/Z4c/CCZ4	Yes	No
FUKA [80, 81]	Yes	No	XCTS	Yes	No
GR-Athena++ [82]	Yes	No	Z4c	Yes	No
GRChombo [83–85]	Yes	No	BSSN+CCZ4	No	Yes
HAD [86–88]	No	No	CCZ4	Yes	Yes
Illinois GRMHD [89, 90]	No	Yes	BSSN	Yes	No
MANGA/NRPy+ [91]	Partially	No	BSSN	Yes	No
MHDuet [92, 93]	No	No	CCZ4	Yes	Yes
SACRA-MPI [94]	No		BSSN+Z4c	Yes	No
SpEC [95, 96]	No	[96, 97]	GHG	Yes	Yes
SpECTRE [98, 99]	Yes	No	GHG	Yes	No
SPHINCS_BSSN [100]		No	BSSN	SPH	No



When you spent your entire PhD porting a legacy code onto GPUs but now NVIDIA has deprecated the CUDA version you learned

Porting requires **human effort**

(but maybe not anymore?)



Part II

Formulations

initial value problem, constraints, well-posedness

*many equations & examples brought from Duez & Zlochower 2019
(<https://arxiv.org/abs/1808.06011>)

“Hey Siri,

$$\begin{aligned}\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i, \\ \partial_t K_{ij} &= -D_i D_j \alpha + \alpha({}^{(3)}R_{ij} - 2K_{ik} K_j^k) \\ &\quad - 8\pi\alpha(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \\ &\quad + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k, \\ 16\pi\rho &= {}^{(3)}R + K^2 - K_{ij} K^{ij}, \\ 8\pi S^i &= D_j (K^{ij} - \gamma^{ij} K),\end{aligned}$$

$$\begin{aligned}\rho &= n_\mu n_\nu T^{\mu\nu}, \\ S^i &= -\gamma^{ij} n^\mu T_{\mu j}, \\ S_{ij} &= T_{ij}, \\ S &= \gamma^{ij} S_{ij}.\end{aligned}$$

and collide two black holes”



???

don't try this at home

until mid 1990s, all NR used 3+1 or ADM equations, but 3D evolution failed

What was missing?

- well-posedness
- constraint growth

C. Making the problem well posed

As will be discussed later, the ADM equations by themselves proved to be unstable for many strong-field problems, including black-hole mergers. With all the difficulties encountered trying to implement the ADM equations in the 1990s, emphasis changed to developing new 3+1 systems and analyzing their well-posedness [61–74].

e.g. Maxwell equations in vacuum

$$\mathcal{C}_E = \vec{\nabla} \cdot \vec{E} = 0,$$
$$\mathcal{C}_B = \vec{\nabla} \cdot \vec{B} = 0.$$

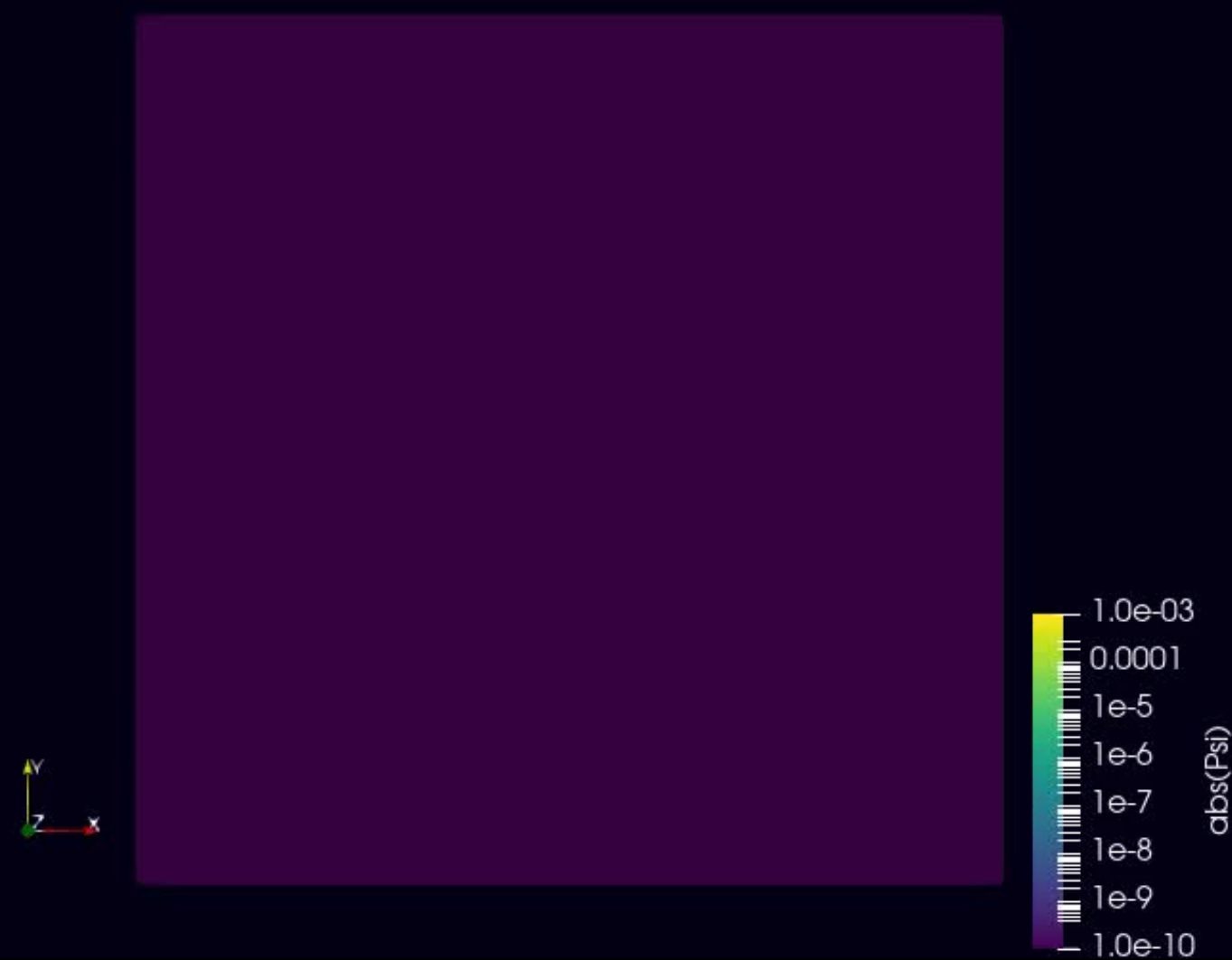
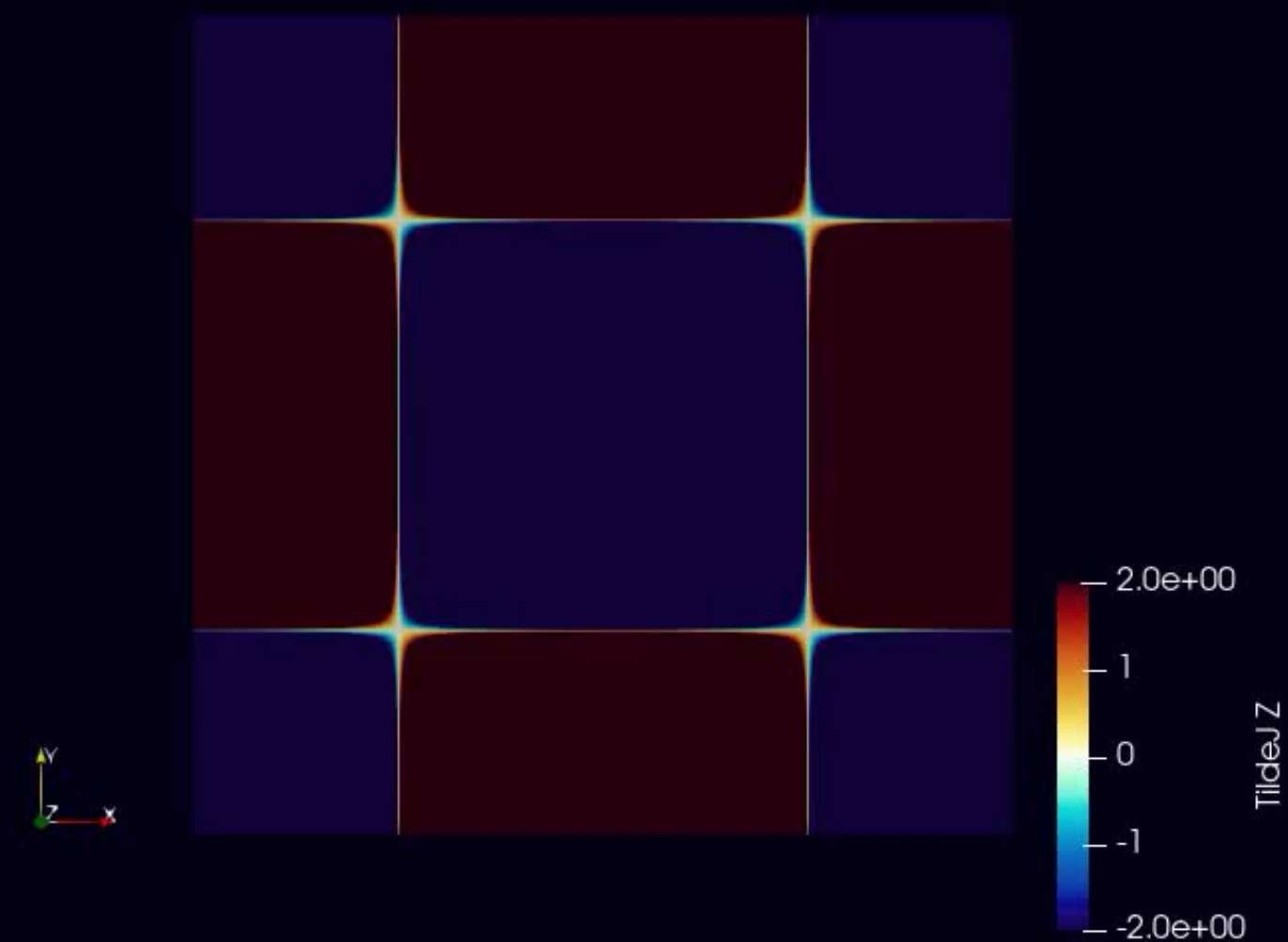
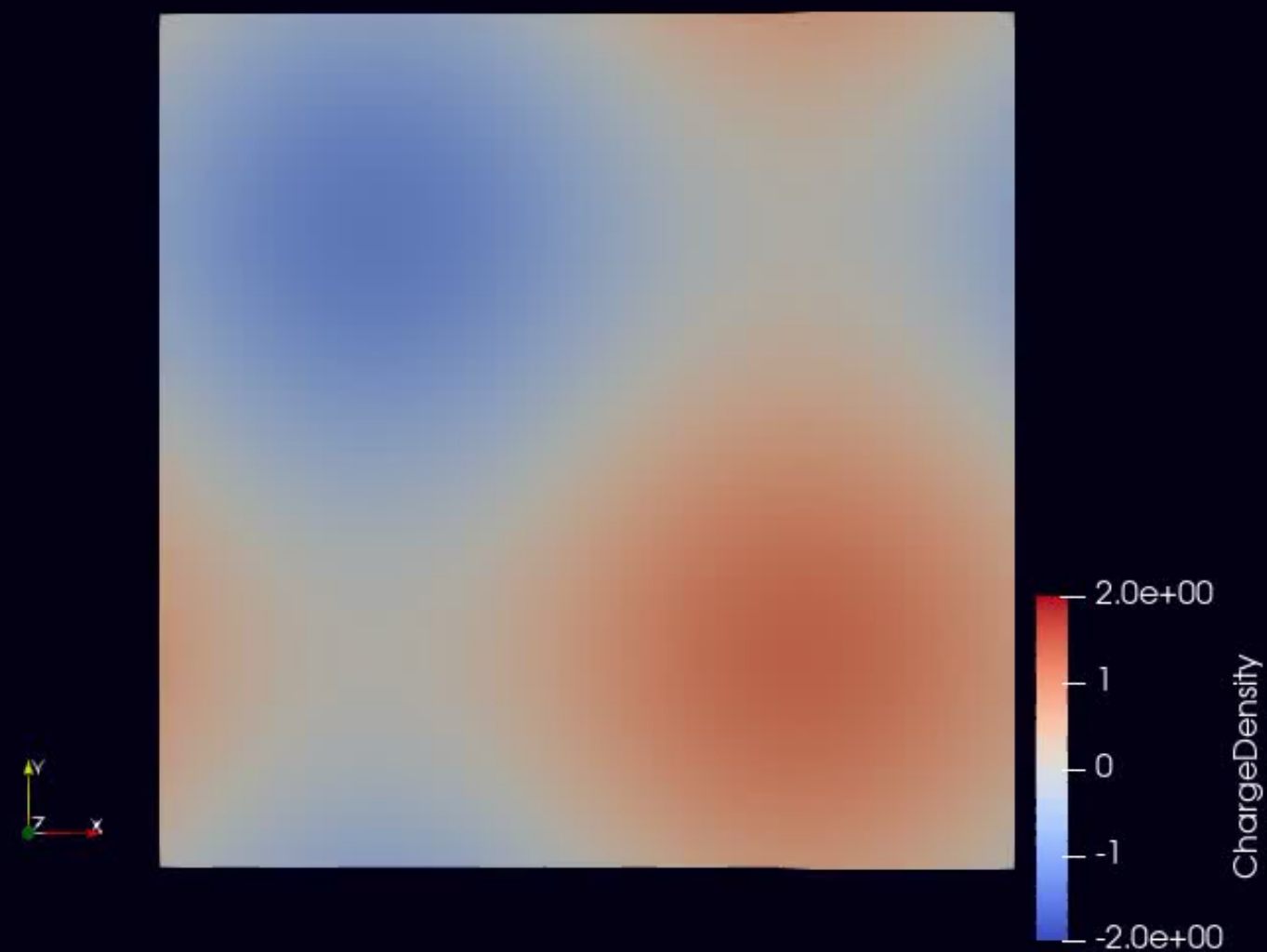
(Gauss constraints)

$$\left(\frac{\partial \vec{E}}{\partial t}\right) = \vec{\nabla} \times \vec{B},$$
$$\left(\frac{\partial \vec{B}}{\partial t}\right) = -\vec{\nabla} \times \vec{E},$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \vec{E} = 0$$
$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \vec{B} = 0$$

$$\left(\frac{\partial \vec{E}}{\partial t}\right) = \vec{\nabla} \times \vec{B} + \vec{\nabla} \psi_E,$$
$$\left(\frac{\partial \vec{B}}{\partial t}\right) = -\vec{\nabla} \times \vec{E} + \vec{\nabla} \psi_B,$$
$$\left(\frac{\partial \psi_E}{\partial t}\right) = -\vec{\nabla} \cdot \vec{E} - \psi_E,$$
$$\left(\frac{\partial \psi_B}{\partial t}\right) = -\vec{\nabla} \cdot \vec{B} - \psi_B.$$

Time: 0.0



Well-posedness

Hadamard (1902) :

- solution exists
- is unique
- depends **continuously** on initial data and boundary conditions

e.g. Hadamard 1923

$$\partial_t^2 u - \partial_x^2 u = 0, \quad \begin{cases} u_0 = 0 \\ \dot{u}_0 = \sin(2\pi n x) / (2\pi n)^p & p > 1 \\ \text{BC : } u = 0 \text{ at } x = 0, 1 \end{cases}$$

Sol: $u(x, t) = \sin(2\pi n t) \sin(2\pi n x) / (2\pi n)^{p+1}$

As $n \rightarrow \infty$, initial data goes to zero and $u(x, t)$ goes to zero.

$\Rightarrow$ well posed

e.g. $\partial_t^2 u + \partial_x^2 u = 0$, same initial data + BC.

$$u(x,t) = \frac{\sinh(2\pi n t) \sin(2\pi n x)}{(2\pi n)^{p+1}}$$

As $n \rightarrow \infty$, initial data $\rightarrow 0$

but $u(x,t) \rightarrow e^n$ blows up

i.e. Given any ID and any BC , can introduce a small perturbation at $t=0$, get an arbitrarily $\uparrow$ solution .

