Lecture Materials

- Slide
  - https://link.neverdie.live/s/YS9g2yZPHYjnXp8
- Lecture Note
  - https://www.overleaf.com/read/vpxgkspvvrhp#7a8e8f

GW-BASIC

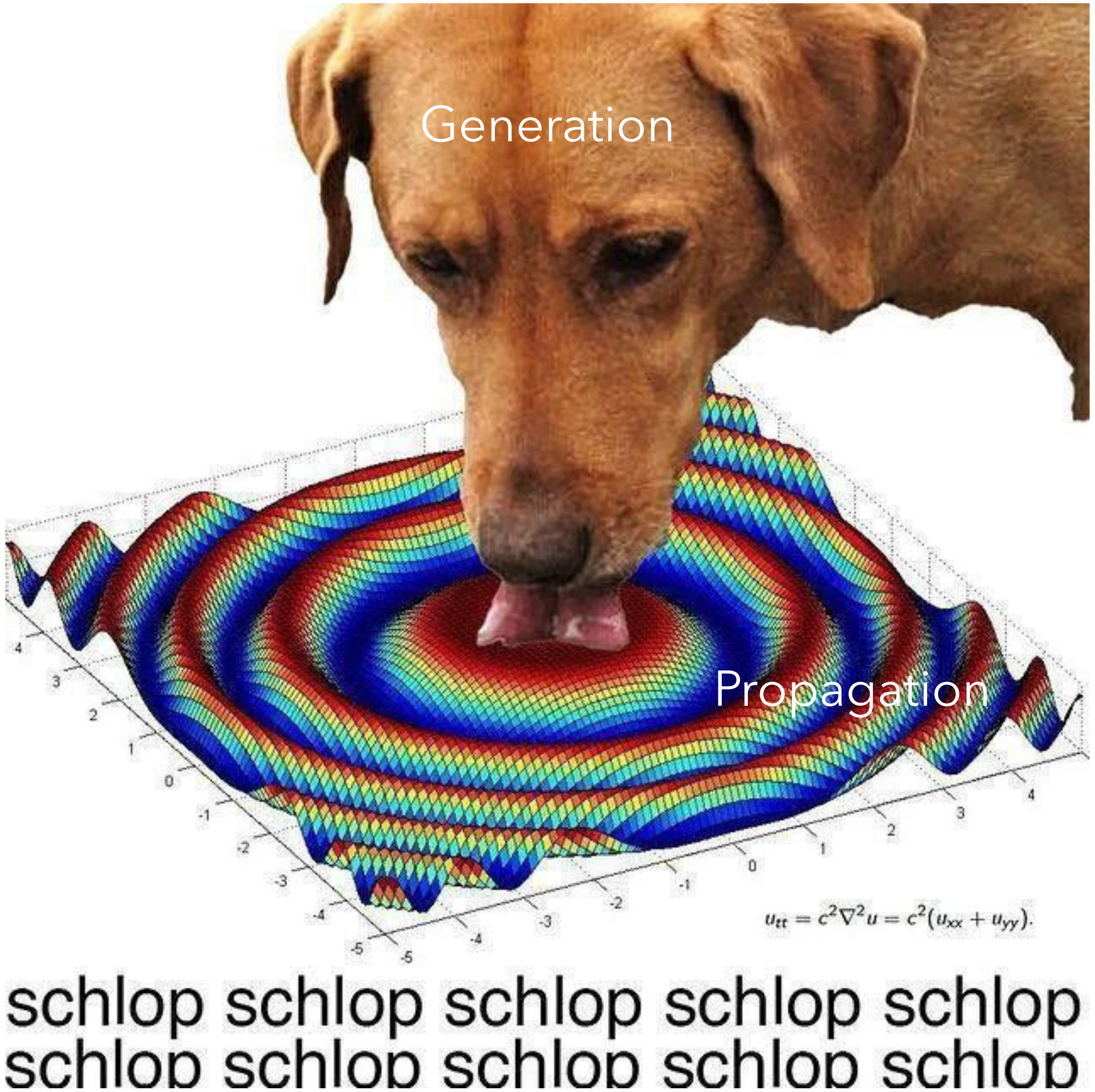
Chan Park (HNAS)

2026.07.28 @ KASI

2026 Summer School on Numerical Relativity and Gravitational Waves

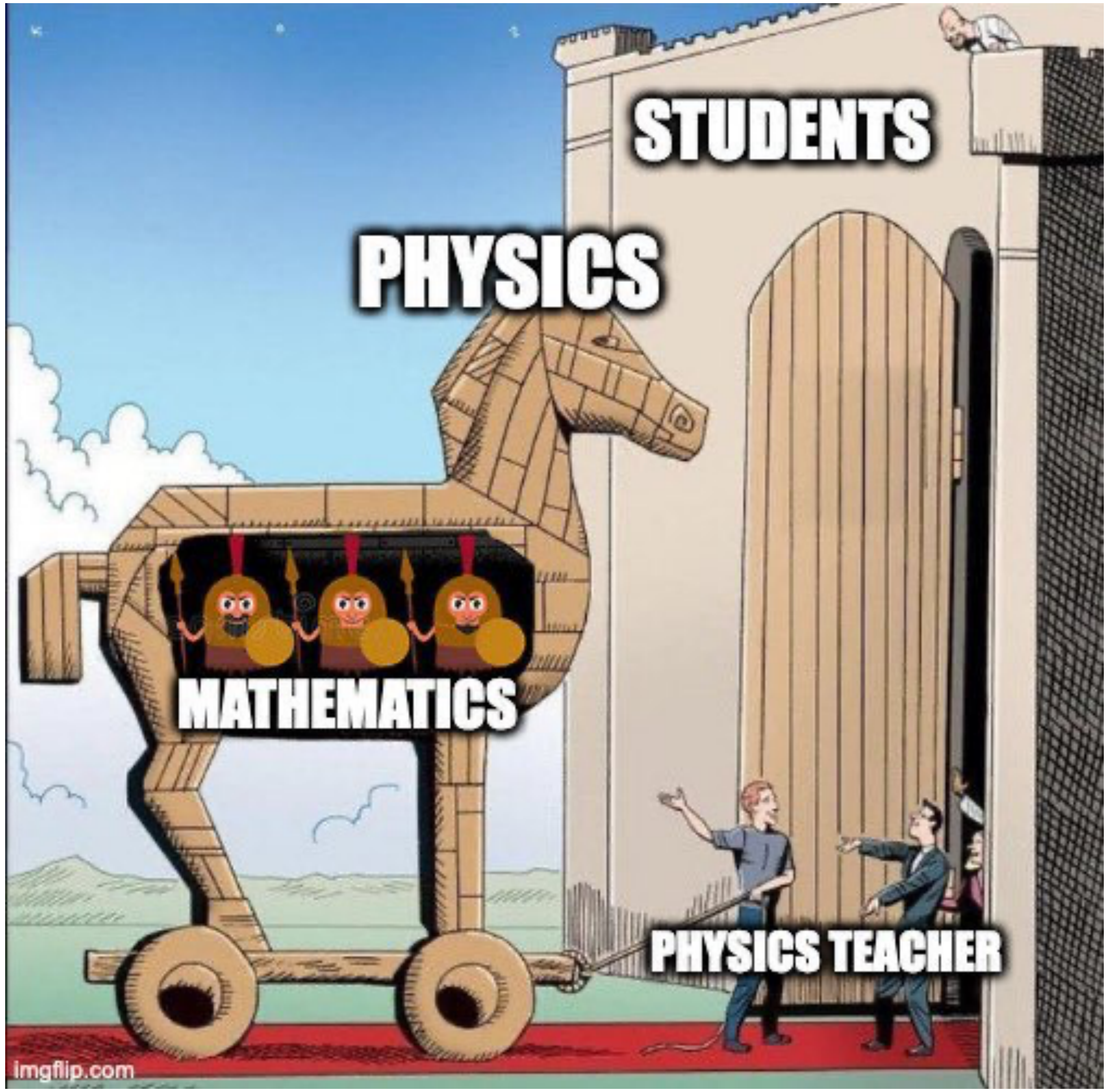
Lecture Contents

- Perturbations
- GW Propagations
- GW Generations
- GW Detections



Lecture Goals

- To understand perturbation gauges.
- To understand how GWs propagate and the physical effects they produce.
- To understand how GWs are generated.
- To understand the principles of GW detection using electromagnetic waves.



I. Perturbations

Spherical Chicken in a Vacuum

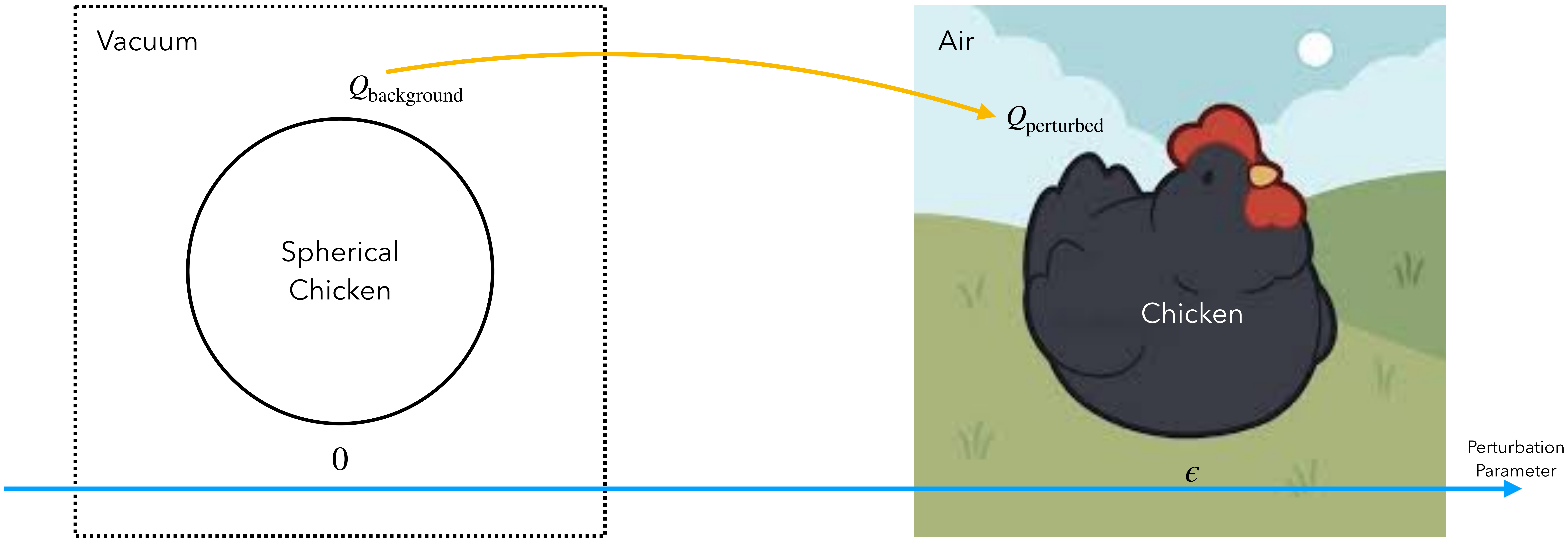
- ... chickens ... won't lay any eggs...
- ... physicists then does some calculations ...
- ... I have a solution but it only works with spherical chickens in a vacuum!

TV Drama "Big Bang Theory"

Perturbations

Perturbation:

$$\delta Q = Q_{\text{perturbed}} - Q_{\text{background}}$$



Background System

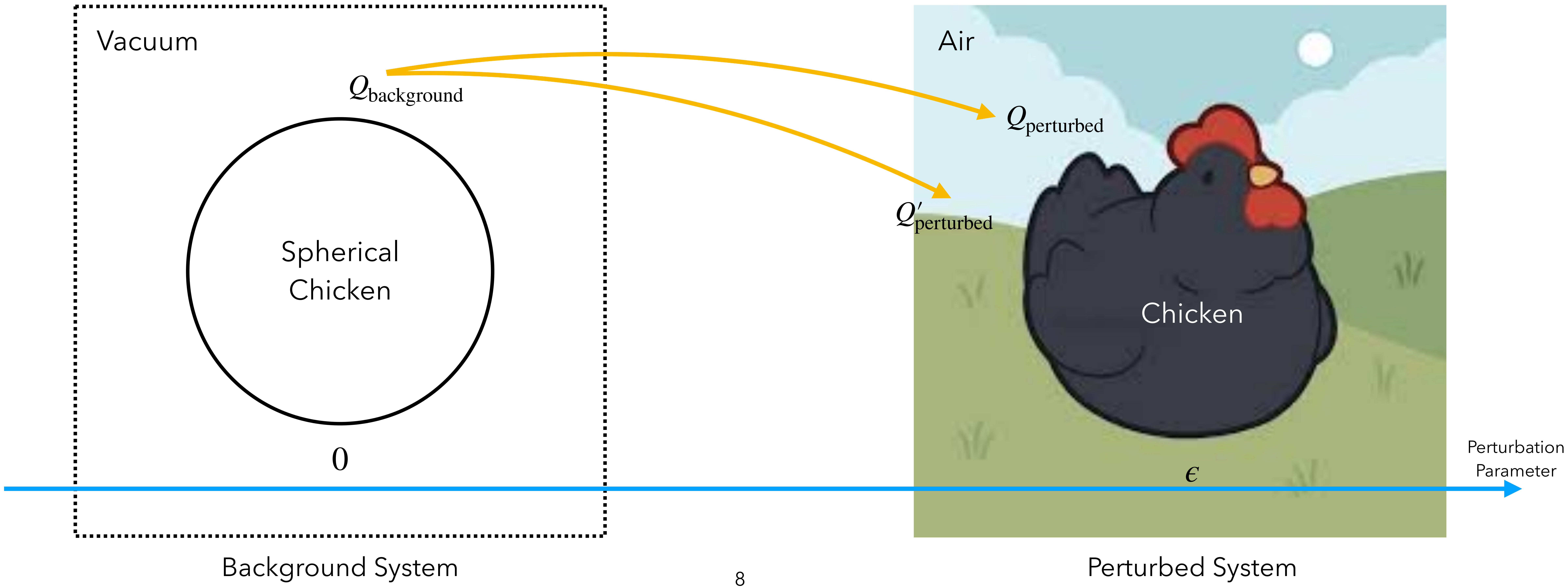
Perturbed System

Gauges in Perturbations

Perturbations in Two Different Gauges:

$$\Delta Q = Q_{\text{perturbed}} - Q_{\text{background}}$$
$$\Delta' Q = Q'_{\text{perturbed}} - Q_{\text{background}}$$

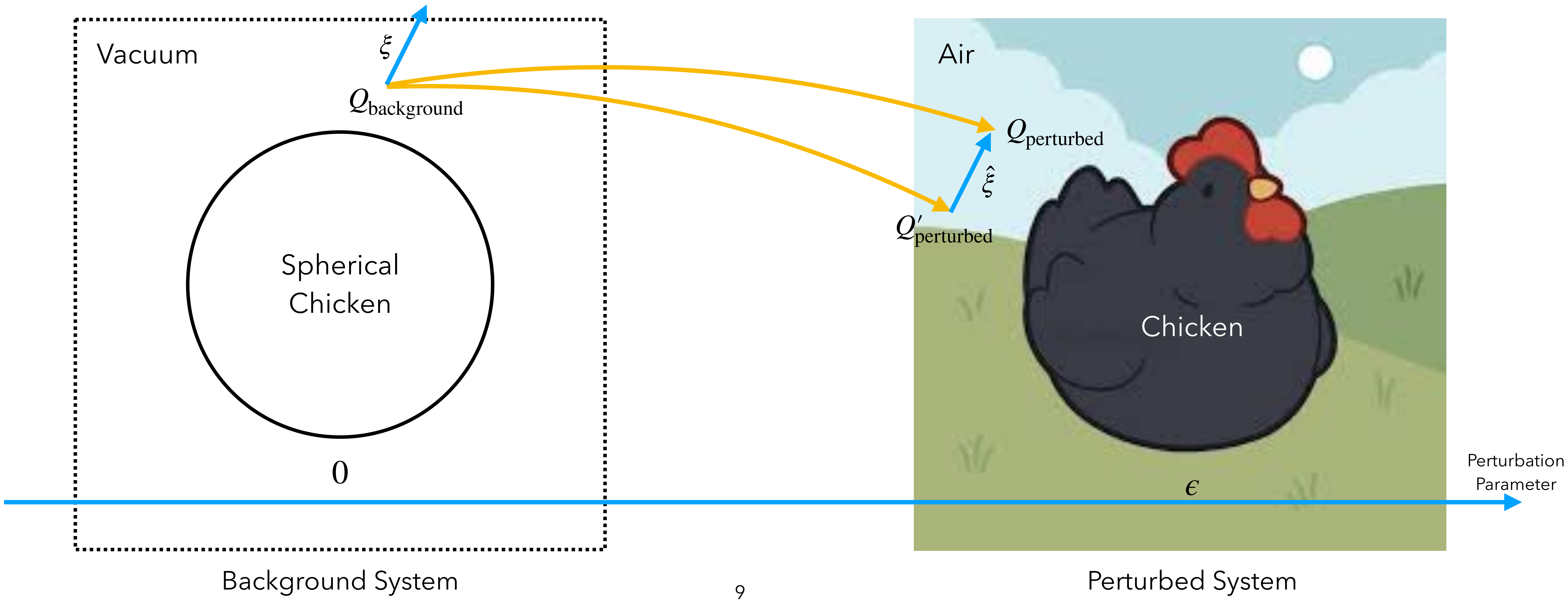
Gauges = Mathematical redundancies for a single physical reality



