GW-BASIC

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Abstract

This lecture introduces a basic theory for gravitational waves (GWs). We derive the wave equation from the Einstein equation using relativistic perturbation theory. The polarization of GWs is explored by an analogy with electromagnetic waves (EMWs). To give a theoretical basis for detection of GWs by EMWs, we introduce the perturbation of geometrical optics and its beyond.

1 Perturbations

1.1 Mathematical Setup

Let $\mathcal{F}$ be a 5-dimensional manifold that is foliated by a one-parameter family of 4-dimensional manifolds $\mathcal{M}_\lambda$ where λ is a parameter. We introduce a one-parameter group of diffeomorphisms $\phi_\epsilon : \mathcal{F} \to \mathcal{F}$ with a parameter ϵ such that $\phi_\epsilon \left[\mathcal{M}_\lambda\right] = \mathcal{M}_{\lambda+\epsilon}$. We define $\tilde{Q}\left(\epsilon\right)$ as a pull-back of a geometrical quantity Q through ϕ_ϵ given by

$$\tilde{Q}\left(\epsilon\right) \equiv \phi^*_{-\epsilon} Q. \tag{1}$$

Its Taylor expansion at $\epsilon = 0$ is given by

$$\tilde{Q}\left(\epsilon\right) = Q + \epsilon \dot{\tilde{Q}}\left(0\right) + \frac{1}{2}\epsilon^2 \ddot{\tilde{Q}}\left(0\right) + O\left(\epsilon^3\right), \tag{2}$$

where $\dot{\tilde{Q}}$ and $\ddot{\tilde{Q}}$ is the first and second order derivative with respect to ϵ, respectively. From eq. (1), we identify that

$$\dot{\tilde{Q}}\left(0\right) = \mathcal{L}_\upsilon Q, \tag{3}$$

$$\ddot{\tilde{Q}}\left(0\right) = \mathcal{L}_\upsilon \mathcal{L}_\upsilon Q, \tag{4}$$

where υ is the generator of ϕ_ϵ as

$$\upsilon\left(f\right) \equiv \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(f \circ \phi_\epsilon - f\right), \tag{5}$$

for an arbitrary scalar field f.

1.2 Definitions

Imposing metrics on M_λ and considering ϵ as the perturbation parameter, we define the background spacetime on $\mathcal{M}_0$ and the perturbed spacetimes on $\mathcal{M}_\epsilon$. The perturbed value for a physical quantity Q on the background is given by $\tilde{Q}\left(\epsilon\right)\Big|_{\lambda=0}$. Notice that the perturbed quantity depends on ϕ_ϵ and we implicitly assume a choice of that.

When $\tilde{Q}\left(\epsilon\right)$ is $O\left(\epsilon^n\right)$ at $\lambda = 0$ where n is nonnegative integer, it is convenient to introduce a quantity Z such that its pull-back $\tilde{Z}\left(\epsilon\right) \equiv \phi^*_{-\epsilon} Z$ is $O\left(1\right)$ at $\lambda = 0$ and satisfies

$$\tilde{Q}\left(\epsilon\right) = \epsilon^n \tilde{Z}\left(\epsilon\right), \tag{6}$$

at $\lambda = 0$. Then, the perturbed quantity is expanded by

$$\tilde{Q}\left(\epsilon\right) = \epsilon^n \left\{Z + \epsilon \dot{\tilde{Z}}\left(0\right) + O\left(\epsilon^2\right)\right\}, \tag{7}$$

at $\lambda = 0$ where we call Z the leading-order value and $\dot{\tilde{Z}}\left(0\right)$ the linear perturbation provided by

$$\dot{\tilde{Z}}\left(0\right) = \mathcal{L}_\upsilon Z, \tag{8}$$

where υ is the generator of ϕ_ϵ.

1.3 Gauges

Notice that there are infinite ways to choose ϕ_ϵ on $\mathcal{M}_0$. A change of map for $\dot{\tilde{Z}}(0)$ at $\lambda = 0$ from ϕ_ϵ to ϕ'_ϵ is given by

$$\mathcal{L}_{\upsilon'} Z - \mathcal{L}_\upsilon Z = \mathcal{L}_\xi Z, \tag{9}$$

where υ and υ' are generators of ϕ_ϵ and ϕ'_ϵ, respectively, and $\xi \equiv \upsilon' - \upsilon$. Moreover, ξ is tangent to $\mathcal{M}_0$ because

$$\xi(\lambda) = \upsilon'(\lambda) - \upsilon(\lambda) = 1 - 1 = 0, \tag{10}$$

where λ is understood as the scalar field on $\mathcal{F}$. Hence, we can evaluate $\mathcal{L}_\xi Z$ on $\mathcal{M}_0$ which means

$$[\mathcal{L}_\xi Z]_{\lambda=0} = \mathcal{L}_{\xi|_{\lambda=0}} [Z|_{\lambda=0}]. \tag{11}$$

By the formulation of ξ, we can generate all possible linear perturbations using eq. (9).

Assume that we performed a measurement of a perturbation for a physical quantity. Can we know what ϕ_ϵ is used for the measurement? No. If we have an ambiguity on the perturbation caused by the choice of ϕ_ϵ, the measurement is garbage because we do not know what ϕ_ϵ is used in the measurement. So we can not get any physical information from that. Meanwhile, let us consider a quantity Z for which eq. (11) vanishes. It is possible only when the quantity Z on $\mathcal{M}_0$ is zero, a constant scalar, or constructed by Kronecker delta with constant coefficients [1, 2]. Then, its linear perturbation $\dot{\tilde{Z}}(0)$ does not have the ambiguity because all perturbations by ϕ_ϵ are identical. In this case, a measurement of $\dot{\tilde{Z}}(0)$ provides proper physical information.

Let us rephrase these by introducing the concept of gauge. Gauges are redundancies of mathematical representation for a physical reality. For a fixed $\mathcal{F}$, a perturbation of a quantity has gauges by the choice of ϕ_ϵ. The gauge transformation is given by eq. (9). For a quantity for which eq. (11) vanishes, its perturbation is gauge-invariant. In terms of measurement, only gauge-invariant perturbations can provide physical information.

1.4 Metric

From now on, we assume that all quantities live in $\mathcal{M}_0$ if there is no explicit mention. The perturbed metric is expanded into

$$\tilde{g}_{ab}(\epsilon) = g_{ab} + \epsilon h_{ab} + O\left(\epsilon^2\right), \tag{12}$$

where h is the linear perturbation of g. The perturbation of identity endomorphism δ vanishes because

$$\begin{aligned} \upsilon^b \mathcal{L}_\upsilon \delta^a{}_b &= \mathcal{L}_\upsilon \left(\delta^a{}_b \upsilon^b\right) - \delta^a{}_b \mathcal{L}_\upsilon \upsilon^b \\ &= 0 \end{aligned} \tag{13}$$

for any vector υ. Then, the perturbation of inverse metric becomes

$$\begin{aligned} \mathcal{L}_\upsilon g^{ab} &= g^{ac} g_{cd} \mathcal{L}_\upsilon g^{db} \\ &= g^{ac} \left\{\mathcal{L}_\upsilon \left(g_{cd} g^{db}\right) - g^{db} \mathcal{L}_\upsilon g_{cd}\right\} \\ &= -h^{ab}. \end{aligned} \tag{14}$$

1.5 Levi-Civita Tensor

The normalization condition of Levi-Civita tensor,

$$-4! = \varepsilon_{abcd} \varepsilon^{abcd} \tag{15}$$

is perturbed as

$$\begin{aligned} 0 &= \mathcal{L}_\upsilon \left(g^{ae} g^{bf} g^{cg} g^{dh} \varepsilon_{abcd} \varepsilon_{efgh}\right) \\ &= 2\varepsilon^{abcd} \mathcal{L}_\upsilon \varepsilon_{abcd} - 4h^{ab} \varepsilon_{acde} \varepsilon_b{}^{cde} \\ &= 2\varepsilon^{abcd} \mathcal{L}_\upsilon \varepsilon_{abcd} + 4! h^a{}_a. \end{aligned} \tag{16}$$

We can prove $\mathcal{L}_\upsilon \varepsilon$ is 4-form using the identity given by

$$X_1^a X_2^b X_3^c X_4^d \mathcal{L}_\upsilon \varepsilon_{abcd} = \mathcal{L}_\upsilon \left(\varepsilon_{abcd} X_1^a X_2^b X_3^c X_4^d\right) - \varepsilon_{abcd} \mathcal{L}_\upsilon \left(X_1^a X_2^b X_3^c X_4^d\right), \tag{17}$$

for arbitrary arguments X_μ. Switching X_1 and X_2 in the above, we get

$$\begin{aligned} X_2^a X_1^b X_3^c X_4^d \mathcal{L}_\upsilon \varepsilon_{abcd} &= \mathcal{L}_\upsilon \left(\varepsilon_{abcd} X_2^a X_1^b X_3^c X_4^d\right) - \varepsilon_{abcd} \mathcal{L}_\upsilon \left(X_2^a X_1^b X_3^c X_4^d\right) \\ &= -\mathcal{L}_\upsilon \left(\varepsilon_{abcd} X_1^a X_2^b X_3^c X_4^d\right) + \varepsilon_{abcd} \mathcal{L}_\upsilon \left(X_1^a X_2^b X_3^c X_4^d\right) \\ &= -X_1^a X_2^b X_3^c X_4^d \mathcal{L}_\upsilon \varepsilon_{abcd}. \end{aligned} \tag{18}$$