3+1 equations are not well-posed (only weakly hyperbolic)

BSSN formalism

Shibata & Nakamura 1995, Baumgarte & Shapiro 1999
+ critical gauge choices (~2005)

Generalized harmonic formulation

Pretorius 2005, SXS collaboration

a 'good theory' should have a 'good formulation'

e.g. higher-order metric derivatives $(\partial^3 g, \partial^4 g, \dots)$ in alternative theories of gravity

Part III

Relativistic hydrodynamics & matter physics

How can we see black holes?
(by the way, we can now 'hear' them)

How can we see black holes?
(by the way, we can now 'hear' them)



black holes can be observed when
interacting with their surroundings

Evolution of fluid

begin with Newtonian hydrodynamics :

Physical quantities describing the fluid at $(\mathbf{x}, t)$

- mass density ρ
- velocity $\mathbf{v}$
- energy density ($\sim$ temperature) ϵ

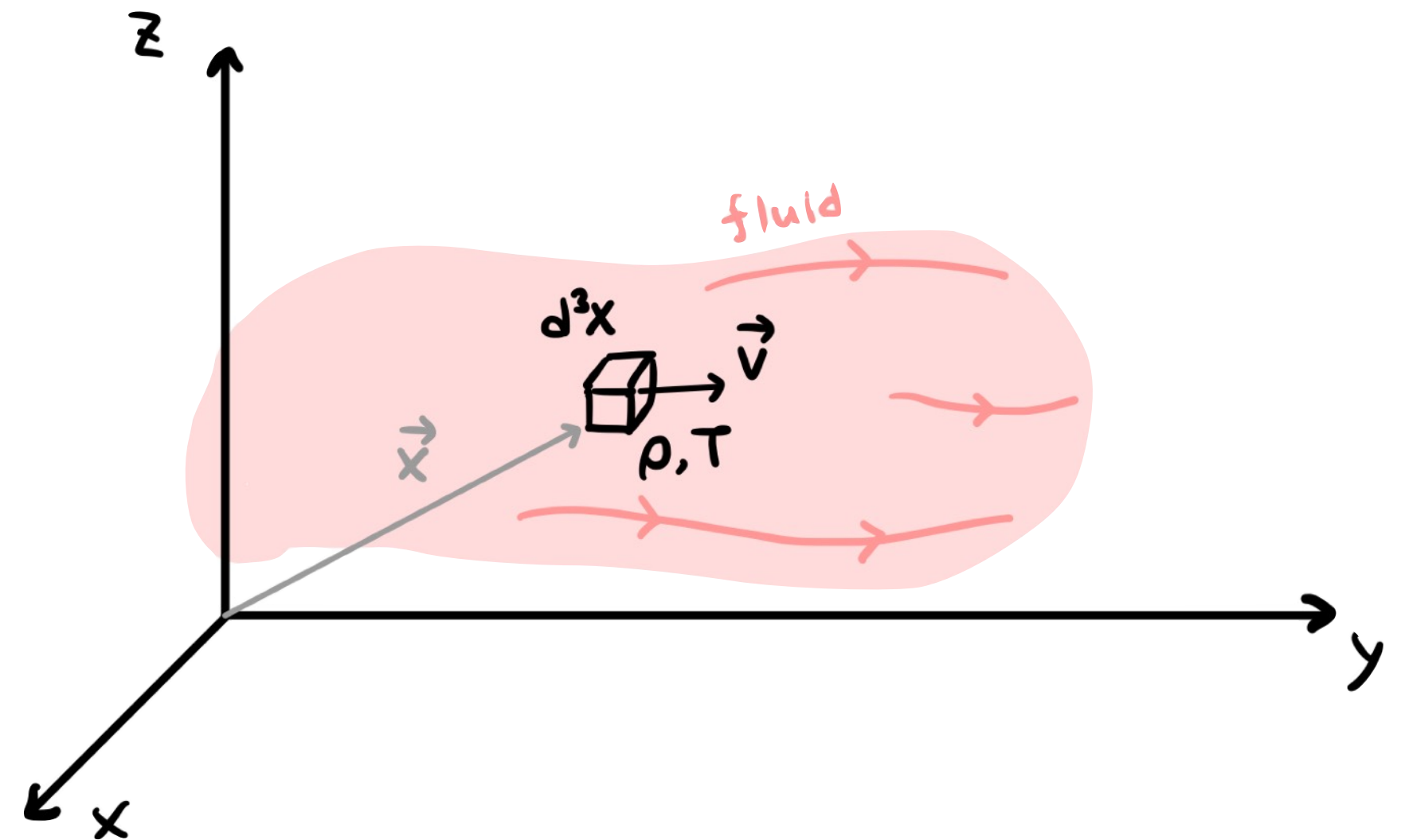
*pressure from $P = P(\rho, \epsilon)$: **equations of state** (EOS)

e.g. Classical ideal gas

$$P = \frac{2}{3}\epsilon \quad (= nk_B T)$$

e.g. Electron degeneracy

$$P = \frac{(3\pi^2)^{2/3} \hbar^2}{5m_e^{8/3}} \rho^{5/3}$$



What laws of physics do we have?

- mass conservation
- $\mathbf{F} = m\mathbf{a}$ (~*momentum conservation*)
- energy conservation

Basic equations in Newtonian hydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{\nabla P}{\rho}$$

$$\frac{\partial \epsilon}{\partial t} + (\mathbf{v} \cdot \nabla) \epsilon = - \frac{P(\nabla \cdot \mathbf{v})}{\rho}$$

note) convective derivative $\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$

makes equations *nonlinear*



Conservative form of PDEs

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F} = \mathbf{S}$$

evolved variables
(conserved variables) flux source

e.g. charge continuity $\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{J} = 0$

- Important for correct shock speeds
- Advantageous when solving numerically
(Lax & Wendroff 1960, Hou & LeFloch 1994)

this is a good form !

Can we write Newtonian hydro equations into this form? **Yes**

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho v^j \\ \rho \epsilon + \frac{1}{2} \rho v^2 \end{bmatrix}, \quad \mathbf{F}^i = \begin{bmatrix} \rho v^i \\ \rho v^i v^j + P \delta^{ij} \\ (\rho \epsilon + \frac{1}{2} \rho v^2 + P) v^i \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

conserved (evolved) variables $\mathbf{U}$ vs primitive variables $\{\rho, v^i, \epsilon\}$

General relativistic hydrodynamics

Physical quantities describing the fluid at $(\mathbf{x}, t)$: $T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu}$ stress-energy tensor (ideal fluid)

- conservation of mass $\nabla_\mu(\rho u^\mu) = 0$
 - conservation of energy and momentum $\nabla_\mu T^{\mu\nu} = 0$
- $$\left. \begin{array}{l} \nabla_\mu(\rho u^\mu) = 0 \\ \nabla_\mu T^{\mu\nu} = 0 \end{array} \right\} \xrightarrow{\text{cast to the conservative form}} \partial_t \mathbf{U} + \nabla \cdot \mathbf{F} = \mathbf{S}$$

- Valencia formulation (Marti+1991; Banyuls+1997)

$$\mathbf{U} = \begin{bmatrix} \tilde{D} \\ \tilde{S}_j \\ \tilde{\tau} \end{bmatrix} = \sqrt{\gamma} \begin{bmatrix} \rho W \\ \rho h W^2 v_j \\ \rho h W^2 - P - \rho W \end{bmatrix}, \quad \mathbf{F}^i = \alpha \begin{bmatrix} \tilde{D} \tilde{v}^i \\ \tilde{S}_j \tilde{v}^i + \sqrt{\gamma} P \delta_j^i \\ \tilde{\tau} \tilde{v}^i + \sqrt{\gamma} P v^i \end{bmatrix}, \quad \mathbf{S} = \alpha \sqrt{\gamma} \begin{bmatrix} 0 \\ T^{\mu\nu} g_{\nu\sigma} \Gamma_{\mu j}^\sigma \\ T^{\mu 0} \partial_\mu \alpha - \alpha T^{\mu\nu} \Gamma_{\mu\nu}^0 \end{bmatrix}$$

$$W = \alpha u^0 = (1 - \gamma_{ij} v^i v^j)^{-1/2} \quad \tilde{v}^i = v^i - \beta^i / \alpha$$

used by current NR codes

See analogies:

Newtonian hydro

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho v^j \\ \rho \epsilon + \frac{1}{2} \rho v^2 \end{bmatrix}$$

$$\mathbf{F}^i = \begin{bmatrix} \rho v^i \\ \rho v^i v^j + P \delta^{ij} \\ (\rho \epsilon + \frac{1}{2} \rho v^2 + P) v^i \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

GR hydro

$$\mathbf{U} = \sqrt{\gamma} \begin{bmatrix} \rho W \\ \rho h W^2 v_j \\ \rho h W^2 - P - \rho W \end{bmatrix}$$

$$\mathbf{F}^i = \alpha \begin{bmatrix} \tilde{D} \tilde{v}^i \\ \tilde{S}_j \tilde{v}^i + \sqrt{\gamma} P \delta_j^i \\ \tilde{\tau} \tilde{v}^i + \sqrt{\gamma} P v^i \end{bmatrix}$$

$$\mathbf{S} = \alpha \sqrt{\gamma} \begin{bmatrix} 0 \\ T^{\mu\nu} g_{\nu\sigma} \Gamma_{\mu j}^\sigma \\ T^{\mu 0} \partial_\mu \alpha - \alpha T^{\mu\nu} \Gamma_{\mu\nu}^0 \end{bmatrix}$$

variables

- mass (1)
- momentum (3)
- energy (1)

~ variables x velocity

relativistic effects

* flat space : $\alpha = 1, \quad \beta^i = 0, \quad \sqrt{\gamma} = 1, \quad \Gamma_{\mu\nu}^\sigma = 0$

* non-relativistic : $v^j \ll 1, \quad \rho \gg \epsilon, P$

General relativistic **magnetohydrodynamics**

$$T^{\mu\nu} = T^{\mu\nu}_{\text{matter}} + \boxed{T^{\mu\nu}_{\text{EM}}} \quad \text{add contributions from EM field}$$

$$T^{\mu\nu}_{\text{EM}} = F^{\mu\sigma} F^{\nu}_{\sigma} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \longrightarrow b^2 \left[u^{\mu} u^{\nu} + \frac{g^{\mu\nu}}{2} \right] - b^{\mu} b^{\nu}$$

$$B^i = - {}^*F^{i\mu} n_{\mu}$$

$$b^{\mu} \equiv - {}^*F^{\mu\nu} u_{\nu}$$

$$b^0 = WB^i v_i / \alpha, \quad b^i = (B^i + \alpha b^0 u^i) / W$$

$$\mathbf{U} = \begin{bmatrix} \tilde{D} \\ \tilde{S}_j \\ \tilde{\tau} \\ \tilde{B}^k \end{bmatrix} = \sqrt{\gamma} \begin{bmatrix} \rho W \\ (\rho h + b^2) W^2 v_j - \alpha b^0 b_j \\ (\rho h + b^2) W^2 - (P + b^2/2) - \rho W - (\alpha b^0)^2 \\ B^k \end{bmatrix},$$

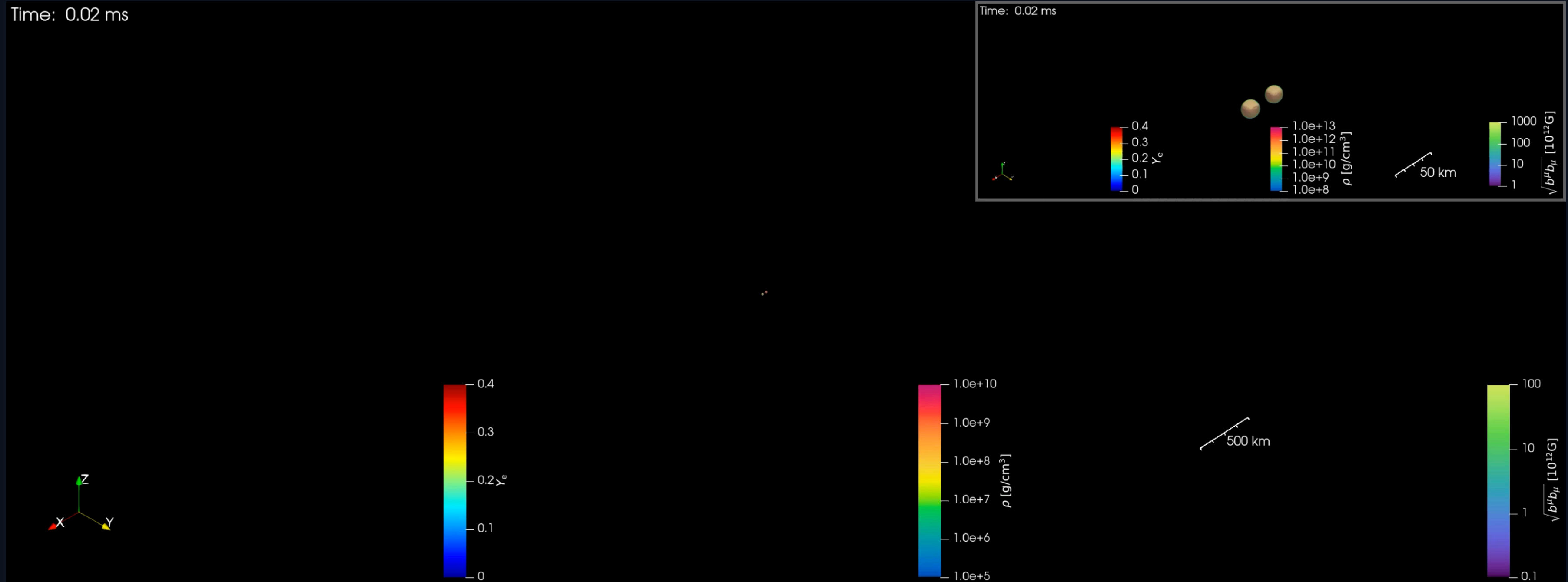
$$\mathbf{F}^i = \alpha \begin{bmatrix} \tilde{D} \tilde{v}^i \\ \tilde{S}_j \tilde{v}^i + \sqrt{\gamma} (P + b^2/2) \delta_j^i - b_j \tilde{B}^i / W \\ \tilde{\tau} \tilde{v}^i + \sqrt{\gamma} (P + b^2/2) v^i - \alpha b^0 \tilde{B}^i / W \\ \tilde{B}^k \tilde{v}^i - \tilde{B}^i \tilde{v}^k \end{bmatrix}, \quad \mathbf{S} = \alpha \sqrt{\gamma} \begin{bmatrix} 0 \\ T^{\mu\nu} g_{\nu\sigma} \Gamma_{\mu j}^{\sigma} \\ T^{\mu 0} \partial_{\mu} \alpha - \alpha T^{\mu\nu} \Gamma_{\mu\nu}^0 \\ 0 \end{bmatrix}$$