Gauge Transformation

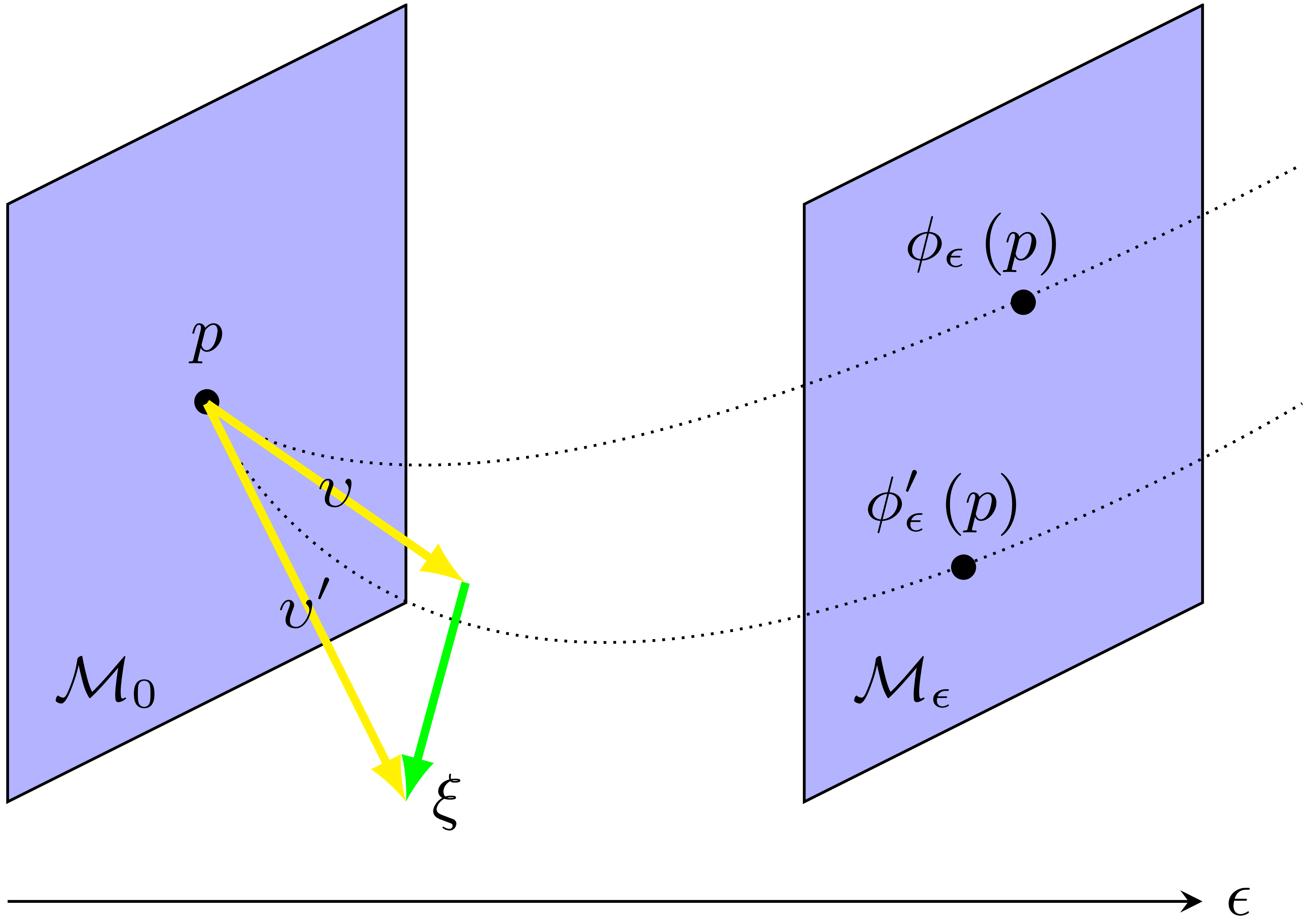
$$\Delta' Q - \Delta Q = Q'_{\text{perturbed}} - Q_{\text{perturbed}} = \epsilon \mathscr{L}_{\hat{\xi}} Q_{\text{perturbed}}$$

$$= \epsilon \mathscr{L}_{\xi} Q_{\text{background}} + \mathcal{O}\left(\epsilon^2\right)$$



Rigorous Mathematical Definition

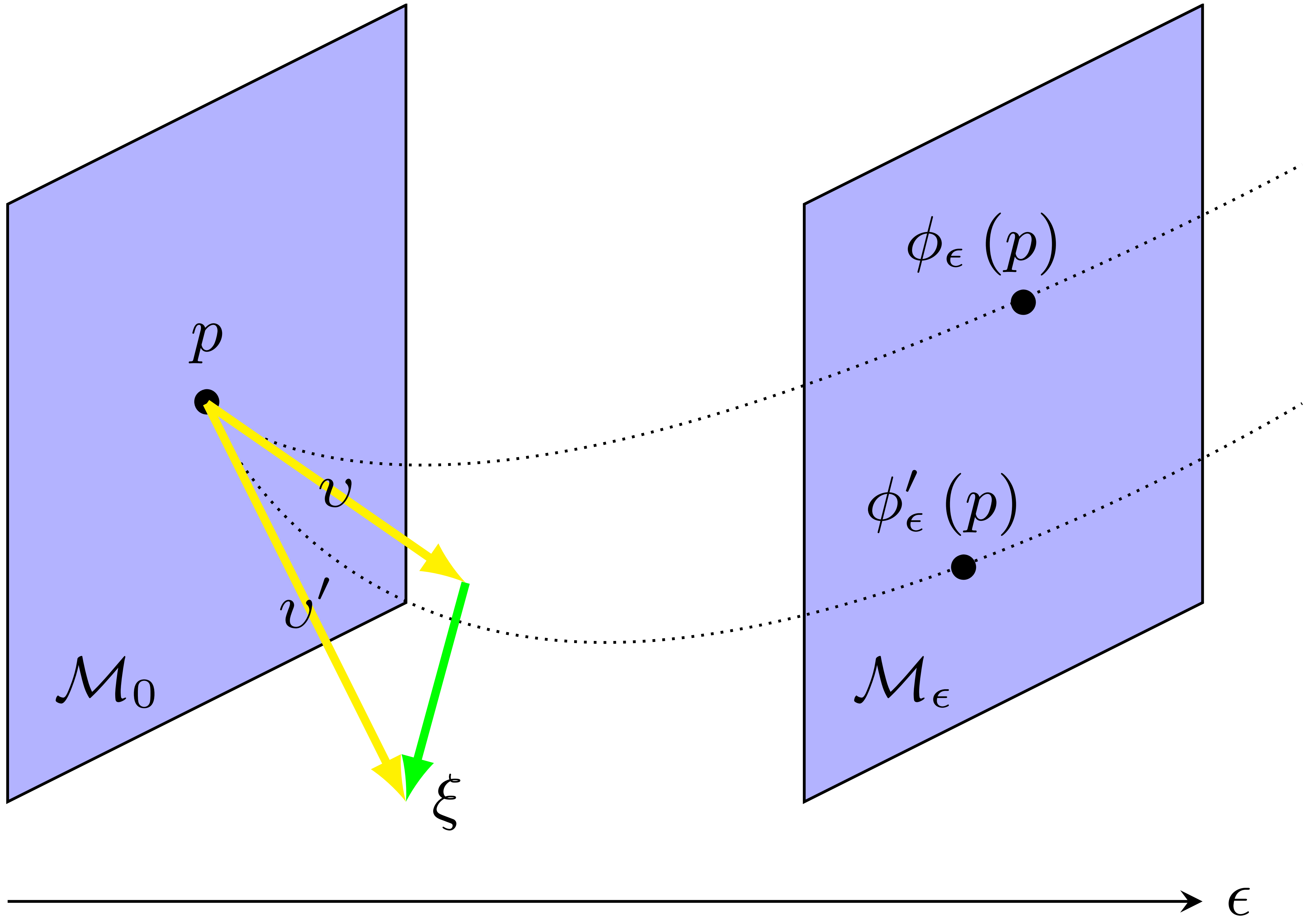
- For a one-parameter foliation of spacetimes $\mathscr{M}_\epsilon$ on 5-dimensional manifold
- ϕ_ϵ: one-parameter group of map between $\mathscr{M}_0$ and $\mathscr{M}_\epsilon$
- Perturbed Quantity in $\mathscr{M}_0$
  - $\phi_\epsilon^* \hat{Q} = Q + \epsilon \mathscr{L}_v \hat{Q} + O\left(\epsilon^2\right)$
  - $\hat{Q}$: quantities on 5D manifold
  - ϕ_ϵ^*: pullback operator by ϕ_ϵ
  - Q: quantities on $\mathscr{M}_0$
  - $\mathscr{L}_v \hat{Q}$: linear perturbation
  - v: generator of ϕ_ϵ



Credit: C. Park / ApJ 940 58

Perturbation Gauges

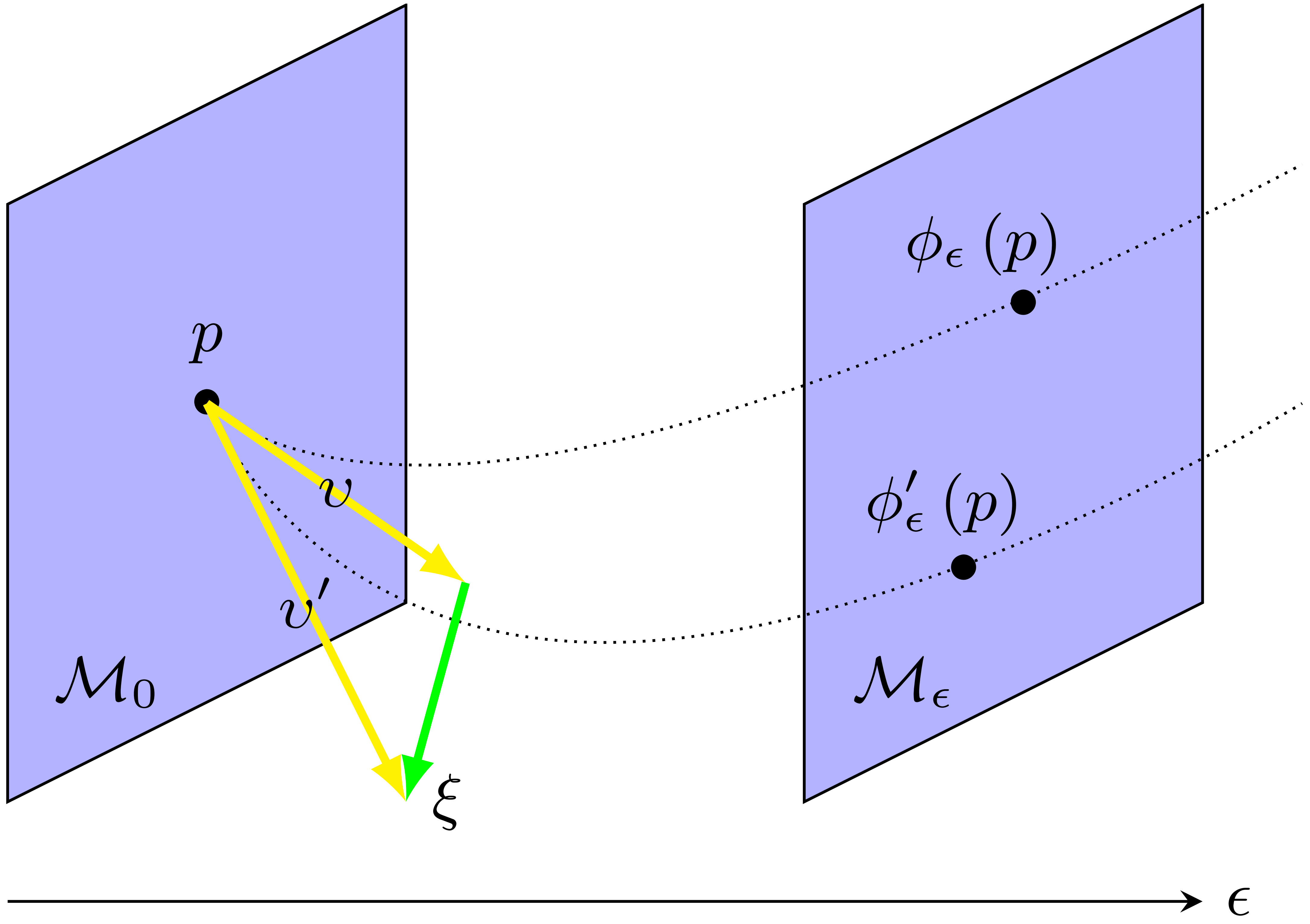
- Gauge freedom = freedom to choose ϕ_ϵ
- Gauge Transformation
  - $\mathscr{L}_{v'}\hat{Q} - \mathscr{L}_{v}\hat{Q} = \mathscr{L}_{\xi}Q$ for $\xi \in \mathscr{T}\left(\mathscr{M}_0\right)$
- Example: Metric Perturbation
  - $h_{ab} = \mathscr{L}_{v}\hat{g}_{ab}$
  - $h'_{ab} = \mathscr{L}_{v'}\hat{g}_{ab}$
  - $h'_{ab} - h_{ab} = \mathscr{L}_{\xi}g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$



Credit: C. Park / ApJ 940 58

Gauge Invariance

- Stewart-Walker theorem
  - $\mathscr{L}_v\hat{Q}$ is gauge-invariant if and only if $\mathscr{L}_\xi Q = 0$ for all ξ.
  - $\mathscr{L}_\xi Q = 0 \leftrightarrow Q$ is zero or a combination of constant scalar and Kronecker delta.
- Example: Perturbation of Flat spacetime
  - $h_{ab} = \mathscr{L}_v\hat{g}_{ab}$ is not gauge invariant because $g_{ab} \neq 0$
  - Linear perturbation of the Riemann tensor $(\delta R)^a{}_{bcd} = \mathscr{L}_v\hat{R}^a{}_{bcd}$ is gauge invariant because $R^a{}_{bcd} = 0$.