In the same way, for all permutations of arguments X_μ, sign changes of $\mathcal{L}_v\varepsilon$ are identical to ε. Therefore, $\mathcal{L}_v\varepsilon$ is 4-form. Because all 4-forms are proportional to ε, we can write

$$\mathcal{L}_v\varepsilon_{abcd} = \alpha\varepsilon_{abcd}. \tag{19}$$

Substituting the above to eq. (16), we get α as

$$\alpha = \frac{1}{2}h^a{}_a. \tag{20}$$

As a result, we get

$$\mathcal{L}_v\varepsilon_{abcd} = \frac{1}{2}h^e{}_e\varepsilon_{abcd}. \tag{21}$$

1.6 Covariant Derivatives

Let us consider a operation $\tilde{\nabla}(\epsilon)$ defined by

$$\tilde{\nabla}(\epsilon) \equiv \phi^*_{-\epsilon}\nabla\phi^*_\epsilon, \tag{22}$$

where ∇ is the Levi-Civita connection associated with the spacetime metric g on $\mathcal{M}_\epsilon$. In fact, this operation is also the Levi-Civita connection associated with the perturbed metric $\tilde{g}(\epsilon)$ on $\mathcal{M}_0$ as shown by

$$\begin{aligned}\tilde{\nabla}(\epsilon)\tilde{g}(\epsilon) &= \phi^*_{-\epsilon}\nabla\phi^*_\epsilon\phi^*_{-\epsilon}g \\ &= 0, \end{aligned}\tag{23}$$

$$\begin{aligned}\tilde{\nabla}_{[a}(\epsilon)\tilde{\nabla}_{b]}(\epsilon) f &= \phi^*_{-\epsilon}\nabla_{[a}\phi^*_\epsilon\phi^*_{-\epsilon}\nabla_{b]}\phi^*_\epsilon f \\ &= \phi^*_{-\epsilon}\nabla_{[a}\nabla_{b]}f\cdot\phi_{-\epsilon} \\ &= 0,\end{aligned}\tag{24}$$

where f is an arbitrary function. The difference between covariant derivatives of a tensor T with respect to $\tilde{\nabla}(\epsilon)$ and ∇ on $\mathcal{M}_0$ is written by (refer [3])

$$\left(\tilde{\nabla}_c(\epsilon) - \nabla_c\right)T^{a_1\cdots a_k}{}_{b_1\cdots b_l} = \sum_{i=1}^k T^{a_1\cdots d\cdots a_k}{}_{b_1\cdots b_l}C^{a_i}{}_{dc}(\epsilon) - \sum_{i=1}^l T^{a_1\cdots a_k}{}_{b_1\cdots d\cdots b_l}C^d{}_{b_i c}(\epsilon), \tag{25}$$

where

$$C^a{}_{bc}(\epsilon) = \frac{1}{2}\tilde{g}^{ad}(\epsilon)\left(\nabla_c\tilde{g}_{bd}(\epsilon) + \nabla_b\tilde{g}_{cd}(\epsilon) - \nabla_d\tilde{g}_{bc}(\epsilon)\right), \tag{26}$$

where $\tilde{g}^{ab}(\epsilon)$ is the inverse of $\tilde{g}_{ab}(\epsilon)$.

Defining $\dot{C}$ and $\dot{\nabla}$ as

$$\dot{C}^a{}_{bc} \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}C^a{}_{bc}(\epsilon), \tag{27}$$

$$\dot{\nabla} \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\tilde{\nabla}(\epsilon) - \nabla\right), \tag{28}$$

we get

$$\begin{aligned}\dot{C}^a{}_{bc} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\frac{1}{2}\tilde{g}^{ad}(\epsilon)\left(\nabla_c\tilde{g}_{bd}(\epsilon) + \nabla_b\tilde{g}_{cd}(\epsilon) - \nabla_d\tilde{g}_{bc}(\epsilon)\right) \\ &= \lim_{\epsilon\to 0}\frac{1}{2}\tilde{g}^{ad}(\epsilon)\frac{1}{\epsilon}\left\{\nabla_c\left(\tilde{g}_{bd}(\epsilon) - g_{bc}\right) + \nabla_b\left(\tilde{g}_{cd}(\epsilon) - g_{cd}\right) - \nabla_d\left(\tilde{g}_{bc}(\epsilon) - g_{bc}\right)\right\} \\ &= \frac{1}{2}g^{ad}\left(\nabla_c h_{bd} + \nabla_b h_{cd} - \nabla_d h_{bc}\right)\end{aligned}\tag{29}$$

$$\dot{\nabla}_c T^{a_1\cdots a_k}{}_{b_1\cdots b_l} = \sum_{i=1}^k T^{a_1\cdots d\cdots a_k}{}_{b_1\cdots b_l}\dot{C}^{a_i}{}_{dc} - \sum_{i=1}^l T^{a_1\cdots a_k}{}_{b_1\cdots d\cdots b_l}\dot{C}^d{}_{b_i c}. \tag{30}$$

As a result, the linear perturbation of covariant derivative for T becomes

$$\begin{aligned}\mathcal{L}_v\nabla T &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\phi^*_{-\epsilon}\nabla T - \nabla T\right] \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\nabla\left(\tilde{T}(\epsilon) - T\right) + \left(\tilde{\nabla}(\epsilon) - \nabla\right)\tilde{T}(\epsilon)\right] \\ &= \nabla\mathcal{L}_v T + \dot{\nabla}T.\end{aligned}\tag{31}$$

1.7 Riemann Curvature

Riemann curvature tensor of the Levi-Civita connection ∇ associated with the spacetime metric g satisfies the Ricci identity as

$$R^a{}_{bcd}X^b = 2\nabla_{[c}\nabla_{d]}X^a, \tag{32}$$

where X is an arbitrary vector. Let us consider the linear perturbation of the above as

$$\begin{aligned}
\mathcal{L}_v\left(R^a{}_{bcd}X^b\right) &= 2\nabla_{[c}\mathcal{L}_v\left(\nabla_{d]}X^a\right) + 2\dot{\nabla}_{[c}\nabla_{d]}X^a \\
&= 2\nabla_{[c}\nabla_{d]}\mathcal{L}_vX^a + 2\nabla_{[c}\dot{\nabla}_{d]}X^a + 2\dot{C}^a{}_{e[c}\nabla_{d]}X^e - 2\dot{C}^e{}_{[dc]}\nabla_eX^a \\
&= R^a{}_{bcd}\mathcal{L}_vX^b + 2\nabla_{[c}\left(\dot{C}^a{}_{|e|d]}X^e\right) + 2\dot{C}^a{}_{e[c}\nabla_{d]}X^e \\
&= R^a{}_{bcd}\mathcal{L}_vX^b + 2X^b\nabla_{[c}\dot{C}^a{}_{d]b}.
\end{aligned} \tag{33}$$

By the Leibniz' rule of Lie derivatives, we get

$$\begin{aligned}
\mathcal{L}_vR^a{}_{bcd} &= 2\nabla_{[c}\dot{C}^a{}_{d]b} \\
&= \nabla_{[c}\nabla_{d]}h^a{}_b + \nabla_{[c}\nabla_{|b|}h_{d]}{}^a - \nabla_{[c}\nabla^ah_{d]b} \\
&= \frac{1}{2}\left(h^e{}_bR^a{}_{ecd} - h^a{}_eR^e{}_{bcd}\right) + \nabla_{[c}\nabla_{|b|}h_{d]}{}^a - \nabla_{[c}\nabla^ah_{d]b}
\end{aligned} \tag{34}$$

1.8 Einstein Equations

The linear perturbation of Ricci tensor $\mathrm{Ric}_{ab} \equiv R^c{}_{acb}$ and scalar $\mathrm{Scal} \equiv \mathrm{Ric}^a{}_a$ are given by

$$\begin{aligned}
\mathcal{L}_v\mathrm{Ric}_{ab} &= \delta^d{}_c\mathcal{L}_vR^c{}_{adb} + R^c{}_{adb}\mathcal{L}_v\delta^d{}_c \\
&= \nabla^c\nabla_{(a}h_{b)c} - \frac{1}{2}\nabla^c\nabla_ch_{ab} - \frac{1}{2}\nabla_b\nabla_ah^c{}_c \\
&= \nabla_{(a}\nabla^ch_{b)c} - \frac{1}{2}\nabla^c\nabla_ch_{ab} - \frac{1}{2}\nabla_b\nabla_ah^c{}_c - R_a{}^c{}_b{}^dh_{cd} + \mathrm{Ric}^c{}_{(a}h_{b)c}
\end{aligned} \tag{35}$$

$$\begin{aligned}
\mathcal{L}_v\mathrm{Scal} &= g^{ab}\mathcal{L}_v\mathrm{Ric}_{ab} - h^{ab}\mathrm{Ric}_{ab} \\
&= \nabla_b\nabla_ah^{ab} - \nabla^b\nabla_bh^a{}_a - h^{ab}\mathrm{Ric}_{ab}
\end{aligned} \tag{36}$$