Part IV

Applications

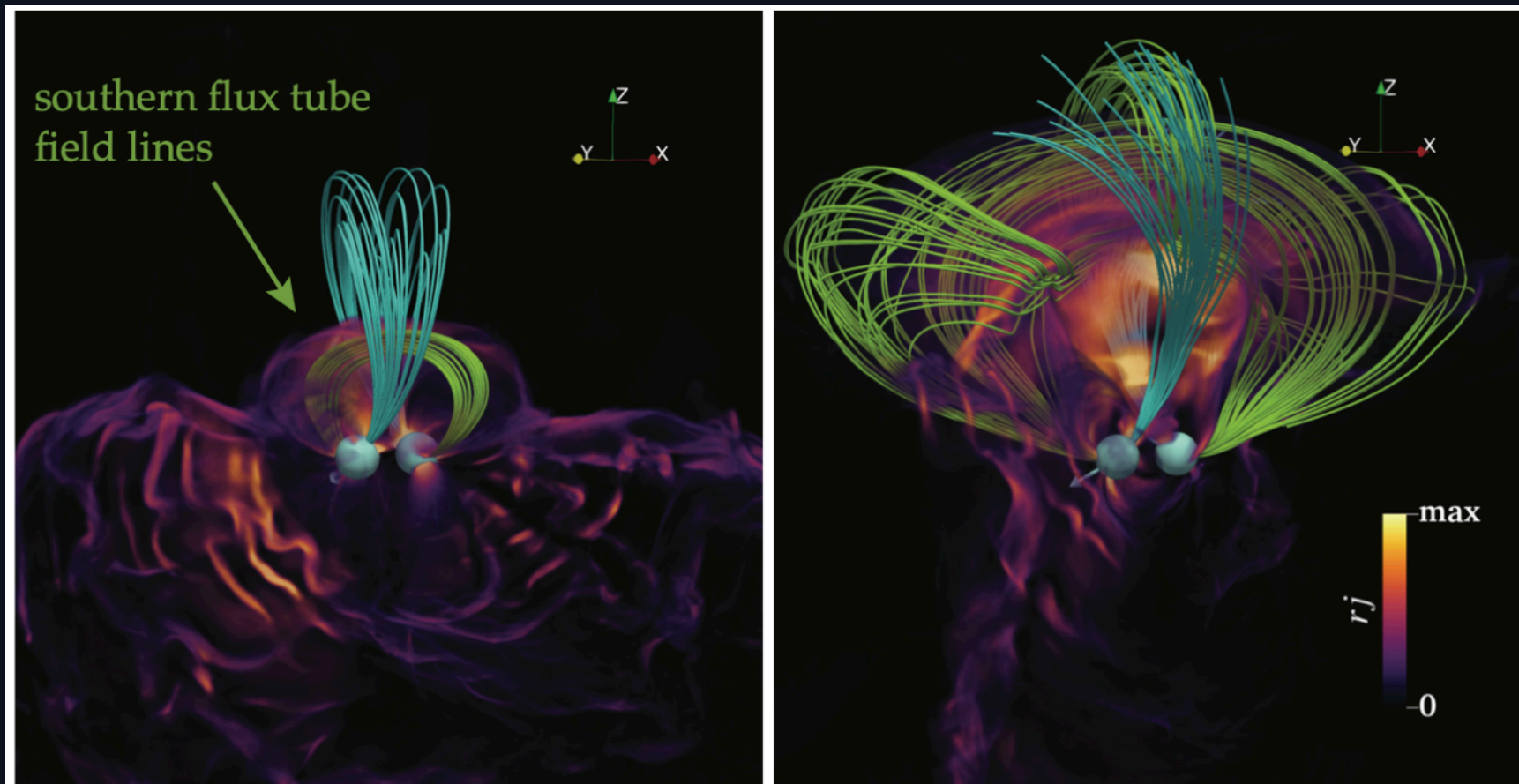
compact binary mergers / accretion disks / jets

Amplification of magnetic fields in BNS merger remnants

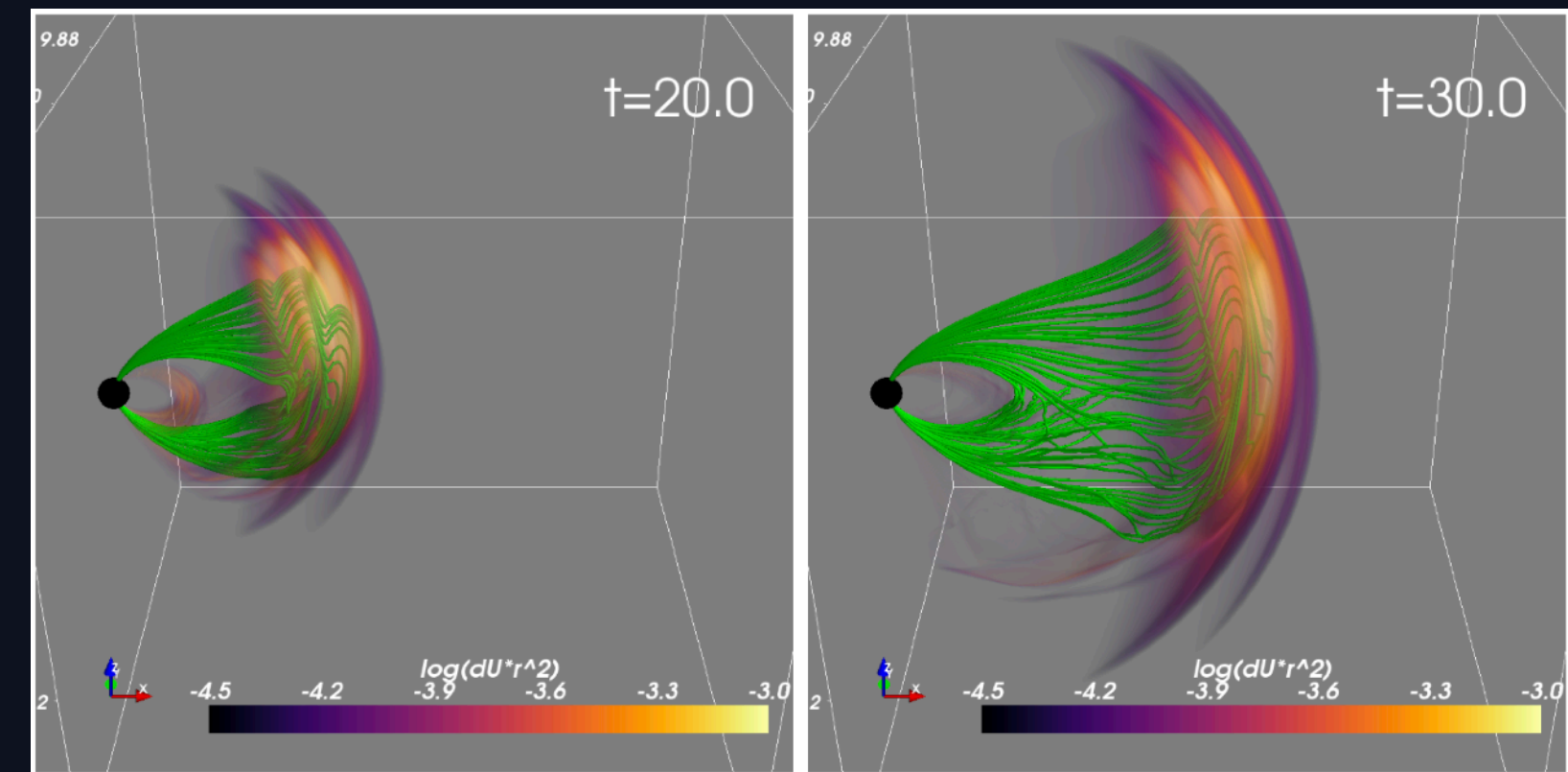


Magnetospheric transient models

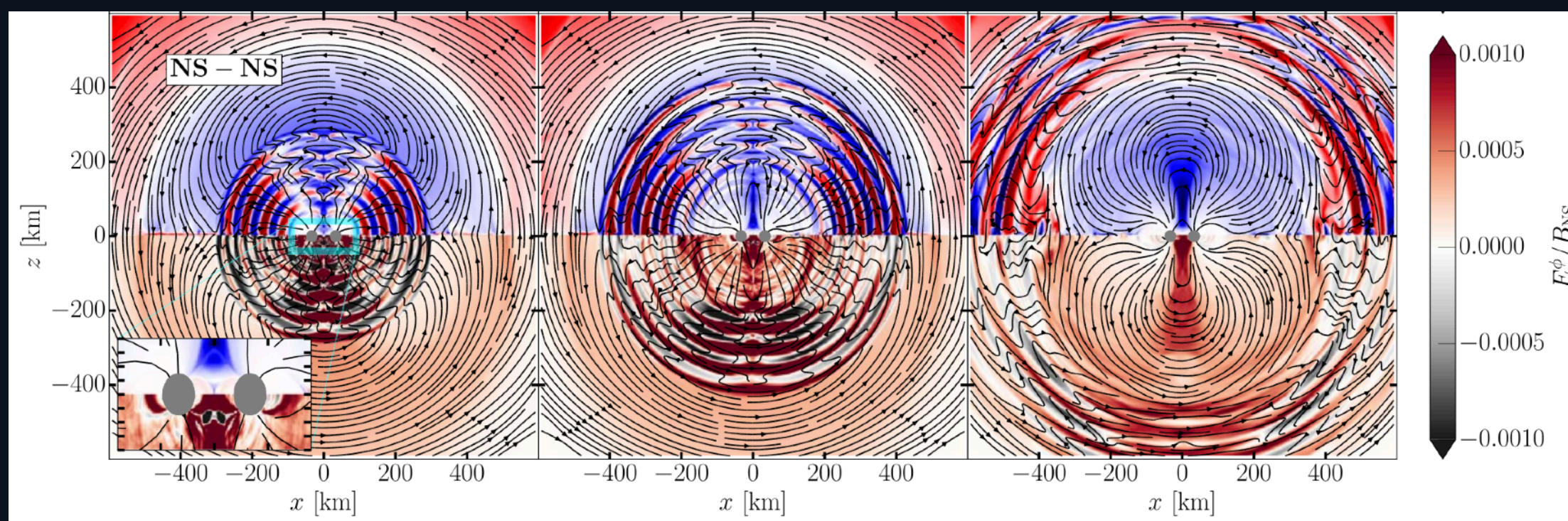
- pulsar emissions
- collapsing neutron stars
- compact binary merger precursors
- magnetar quakes



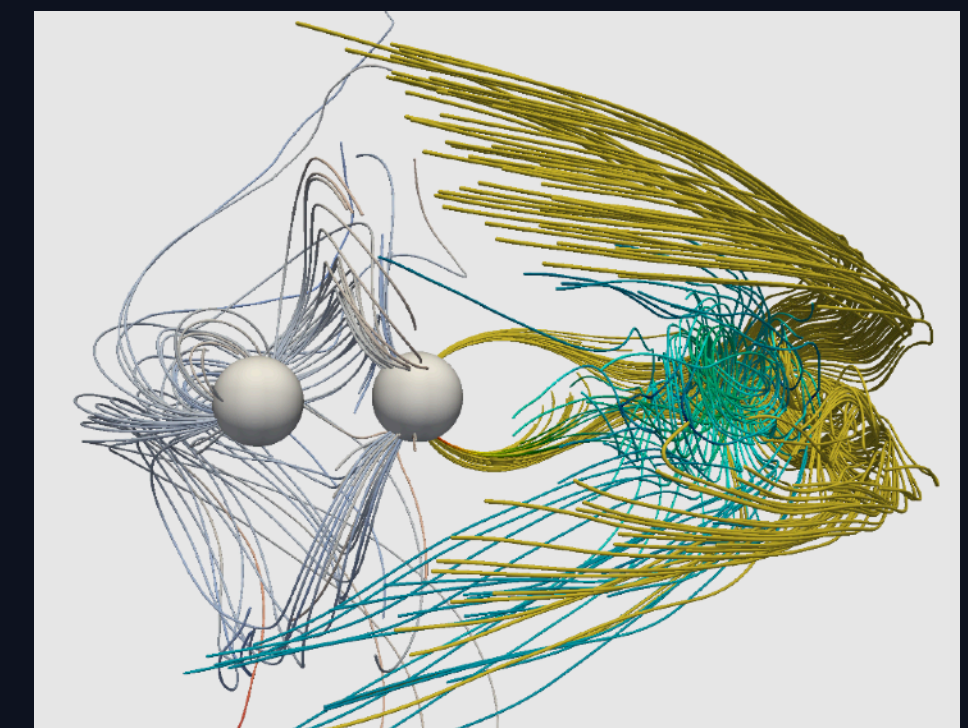
Parsons+2026



Yuan+2022

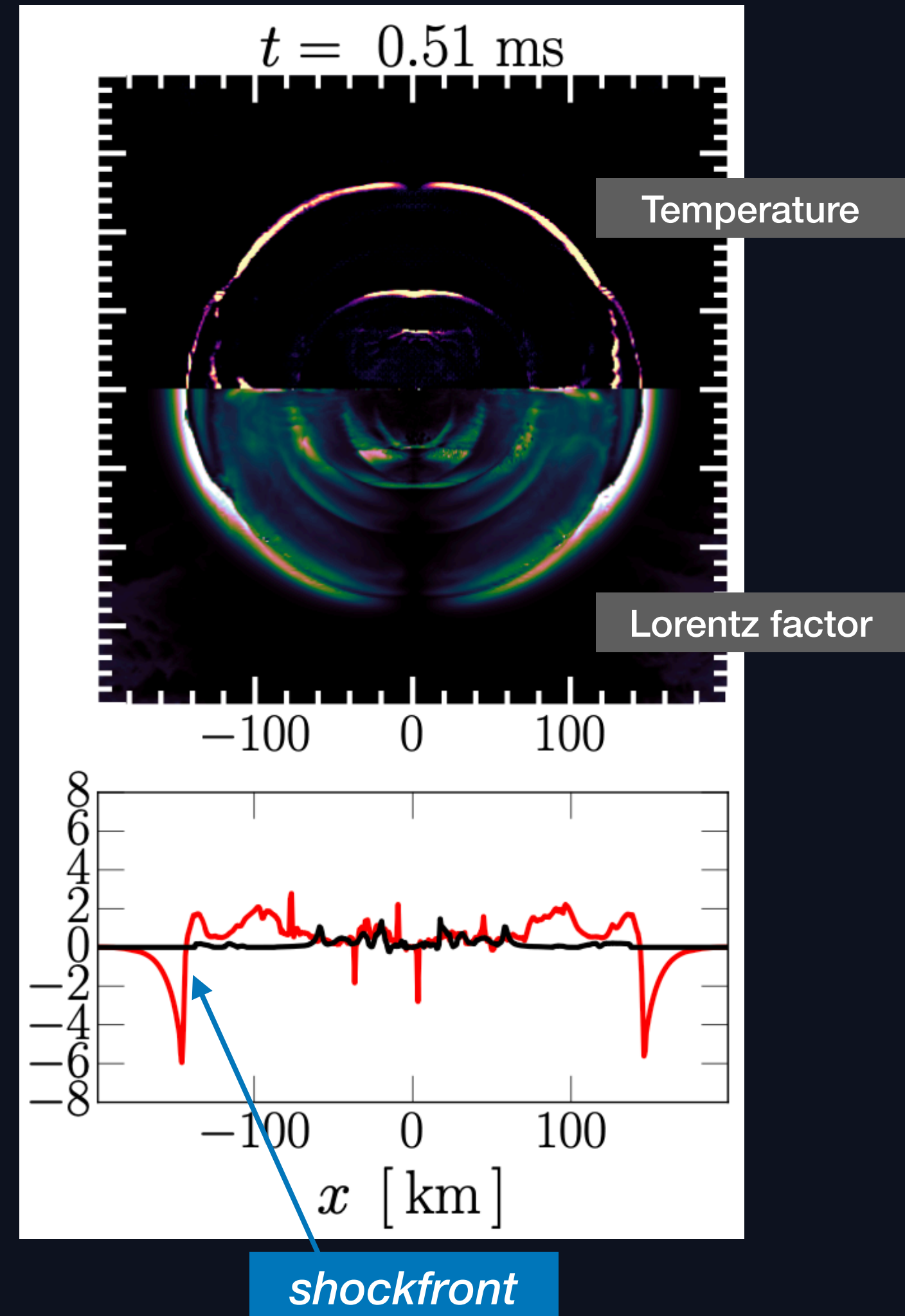
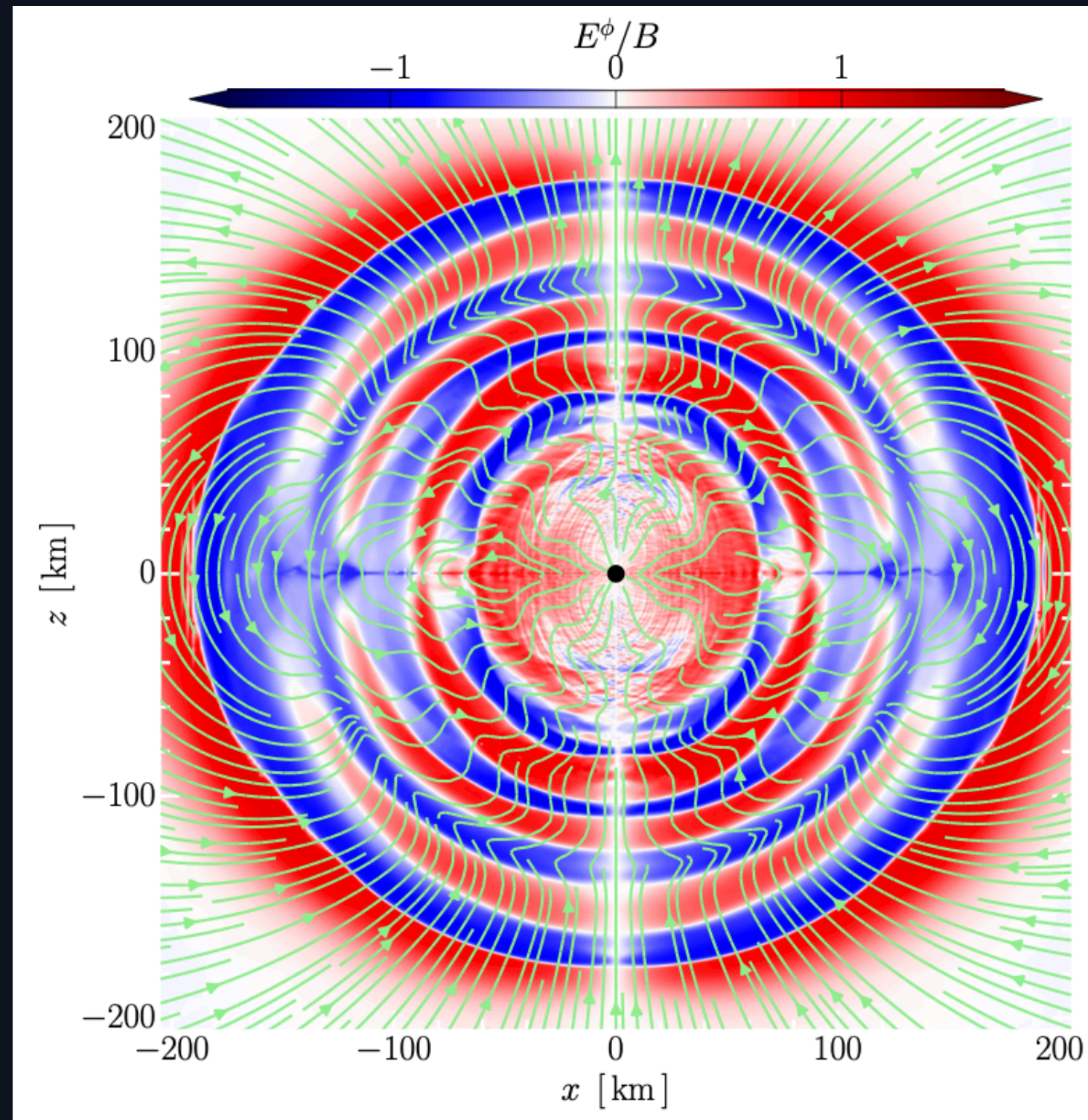