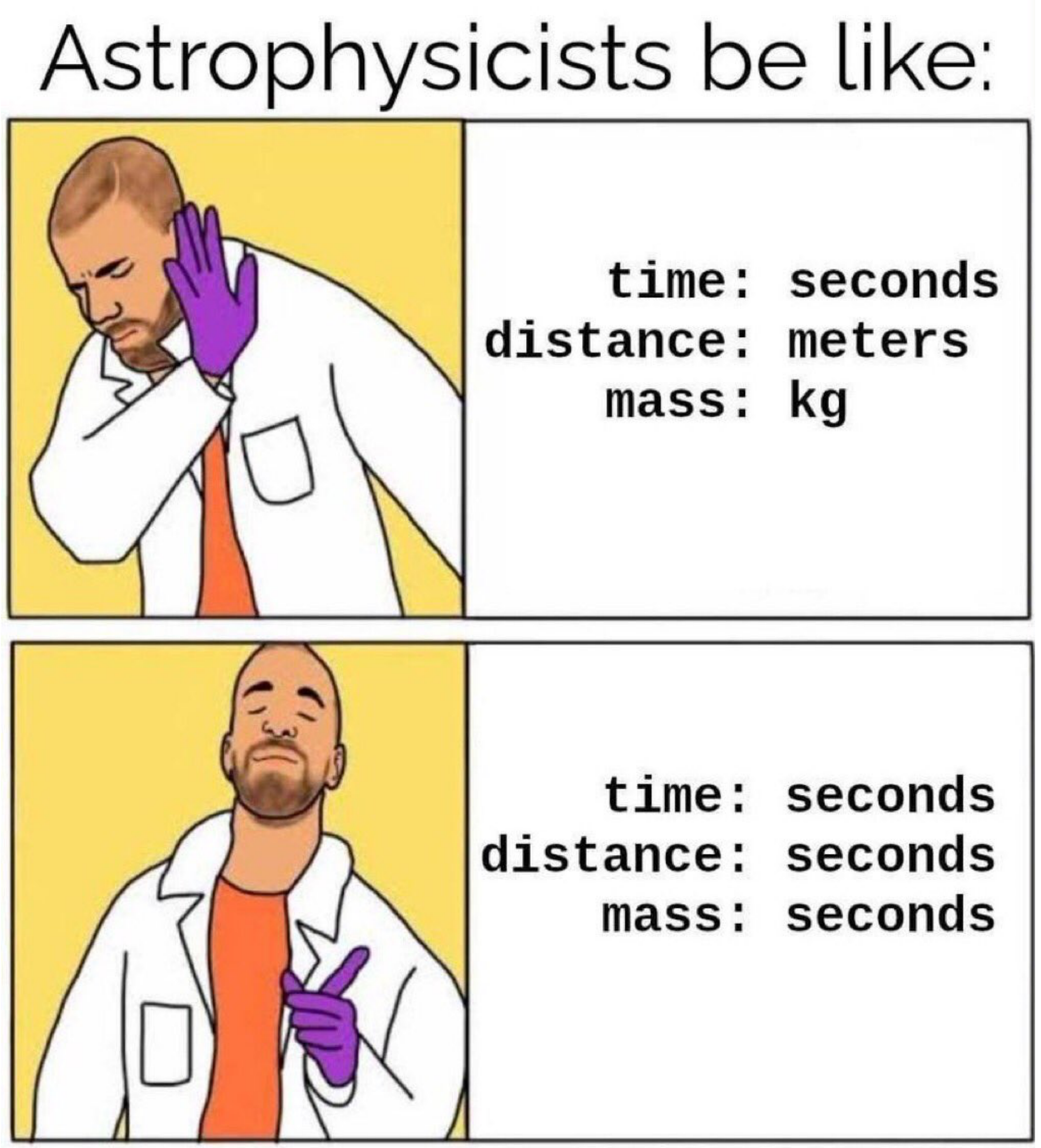


Credit: C. Park / ApJ 940 58

II. Propagation of GWs

Geometrized Unit System

- $c = 299792458\,\mathrm{m/s} = 1$
  - $1\,\mathrm{s} = 299792458\,\mathrm{m}$
- $G = 6.6743 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}} = 1$
  - $1\,\mathrm{kg} = \dfrac{6.6743 \times 10^{-11}}{299792458^2}\,\mathrm{m}$



Wave Equations

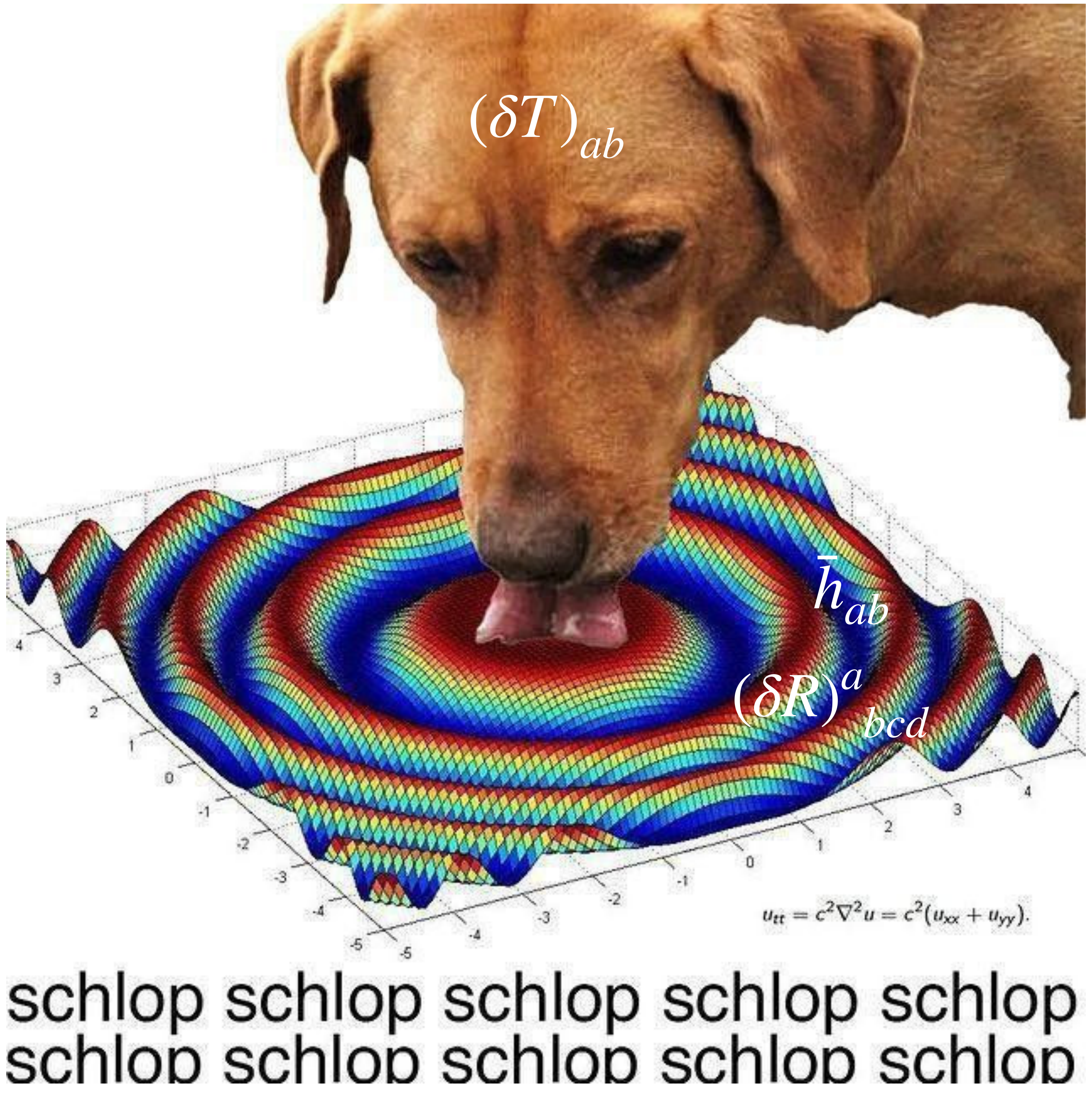
- Plane Wave Equation in Flat Spacetime
  - $0 = -\dfrac{\partial^2\psi}{\partial t^2} + \dfrac{\partial^2\psi}{\partial x^2} = 0$
- Wave Equation in Flat Spacetime
  - $0 = \Box\psi = \partial^\mu\partial_\mu\psi = g^{\mu\nu}\partial_\mu\partial_\nu\psi \quad (\mu, \nu = 0,1,2,3)$
- Scalar Wave Equation in Curved Spacetime
  - $0 = \nabla^a\nabla_a\psi = g^{ab}\nabla_a\nabla_b\psi$ (valid only for scalar field)
- Wave Equation in Curved Spacetime
  - $0 = \Delta_{\mathrm{dR}}\omega = -\left(d\delta + \delta d\right)\omega$ (for differential forms)

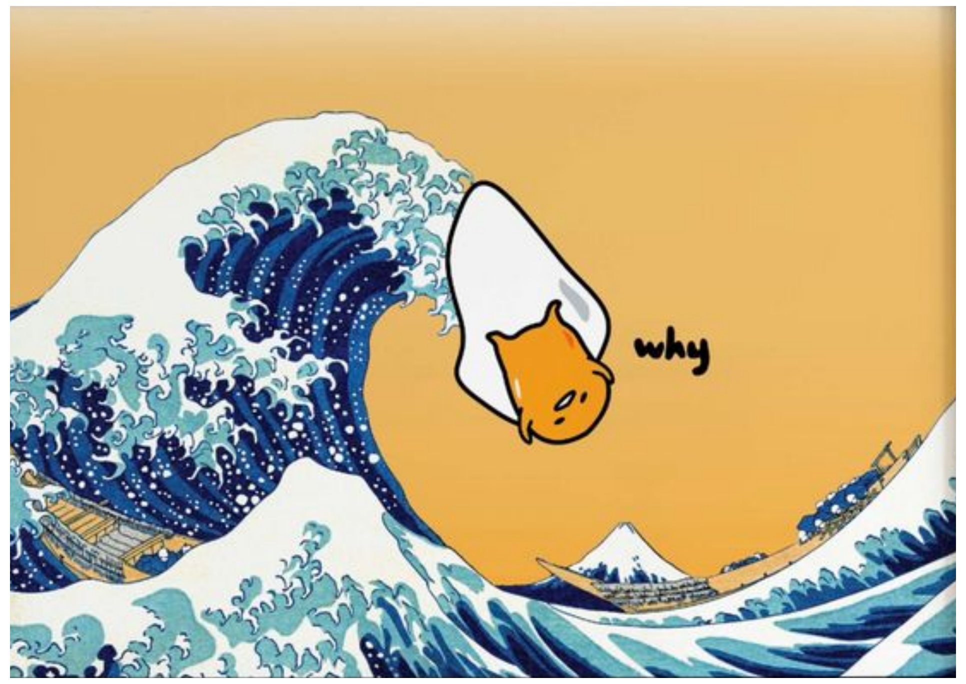
Wave Equation for Metric Perturbation

- Einstein's Equations
  - $8\pi T_{ab} = G_{ab}$
- Perturbations on Flat Background Spacetime in Lorenz gauge
  - $8\pi (\delta T)_{ab} = \mathscr{L}_v \hat{G}_{ab} = -\dfrac{1}{2} \nabla^c \nabla_c \bar{h}_{ab}$
  - $\bar{h}_{ab} \equiv h_{ab} - \dfrac{1}{2} g_{ab} h^{cd} g_{cd}$
  - Lorenz gauge condition: $\nabla^b \bar{h}_{ab} = 0$
- Riemann tensor perturbation
  - $(\delta R)^{ab}{}_{cd} = -2 \nabla^{[a} \nabla_{[c} h^{b]}{}_{d]}$

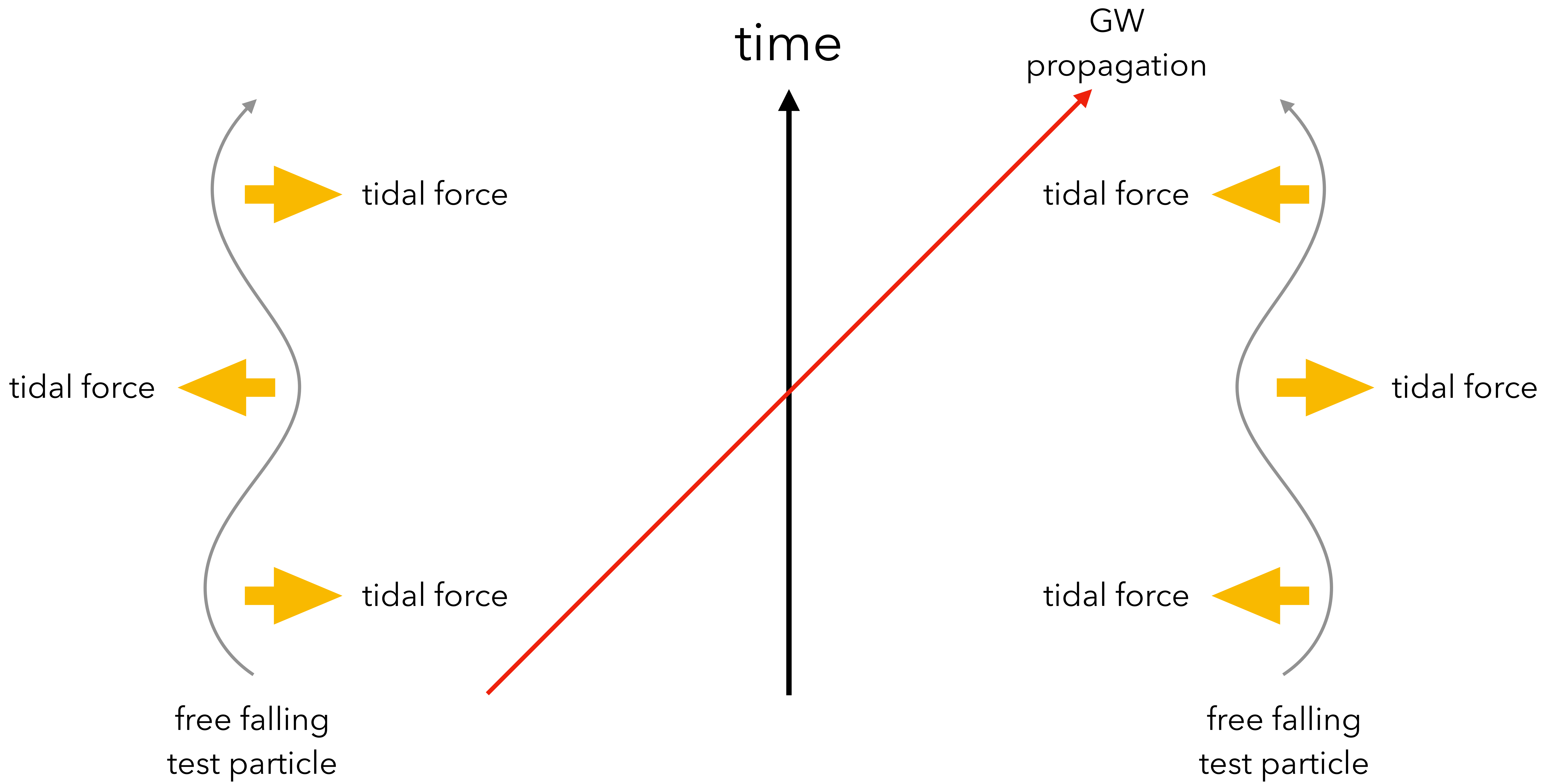
Gravitational Waves

- Wave propagations in vacuum region
  - $(\delta T)_{ab} = 0$
- Gravitational Waves
  - $\nabla^c \nabla_c \bar{h}_{ab} = 0$
  - $\nabla^e \nabla_e (\delta R)^a{}_{bcd} = 0$