For the Einstein tensor, we get

$$\begin{aligned}
\mathcal{L}_vG_{ab} &= \mathcal{L}_v\mathrm{Ric}_{ab} - \frac{1}{2}h_{ab}\mathrm{Scal} - \frac{1}{2}g_{ab}\mathcal{L}_v\mathrm{Scal} \\
&= -\frac{1}{2}\nabla^c\nabla_ch_{ab} + \nabla_c\nabla_{(a}h_{b)}{}^c - \frac{1}{2}g_{ab}\nabla_d\nabla_ch^{cd} - \frac{1}{2}\nabla_b\nabla_ah^c{}_c + \frac{1}{2}g_{ab}\nabla^d\nabla_dh^c{}_c + \frac{1}{2}g_{ab}h^{cd}\mathrm{Ric}_{cd} - \frac{1}{2}h_{ab}\mathrm{Scal}.
\end{aligned} \tag{37}$$

Spacetime is governed by the Einstein equation given by

$$G_{ab} = 8\pi T_{ab}, \tag{38}$$

where T is the stress-energy and we introduce geometrized unit, $c = 1$ and $G = 1$. Its linear perturbation is given by

$$\mathcal{L}_vG_{ab} = 8\pi\mathcal{L}_vT_{ab}. \tag{39}$$

1.9 Geodesics

Why a test mass follows geodesic? The proof is given by the Einstein equation. Let us consider a dust having stress-energy tensor as

$$T_{ab} = \rho v_av_b, \tag{40}$$

where ρ is the energy density and v is the 4-velocity such that

$$v \cdot v = -1 \tag{41}$$

Perturbed quantities are expanded by

$$\tilde{\rho}(\epsilon) = \rho + \epsilon(\delta\rho) + O\left(\epsilon^2\right), \tag{42}$$

$$\tilde{v}^a(\epsilon) = v^a + \epsilon(\delta v)^a + O\left(\epsilon^2\right), \tag{43}$$

$$\tilde{T}_{ab}(\epsilon) = T_{ab} + \epsilon(\delta T)_{ab} + O\left(\epsilon^2\right). \tag{44}$$

We assume that background spacetime is vacuum as

$$\rho = 0, \tag{45}$$
$$T_{ab} = 0. \tag{46}$$

In this case, v is not well-defined on the background. We use a definition for v as

$$v^a \equiv \lim_{\epsilon \to 0} \phi^*_{-\epsilon} v^a. \tag{47}$$

Then, δT is given by

$$(\delta T)_{ab} = (\delta\rho)\, v_a v_b \tag{48}$$

We would like to know a motion of infinitesimal dust with the energy density $\delta\rho$ in the limit of the vacuum background, i.e., v. The contracted Bianchi identity derived by the Einstein equation will give the answer:

$$\begin{aligned} 0 &= \delta\left(\nabla^b T_{ab}\right) && (49)\\ &= \nabla^b (\delta T)_{ab} - T_{cb} C^c{}_a{}^b - T_{ac} C^c{}_b{}^b - h^{bc} \nabla_c T_{ab} && (50)\\ &= \nabla^b (\delta T)_{ab} && (51)\\ &= \nabla^b \left((\delta\rho)\, v_a v_b\right) && (52)\\ &= v_a \nabla_b \left((\delta\rho)\, v^b\right) + (\delta\rho)\, \nabla_v v_a. && (53) \end{aligned}$$

Note that $\nabla_v v^a$ is orthogoanl to v because

$$\begin{aligned} 0 &= \nabla_v (v \cdot v) && (54)\\ &= 2 v_a \nabla_v v^a. && (55) \end{aligned}$$

By the orthogonal decomposition of eq. (53), we obtain

$$0 = \nabla_a \left((\delta\rho)\, v^a\right), \tag{56}$$
$$0 = \nabla_v v^a. \tag{57}$$

The first equation is the conservation of energy flux and the second equation is the geodesic equation.

2 Gravitational Waves

In this lecture, we only consider perturbations with Minkowski background. It means that $(\mathcal{M}_0, g)$ is Minkowski spacetime where g is the flat metric.

2.1 Minkowski Background

The Riemann tensor of Minkowski spacetime vanishes:

$$R^a{}_{bcd} = 0. \tag{58}$$

Thus, the Levi-Civita connection for any tensors is commute:

$$\nabla_{[a} \nabla_{b]} = 0. \tag{59}$$

Assuming $\tilde{T}(\epsilon) = O\left(\epsilon^2\right)$, we get the linear perturbation of the Einstein equation as

$$0 = -\frac{1}{2} \nabla^c \nabla_c h_{ab} + \nabla_c \nabla_{(a} h_{b)}{}^c - \frac{1}{2} g_{ab} \nabla_d \nabla_c h^{cd} - \frac{1}{2} \nabla_b \nabla_a h^c{}_c + \frac{1}{2} g_{ab} \nabla^d \nabla_d h^c{}_c \tag{60}$$

Defining $\bar{h}$ as

$$\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2} g_{ab} h^c{}_c, \tag{61}$$

we get

$$0 = -\frac{1}{2} \nabla^c \nabla_c \bar{h}_{ab} + \nabla_{(a} \nabla^c \bar{h}_{b)c} - \frac{1}{2} g_{ab} \nabla^c \nabla^d \bar{h}_{cd}. \tag{62}$$

2.2 Lorenz Gauge

We impose a gauge condition, so-called Lorenz gauge, given by

$$\nabla^b \bar{h}_{ab} = 0. \tag{63}$$

Then, the perturbed Einstein equation becomes the wave equation:

$$0 = \nabla^c \nabla_c \bar{h}_{ab}. \tag{64}$$

Note that eq. (63) does not determine gauge uniquely. We can generate a set of gauges satisfying the Lorenz gauge condition by the gauge transformation given in eq. (9):

$$h'_{ab} = h_{ab} + \mathcal{L}_\xi g_{ab}, \tag{65}$$

where h and h' are metric perturbations in given gauge and new gauge, respectively, and ξ is a vector. Their relation in $\bar{h}$ becomes

$$\bar{h}'_{ab} = \bar{h}_{ab} + \nabla_a \xi_b + \nabla_b \xi_a - g_{ab} \nabla^c \xi_c. \tag{66}$$

If both gauges satisfy the Lorenz gauge condition, we obtain

$$\nabla^b \nabla_b \xi^a = 0. \tag{67}$$

2.3 Wave Solution

Let us consider a wave solution of the metric perturbation with the wave equation in eq. (64) given by

$$h_{ab} = \int_{\mathcal{N}} d^3\mathcal{N}\left(k\right)\, \tilde{h}_{ab}\left(k\right) e^{iP(;k)}, \tag{68}$$

where $\mathcal{N} \equiv \{k : k \cdot k = 0\} - \{0\}$ is the set of null vectors, $\tilde{h}\, d^3\mathcal{N}$ is the infinitesimal amplitude, and P is the phase such that $k^a = \nabla^a P$. $\mathcal{N}$ is decomposed into the future-directed subset $\mathcal{N}^+$ and the past-directed subset $\mathcal{N}^-$, respectively. Because h is real, $\mathcal{N}^+$ and $\mathcal{N}^-$ have one-to-one correspondence given by

$$\tilde{h}_{ab}\left(-k\right) = \tilde{h}^*_{ab}\left(k\right). \tag{69}$$

When we introduce a globally inertial coordinate system $\{t, \vec{x}\}$, components of k and the phase P are given by

$$k^\mu = \omega\left(1, \kappa_x, \kappa_y, \kappa_z\right), \tag{70}$$

$$P\left(t, \vec{x}; \omega, \kappa\right) = \omega\left(-t + \kappa \cdot \vec{x}\right), \tag{71}$$

where ω is the frequency and κ is the unit spatial vector for the propagation. Then, GWs become

$$h_{ab}\left(t, \vec{x}\right) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d^2\kappa\, \tilde{h}_{ab}\left(\omega, \kappa\right) e^{i\omega(-t+\kappa\cdot\vec{x})}, \tag{72}$$

where $\int d^2\kappa$ is the integration over all directions. When we introduce parameters (θ, ϕ) of the spherical coordinate for κ as

$$\kappa\left(\theta, \phi\right) = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}, \tag{73}$$

GWs become

$$h_{ab}\left(t, \vec{x}\right) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{2\pi} d\phi \int_{-1}^{1} d\left(\cos\theta\right)\, \tilde{h}_{ab}\left(\omega, \theta, \phi\right) e^{i\omega(-t+\kappa(\theta,\phi)\cdot\vec{x})}. \tag{74}$$

2.4 Traceless Gauge

Let us investigate possibilities of the further gauge restriction given by

$$h^a{}_a = 0, \tag{75}$$

which is the traceless gauge. For the above, we have to transform gauge with ξ satisfying

$$0 = h^a{}_a + 2\nabla_a \xi^a, \tag{76}$$

where h is a metric perturbation in given gauge. Because we only consider gauges satisfying the Lorenz gauge condition, ξ is the solution of eq. (67). Thus, it has form of

$$\xi^a = \int_{\mathcal{N}} d^3\mathcal{N}\left(k\right)\, \tilde{\xi}^a\left(k\right) e^{iP(;k)}. \tag{77}$$

Then, eq. (76) becomes

$$0 = \int_{\mathcal{N}} d^3\mathcal{N} \left(\tilde{h}^a{}_a + 2ik_a\tilde{\xi}^a \right) e^{iP}. \tag{78}$$

Choices of $\tilde{\xi}$ to vanish the integrand of the above for all $k \in \mathcal{N}$ realize the traceless gauge.