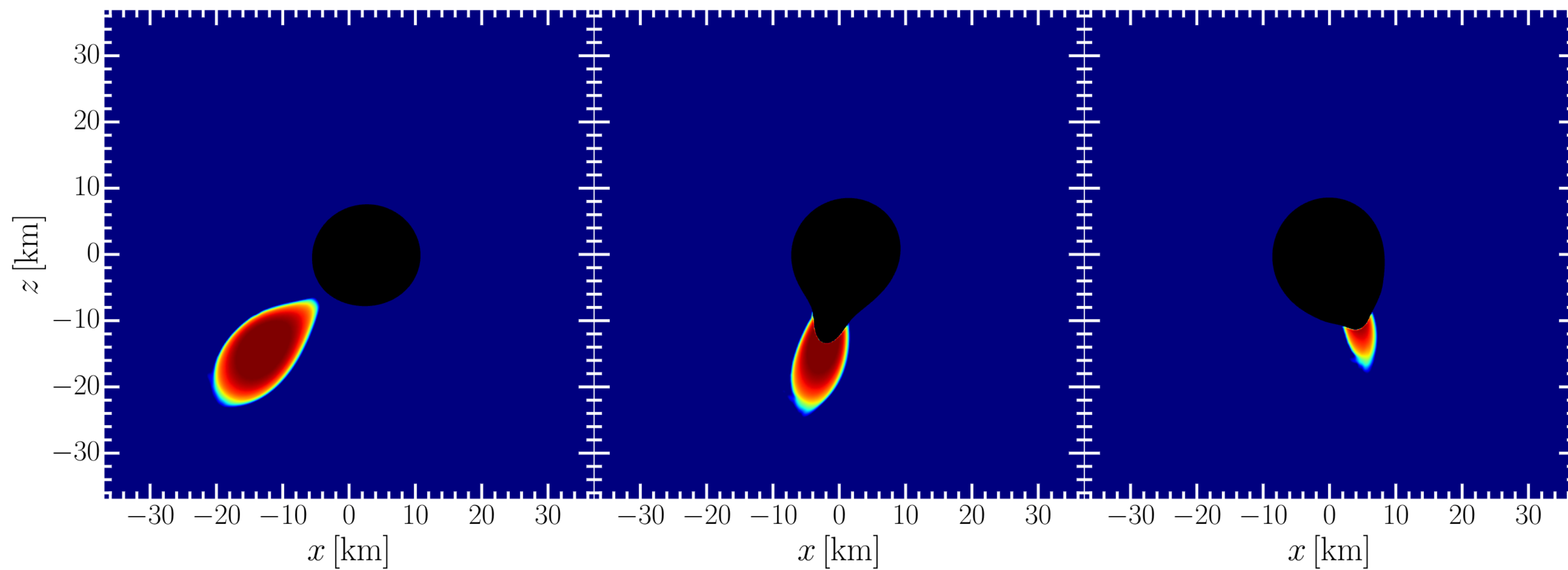
Most+ (incl. YK) 2024



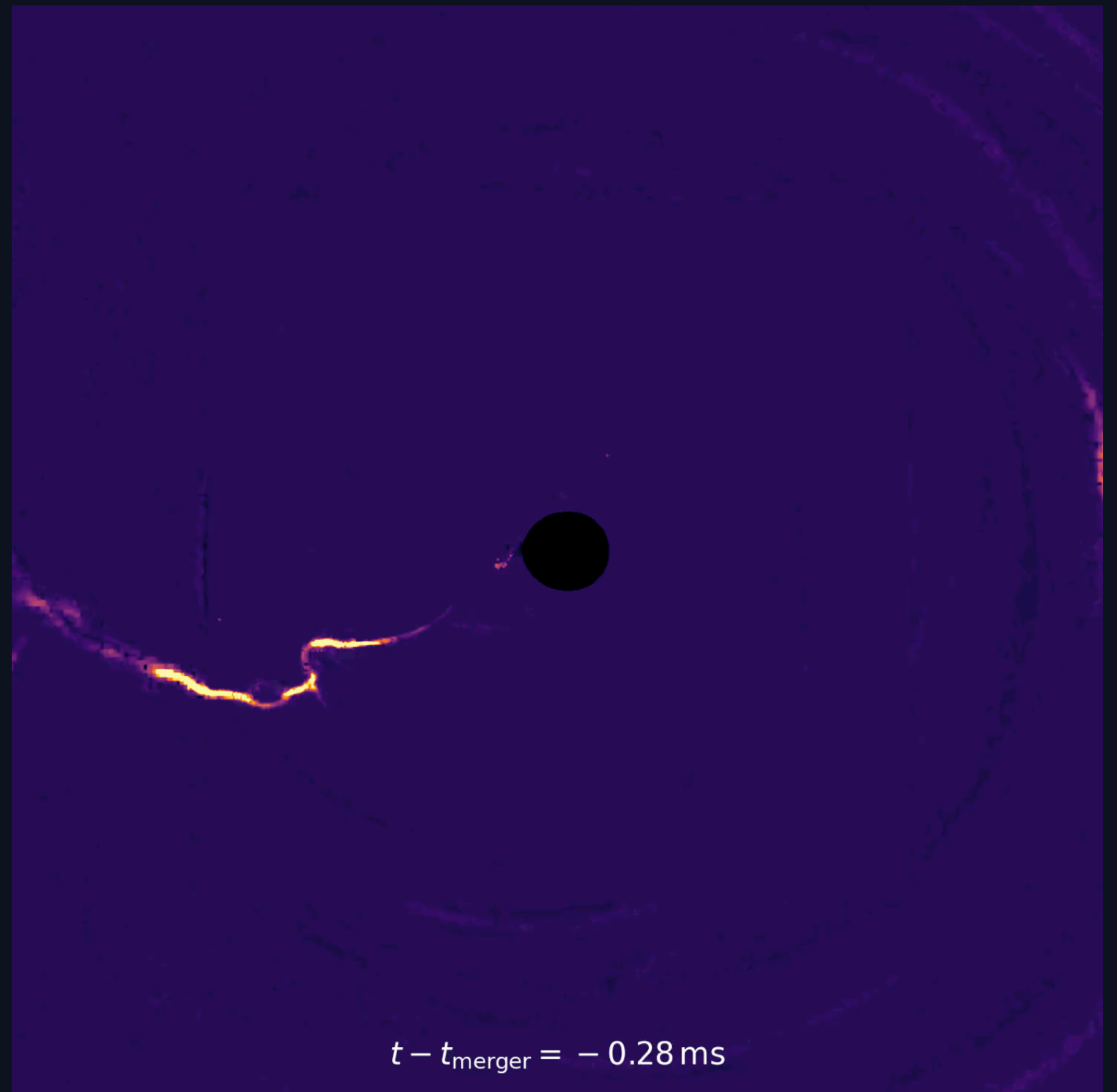
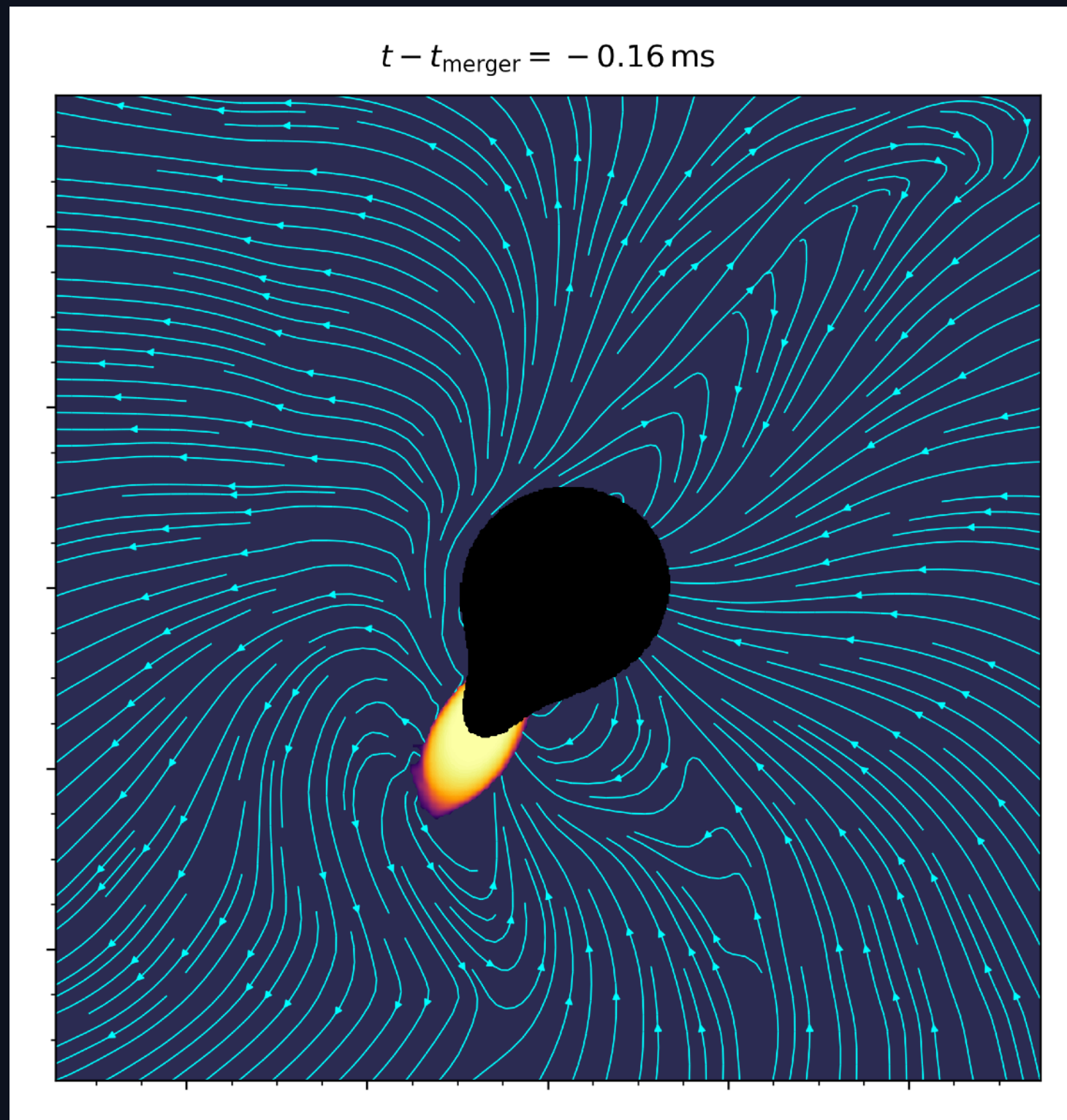
Most & Philippov 2023

Magnetospheric shockwaves from collapsing magnetars (Most+2024)