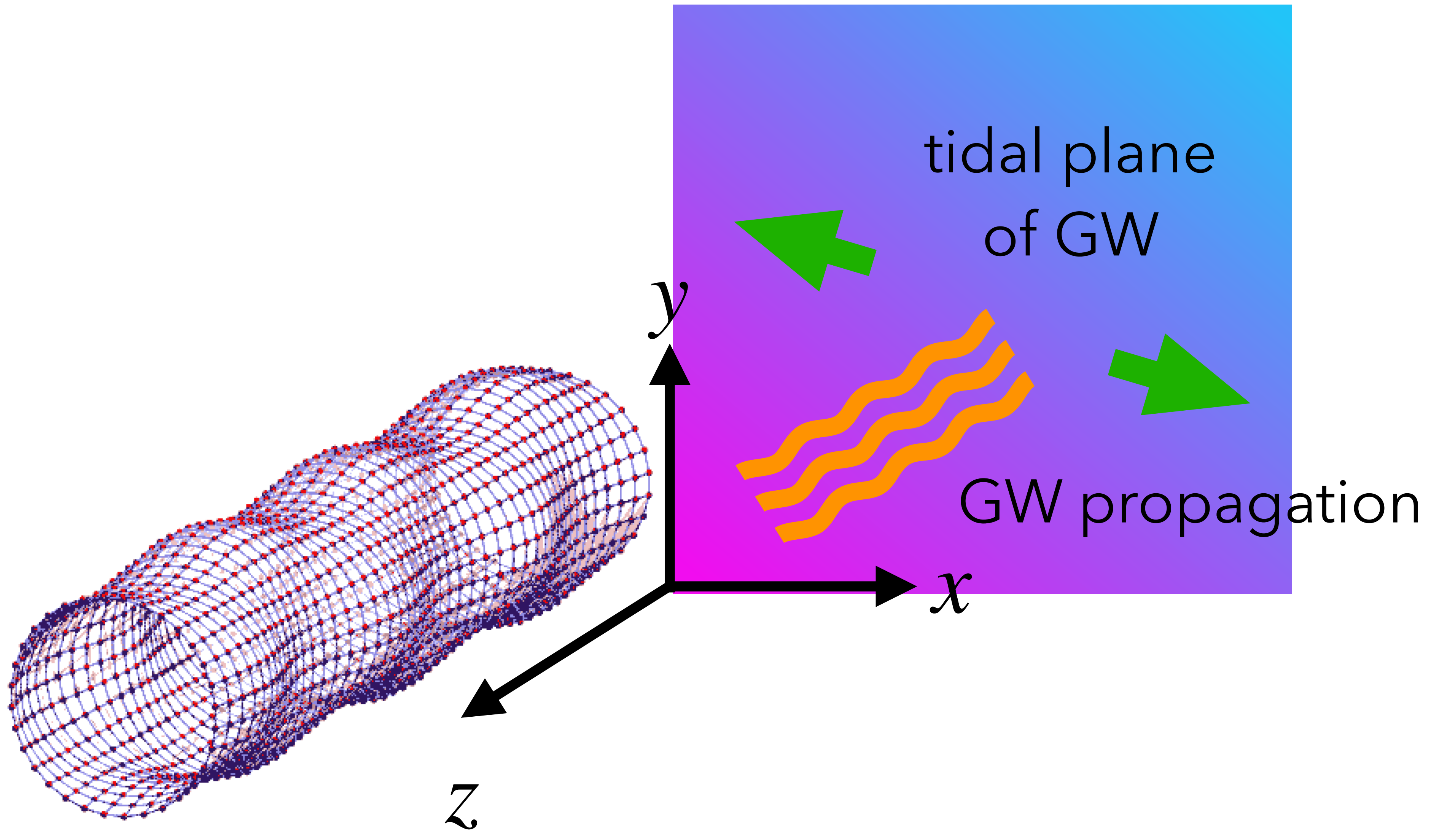


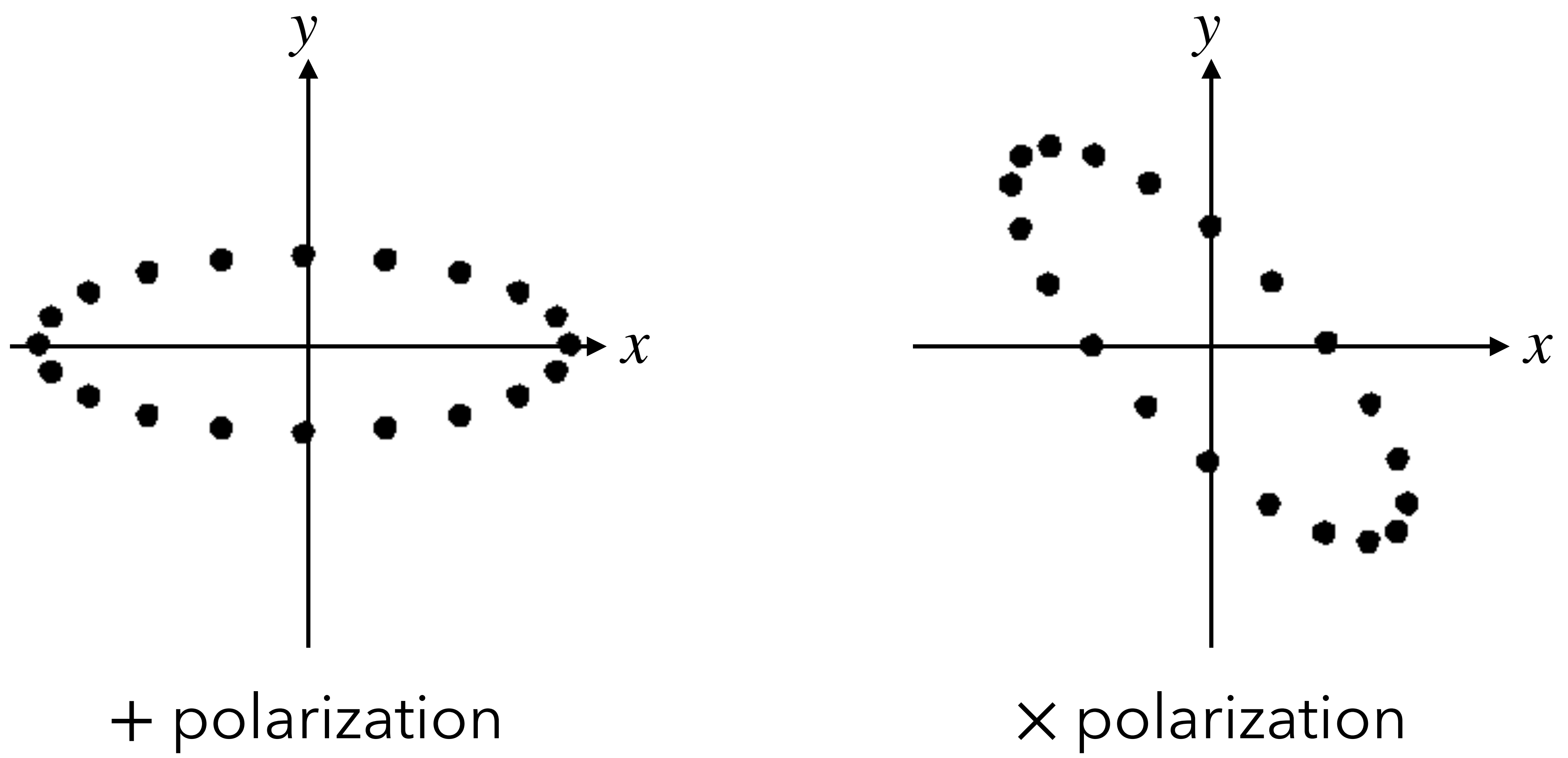
Effects of Gravitational Waves



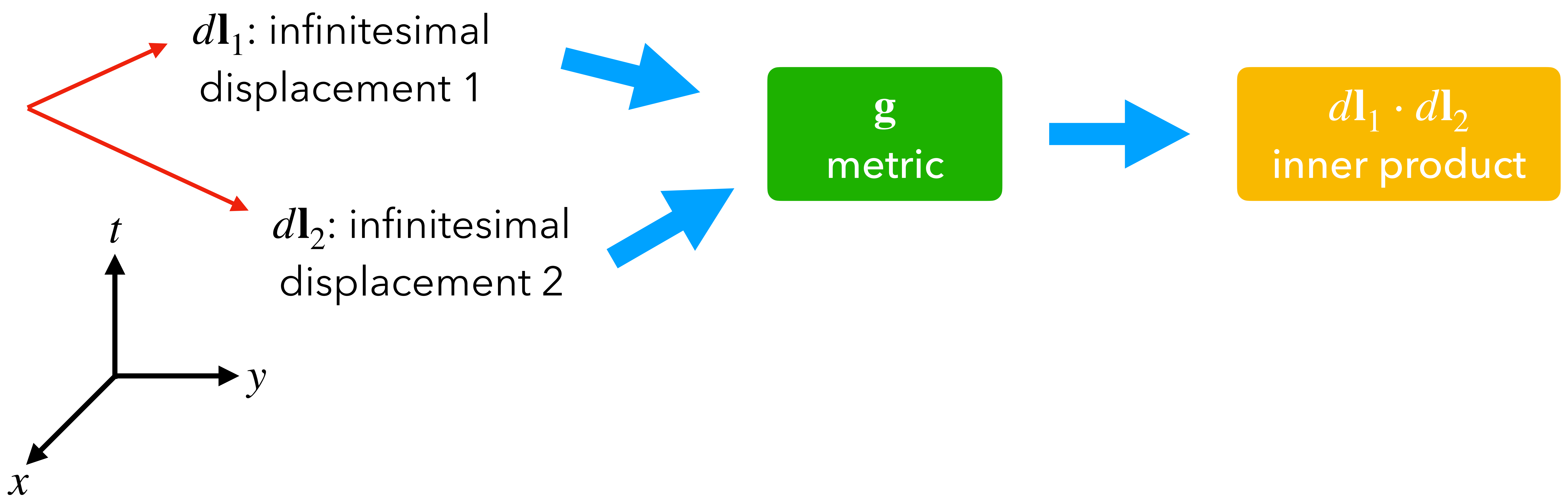
Properties of Gravitational Waves (GWs)

- Propagation speed: speed of light
- Transverse wave: propagation direction ⊥ tidal direction
- Two polarization modes: plus polarization and cross polarization





Metric



$d\mathbf{l}_1$: infinitesimal displacement 1
$d\mathbf{l}_2$: infinitesimal displacement 2
t
y
x
$\mathbf{g}$ metric
$d\mathbf{l}_1 \cdot d\mathbf{l}_2$ inner product

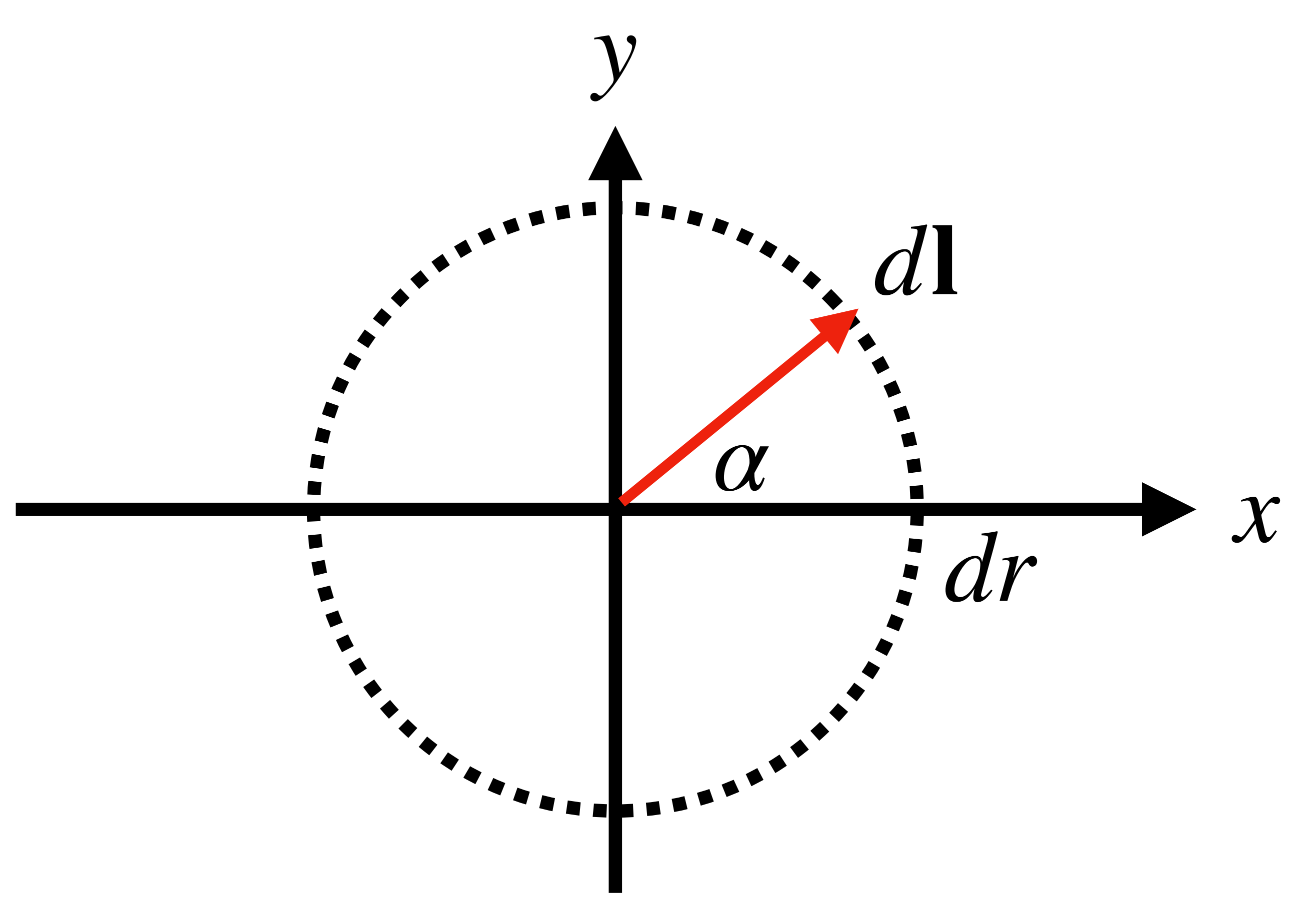
Metric in Components

- Infinitesimal Displacement
  - $d\mathbf{l}_1 = \left(dt_1, dx_1, dy_1, dz_1\right)$
  - $d\mathbf{l}_2 = \left(dt_2, dx_2, dy_2, dz_2\right)$
- Inner product
  - $$d\mathbf{l}_1 \cdot d\mathbf{l}_2 = g_{\mu\nu} dx_1^{\mu} dx_2^{\nu} = \begin{bmatrix} dt_1 & dx_1 & dy_1 & dz_1 \end{bmatrix} \begin{bmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{bmatrix} \begin{bmatrix} dt_2 \\ dx_2 \\ dy_2 \\ dz_2 \end{bmatrix}$$
  - where $g_{\mu\nu}$ is symmetric such that $g_{\mu\nu} = g_{\nu\mu}$.

Metric Perturbation of GWs

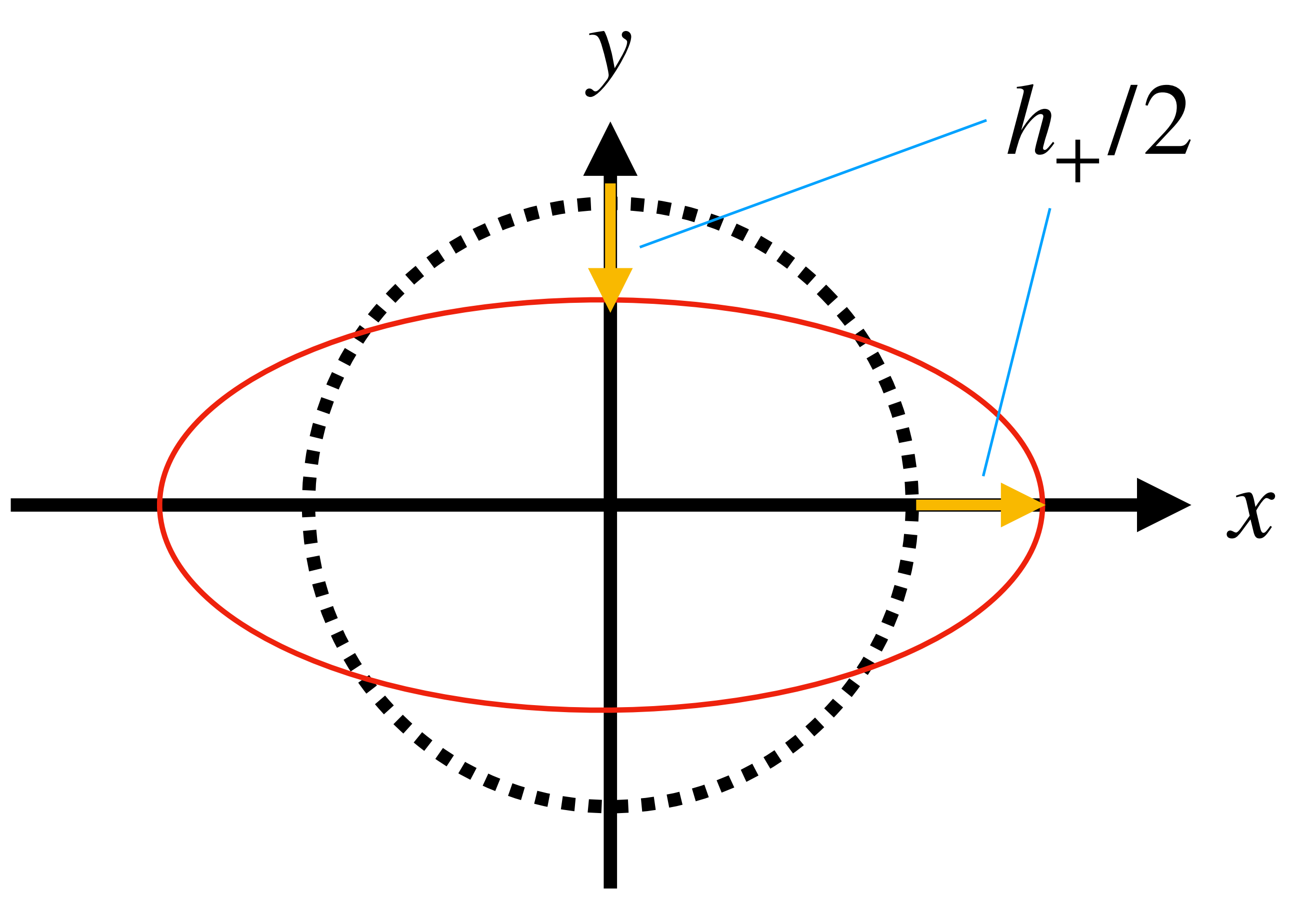
- Redefinition of h: $h_{ab} = \epsilon \mathscr{L}_v g_{ab} \ll 1$
- Metric perturbation in TT (Transverse Traceless) gauge
  - $g^{ab} h_{ab}^{\mathrm{TT}} = 0 \qquad \nabla^b h_{ab}^{\mathrm{TT}} = 0$
  - $h_{0\mu}^{\mathrm{TT}} = 0$
- Then, $\nabla^a \nabla_a h_{ij}^{\mathrm{TT}} = 0 \qquad (i, j = 1,2,3)$
- Components of metric
  - $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{\mathrm{TT}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- Let us consider the coordinate displacement
  $d\mathbf{l} = \left(0, dr\cos\alpha, dr\sin\alpha, 0\right)$