In summary, our gauge choice have been

$$0 = \nabla^b h_{ab}, \tag{79}$$
$$0 = h^a{}_a, \tag{80}$$

which implies

$$0 = k^b \tilde{h}_{ab}(k), \tag{81}$$
$$0 = \tilde{h}^a{}_a(k), \tag{82}$$

for all $k \in \mathcal{N}$. Among gauges satisfying the above condition, the gauge transformations are given by ξ satisfying

$$\begin{aligned} 0 &= h'^a{}_a \\ &= h^a{}_a + 2\nabla_a\xi^a \\ &= \nabla_a\xi^a \end{aligned} \tag{83}$$

which implies

$$k_a\tilde{\xi}^a(k) = 0, \tag{84}$$

for all $k \in \mathcal{N}$.

2.5 Riemann Tensor

From eq. (34), we obtain

$$\mathcal{L}_v R^{ab}{}_{cd} = -2\nabla^{[a}\nabla_{[c}h^{b]}{}_{d]} \tag{85}$$
$$= \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\, 4k^{[a}k_{[c}\tilde{h}^{b]}{}_{d]}e^{iP}. \tag{86}$$

Note that the perturbation of Reimann tensor is gauge-invariant because its background value vanishes.

2.6 Introducing Observer

Let us consider a time coordinate t in a globally inertial coordinate system $\{t, \vec{x}\}$ on Minkowski background spacetime. Its Eulerian observer have 4-velocity u given by

$$u^a = -g^{ab}(dt)_b \tag{87}$$

The 3+1 decomposition of a wave vector $k \in \mathcal{N}$ is given by

$$k^a = \omega(u^a + \kappa^a), \tag{88}$$

where $\omega = -u \cdot k$ is the frequency and $\kappa = k/\omega - u$ is the spatial unit vector of propagation. We define a projection operator P onto the vector subspace orthogonal to n and κ as

$$P^a{}_b = \delta^a{}_b + u^a u_b - \kappa^a\kappa_b. \tag{89}$$

Then, it is idempotent,

$$P^a{}_b = P^a{}_c P^c{}_b, \tag{90}$$

and its trace is

$$P^a{}_a = 2. \tag{91}$$

Let us impose an additional gauge condition as

$$h_{ab}u^b = 0. \tag{92}$$

The the gauge conditions we have chosen including the above is so-called the transverse-traceless (TT) gauge condition. All gauge conditions we impose are summarized in

$$0 = u^b \tilde{h}_{ab}(k), \tag{93}$$
$$0 = \kappa^b \tilde{h}_{ab}(k), \tag{94}$$
$$0 = \tilde{h}^a{}_a(k), \tag{95}$$

for all $k \in \mathcal{N}$.

Let us show that the TT gauge condition determines a gauge uniquely. Orthogonal decomposision of $\tilde{\xi}$ for traceless gauges is given by

$$\tilde{\xi}^a = \tilde{\alpha} u^a + \tilde{\beta} \kappa^a + \tilde{X}^a, \tag{96}$$

where $\tilde{X}^a = P^a{}_b \tilde{\xi}^b$. Because eq. (84) implies $\tilde{\alpha} = \tilde{\beta}$, we get form of

$$\tilde{\xi}^a = \tilde{\gamma} k^a + \tilde{X}^a. \tag{97}$$

For two gauges satisfying the TT gauge condition have a transformation,

$$\begin{aligned} 0 &= h'_{ab} u^b \\ &= (h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a) u^b \\ &= (\nabla_a \xi_b + \nabla_b \xi_a) u^b, \end{aligned} \tag{98}$$

with $\tilde{\xi}$ satisfying

$$0 = i\left(k_a \tilde{\xi}_b + \tilde{\xi}_a k_b\right) u^b \tag{99}$$
$$= i\left\{k_a \tilde{\gamma}(k \cdot u) + \left(\tilde{\gamma} k_a + \tilde{X}_a\right)(k \cdot u)\right\} \tag{100}$$
$$= i\left(2\tilde{\gamma} k_a + \tilde{X}_a\right)(k \cdot u). \tag{101}$$

It implies that $\tilde{\gamma} = 0$ and $\tilde{X} = 0$. As a result, $\tilde{\xi}$ vanishes and both gauges are indentical.

2.7 Polarization of Amplitude

We consider a set of rank $(0, 2)$ tensors $\mathcal{T}$ satisfying eqs. (93) to (95). It has projection operator Λ given by

$$\Lambda^{ab}{}_{cd} \equiv P^a{}_{(c} P^b{}_{d)} - \frac{1}{2} P^{ab} P_{cd}. \tag{102}$$

A right-handed orthonormal basis $\left\{e^A : A = +, \times\right\}$ for $\mathcal{T}$ has properties,

$$e^A \cdot e^B = \delta^{AB}, \tag{103}$$
$$e^A_{ab} e^B_{cd} g^{ac} \varepsilon^{bd}{}_{ef} u^e \kappa^f = \varepsilon^{AB} \tag{104}$$

where δ^{AB} is the Kronecker delta, ε^{AB} is the Levi-Civita symbol for two dimension, ε_{abcd} is the spacetime Levi-Civita tensor, and $\cdot$ is the inner product for $\mathcal{T}$ defined by

$$x \cdot y = g^{ac} g^{bd} x_{ab} y_{cd}, \tag{105}$$

for $x, y \in \mathcal{T}$. The collection of right-handed orthonormal bases are parametrized by ϑ with the relation,

$$e^+_{ab}(\vartheta) = \cos(2\vartheta) e^+_{ab}(0) + \sin(2\vartheta) e^\times_{ab}(0), \tag{106}$$
$$e^\times_{ab}(\vartheta) = -\sin(2\vartheta) e^+_{ab}(0) + \cos(2\vartheta) e^\times_{ab}(0), \tag{107}$$

where $\left\{e^A(0)\right\}$ is a fiducial basis.

Let us consider a phase adjustment of $\tilde{h}$ by α as

$$\tilde{h}_{ab} = \left\{\Re\left(\tilde{h}_{ab} e^{-i\alpha}\right) + i\Im\left(\tilde{h}_{ab} e^{-i\alpha}\right)\right\} e^{i\alpha}, \tag{108}$$

such that the real part and imaginary part are orthogonal to each other. Note that α for the orthogonalization is not unique. So, we introduce a "standard" process for the orthogonalization given by

$$-\pi/2 \le \alpha = \frac{1}{2}\operatorname{Arg}\left(\tilde{h}\cdot\tilde{h}\right) \le \pi/2, \tag{109}$$

$$\tilde{h}_+ = \sqrt{\Re\left(\tilde{h}e^{-i\alpha}\right)\cdot\Re\left(\tilde{h}e^{-i\alpha}\right)} > 0, \tag{110}$$

$$e^+ = \frac{1}{\tilde{h}_+}\Re\left(\tilde{h}e^{-i\alpha}\right) \in \mathcal{T}. \tag{111}$$

Then, $e^\times$ is uniquely determined by the right-handedness of the basis $\left\{e^A : A = +, \times\right\}$ for $\mathcal{T}$ and

$$\tilde{h}_\times = i\Im\left(\tilde{h}e^{-i\alpha}\right)\cdot e^\times. \tag{112}$$

Finally, we get the form

$$\tilde{h}_{ab} = \tilde{h}_A e^A_{ab} e^{i\alpha}. \tag{113}$$

This is an analogy to the form for elliptically polarized electromagnetic waves given in [4]. Introducing unit eigenvectors x and y for e^+ with positive and negative eiginvalues, respectively, we get

$$e^+_{ab} = \frac{1}{\sqrt{2}}\left(x_a x_b - y_a y_b\right), \tag{114}$$

$$e^\times_{ab} = \frac{1}{\sqrt{2}}\left(x_a y_b + y_a x_b\right). \tag{115}$$

Exercise: Show that $\left|\tilde{h}_+\right| > \left|\tilde{h}_\times\right|$ in the above process.

2.8 Changing Observer for Metric Perturbation in TT Gauge

A metric perturbation in the Lorenz and traceless gauge has decomposition as

$$\tilde{h}_{ab} = \tilde{A}\hat{k}_a\hat{k}_b + \tilde{B}_a\hat{k}_b + \hat{k}_a\tilde{B}_b + \tilde{C}_{ab}, \tag{116}$$

where

$$\hat{k}^a = u^a + \kappa^a \tag{117}$$

$$\tilde{A} = \tilde{h}_{ab}u^a u^b, \tag{118}$$

$$\tilde{B}_a = -\tilde{h}_{ac}u^a P^c{}_b, \tag{119}$$

$$\tilde{C}_{ab} = \tilde{h}_{cd}P^c{}_a P^d{}_b. \tag{120}$$

We find that the metric perturbation in the TT gauge is only taking $\tilde{C}$.