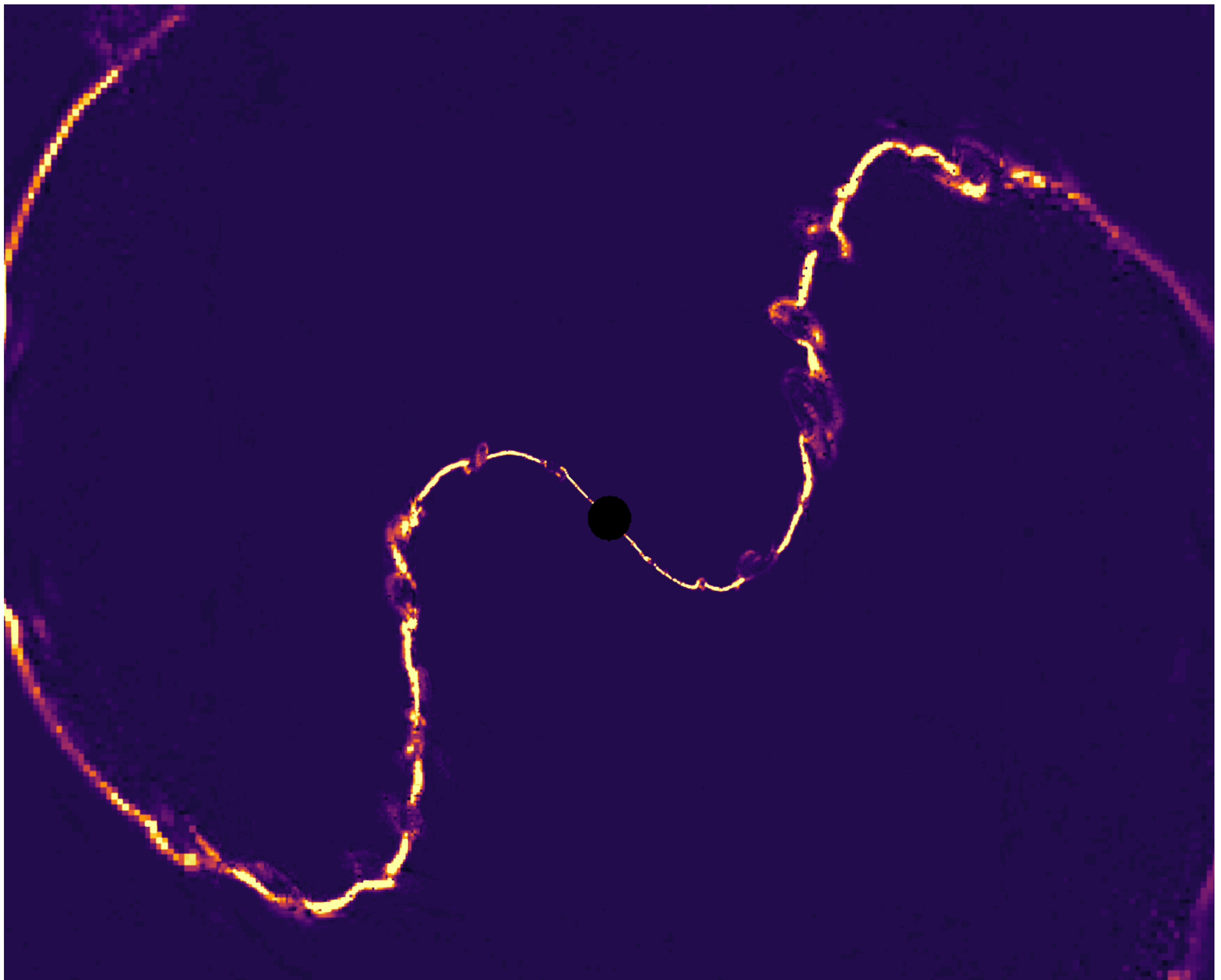




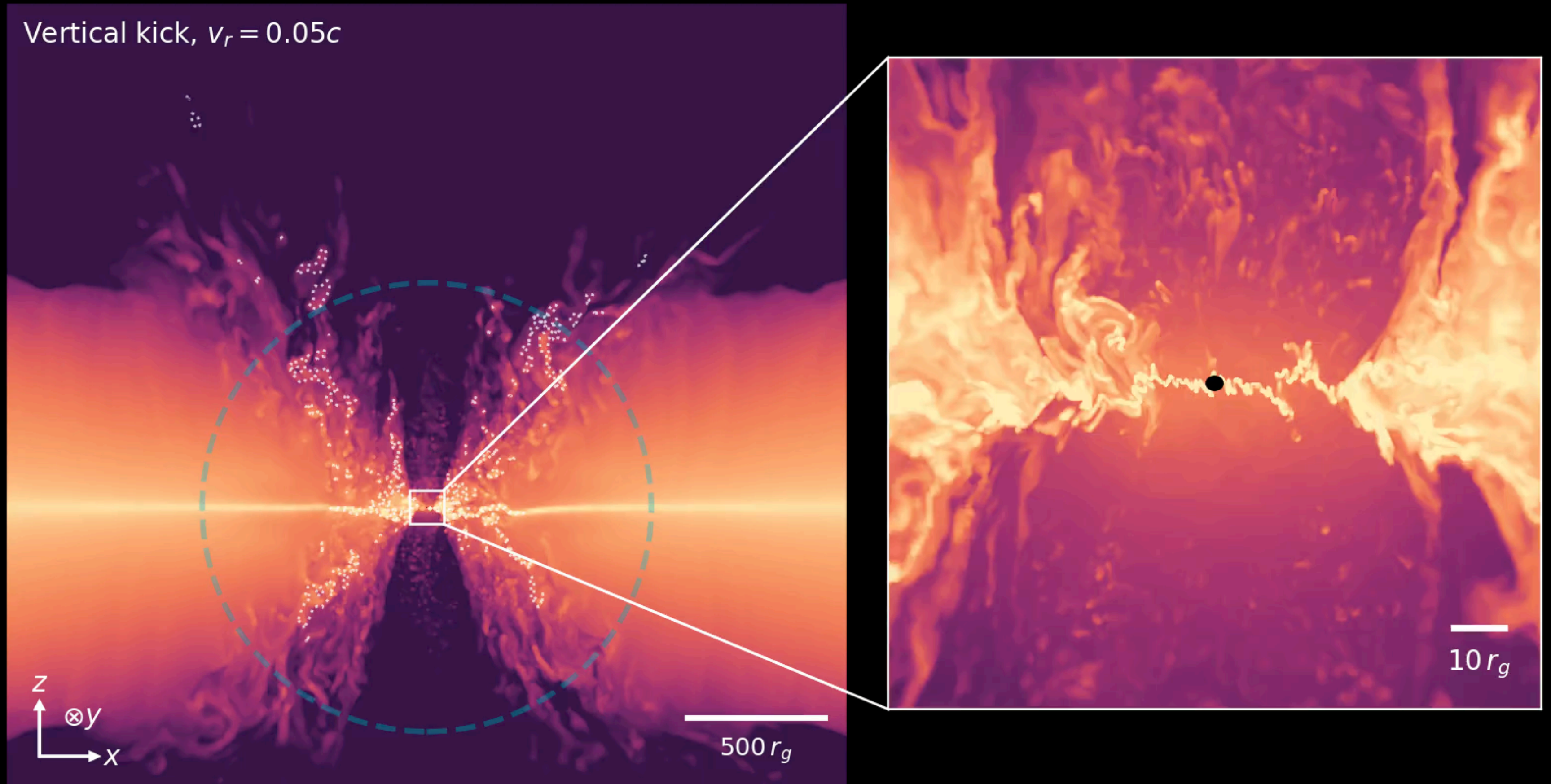
Formation of a transient ‘black hole pulsar’ from BH-NS merger