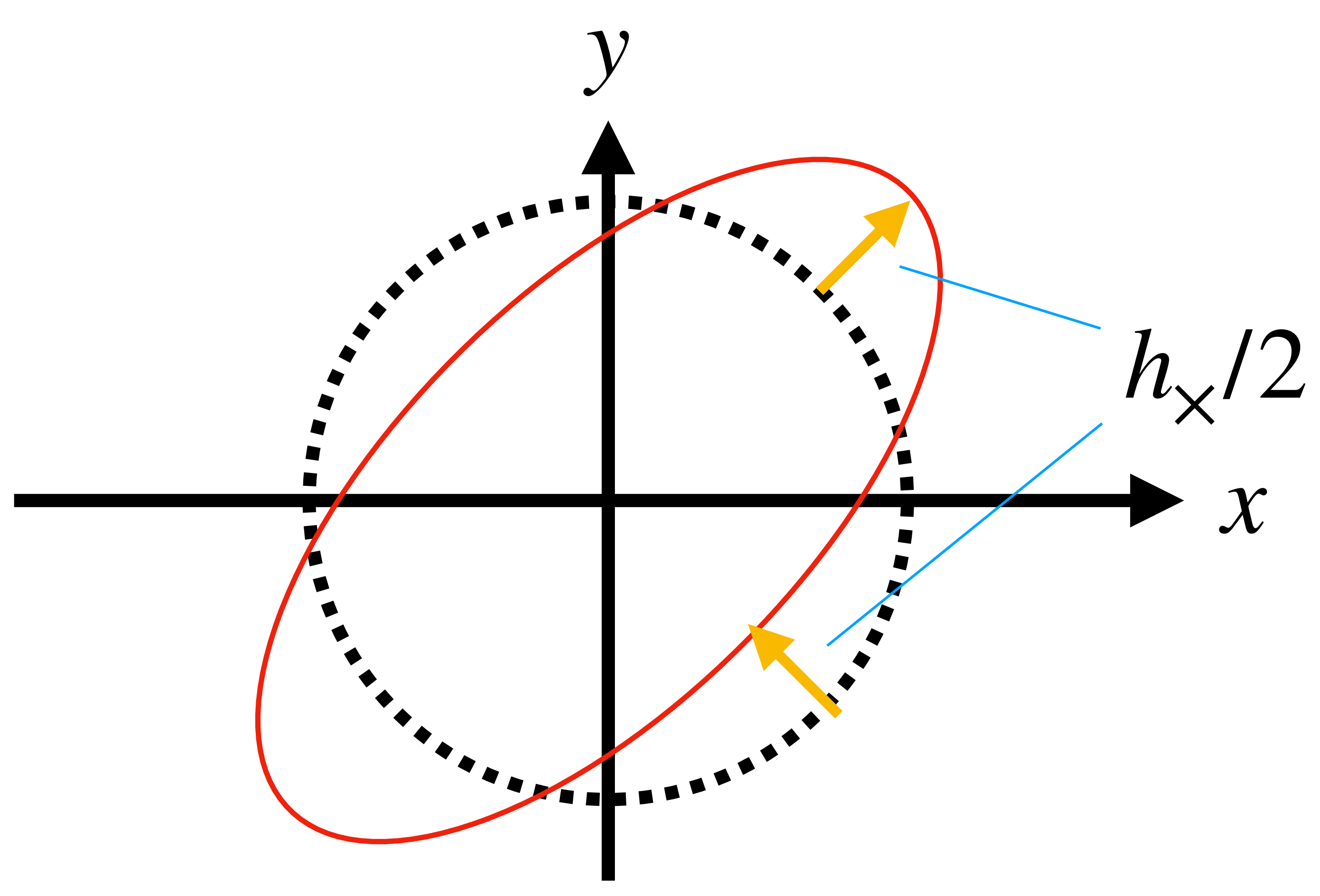


Metric Perturbation of GWs

- The physical length of $d\mathbf{l}$ is given by
- $$d\mathbf{l}\cdot d\mathbf{l} = \begin{bmatrix}0 & dr\cos\alpha & dr\sin\alpha & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1+h_+ & h_\times & 0\\ 0 & h_\times & 1-h_+ & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ dr\cos\alpha\\ dr\sin\alpha\\ 0\end{bmatrix}$$
- $$\sqrt{d\mathbf{l}\cdot d\mathbf{l}} = dr\left\{1+\frac{1}{2}h_+\cos(2\alpha)+\frac{1}{2}h_\times\sin(2\alpha)\right\}$$



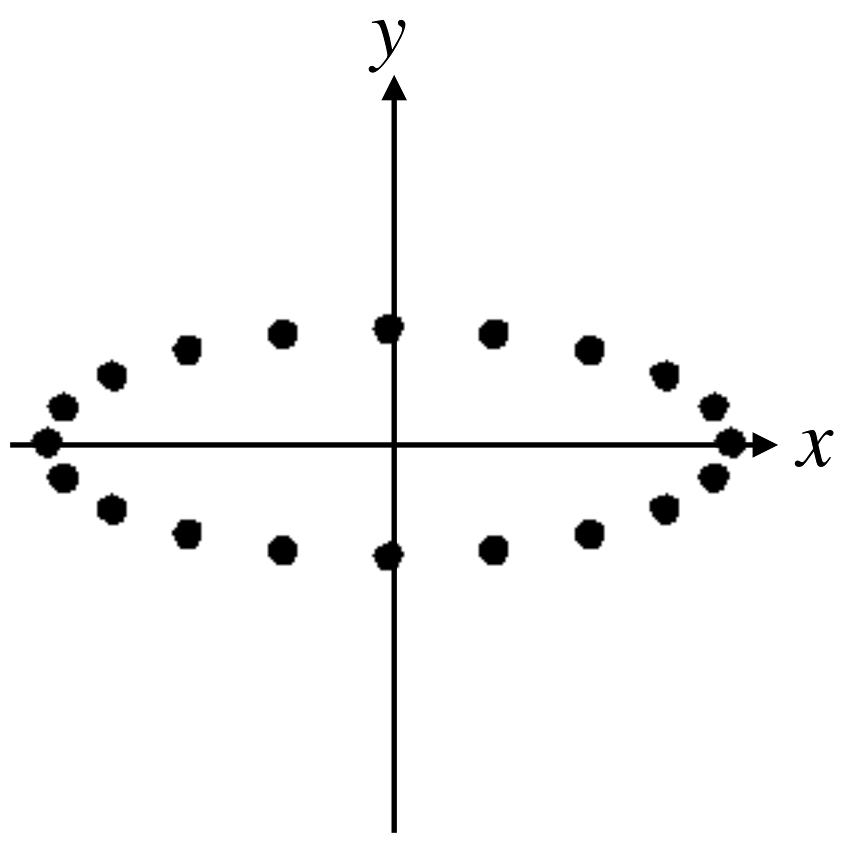
+ polarization



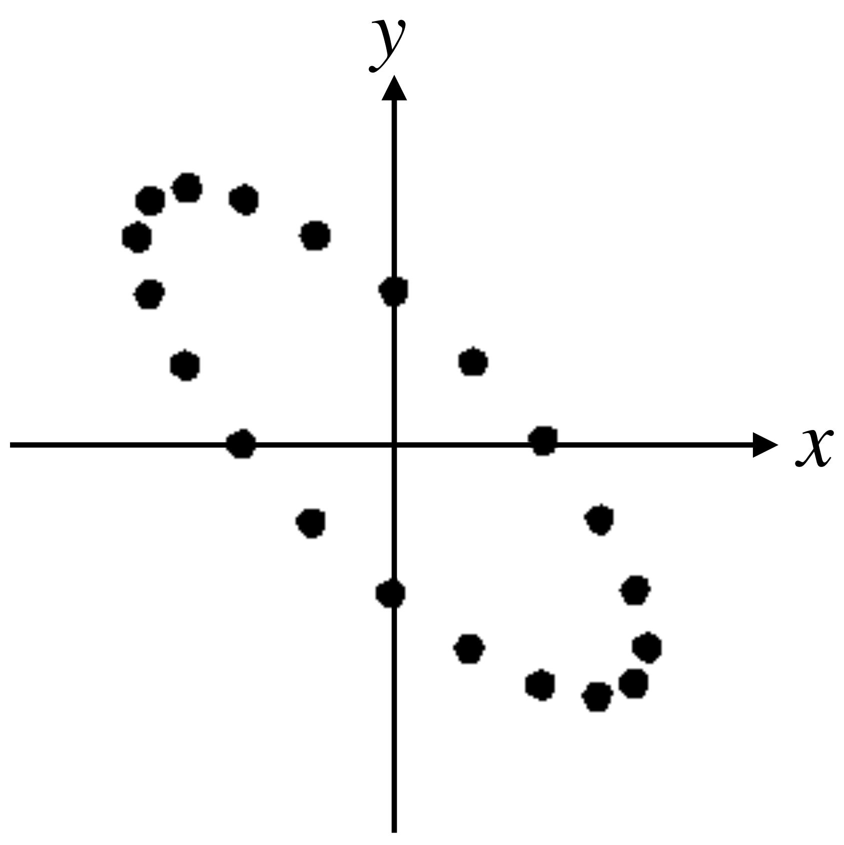
x polarization

Gravitational Waves

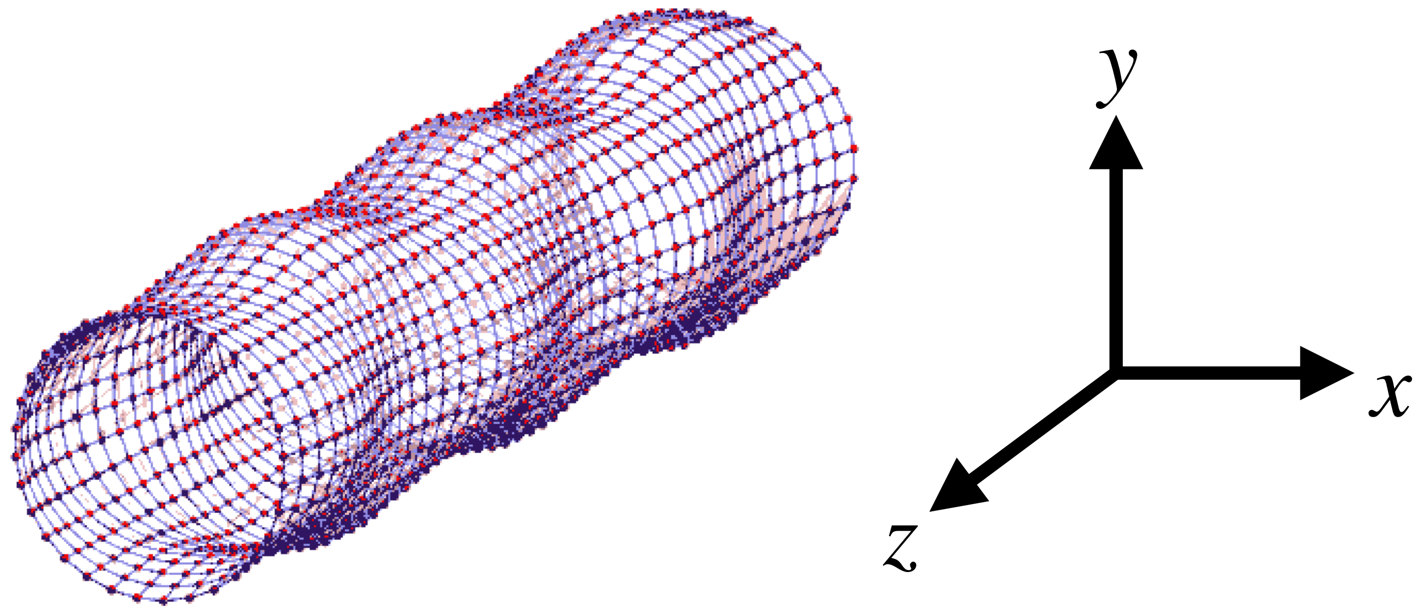
- Monochromatic Plane GWs propagating to $+z$ axis
- $$g_{\mu\nu} = \eta_{\mu\nu} + h^{\mathrm{TT}}_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cos\left\{\omega_{\mathrm{g}}\left(-t+z\right)+\phi\right\}$$



+ polarization



× polarization



propagation of GWs

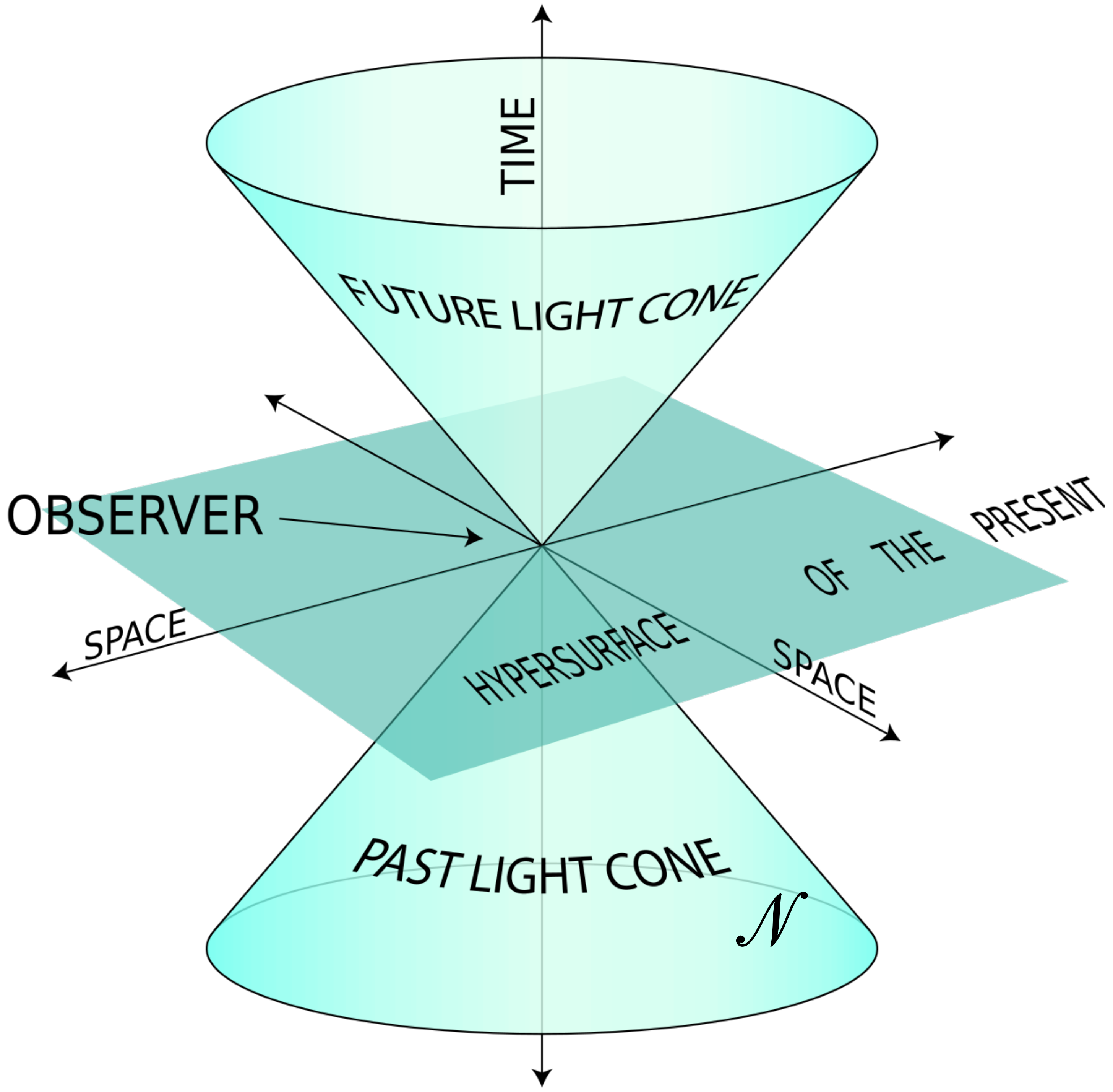
III. Generation of GWs

Solution with δT

- Perturbed Einstein's equations
  - $\nabla^c \nabla_c \bar{h}_{ab} = -16\pi (\delta T)_{ab}$
  - Note that δT is gauge invariant.
- Green function solution