In another observer with 4-velocity u', we have

$$\tilde{h}_{ab} = \tilde{A}'\hat{k}'_a\hat{k}'_b + \tilde{B}'_a\hat{k}'_b + \hat{k}'_a\tilde{B}'_b + \tilde{C}'_{ab}. \tag{121}$$

Then, we get

$$\tilde{C}'_{ab} = \tilde{h}_{cd}P'^c{}_a P'^d{}_b \tag{122}$$

$$= \left(\tilde{A}\hat{k}_a\hat{k}_b + \tilde{B}_a\hat{k}_b + \hat{k}_a\tilde{B}_b + \tilde{C}_{ab}\right)P'^c{}_a P'^d{}_b \tag{123}$$

$$= \tilde{C}_{cd}P'^c{}_a P'^d{}_b, \tag{124}$$

where P' is the projection operator orthogonal to u' and κ' as defined in eq. (89). Note that $\hat{k}$ and $\hat{k}'$ are proportional to wave vector k, thereby, proportional to each other. As a result, $P'^a{}_b\hat{k}^b = 0$. It is the transformation rule for metric perturbations in the TT gauge.

2.9 Perturbation of Observer

In our setting for u, **????** implies that ρ is a constant of motion as

$$0 = u^a\nabla_a\rho. \tag{125}$$

In the TT gauge, **??** implies that v is spatial as

$$0 = v\cdot u, \tag{126}$$

and **??** becomes

$$0 = u_a \left(u^b \nabla_b \sigma + \rho \nabla_b v^b \right) + \rho u^b \nabla_b v_a. \tag{127}$$

Then, we get

$$u^a \nabla_a \sigma = -\rho \nabla_a v^a, \tag{128}$$
$$u^b \nabla_b v^a = 0. \tag{129}$$

If $\sigma = 0$ and $v = 0$ before arrival of GWs, $\sigma = 0$ and $v = 0$ are maintained even though the GWs are passing. So we set

$$\sigma = 0, \tag{130}$$
$$v = 0, \tag{131}$$

over the spacetime. It means that test masses are fixed on the spatial coordinate $\{\vec{x}\}$ and their masses are constant of motion.

3 Generation of Gravitational Waves by a Weak Source

3.1 Weak Source

We put a weak source that does not spoil Minkowski background as

$$\tilde{T}_{ab}(\epsilon) = \epsilon \left\{ T_{ab} + O(\epsilon) \right\}. \tag{132}$$

From eq. (39), we get the in homogeneous wave equation as

$$\nabla^c \nabla_c \bar{h}_{ab} = -16\pi T_{ab}, \tag{133}$$

in Lorenz gauge. Note that, by the contracted Bianchi identity and the Einstein equation, we have the conservation law, $\nabla^b T_{ab} = 0$, that is consistent with the above due to the Lorenz gauge condition, $\nabla^b \bar{h}_{ab} = 0$. Using Green's function satisfying

$$\Box_x G(x - x') = \delta^4 (x - x'), \tag{134}$$

where $\Box_x$ is the d'Alembertian for the coordinate x and δ^4 is the 4-dimensional delta function, we get the solution of eq. (133) as

$$\bar{h}_{ab}(x) = -16\pi \int d^4x' \, G(x - x') \, T_{ab}(x'). \tag{135}$$

The solution of eq. (134) depends on the boundary conditions. By imposing Kirchoff-Sommerfeld no-incoming radiation boundary condition, we get

$$G(x - x') = -\frac{1}{4\pi |\vec{x} - \vec{x}'|} \delta(t - |\vec{x} - \vec{x}'| - t'). \tag{136}$$

Then, eq. (135) becomes

$$\bar{h}_{ab}(t, \vec{x}) = 4 \int d^3x' \, \frac{T_{ab}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{137}$$

3.2 GWs from Slow Source at Far Zone

Using Fourier transformation of T, we get

$$T_{ab}(t, \vec{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{T}_{ab}(\omega, \vec{x}) \, e^{-i\omega t}. \tag{138}$$

Conservation of T provides

$$\begin{aligned} 0 &= \partial_\mu T^{\alpha\mu} \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(-i\omega \hat{T}^{\alpha 0}(\omega, \vec{x}) + \partial_i \hat{T}^{\alpha i}(\omega, \vec{x}) \right) e^{-i\omega t}, \end{aligned} \tag{139}$$

that implies

$$\partial_i \hat{T}^{\alpha i}(\omega, \vec{x}) = i\omega \hat{T}^{\alpha 0}(\omega, \vec{x}). \tag{140}$$

Substituting eq. (138) into eq. (137), we get

$$\bar{h}_{ab}(t,\vec{x}) = 4\int d^3x' \, \frac{T_{ab}\left(t-|\vec{x}-\vec{x}'|,\vec{x}'\right)}{|\vec{x}-\vec{x}'|} \tag{141}$$

$$= 4\int d^3x' \, \frac{1}{|\vec{x}-\vec{x}'|}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\hat{T}_{ab}(\omega,\vec{x}')\,e^{-i\omega\left(t-|\vec{x}-\vec{x}'|\right)} \tag{142}$$

$$= 4\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\int d^3x' \, \frac{e^{i\omega|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\hat{T}_{ab}(\omega,\vec{x}')\,. \tag{143}$$

We assume that the source is compact and the frequencies of interests are sufficiently small such that $\omega\,|\vec{x}-\vec{x}'|$ is almost constant over the source. Then, $\bar{h}$ becomes

$$\bar{h}_{ab}(t,\vec{x}) = \frac{4}{r}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega(-t+r)}\int_{\mathcal{V}} d^3x'\,\hat{T}_{ab}(\omega,\vec{x}')\,, \tag{144}$$

where $r \equiv |\vec{x}|$ and $\mathcal{V}$ is the spatial domain of the source. Introducing a globally inertial coordinate system $\{t,\vec{x}\}$, we consider the integration of $\hat{T}$ only for spatial components as

$$\int_{\mathcal{V}} d^3x\,\hat{T}^{ij}(\omega,\vec{x}) = \int_{\mathcal{V}} d^3x\,\left\{\partial_k\left(x^i\hat{T}^{kj}\right) - x^i\partial_k\hat{T}^{kj}\right\} \tag{145}$$

$$= -i\omega\int_{\mathcal{V}} d^3x\,x^i\hat{T}^{0j} \tag{146}$$

$$= -\frac{1}{2}i\omega\int_{\mathcal{V}} d^3x\,\left(x^i\hat{T}^{0j} + x^j\hat{T}^{0i}\right) \tag{147}$$

$$= -\frac{1}{2}i\omega\int_{\mathcal{V}} d^3x\,\left\{\partial_k\left(x^ix^j\hat{T}^{0k}\right) - x^ix^j\partial_k\hat{T}^{0k}\right\} \tag{148}$$

$$= -\frac{1}{2}\omega^2\int_{\mathcal{V}} d^3x\,x^ix^j\hat{T}^{00}(\omega,\vec{x}) \tag{149}$$

Let us define the second moment of T^{00} as

$$M^{ij}(t) \equiv \int_{\mathcal{V}} d^3x\,x^ix^jT^{00}(t,\vec{x})\,, \tag{150}$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\int_{\mathcal{V}} d^3x\,x^ix^j\hat{T}^{00}(\omega,\vec{x}) \tag{151}$$

Its second-order differentiation becomes

$$\ddot{M}^{ij}(t) = -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega^2e^{-i\omega t}\int_{\mathcal{V}} d^3x\,x^ix^j\hat{T}^{00}(\omega,\vec{x}) \tag{152}$$

$$= 2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\int_{\mathcal{V}} d^3x\,\hat{T}^{ij}(\omega,\vec{x})\,. \tag{153}$$

As a result,

$$\bar{h}^{ij}(t,\vec{x}) = \frac{2}{r}\ddot{M}^{ij}(t-r)\,. \tag{154}$$

Using eq. (102), we get the metric perturbation in TT gauge as

$$h^{TT}_{ab}(t,\vec{x}) = \frac{2}{r}\Lambda^{cd}{}_{ab}(\hat{r})\,\ddot{Q}_{cd}(t-r)\,, \tag{155}$$

where $\hat{r} \equiv \vec{x}/r$ and Q is the quadrupole moment defined by

$$Q_{ab} \equiv M_{ab} - \frac{1}{3}\gamma_{ab}M^a{}_a\,, \tag{156}$$

where $\gamma_{ab} \equiv g_{ab} + u_au_b$.

4 Detection of Gravitational Waves by Light

4.1 Maxwell's Equations

Maxwell's equations for the 4-potential A become

$$2\nabla^b\nabla_{[a}A_{b]} = 4\pi j_a, \tag{157}$$

where j is the electromagnetic 4-current and we introduce Gaussian unit system as $\epsilon_0 = 1/4\pi$ and $\mu_0 = 4\pi$. Using Lorenz gauge,

$$\nabla^a A_a = 0, \tag{158}$$

we get

$$\nabla^b \nabla_b A_a = \mathrm{Ric}^b{}_a A_b - 4\pi j_a. \tag{159}$$

Exercise: As you know, Maxwell's equations consist of 4 equations. However, with the EM 4-potential A, we only need Gauss's law and Ampère's law as in eq. (157). Why?

Exercise: Prove that the conservation of electric charge, $\nabla^a j_a = 0$.