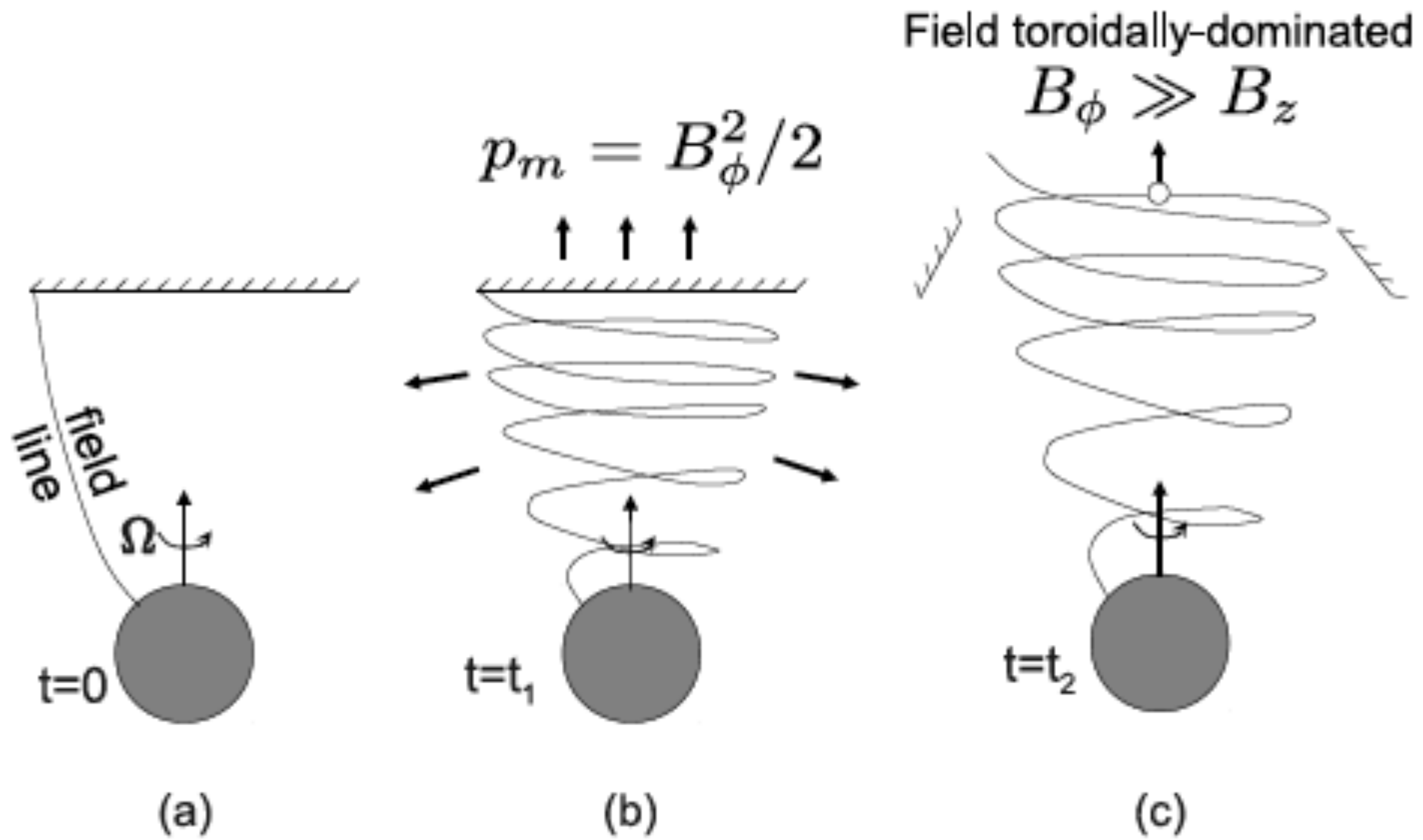
Y Kim+2025



Accretion onto a recoiled SMBH merger remnant



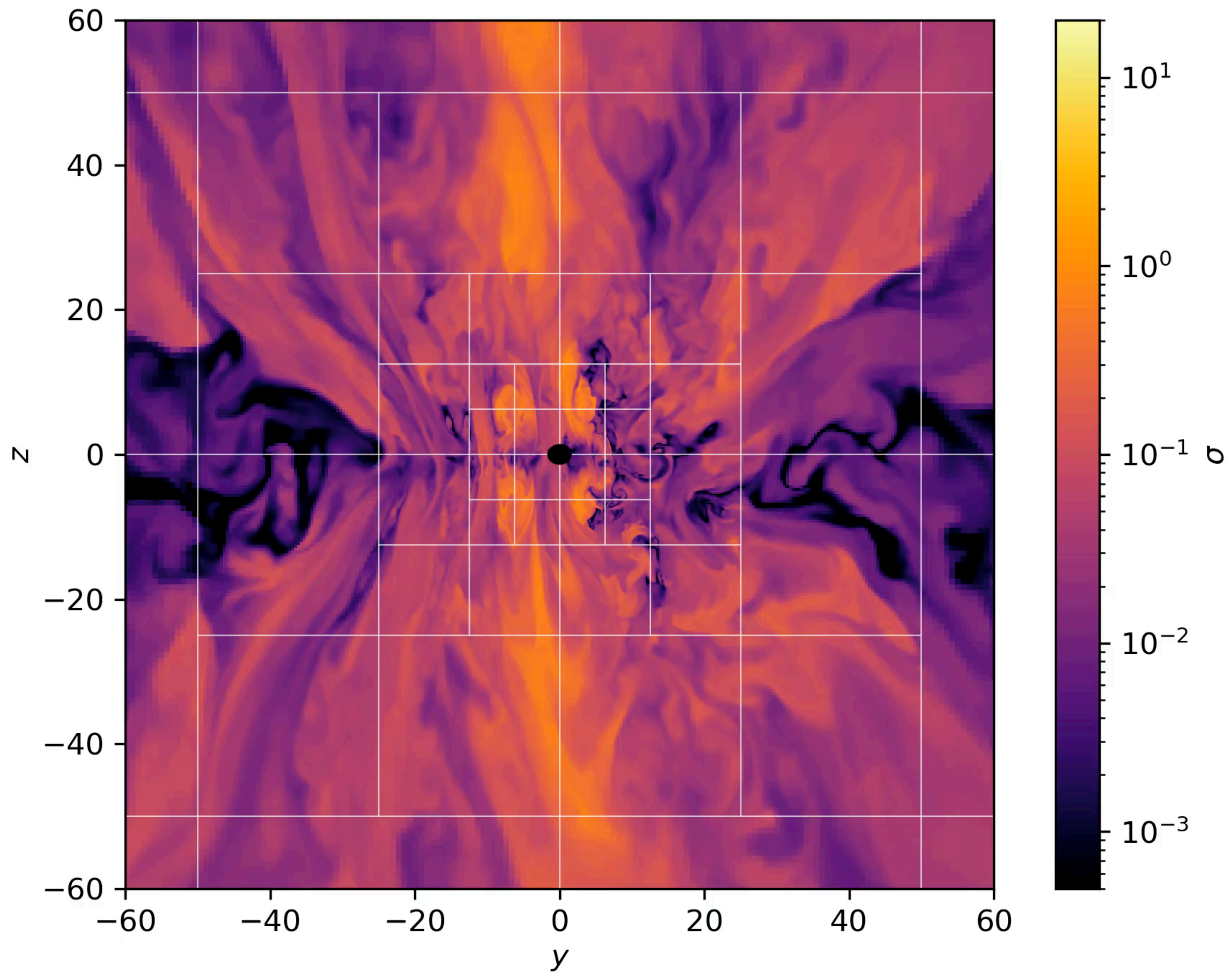
dashed line: $(h + b^2/\rho) u_t < -1$



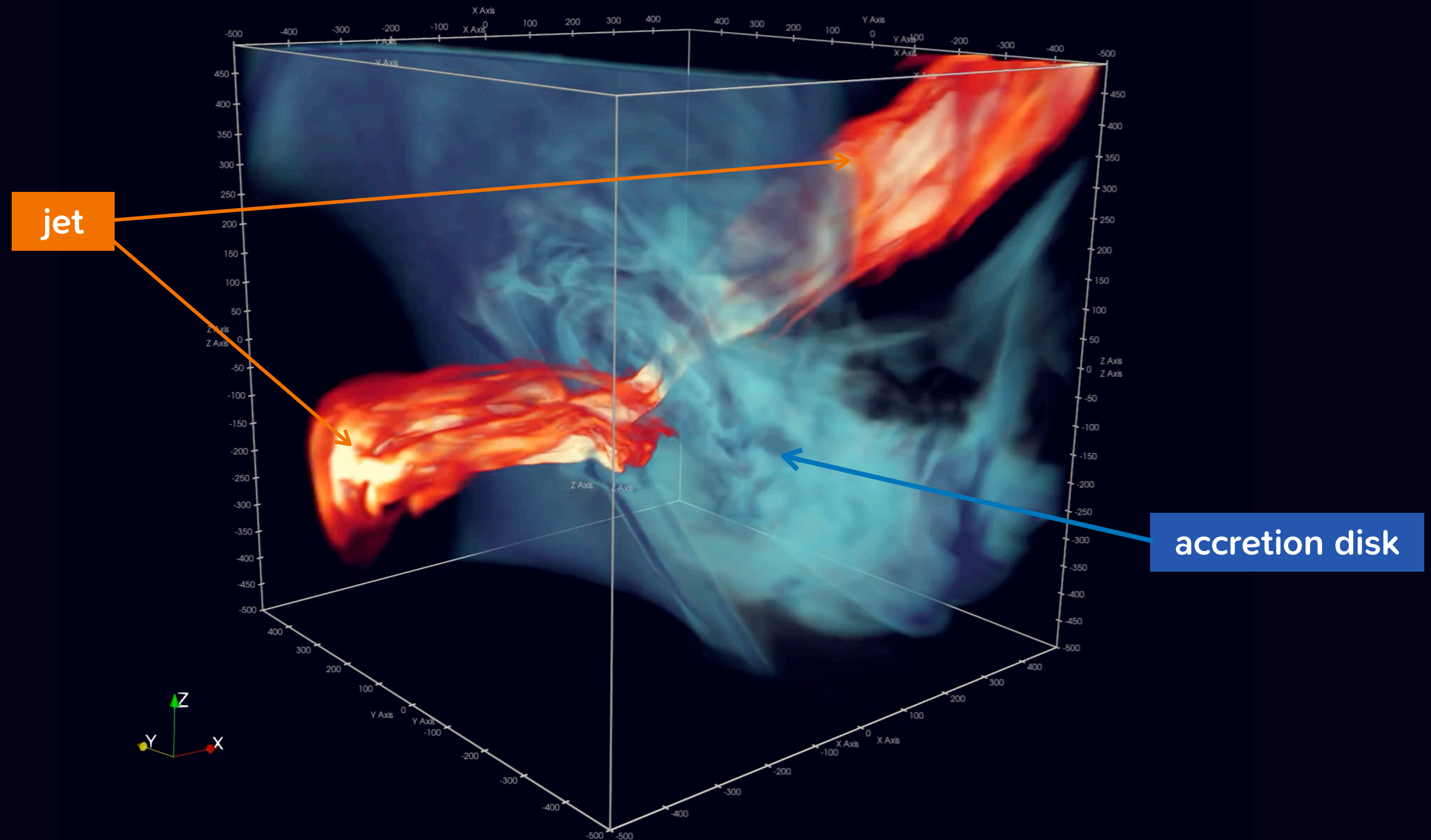


CREDIT
NASA/JPL-Caltech

spinning black holes can launch bipolar outflows : **jets**



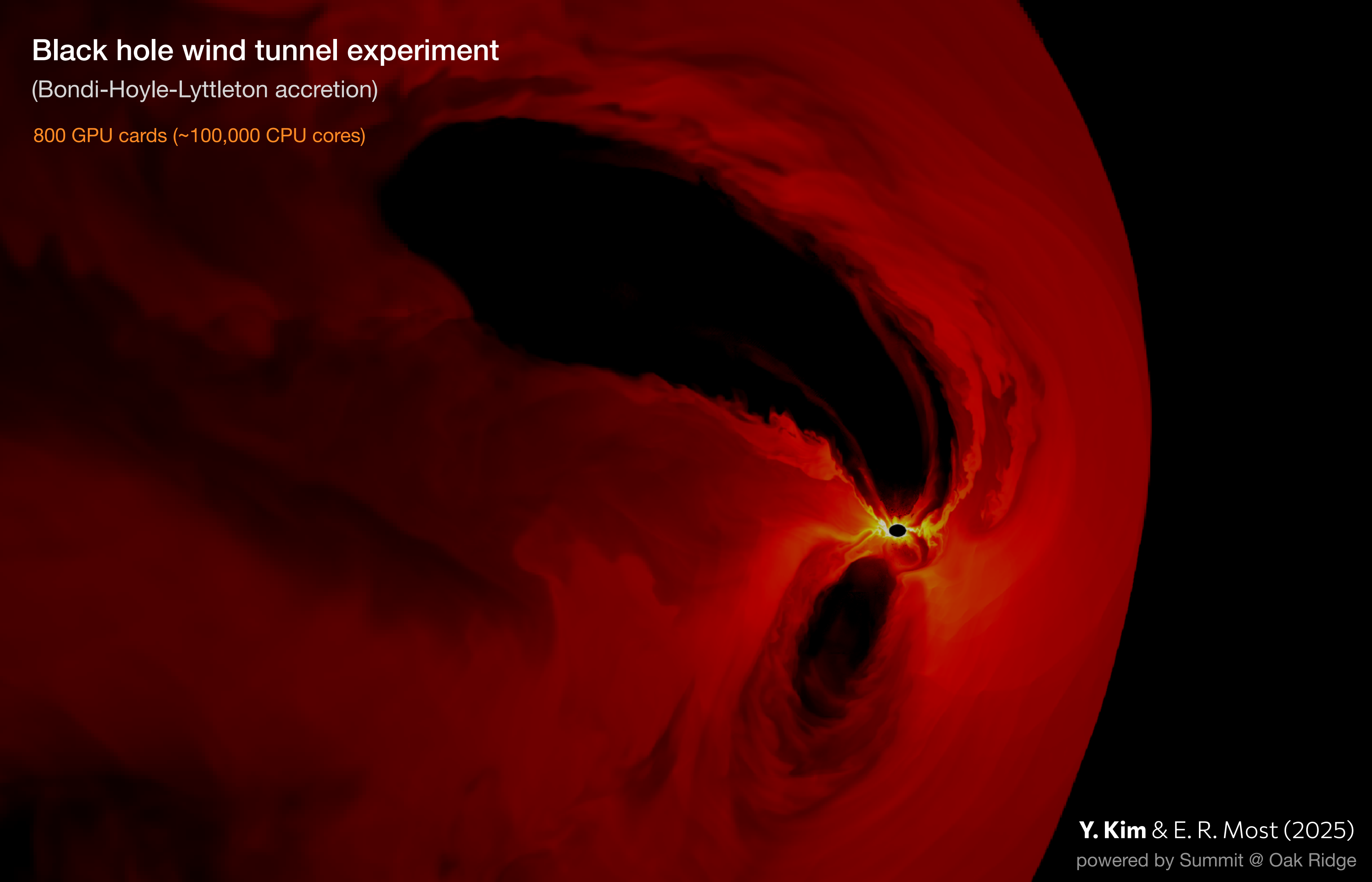
GRMHD simulation of tilted accretion disk and jets



Black hole wind tunnel experiment

(Bondi-Hoyle-Lyttleton accretion)

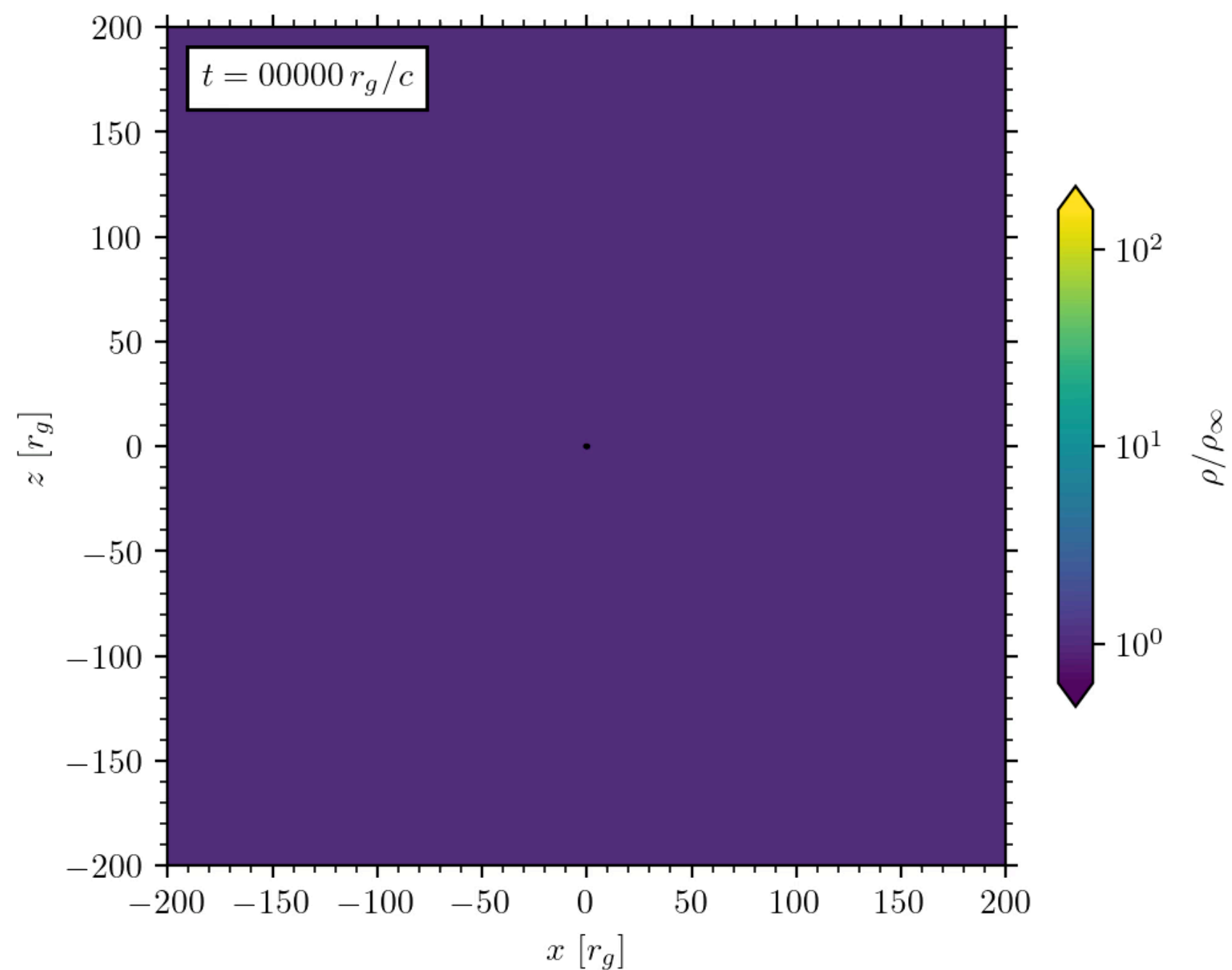
800 GPU cards (~100,000 CPU cores)



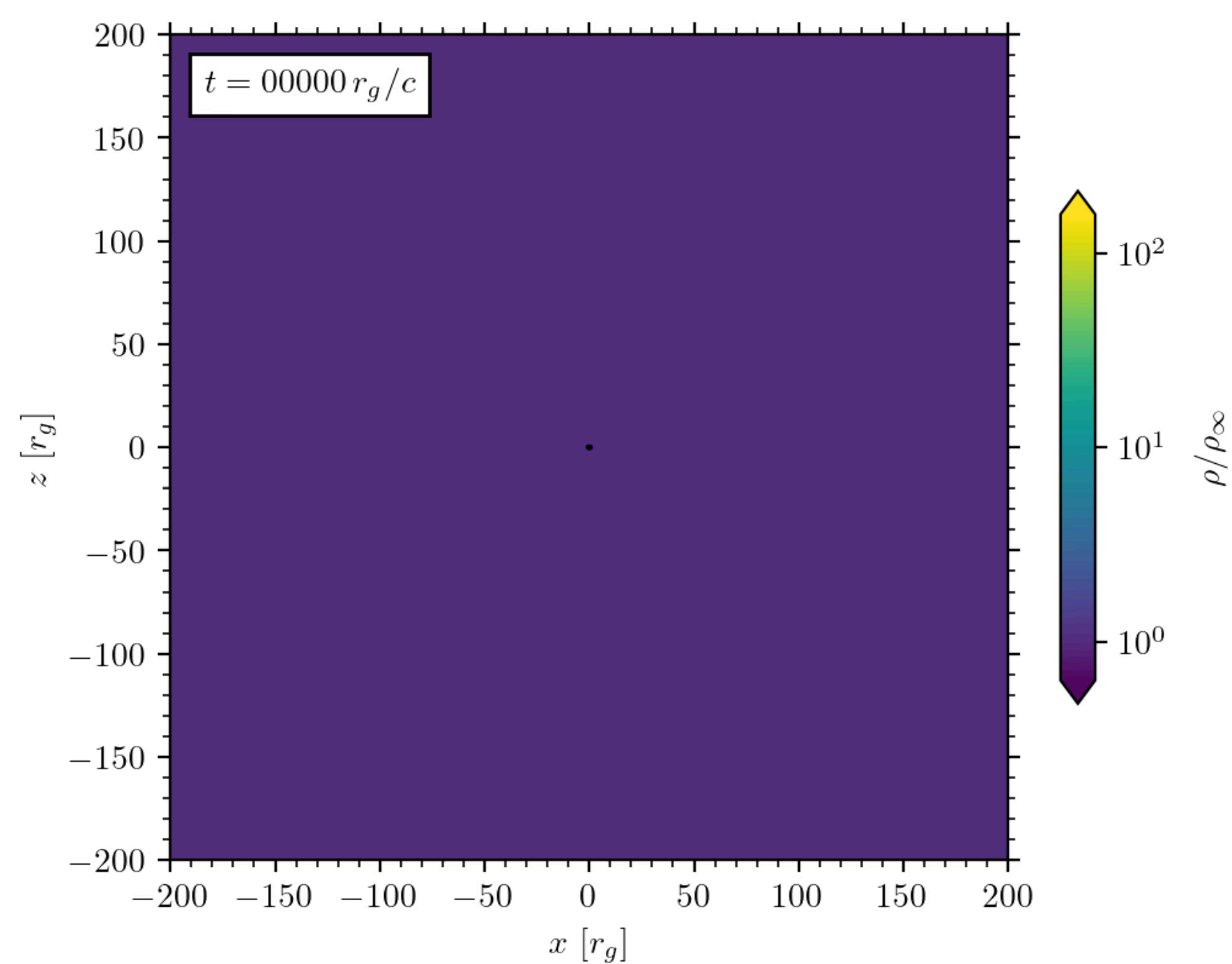
Y. Kim & E. R. Most (2025)