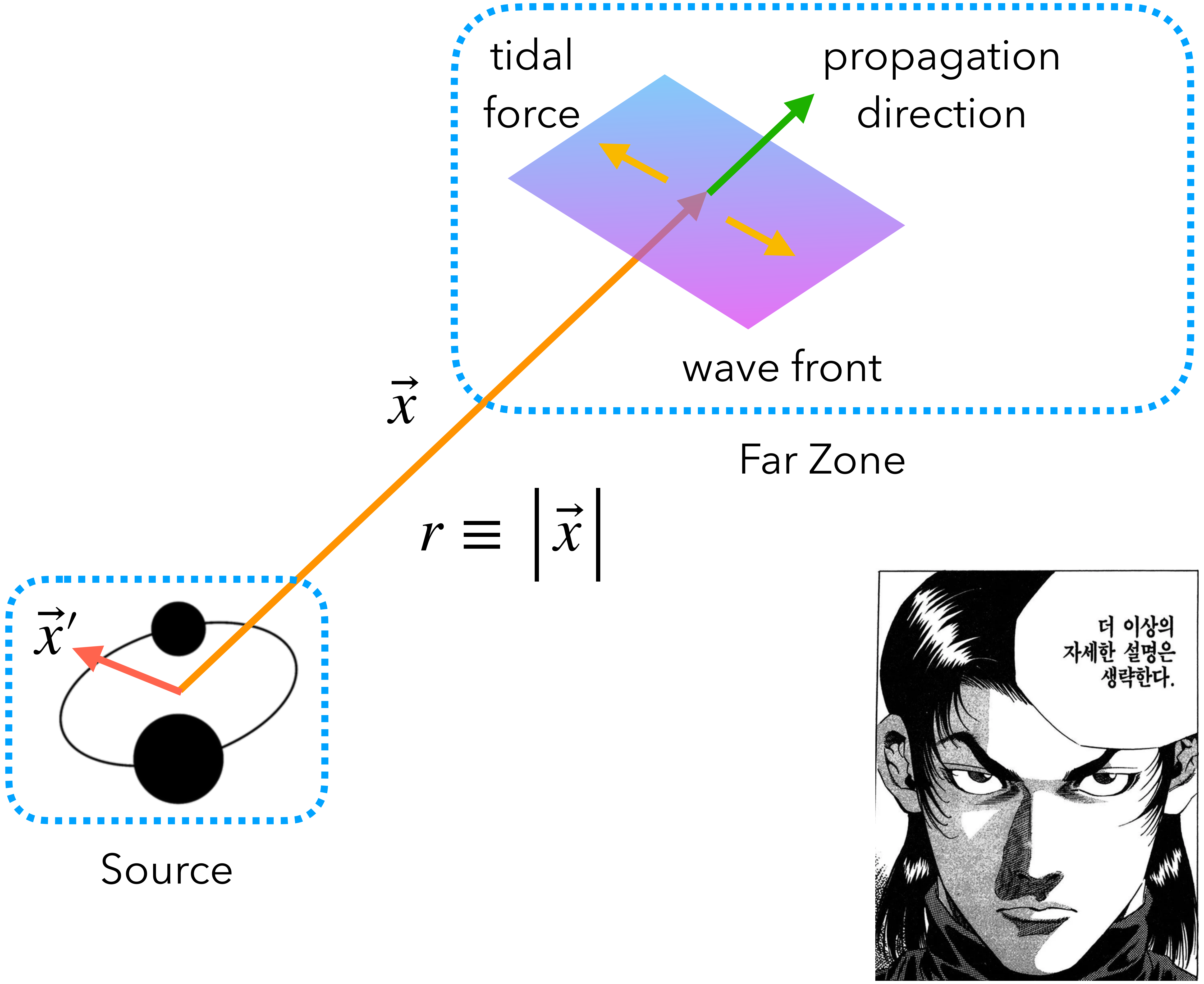
$$\bar{h}_{ab}\left(t, \vec{x}\right) = 4 \int_{\mathcal{N}} d\mathcal{N} \, \frac{1}{r} (\delta T)_{ab}$$

- $$= 4 \int d^3x' \, \frac{(\delta T)_{ab}\left(t - \left|\vec{x} - \vec{x}'\right|, \vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|}$$
- $\mathcal{N}$: past null cone



GWs from Slowly Varying Source

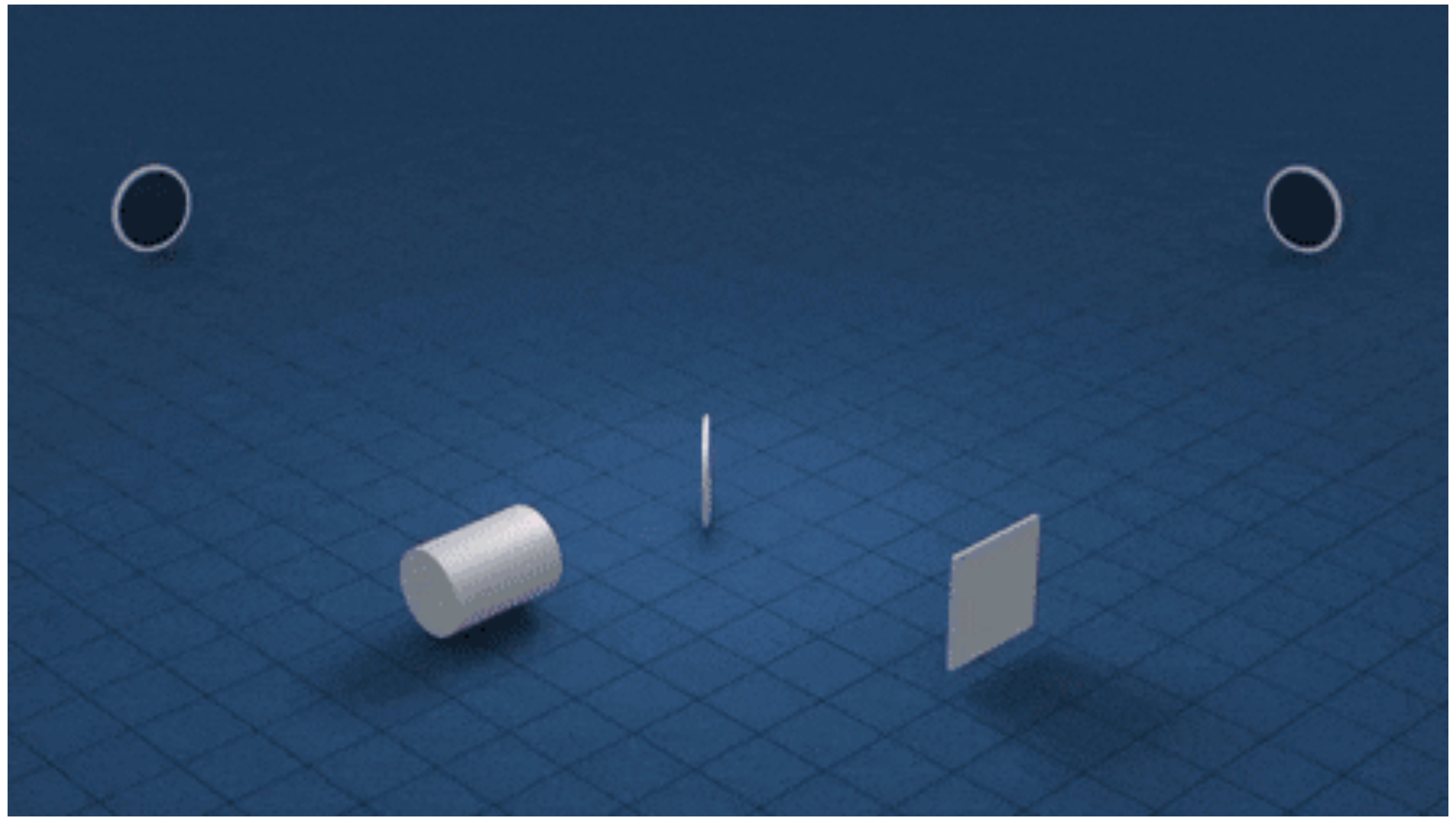
- Second Moment of Energy
  - $M^{ij}(t) = \int d^3x' \, (\delta T)^{00}\left(t, \vec{x}'\right) x'^i x'^j$
- At far zone with slowly varying source,
  - $\bar{h}^{ij}\left(t, \vec{x}\right) = \frac{2}{r}\ddot{M}^{ij}(t-r)$
- h^{TT} with Quadrupole moment
  - $Q_{ij} \equiv M_{ij} - \frac{1}{3}\delta_{ij}M^k{}_k$
  - $h^{\mathrm{TT}}_{ij}\left(t, \vec{x}\right) = \frac{2}{r}\Lambda^{kl}{}_{ij}\ddot{Q}_{kl}(t-r)$
  - Λ: traceless projection to the tidal plane