4.2 Geometrical Optics

Let us consider a local Lorentz frame $\{t, \vec{x}\}$. In the frame, $\mathcal{L}$ is defined as a typical length over which the waves vary and $\mathcal{R}$ is defined as a typical components of the Riemann curvature tensor, which is the curvature radius scale. When the frequency of electromagnetic waves ω is much larger than $1/L$ where $L \equiv \min(\mathcal{L}, \mathcal{R})$, geometrical optics is valid. Let us consider electromagnetic waves given by

$$A_a(t, \vec{x}) = 2\Re\left[\left\{\tilde{A}_a + \omega^{-1}\tilde{B}_a + O\left(\omega^{-2}\right)\right\} e^{i\omega q(t,\vec{x})}\right], \tag{160}$$

such that $\hat{l}^a \equiv \nabla^a q$ is future-directed. In the vacuum, $j = 0$, eq. (159) becomes the wave equation for A as

$$\nabla^b \nabla_b A_a = 0 \tag{161}$$

Through,

$$\begin{aligned}
\nabla_b A_a &= 2\Re\left[\left\{i\omega\hat{l}_b\left(\tilde{A}_a + \omega^{-1}\tilde{B}_a\right) + \nabla_b\tilde{A}_a + O\left(\omega^{-1}\right)\right\} e^{i\omega q}\right] \\
&= 2\Re\left[\left\{i\omega\hat{l}_b\tilde{A}_a + i\hat{l}_b\tilde{B}_a + \nabla_b\tilde{A}_a + O\left(\omega^{-1}\right)\right\} e^{i\omega q}\right], \qquad (162)\\
\nabla^a A_a &= 2\Re\left[\left\{i\omega\left(\hat{l}\cdot\tilde{A}\right) + i\left(\hat{l}\cdot\tilde{B}\right) + \nabla^a\tilde{A}_a + O\left(\omega^{-1}\right)\right\} e^{i\omega q}\right], \qquad (163)
\end{aligned}$$

$$\begin{aligned}
\nabla_c\nabla_b A_a &= 2\Re\left[\left\{i\omega\hat{l}_c\left(i\omega\hat{l}_b\tilde{A}_a + i\hat{l}_b\tilde{B}_a + \nabla_b\tilde{A}_a\right) + i\omega\nabla_c\left(\hat{l}_b\tilde{A}_a\right) + O(1)\right\} e^{i\omega q}\right] \\
&= 2\Re\left[\left\{-\omega^2\hat{l}_c\hat{l}_b\tilde{A}_a + \omega\left(-\hat{l}_b\hat{l}_c\tilde{B}_a + i\hat{l}_c\nabla_b\tilde{A}_a + i\nabla_c\left(\hat{l}_b\tilde{A}_a\right)\right) + O(1)\right\} e^{i\omega q}\right], \qquad (164)\\
\nabla^b\nabla_b A_a &= 2\Re\left[\left\{-\omega^2\left(\hat{l}\cdot\hat{l}\right)\tilde{A}_a + \omega\left(-\left(\hat{l}\cdot\hat{l}\right)\tilde{B}_a + 2i\hat{l}^b\nabla_b\tilde{A}_a + i\tilde{A}_a\nabla_b\hat{l}^b\right) + O(1)\right\} e^{i\omega q}\right], \qquad (165)
\end{aligned}$$

we get

$$0 = \hat{l}\cdot\hat{l}, \tag{166}$$

in the leading-order of ω and

$$\begin{aligned}
0 &= \hat{l}\cdot\tilde{A}, \qquad (167)\\
0 &= 2\hat{l}^b\nabla_b\tilde{A}_a + \tilde{A}_a\nabla_b\hat{l}^b \qquad (168)
\end{aligned}$$

in the next-to-leading-order. Rewriting results in $Q \equiv \omega q$ and $l \equiv \omega\hat{l}$, we obtain the evolution equations along l as

$$\begin{aligned}
l^a\nabla_a Q &= l\cdot l \\
&= 0, \qquad (169)\\
l^b\nabla_b l^a &= g^{ac}l^b\nabla_b\nabla_c Q \\
&= g^{ac}l^b\nabla_c\nabla_b Q \\
&= \frac{1}{2}g^{ac}\nabla_c(l\cdot l) \\
&= 0, \qquad (170)
\end{aligned}$$

in the leading-order and

$$\begin{aligned}
l^b\nabla_b\tilde{A}_a &= -\frac{1}{2}\tilde{A}_a\nabla_b l^b \qquad (171)\\
l\cdot\tilde{A} &= 0 \qquad (172)
\end{aligned}$$

in the next-to-leading-order. Introducing the real amplitude $\mathcal{A} \equiv \sqrt{\tilde{A} \cdot \tilde{A}^*}$ and polarization vector $\tilde{f}_a \equiv \tilde{A}_a/\mathcal{A}$, we get

$$0 = \nabla_b \left(\mathcal{A}^2 l^b\right), \tag{173}$$

$$0 = l^b \nabla_b \tilde{f}_a, \tag{174}$$

$$0 = l \cdot \tilde{f}. \tag{175}$$

The first equation is conservation of the number of light rays, the second equation is the parallel transport of polarization, and the third equation is the transverse condition of polarization.

4.3 Perturbation of Rays

We introduce the Minkowski background spacetime and monochromatic plane electromagnetic wave given by

$$A_a = 2\Re\left[\tilde{A}_a e^{iQ}\right], \tag{176}$$

where $\tilde{A}$ and $l_a \equiv \nabla_a Q$ are constant over spacetime. We impose Lorenz gauge and radiation gauge as

$$0 = \nabla^a A_a, \tag{177}$$

$$0 = u \cdot A, \tag{178}$$

implies

$$0 = \tilde{A} \cdot l, \tag{179}$$

$$0 = \tilde{A} \cdot u. \tag{180}$$

Meanwhile, GWs are in the TT gauge given by

$$0 = k^a \tilde{h}_{ab}(k), \tag{181}$$

$$0 = \tilde{h}^a{}_a(k), \tag{182}$$

$$0 = u^a \tilde{h}_{ab}(k), \tag{183}$$

for all $k \in \mathcal{N}$.

Perturbed phase is given by

$$\tilde{Q}(\epsilon) = Q + \epsilon S + O\left(\epsilon^2\right). \tag{184}$$

The linear perturbation of l becomes

$$\begin{aligned}\mathcal{L}_v l^a &= \mathcal{L}_v \left(g^{ab} \nabla_b Q\right) \\ &= -h^{ab} l_b + \nabla^a S.\end{aligned} \tag{185}$$

Then, eq. (169) provides

$$0 = \left(-h^{ab} l_b + \nabla^a S\right) l_a + l^a \nabla_a S, \tag{186}$$

implies

$$l^a \nabla_a S = \frac{1}{2} h_{ab} l^a l^b. \tag{187}$$

The general solution of S is decomposed into the particular solution and the homogeneous solution as

$$S = S^{\mathrm{p}} + S^{\mathrm{h}}. \tag{188}$$

The particular solution can be solved using

$$S^{\mathrm{p}} = \int_{\mathcal{N}} d^3\mathcal{N}(k)\ \tilde{S}^{\mathrm{p}}(k)\, e^{iP(;k)}. \tag{189}$$

Plugging it into eq. (227), we get

$$\tilde{S}^{\mathrm{p}} = -\frac{1}{2} i \frac{1}{(k \cdot l)} \tilde{h}_{ab}(k)\, l^a l^b. \tag{190}$$

The homogeneous solution have to be satisfied

$$l^a \nabla_a S^{\mathrm{h}} = 0, \tag{191}$$

implies $S^{\rm h}\left(t,x,y,z\right)=X\left(q=-t+x,y,z\right)$.

Let us consider the frequency as

$$\omega\equiv-u^a\nabla_a Q. \tag{192}$$

Its perturbed value becomes

$$\tilde{\omega}\left(\epsilon\right)=\omega+\epsilon\alpha+O\left(\epsilon^2\right), \tag{193}$$

where

$$\begin{aligned}\alpha&=-v^a l_a-u^a\nabla_a S\\&=-u^a\nabla_a S.\end{aligned} \tag{194}$$

Note that α is gauge-invariant because ω is constant on $\mathcal{M}_0$. We give boundary condition at the 3-dimensional timelike plane $\mathcal{P}$ that is the congruence of emitters as

$$0=\left[\alpha\right]_{\mathcal{P}} \tag{195}$$

$$=\left[-u^a\nabla_a S^{\rm p}-u^a\nabla_a S^{\rm h}\right]_{\mathcal{P}} \tag{196}$$

$$=\left[-u^a\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,ik_a\left(-\frac{1}{2}i\frac{1}{k\cdot l}\tilde{h}_{bc}l^bl^c\right)e^{iP(;k)}+\partial_q X\left(q=-t+x,y,z\right)\right]_{\mathcal{P}}, \tag{197}$$

$$=\left[\frac{1}{2}\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{1}{\hat{k}\cdot l}\tilde{h}_{ab}l^al^be^{i\omega_{\rm g}\left(-t+\kappa\cdot\left(y\hat{y}+z\hat{z}\right)\right)}+\partial_q X\left(q=-t,y,z\right)\right]_{\mathcal{P}}, \tag{198}$$

implies

$$\partial_q X\left(q,y,z\right)=-\frac{1}{2}\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{1}{\hat{k}\cdot l}\tilde{h}_{ab}l^al^be^{i\omega_{\rm g}\left(q+\kappa\cdot\left(y\hat{y}+z\hat{z}\right)\right)}, \tag{199}$$

$$X\left(q,y,z\right)=\frac{1}{2}i\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{1}{k\cdot l}\tilde{h}_{ab}\left(k\right)l^al^be^{i\omega_{\rm g}\left(q+\kappa\cdot\left(y\hat{y}+z\hat{z}\right)\right)}+C\left(y,z\right) \tag{200}$$

$$S^{\rm h}\left(t,\vec{x}\right)=\frac{1}{2}i\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{1}{k\cdot l}\tilde{h}_{ab}\left(k\right)l^al^be^{i\omega_{\rm g}\left(-t+\lambda\cdot\vec{x}+\kappa\cdot\left(\vec{x}-\left(\lambda\cdot\vec{x}\right)\lambda\right)\right)}+C\left(y,z\right). \tag{201}$$