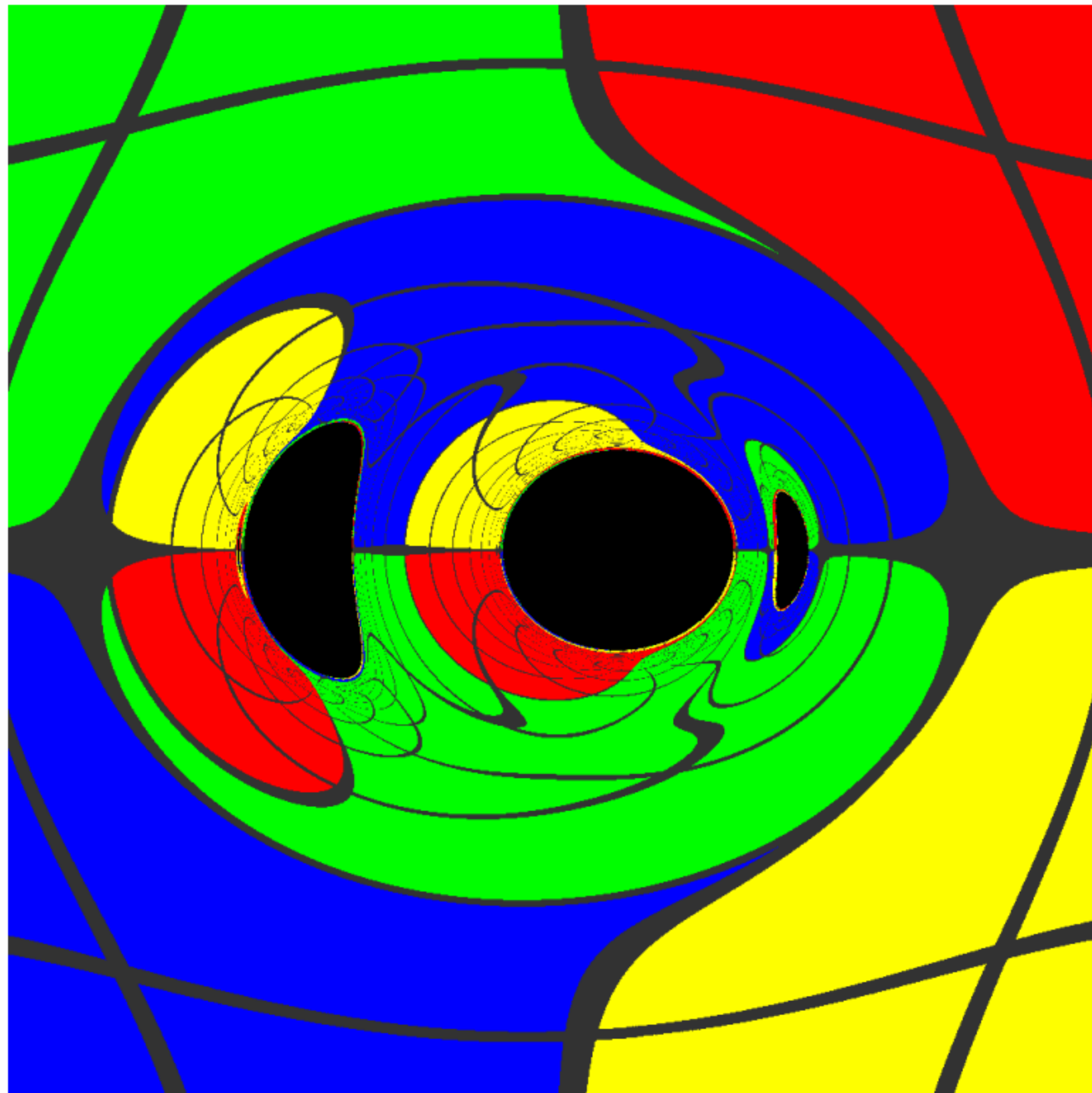
powered by Summit @ Oak Ridge

Mass density

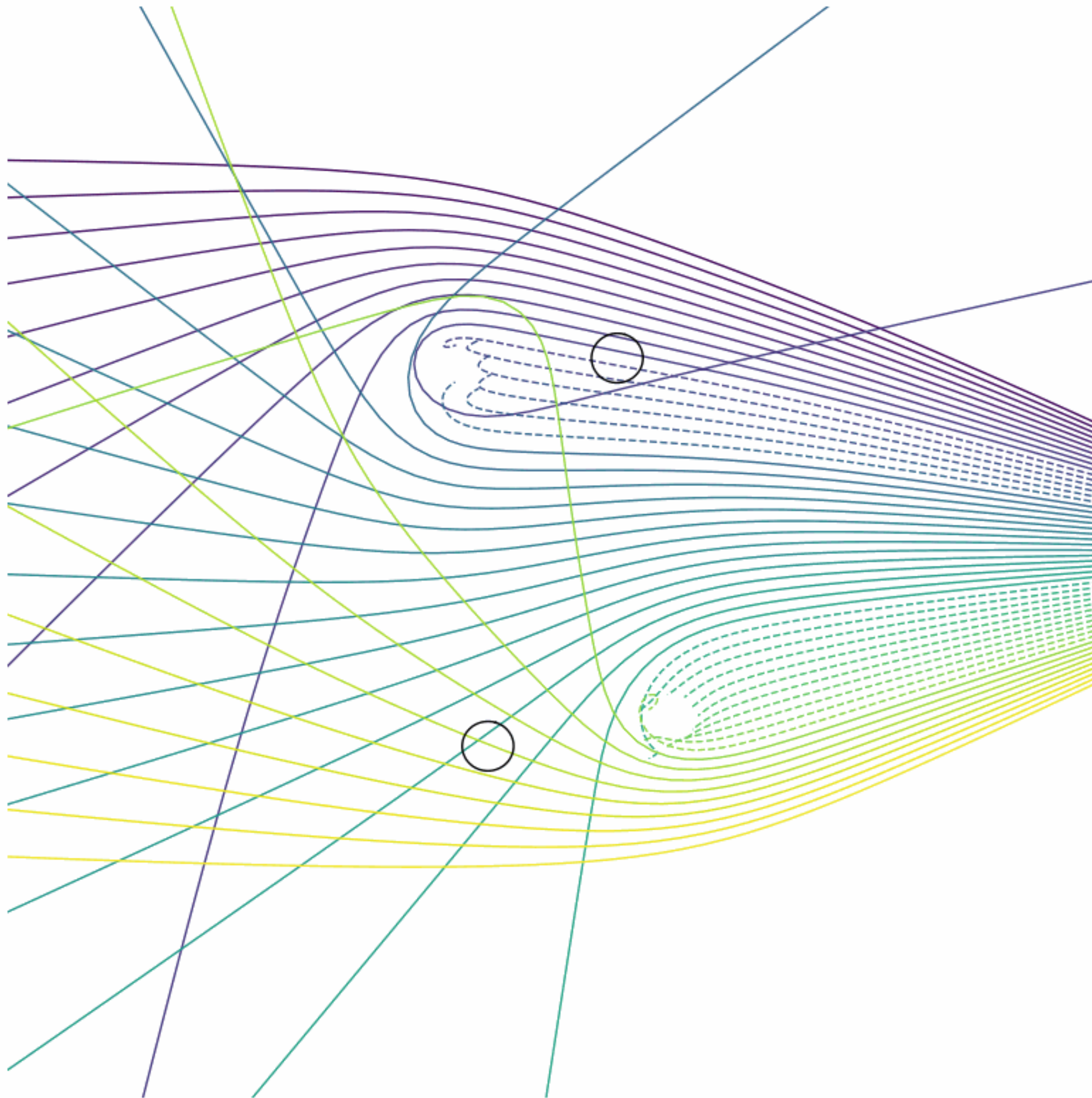


Magnetization





Courtesy of Nils Vu



Courtesy of Nils Vu

some other stuff

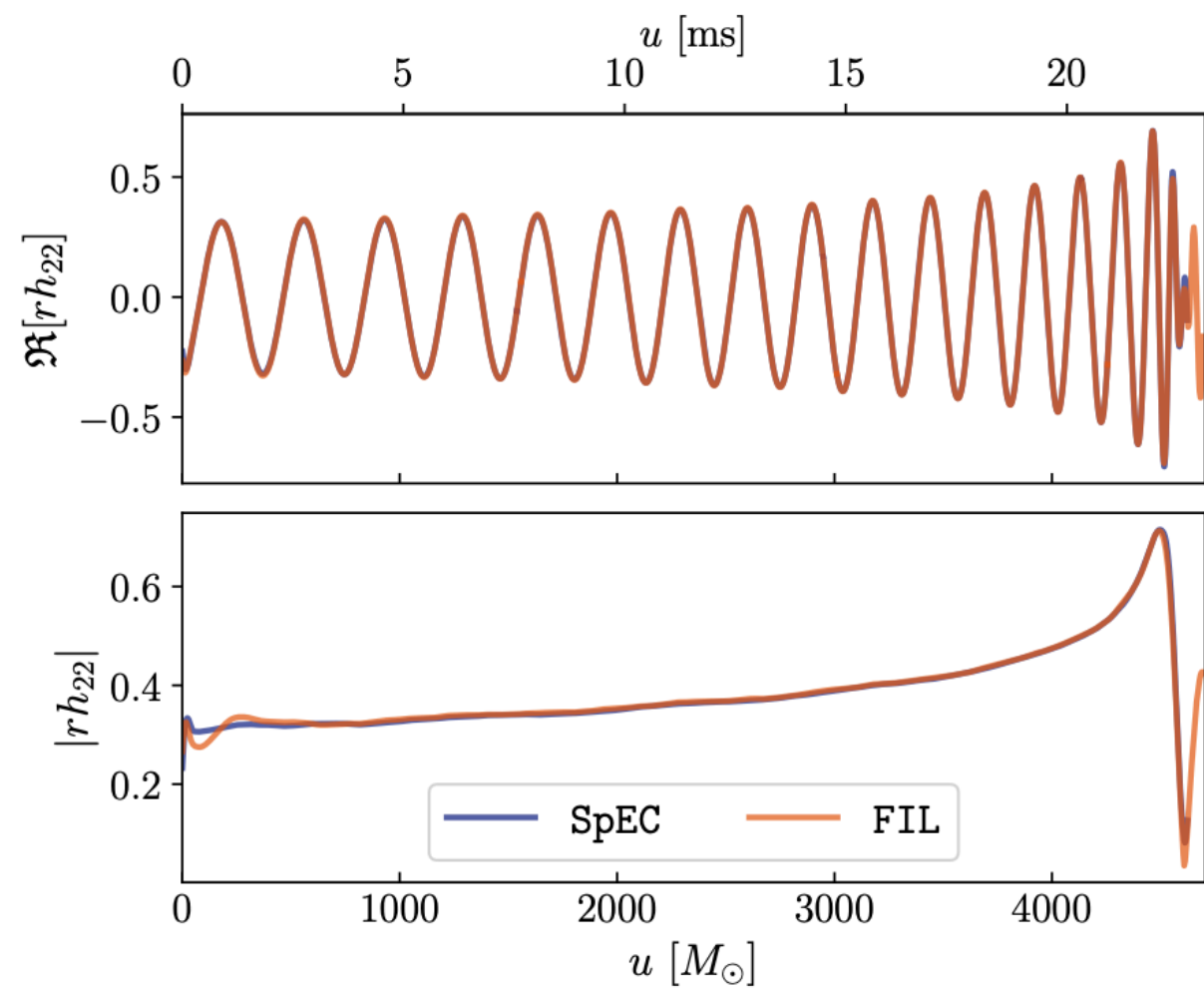
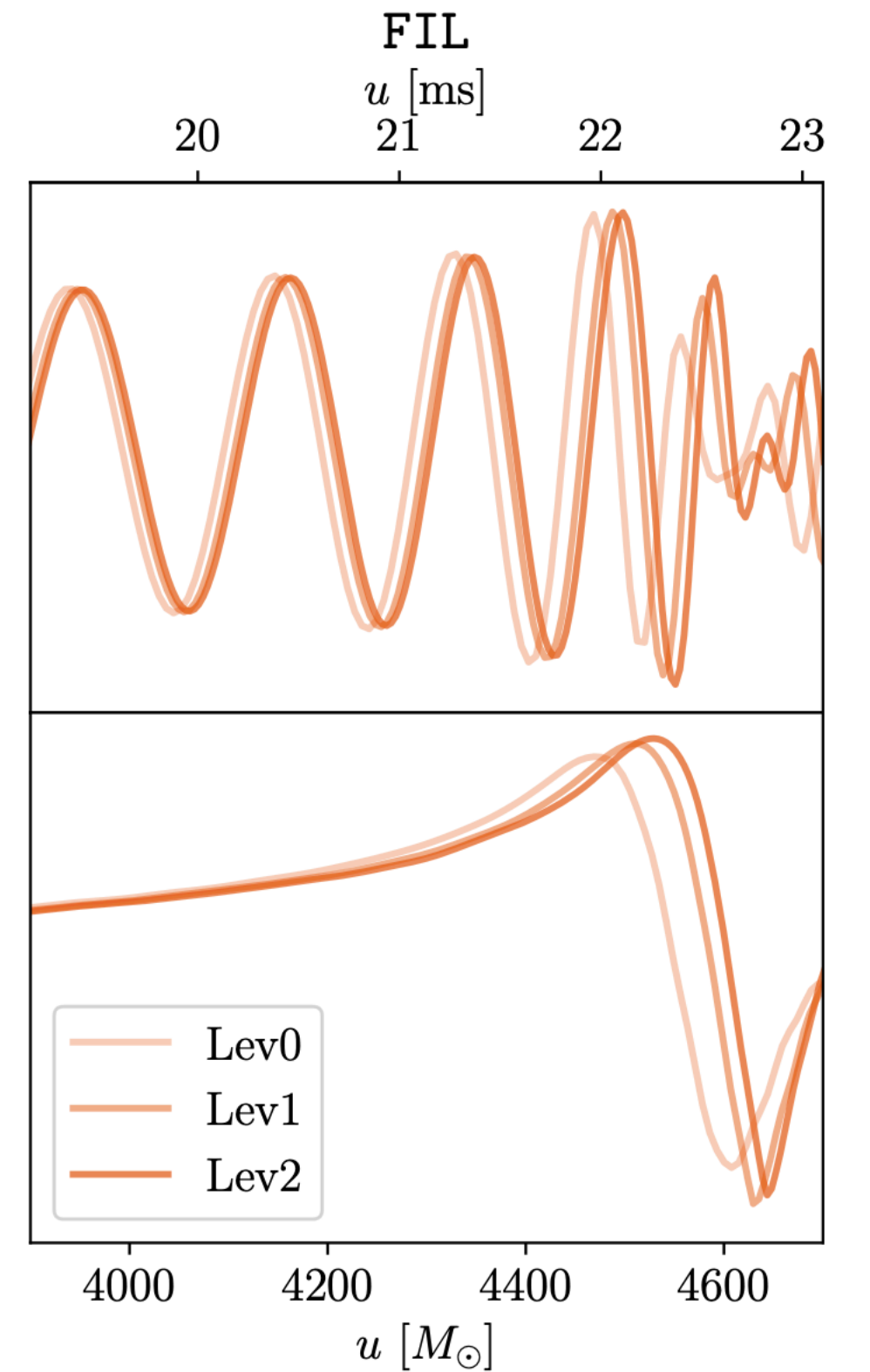
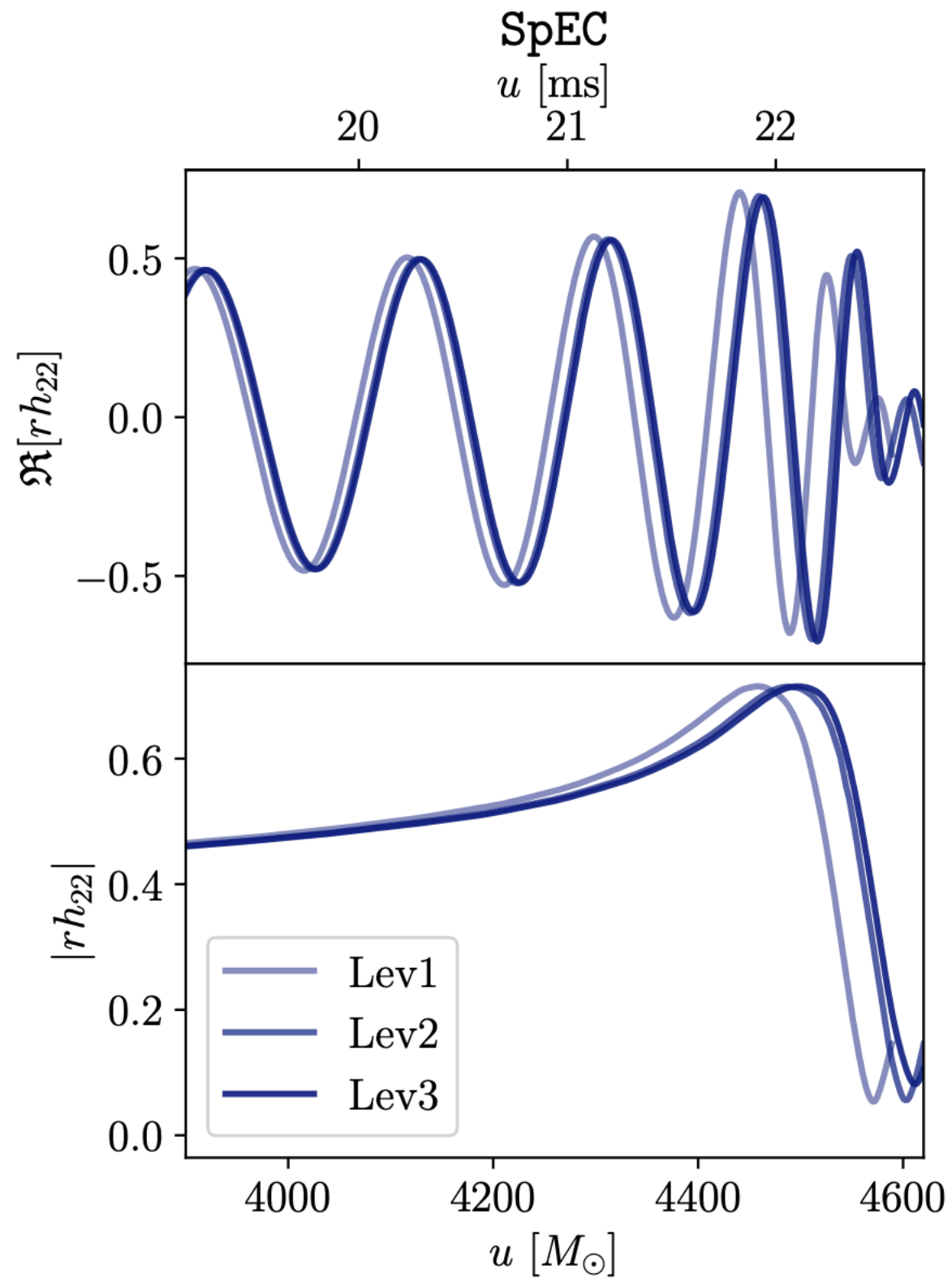
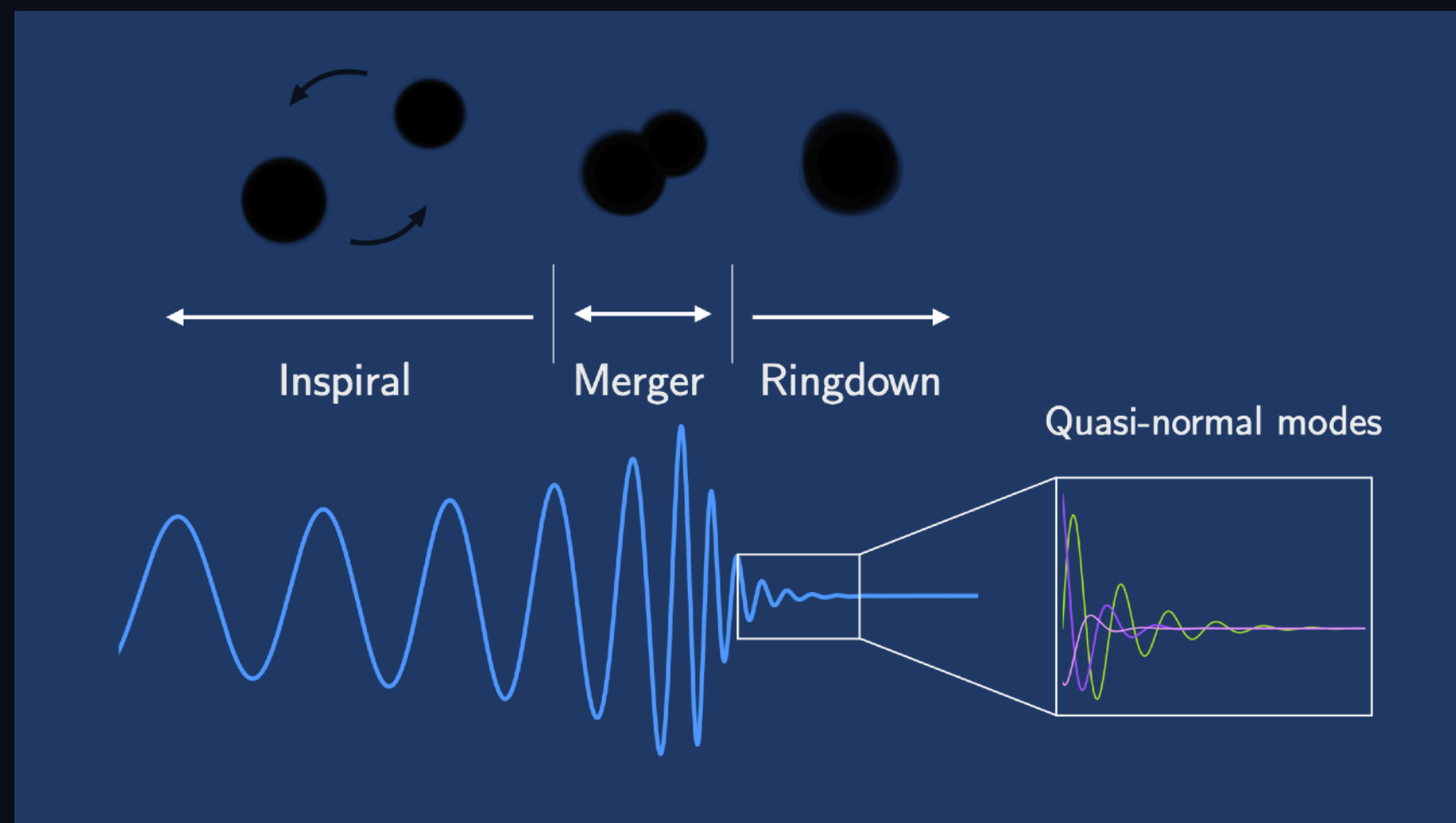