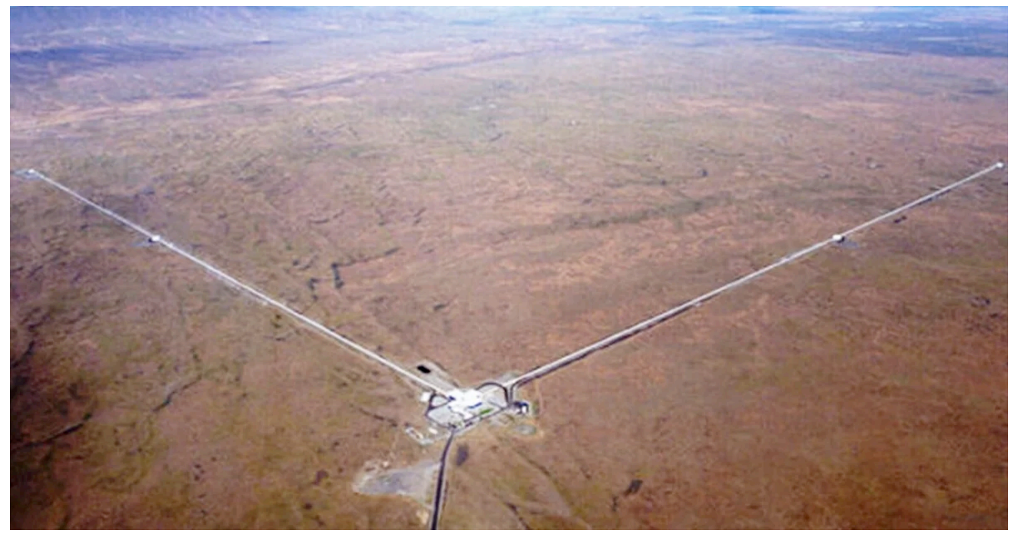
IV. Detection of GWs

Electromagnetic Detector

- Interferometer



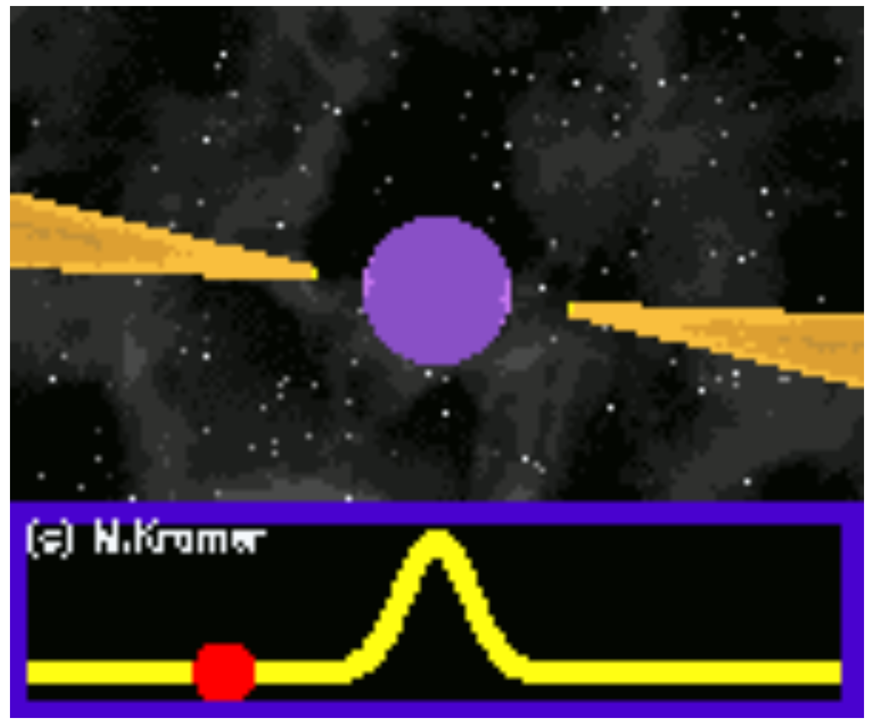
Interferometric Detector



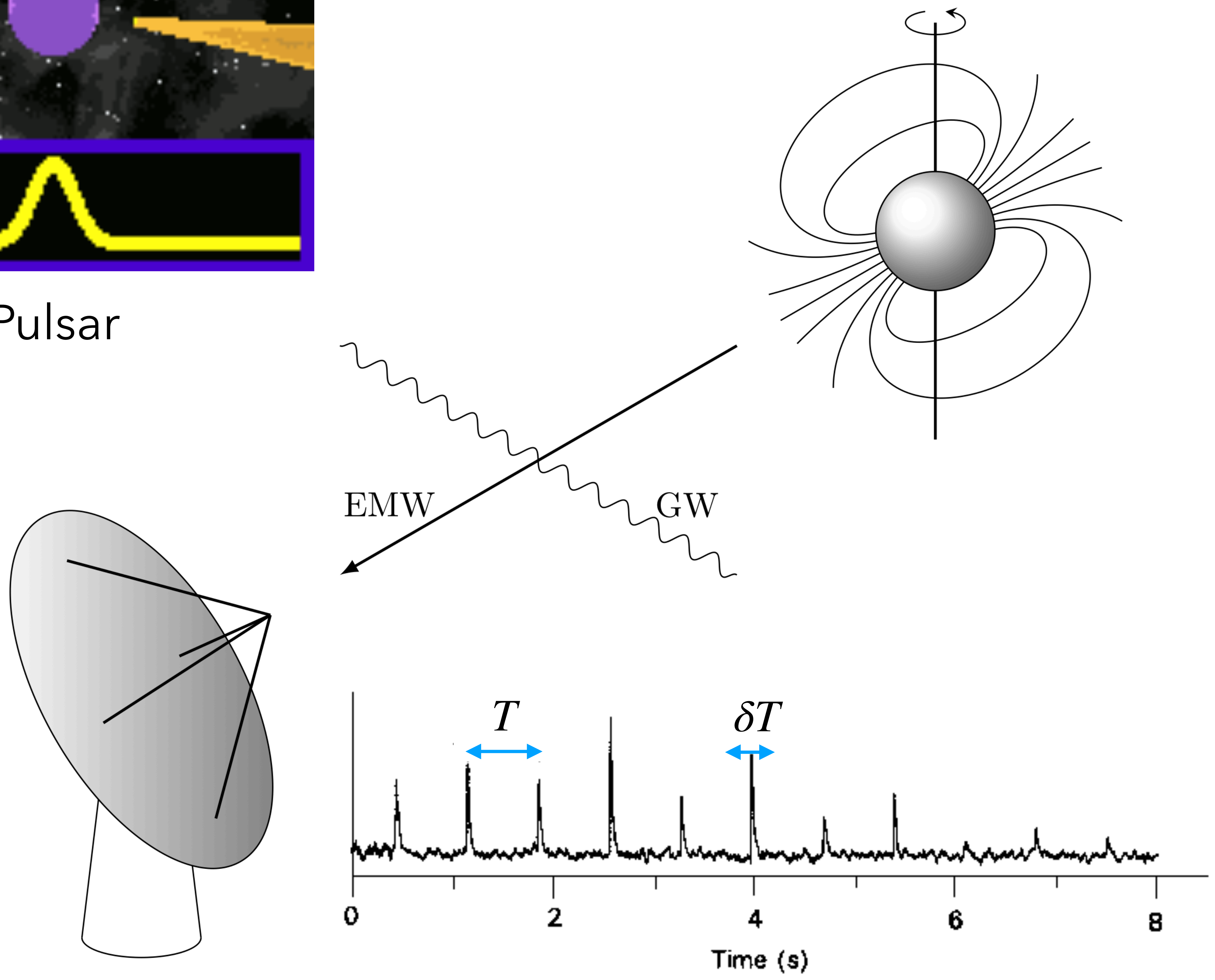
LIGO Hanford

LIGO Livingston

- Pulsar Timing Array



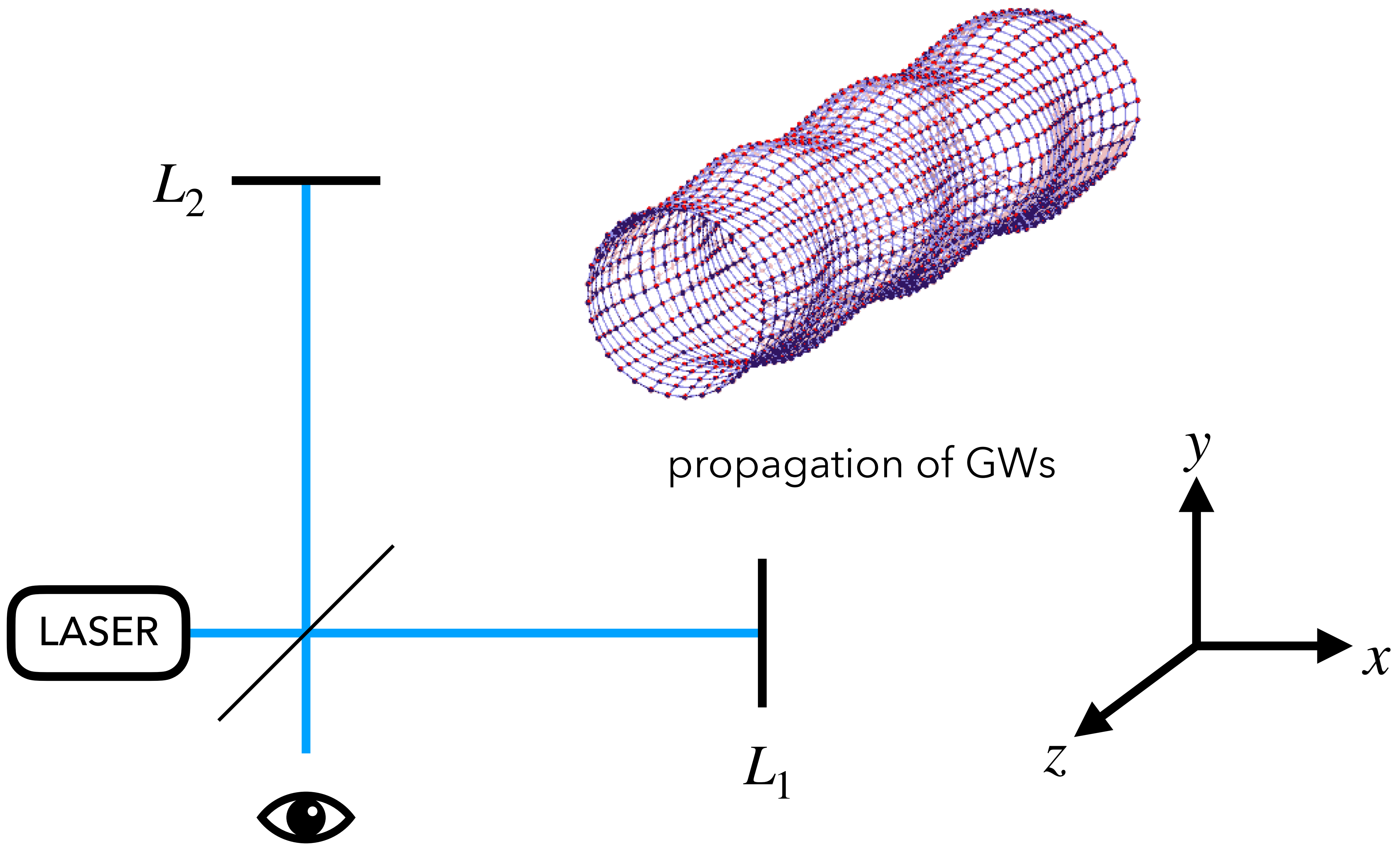
Pulsar



Pulsar Timing Array

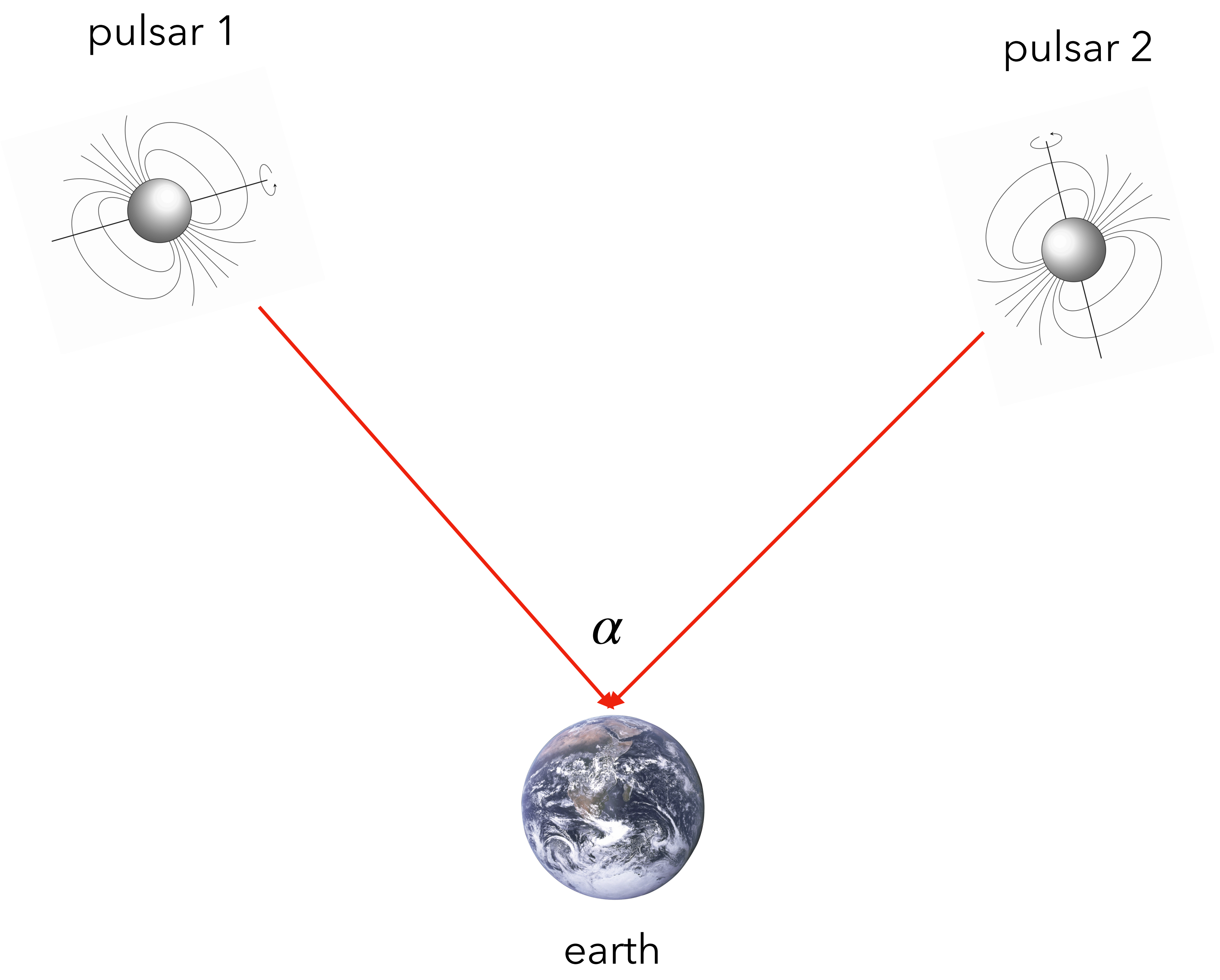
Interferometric GW detector

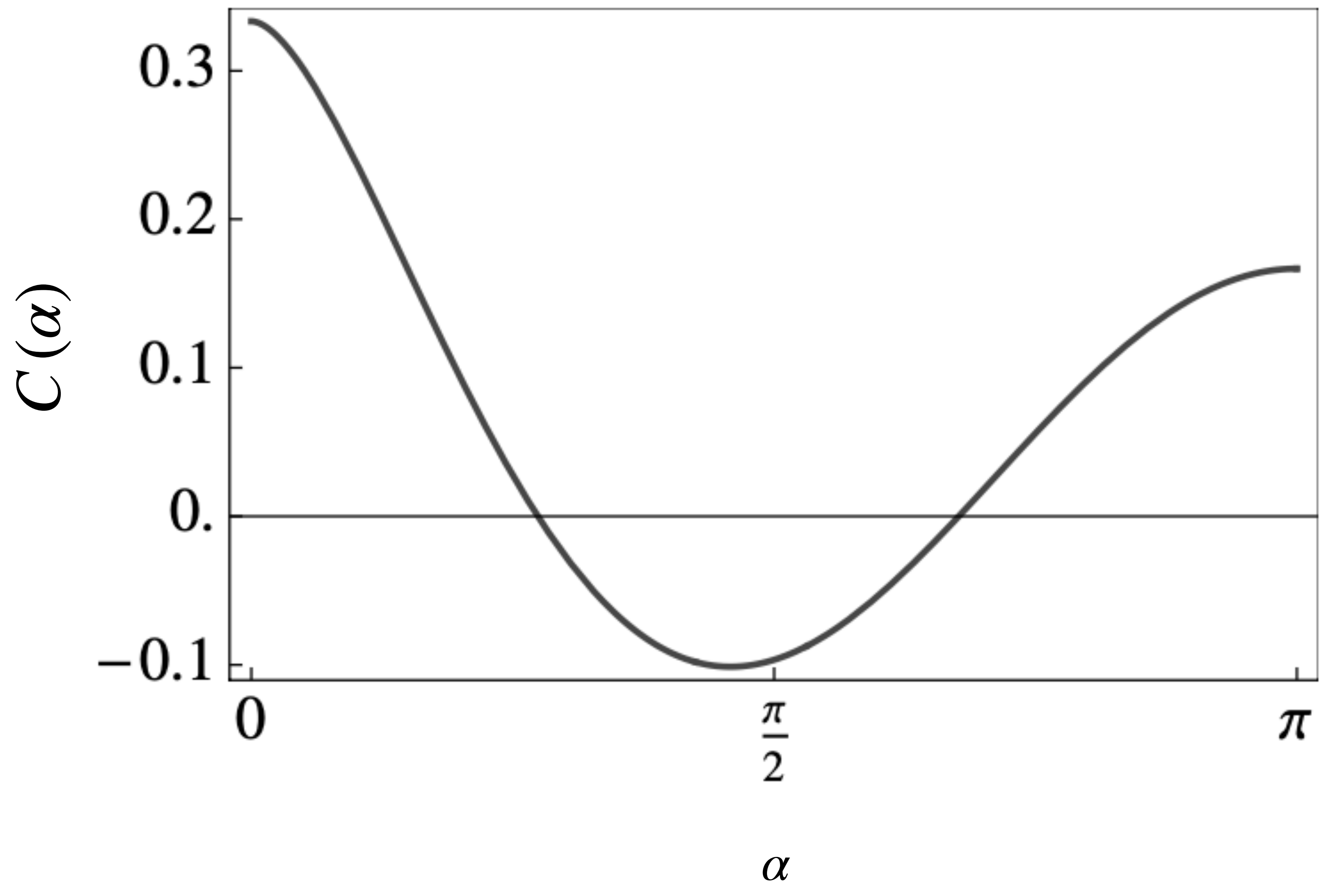
- GWs in + polarization at interferometer (at $z = 0$)
  - $h_{\mu\nu} = \begin{bmatrix} h_+ & 0 \\ 0 & -h_+ \end{bmatrix} \cos\left(-\omega_g t + \phi\right)$
- Phase difference without GWs
  - $\Delta = 2\omega_e \left(L_2 - L_1\right)$
- Phase difference with GWs
  - $L_1'(t) = L_1 \left\{1 + \frac{1}{2} h_+ \cos\left(-\omega_g t + \phi\right)\right\}$
  - $L_2'(t) = L_2 \left\{1 - \frac{1}{2} h_+ \cos\left(-\omega_g t + \phi\right)\right\}$
  - $\Delta'(t) = \Delta + \omega_e \left(L_1 + L_2\right) h_+ \cos\left(-\omega_g t + \phi\right)$



Pulsar Timing Array (PTA)

- Correlation of measurements from two pulsars
  - $\left\langle \frac{\delta T_1}{T_1} \frac{\delta T_2}{T_2} \right\rangle = C(\alpha) \left\langle h_{ab} h^{ab} \right\rangle$





Hellings and Downs / ApJ (1983)

Geometrical Optics
in Perturbed Spacetime

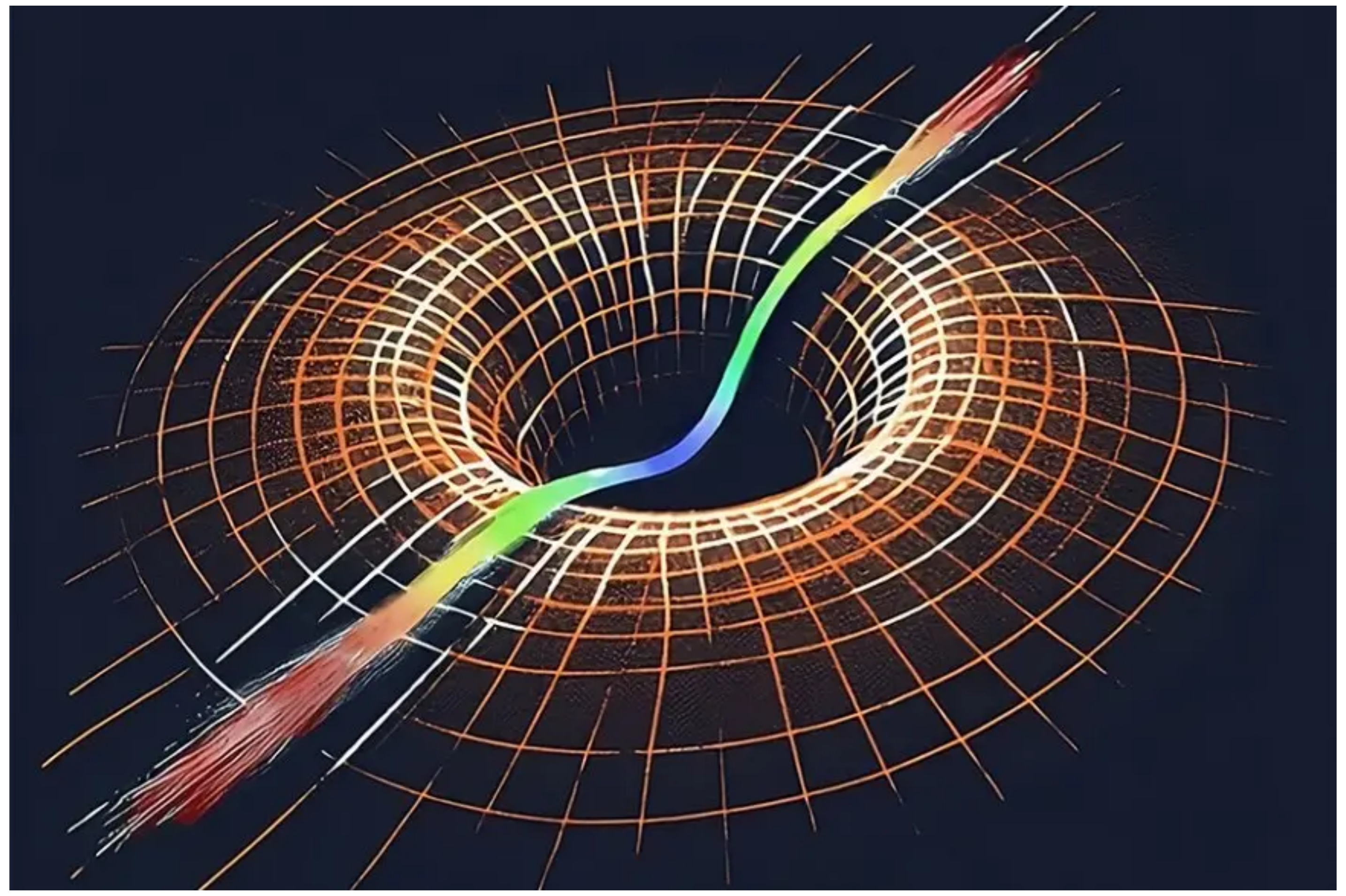
Electromagnetism in Curved Spacetime

- Maxwell's equations
  - $\delta F = \nabla^b F_{ab} = 4\pi j$
  - $dF = 3\,\nabla_{[a}F_{bc]} = 0$
  - F: field strength tensor
  - j: electric 4-current
  - d: exterior derivative
  - δ: codifferential
  - $\epsilon = 1/4\pi$ and $\mu_0 = 4\pi$: Gaussian unit