We conclude that

$$S=-\frac{1}{2}i\int_{\mathcal{N}}d^3\mathcal{N}\,\frac{1}{k\cdot l}\tilde{h}_{ab}\left(k\right)l^al^b\left(1-e^{iD(;k,l)}\right)e^{iP(;k)}+C\left(y,z\right), \tag{202}$$

where

$$D\left(t,\vec{x};k,l\right)=\omega_{\rm g}\left(1-\kappa\cdot\lambda\right)\left(\lambda\cdot\vec{x}\right). \tag{203}$$

Before arrival of GWs, we assume no perturbation as $S\left(t<t_0,0<x<L,y,z\right)=C\left(y,z\right)=0$. Therefore,

$$S=-\frac{1}{2}i\int_{\mathcal{N}}d^3\mathcal{N}\,\frac{1}{k\cdot l}\tilde{h}_{ab}l^al^b\left(e^{iP(;k)}-e^{i(P+D)(;k,l)}\right), \tag{204}$$

where

$$\left(P+D\right)\left(t,\vec{x};k,l\right)=P\left(t-\lambda\cdot\vec{x},\vec{x}-\lambda\left(\lambda\cdot\vec{x}\right);k\right). \tag{205}$$

Perturbed amplitude is given by

$$\tilde{\tilde{A}}_a\left(\epsilon\right)=\tilde{A}_a+\epsilon\tilde{B}_a+O\left(\epsilon^2\right). \tag{206}$$

The evolution of the amplitude, eq. (171), provides

$$l^b\nabla_b\tilde{B}_a-\dot{C}^c{}_{ab}\tilde{A}_cl^b=-\frac{1}{2}\tilde{A}_a\left\{\nabla_b\left(-h^{bc}l_c+\nabla^bS\right)+l^c\dot{C}^b{}_{cb}\right\}, \tag{207}$$

$$l^b\nabla_b\tilde{B}_a=-\frac{1}{2}\tilde{A}_a\nabla_b\nabla^bS+\frac{1}{2}\left(\nabla_ah^c{}_b+\nabla_bh^c{}_a-\nabla^ch_{ab}\right)\tilde{A}_cl^b. \tag{208}$$

It gives

$$\tilde{\tilde{B}}_a=\frac{1}{2\left(k\cdot l\right)}\left[\frac{1}{2}\frac{1}{k\cdot l}\tilde{A}_a\tilde{h}_{bc}l^bl^c\left\{k-\left(k\cdot\hat{l}\right)\lambda\right\}\cdot\left\{k-\left(k\cdot\hat{l}\right)\lambda\right\}e^{iD(;k,l)}+\left(k_a\tilde{h}^c{}_b+k_b\tilde{h}^c{}_a-k^c\tilde{h}_{ab}\right)\tilde{A}_cl^b\right]e^{iP(;k)} \tag{209}$$

$$=\frac{1}{2\left(k\cdot l\right)}\left[\frac{1}{2}\tilde{A}_a\tilde{h}_{bc}l^b\hat{l}^ck\cdot\left(u-\lambda\right)e^{iD(;k,l)}+\left(k_a\tilde{h}^c{}_b+k_b\tilde{h}^c{}_a-k^c\tilde{h}_{ab}\right)\tilde{A}_cl^b\right]e^{iP(;k)} \tag{210}$$

$$\tilde{B}_a = \int_{\mathcal{N}} d^3\mathcal{N}\, \frac{1}{2(k\cdot l)} \left[\frac{1}{2}\tilde{A}_a \tilde{h}_{bc} l^b \hat{l}^c k\cdot(u-\lambda)\, e^{iD(;k,l)} + \left(k_a \tilde{h}^c{}_b + k_b \tilde{h}^c{}_a - k^c \tilde{h}_{ab}\right)\tilde{A}_c l^b\right] e^{iP(;k)} + \tilde{C}_a, \tag{211}$$

where $\tilde{C}_a(t,x,y,z) = \tilde{Y}_a(q=-t+x,y,z)$ is quantity satisfying $l^b\nabla_b\tilde{C}_a = 0$. The Lorenz gauge, eq. (172), provides

$$0 = \left(-h^{ab}l_b + \nabla^a S\right)\tilde{A}_a + l\cdot\tilde{B} \tag{212}$$

$$= \left[-h^{ab}l_b + \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{1}{k\cdot l}\tilde{h}_{bc}(k)\, l^b l^c \left\{k^a e^{iP(;k)} - \left(k^a - \left(k\cdot\hat{l}\right)\lambda^a\right) e^{i(P+D)(;k,l)}\right\}\right]\tilde{A}_a$$
$$+ \int_{\mathcal{N}} d^3\mathcal{N}\,\tilde{h}_{ab}\left(\tilde{A}^a - \frac{k\cdot\tilde{A}}{2(k\cdot l)} l^a\right) l^b e^{iP(;k)} + l\cdot\tilde{C} \tag{213}$$

$$= -\frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot\tilde{A}}{k\cdot l}\tilde{h}_{ab}(k)\, l^a l^b e^{i(P+D)(;k,l)} + l\cdot\tilde{C} \tag{214}$$

$$\tilde{C}_a = \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot\tilde{A}}{k\cdot l}\tilde{h}_{ab} l^b e^{i(P+D)(;k,l)} + C(q=-t+x,y,z)\,\hat{l}_a \tag{215}$$

The radiation gauge provides

$$0 = \left(\tilde{B} + iS\tilde{A}\right)\cdot u \tag{216}$$

$$= \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot u}{k\cdot l}\tilde{h}_{ab}\tilde{A}^a l^b e^{iP(;k)} - C(q=-t+x,y,z) \tag{217}$$

$$C = \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot u}{k\cdot l}\tilde{h}_{ab}\tilde{A}^a l^b e^{iP(;k)} \tag{218}$$

As a result,

$$\tilde{B}_a = \int_{\mathcal{N}} d^3\mathcal{N}\,\frac{1}{2(k\cdot l)}\left[\frac{1}{2}\tilde{A}_a\tilde{h}_{bc} l^b\hat{l}^c k\cdot(u-\lambda)\, e^{i(P+D)(;k,l)} + \left(k_a\tilde{h}^c{}_b + k_b\tilde{h}^c{}_a - k^c\tilde{h}_{ab}\right)\tilde{A}_c l^b e^{iP(;k)}\right]$$
$$+ \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot\tilde{A}}{k\cdot l}\tilde{h}_{ab}(k)\, l^b e^{i(P+D)(;k,l)}$$
$$+ \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot u}{k\cdot l}\tilde{h}_{bc}\tilde{A}^b l^c e^{iP(;k)}\hat{l}_a$$
$$= \int_{\mathcal{N}} d^3\mathcal{N}\,\frac{1}{2(k\cdot l)}\left[\left\{\frac{1}{2}\tilde{A}_a\tilde{h}_{bc} l^b\hat{l}^c k\cdot(u-\lambda) + \left(k\cdot\tilde{A}\right)\tilde{h}_{ab} l^b\right\} e^{i(P+D)(;k,l)}\right.$$
$$\left. + \left\{\left(k_a\tilde{h}^c{}_b + k_b\tilde{h}^c{}_a - k^c\tilde{h}_{ab}\right)\tilde{A}_c l^b + \hat{l}_a(k\cdot u)\,\tilde{h}_{bc}\tilde{A}^b l^c\right\} e^{iP(;k)}\right] \tag{219}$$

4.4 Perturbation of Rays in Double Null Gauge

We introduce the Minkowski background spacetime and monochromatic plane electromagnetic wave given by

$$A_a = 2\Re\left[\tilde{A}_a e^{iQ}\right], \tag{220}$$

where $\tilde{A}$ and $l_a \equiv \nabla_a Q$ are constant over spacetime. Let us consider GWs in the double null gauge given by

$$0 = k^a\tilde{h}_{ab}(k), \tag{221}$$

$$0 = l^a\tilde{h}_{ab}(k), \tag{222}$$

$$0 = \tilde{h}^a{}_a(k), \tag{223}$$

for all $k\in\mathcal{N}$.