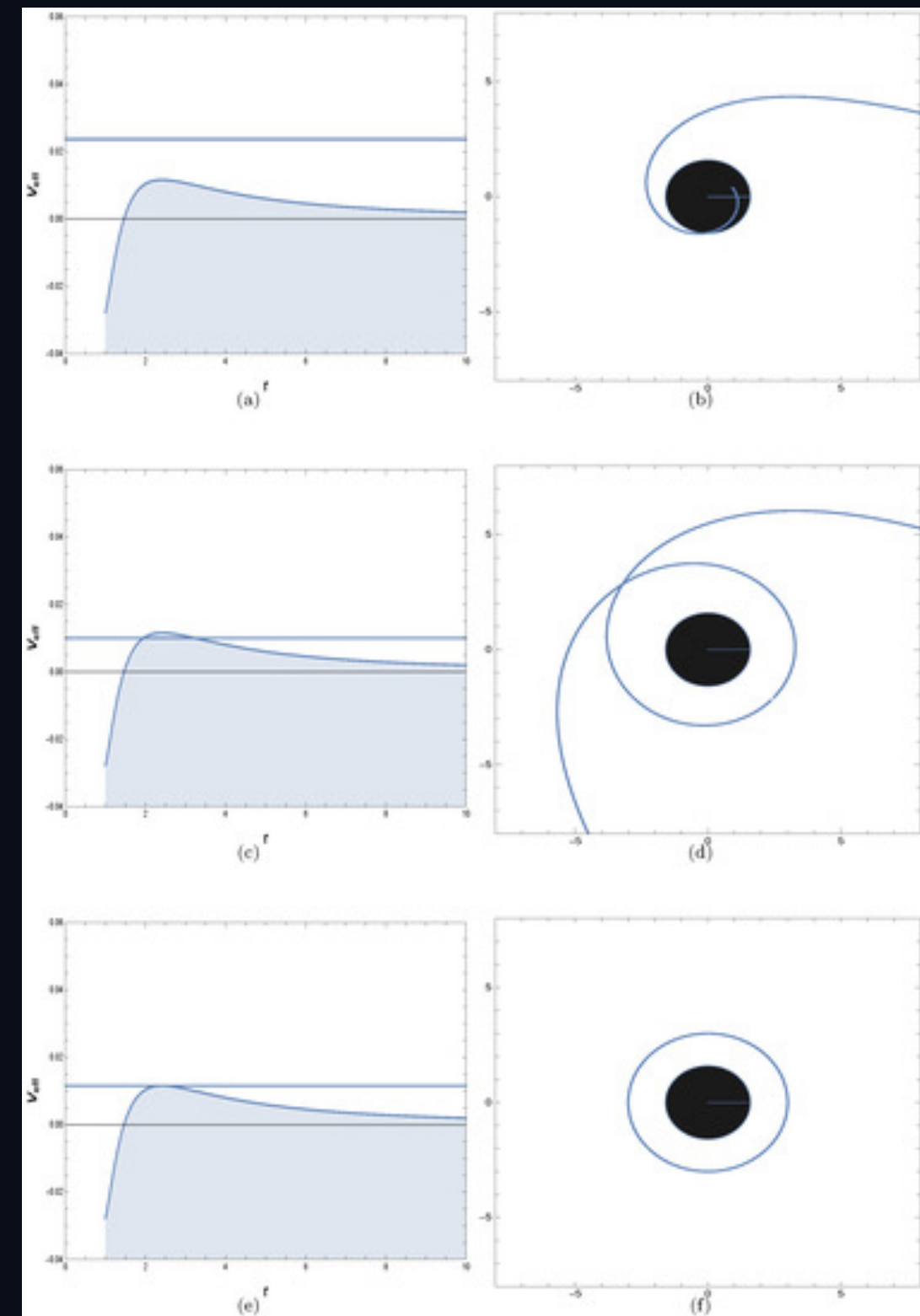
Figure 1. Real part of the $l = 2, m = 2$ strain mode $\Re(rh_{22})$ and amplitude of the $l = 2, m = 2$ strain mode $|rh_{22}|$ for the highest resolution runs available from each code (SpEC Lev3, and FIL Lev2). Runs are aligned to SpEC Lev3 with a time and phase shift. All waveforms generally show close agreement during inspiral and reach merger at similar times, having similar cycle evolutions, peak amplitudes, and inspiral lengths.



Not all problems require numerical relativity



<https://kiaa.pku.edu.cn/info/1031/10642.htm>

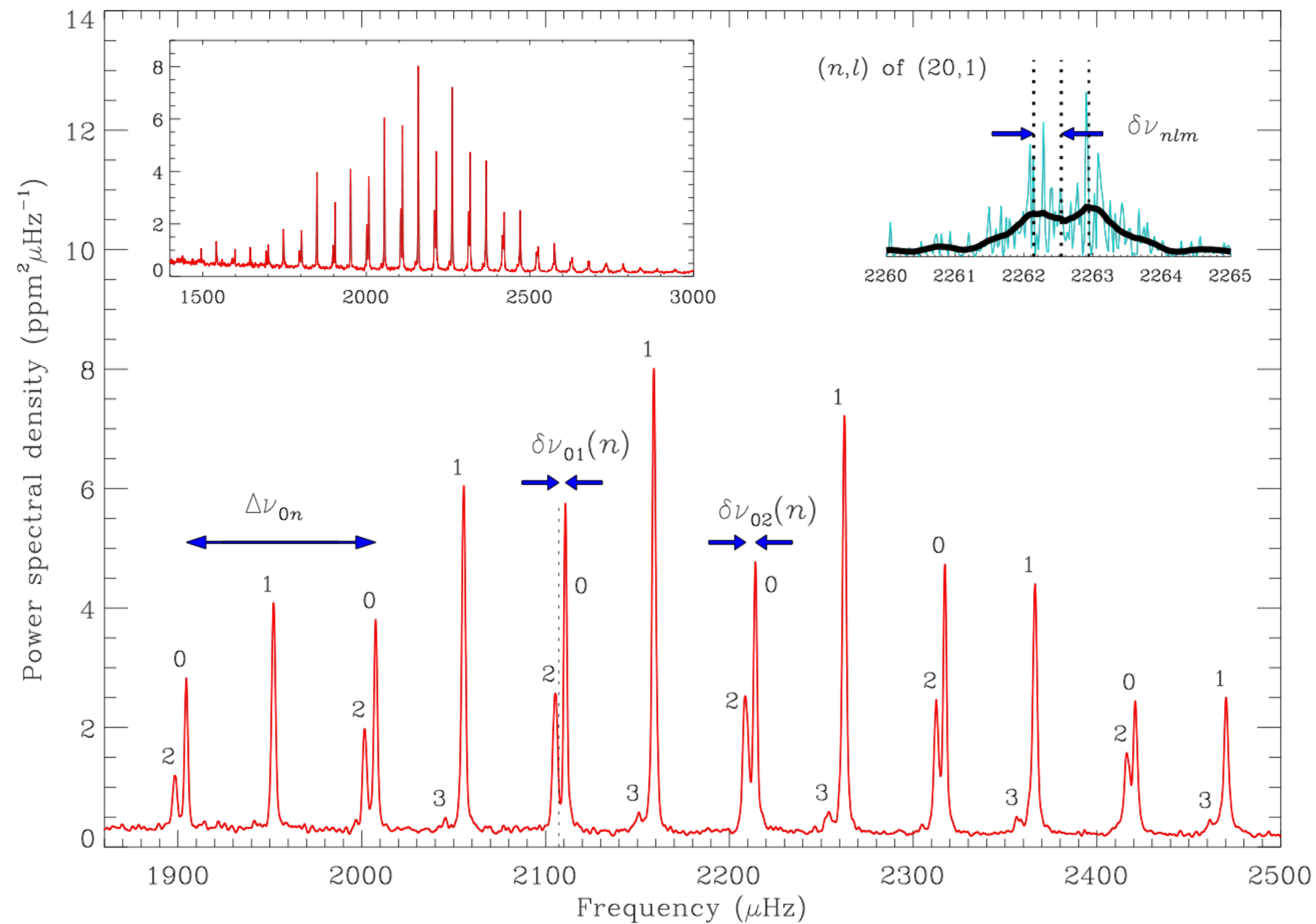


<https://www.mdpi.com/2073-8994/14/2/309>

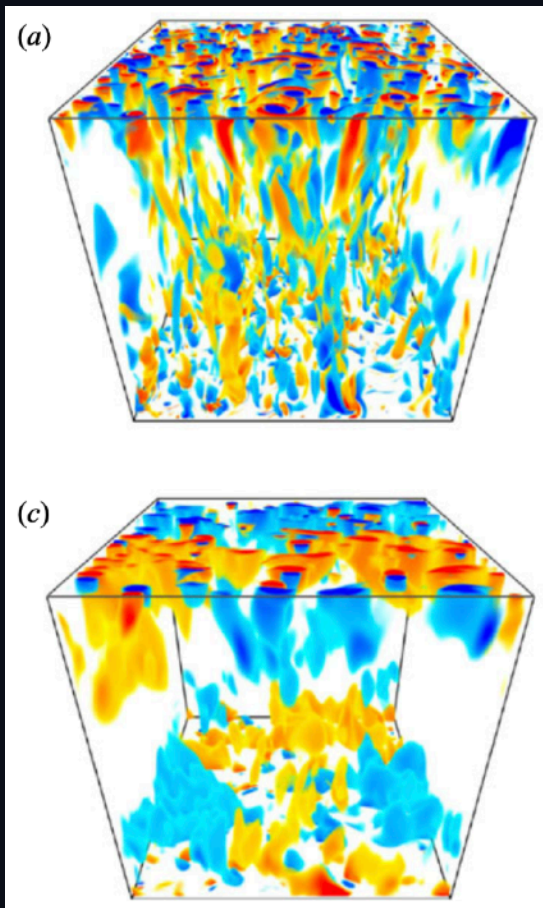
Not all problems **require numerical approaches**

82

Asteroseismology of Solar-Type and Red-Giant Stars



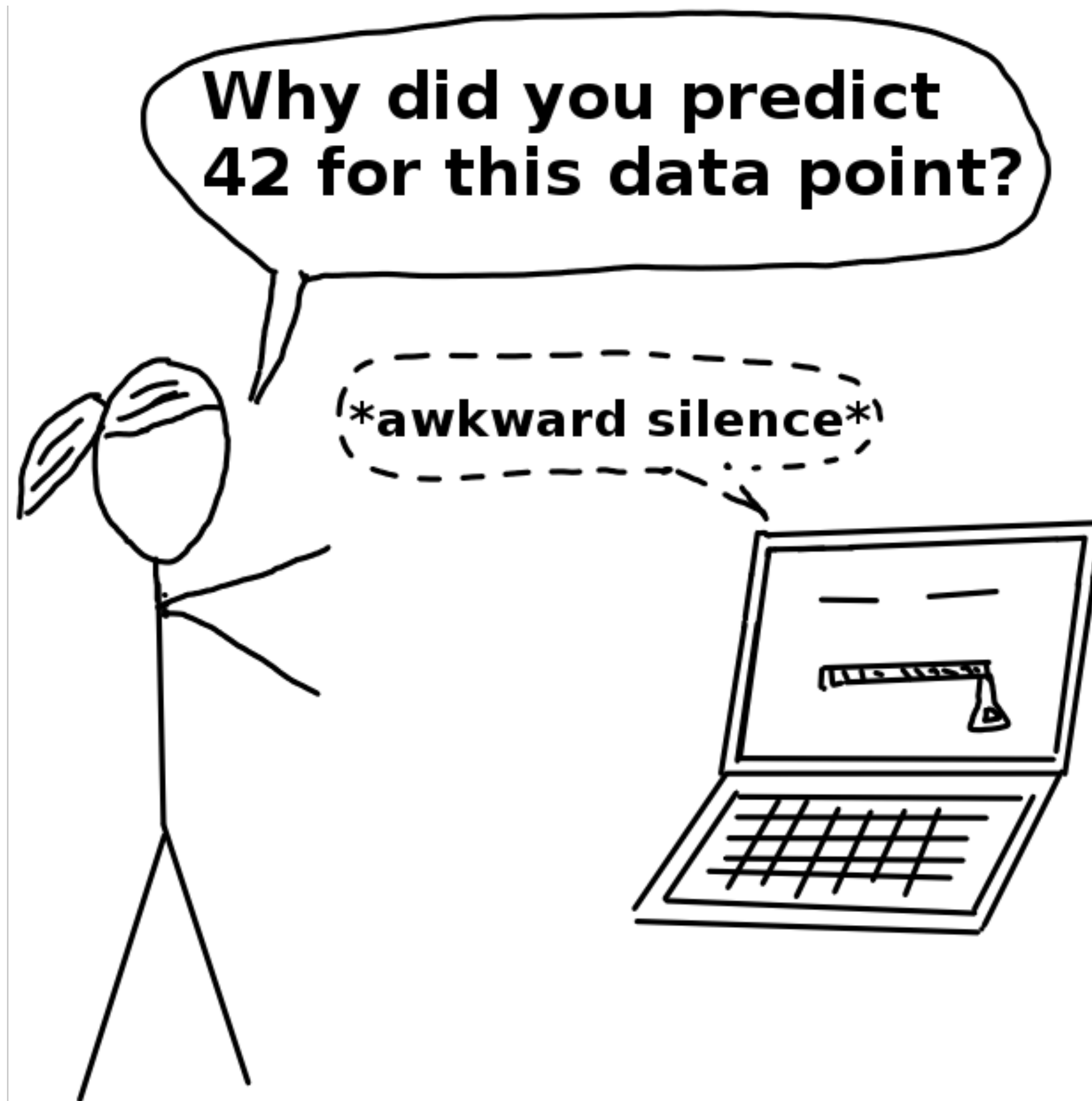
Not all problems can be **done numerically**



The previous discussion has emphasised that the current status of the theory of turbulent dynamos is uncertain, both for the geodynamo and for its astrophysical counterparts. Despite the heroic efforts to solve the relevant equations numerically, computational efforts are limited to parameter regimes that are far from those that pertain to naturally

occurring dynamos. Perhaps this is best emphasised by a recent calculation by P. Käpylä (private communication). He demonstrates that, even if the computational resources were available to undertake numerical simulation of turbulent dynamos, the power required to simulate a star such as the Sun is 10^{22} W. This is equivalent to the luminosity of a M9V main sequence red dwarf. Although this is a fairly cool star, the point is well made.

simulating the sun requires the energy output of another star



Source: Interpretable Machine Learning by Christoph Molnar