Perturbed phase is given by

$$\tilde{Q}(\epsilon) = Q + \epsilon S + O\left(\epsilon^2\right). \tag{224}$$

The linear perturbation of l becomes

$$\mathcal{L}_v l^a = \mathcal{L}_v\left(g^{ab}\nabla_b Q\right)$$
$$= -h^{ab}l_b + \nabla^a S.$$
$$= \nabla^a S. \tag{225}$$

Then, eq. (169) provides

$$0=\left(-h^{ab}l_b+\nabla^a S\right)l_a+l^a\nabla_a S \tag{226}$$

implies

$$l^a\nabla_a S=0. \tag{227}$$

The solution have the form of

$$S\left(t,x,y,z\right)=X\left(q=-t+x,y,z\right). \tag{228}$$

Let us consider the frequency $\omega\equiv-u^a\nabla_a Q$. Its perturbed value become

$$\tilde{\omega}\left(\epsilon\right)=\omega+\epsilon\alpha+O\left(\epsilon^2\right), \tag{229}$$

where

$$\begin{aligned}\alpha&=-v^a l_a-u^a\nabla_a S\\ &=-\frac{1}{2}\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{k\cdot l}{k\cdot u}\tilde{h}_{ab}u^a u^b e^{iP(;k)}-u^a\nabla_a S.\end{aligned} \tag{230}$$

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$$0=u^b\nabla_b u^a \tag{231}$$

$$0=u^b\nabla_b v^a+u^b\dot{C}^a{}_{cb}u^c \tag{232}$$

$$i\left(k\cdot u\right)\tilde{v}^a=-\frac{1}{2}i\left(k_b\tilde{h}^a{}_c+k_c\tilde{h}^a{}_b-k^a\tilde{h}_{bc}\right)u^b u^c \tag{233}$$

$$\tilde{v}^a=-\tilde{h}^a{}_b u^b+\frac{1}{2\left(k\cdot u\right)}k^a\tilde{h}_{bc}u^b u^c \tag{234}$$

$$v^a=-h^a{}_b u^b+\frac{1}{2}\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{1}{k\cdot u}k^a\tilde{h}_{bc}u^b u^c e^{iP(;k)} \tag{235}$$

———-

Note that α is gauge-invariant because ω is constant on $\mathcal{M}_0$. We give boundary condition at the 3-dimensional timelike plane $\mathcal{P}$ that is the congruence of emitters as

$$0=\left[\alpha\right]_{\mathcal{P}} \tag{236}$$

$$=\left[-\frac{1}{2}\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{k\cdot l}{k\cdot u}\tilde{h}_{ab}u^a u^b e^{iP(;k)}-u^a\nabla_a S\right]_{\mathcal{P}} \tag{237}$$

$$=\left[-\frac{1}{2}\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{k\cdot l}{k\cdot u}\tilde{h}_{ab}u^a u^b e^{i\omega_{\mathrm{g}}(-t+\kappa\cdot\vec{x})}+\partial_q X\left(q=-t+x,y,z\right)\right]_{\mathcal{P}}, \tag{238}$$

$$=-\frac{1}{2}\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{k\cdot l}{k\cdot u}\tilde{h}_{ab}u^a u^b e^{i\omega_{\mathrm{g}}(-t+\kappa\cdot(y\hat{y}+z\hat{z}))}+\partial_q X\left(q=-t,y,z\right), \tag{239}$$

implies

$$\partial_q X\left(q,y,z\right)=\frac{1}{2}\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{k\cdot l}{k\cdot u}\tilde{h}_{ab}u^a u^b e^{i\omega_{\mathrm{g}}(q+\kappa\cdot(y\hat{y}+z\hat{z}))} \tag{240}$$

$$X\left(q,y,z\right)=\frac{1}{2}i\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{k\cdot l}{\left(k\cdot u\right)^2}\tilde{h}_{ab}u^a u^b e^{i\omega_{\mathrm{g}}(q+\kappa\cdot(y\hat{y}+z\hat{z}))}+C\left(y,z\right) \tag{241}$$

$$S\left(t,\vec{x}\right)=\frac{1}{2}i\int_{\mathcal{N}}d^3\mathcal{N}\left(k\right)\,\frac{k\cdot l}{\left(k\cdot u\right)^2}\tilde{h}_{ab}u^a u^b e^{i\omega_{\mathrm{g}}(-t+\lambda\cdot\vec{x}+\kappa\cdot(\vec{x}-(\lambda\cdot\vec{x})\lambda))}+C\left(y,z\right) \tag{242}$$

We conclude that

$$S=\frac{1}{2}i\int_{\mathcal{N}}d^3\mathcal{N}\,\frac{k\cdot l}{\left(k\cdot u\right)^2}\tilde{h}_{ab}u^a u^b\left(1-e^{iD(;k,l)}\right)e^{iP(;k)}+C\left(y,z\right), \tag{243}$$

where

$$D\left(t,\vec{x};k,l\right)=\omega_{\mathrm{g}}\left(1-\kappa\cdot\lambda\right)\left(\lambda\cdot\vec{x}\right). \tag{244}$$

Before arrival of GWs, we assume no perturbation as $S\left(t<t_0, 0<x<L, y, z\right)=C\left(y, z\right)=0$. Therefore,

$$S=\frac{1}{2}i\int_{\mathcal{N}}d^3\mathcal{N}\,\frac{k\cdot l}{\left(k\cdot u\right)^2}\tilde{h}_{ab}u^au^b\left(e^{iP(;k)}-e^{i(P+D)(;k,l)}\right), \tag{245}$$

where

$$\left(P+D\right)\left(t,\vec{x};k,l\right)=P\left(t-\lambda\cdot\vec{x},\vec{x}-\lambda\left(\lambda\cdot\vec{x}\right);k\right). \tag{246}$$

Perturbed amplitude is given by

$$\tilde{\tilde{A}}_a\left(\epsilon\right)=\tilde{A}_a+\epsilon\tilde{B}_a+O\left(\epsilon^2\right). \tag{247}$$

eq. (171) provides

$$l^b\nabla_b\tilde{B}_a-\dot{C}^c{}_{ab}\tilde{A}_cl^b=-\frac{1}{2}\tilde{A}_a\left(\nabla_b\nabla^bS+l^c\dot{C}^b{}_{cb}\right), \tag{248}$$

implies

$$l^b\nabla_b\tilde{B}_a=\frac{1}{2}\tilde{A}_bl^c\nabla_ch^b{}_a, \tag{249}$$

gives

$$\tilde{B}_a=\frac{1}{2}\tilde{A}_bh^b{}_a+\tilde{C}_a, \tag{250}$$

where $\tilde{C}_a\left(t,x,y,z\right)=\tilde{Y}_a\left(q=-t+x,y,z\right)$ is quantity satisfying $l^b\nabla_b\tilde{C}_a=0$. eq. (172) provides

$$0=l\cdot\tilde{B}, \tag{251}$$

implies

$$0=l\cdot\tilde{C}. \tag{252}$$

Let us consider the intensity,

$$\begin{aligned}
I&\equiv E^2\\
&=4g^{ac}\nabla_{[a}A_{b]}u^b\nabla_{[c}A_{d]}u^d\\
&=\omega^22\Re\left(i\tilde{A}_ae^{iQ}\right)2\Re\left(i\tilde{A}^ae^{iQ}\right) && (253)\\
&=\omega^2\left(i\tilde{A}_ae^{iQ}-i\tilde{A}^*_ae^{-iQ}\right)\left(i\tilde{A}^ae^{iQ}-i\tilde{A}^{a*}e^{-iQ}\right) && (254)\\
&=\omega^22\Re\left[-\left(\tilde{A}\cdot\tilde{A}\right)e^{2iQ}+\tilde{A}\cdot\tilde{A}^*\right]. && (255)
\end{aligned}$$

Its perturbed value become

$$\tilde{I}\left(\epsilon\right)=I+\epsilon J+O\left(\epsilon^2\right), \tag{256}$$

where

$$\begin{aligned}
J&=\omega^22\Re\left[-\left(2\tilde{A}\cdot\tilde{B}-h^{ab}\tilde{A}_a\tilde{A}_b\right)e^{2iQ}+2\tilde{A}\cdot\tilde{B}^*-h^{ab}\tilde{A}_a\tilde{A}^*_b\right] && (257)\\
&=\omega^22\Re\left[-2\left(\tilde{A}\cdot\tilde{C}\right)e^{2iQ}+2\tilde{A}\cdot\tilde{C}^*\right] && (258)
\end{aligned}$$

Note that J is gauge-invariant because I is constant on $\mathcal{M}_0$. We give boundary condition at the 3-dimensional timelike plane $\mathcal{P}$ that is the congruence of emitters as

$$\begin{aligned}
0&=\left[J\right]_{\mathcal{P}} && (259)\\
&=\omega^22\Re\left\{-2\left(\tilde{A}\cdot\tilde{Y}\left(q=-t,y,z\right)\right)e^{2i\omega(q+\lambda\cdot(y\hat{y}+z\hat{z}))}+2\tilde{A}\cdot\tilde{Y}^*\left(q,y,z\right)\right\}, && (260)
\end{aligned}$$

implies

$$\tilde{A}\cdot\tilde{C}=0, \tag{261}$$

$$\tilde{A}\cdot\tilde{C}^*=0. \tag{262}$$

$$0=\tilde{a}\tilde{c}+\tilde{b}\tilde{d} \tag{263}$$

$$0=\tilde{a}\tilde{c}^*+\tilde{b}\tilde{d}^* \tag{264}$$

4.5 Beyond Geometrical Optics

Please refer [5, 6, 7].

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