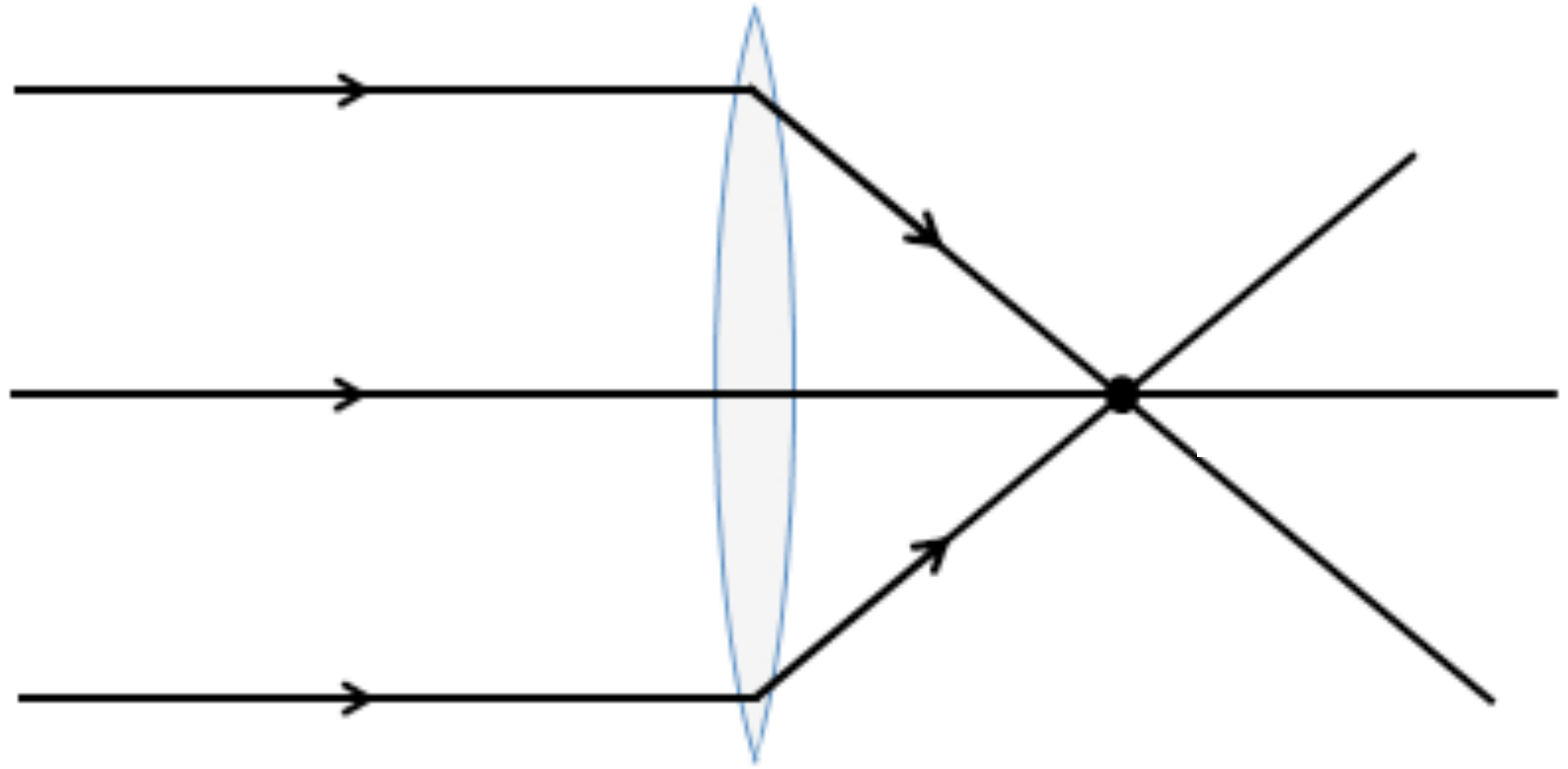
- EM potential
  - We can introduce EM potential A such that $F = dA = 2\,\nabla_{[a}A_{b]}$ at least locally.
  - Then,
  - $$\begin{aligned}\Delta_{\mathrm{dR}}A &= -\left(d\delta + \delta d\right)A \\ &= -\nabla^b\nabla_b A_a + \mathrm{Ric}^b{}_a A_b = -4\pi j\end{aligned}$$
  - with Lorenz gauge $\delta A = -\nabla^a A_a = 0$

Geometrical Optics in Curved Spacetime

- In a local Lorentz frame $\{t, \vec{x}\}$, if the frequency of electromagnetic waves are large enough such that
  - $\alpha \equiv 1/L\omega \ll 1$ $\quad$ $L \equiv \min\left(\mathscr{R}, \mathscr{L}\right)$
  - $\mathscr{R}$: curvature radius scale $\quad$ $\mathscr{L}$: medium variation length scale
- EM potential A can be described as
  - $A_a\left(t, \vec{x}\right) = 2\Re\left[\tilde{A}_a\left(t, \vec{x}\right) e^{iQ\left(t, \vec{x}\right)}\right]$
  - $\tilde{A}$: amplitude $\quad$ Q: phase
  - $l^a \equiv \nabla^a Q$: wave vector $\quad$ $\omega = -l_0 = -\partial_t Q > 0$: frequency
- It is the EMW in the geometrical optics regime.
- Rays can be introduced in geometrical optics.



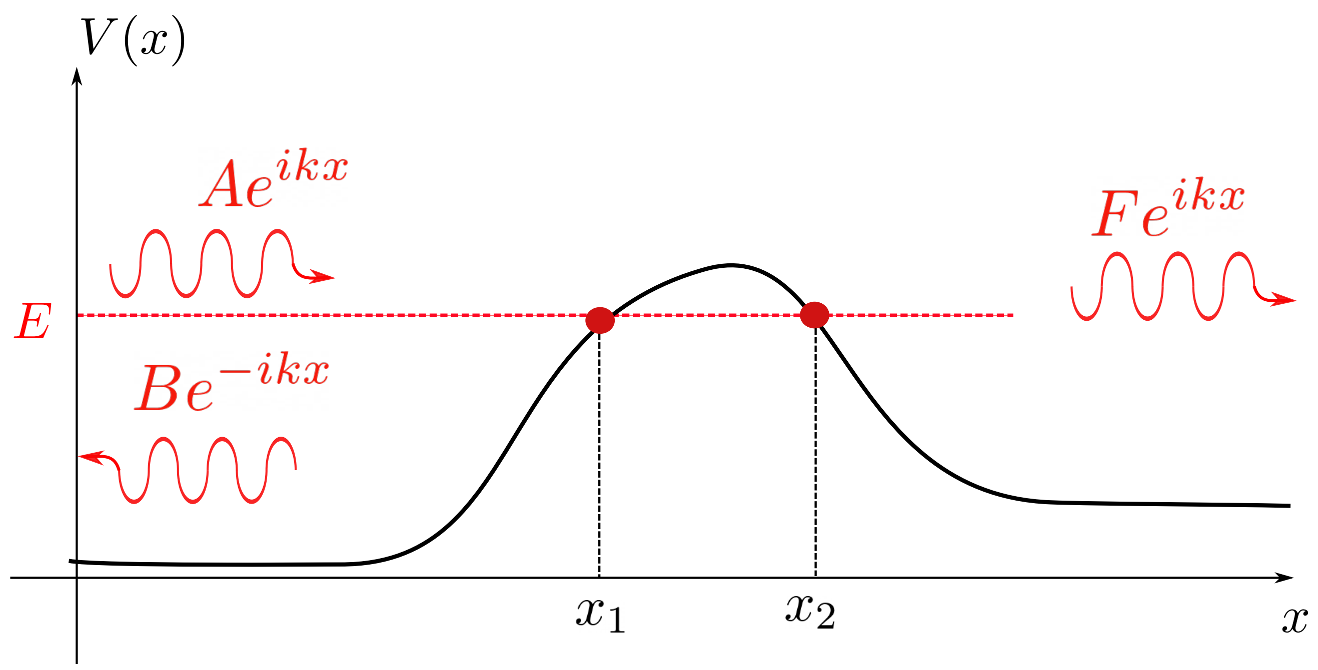
Credit: Matias Koivurova



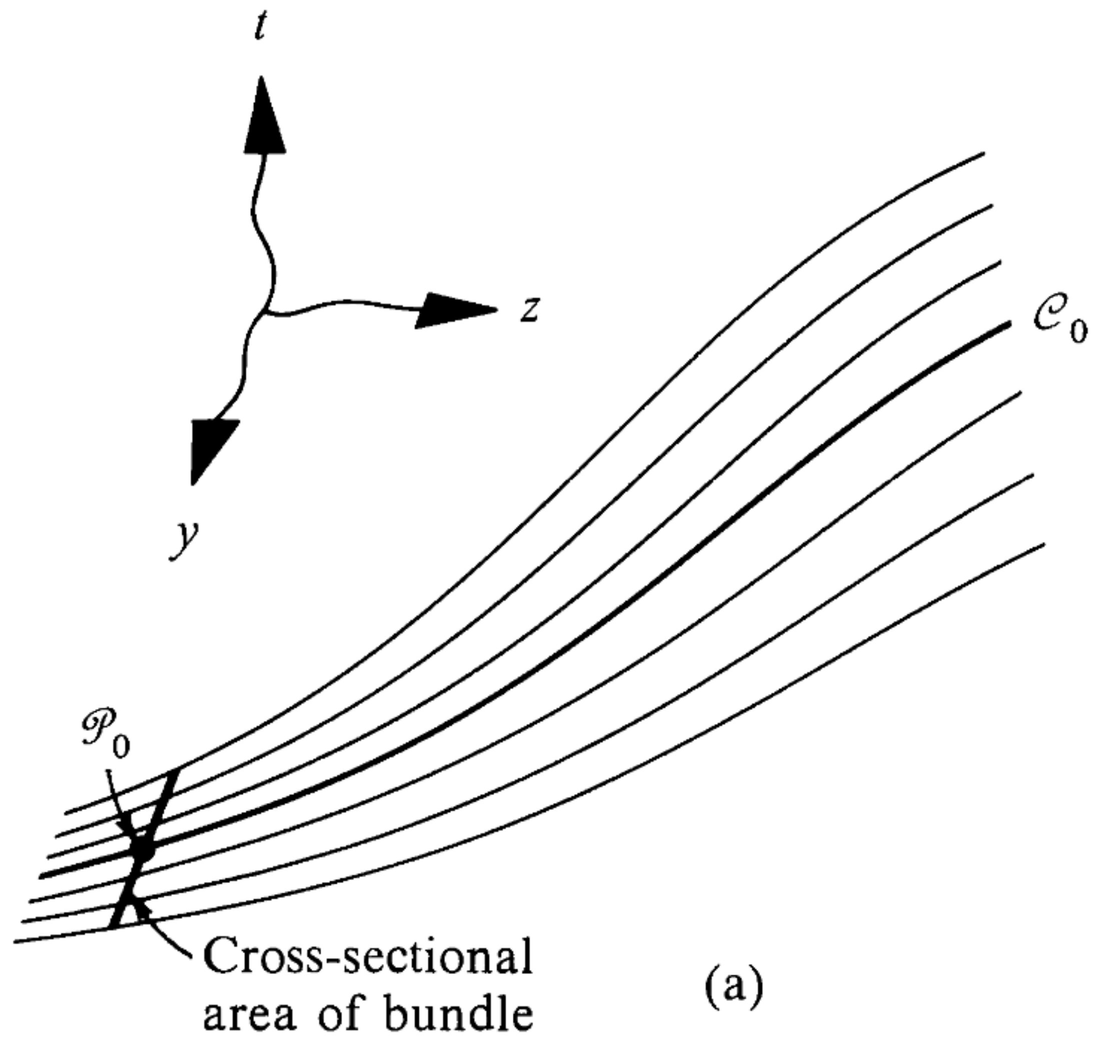
Geometrical Optics with Lens

Geometrical Optics in Curved Spacetime

- By introducing an order analysis parameter $\alpha \to 0$ such that
  - $l_\alpha = O\left(\alpha^{-1}\right) \qquad \tilde{A}_\alpha = O\left(\alpha^{0}\right)$
  - which means that phase changes much faster than the amplitude.
- Maxwell's equations in the leading-order give
  - $l \cdot l = 0$: wave is propagating to null direction
  - $l^a \nabla_a Q = 0$: phase is constant along wave propagation
  - $l^b \nabla_b l^a = 0$: waves are propagating along null geodesic
- From the next-to-leading-order
  - $0 = \nabla_a \left\{ \left(\tilde{A} \cdot \tilde{A}^* \right) l^a \right\}$: number of rays are conserved → photon number conservation
  - $0 = l \cdot \tilde{A}$: transverse wave



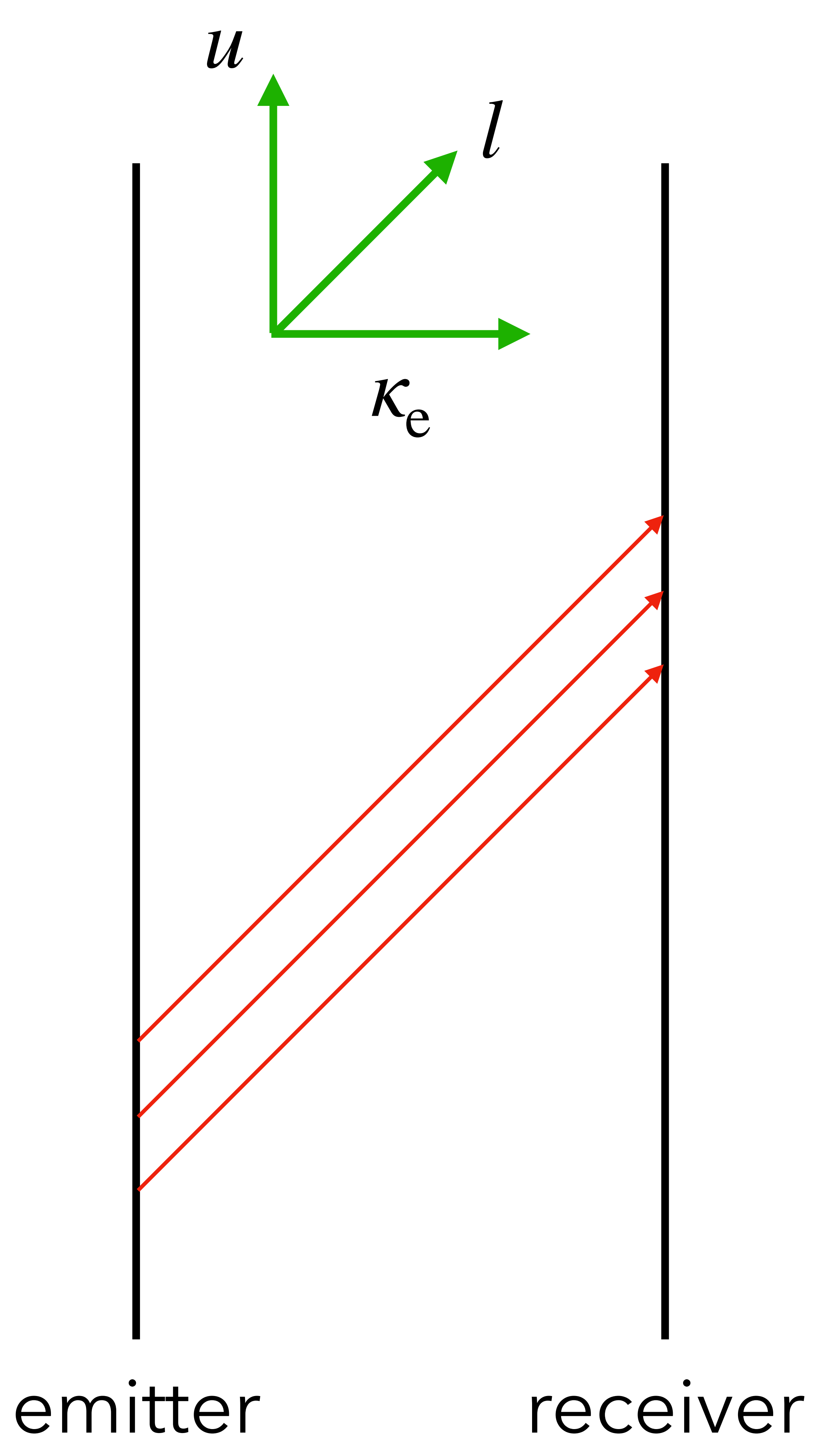
WKB Approximation in Quantum Mechanics / Credit: TU Delft



MTW Figure 22.1

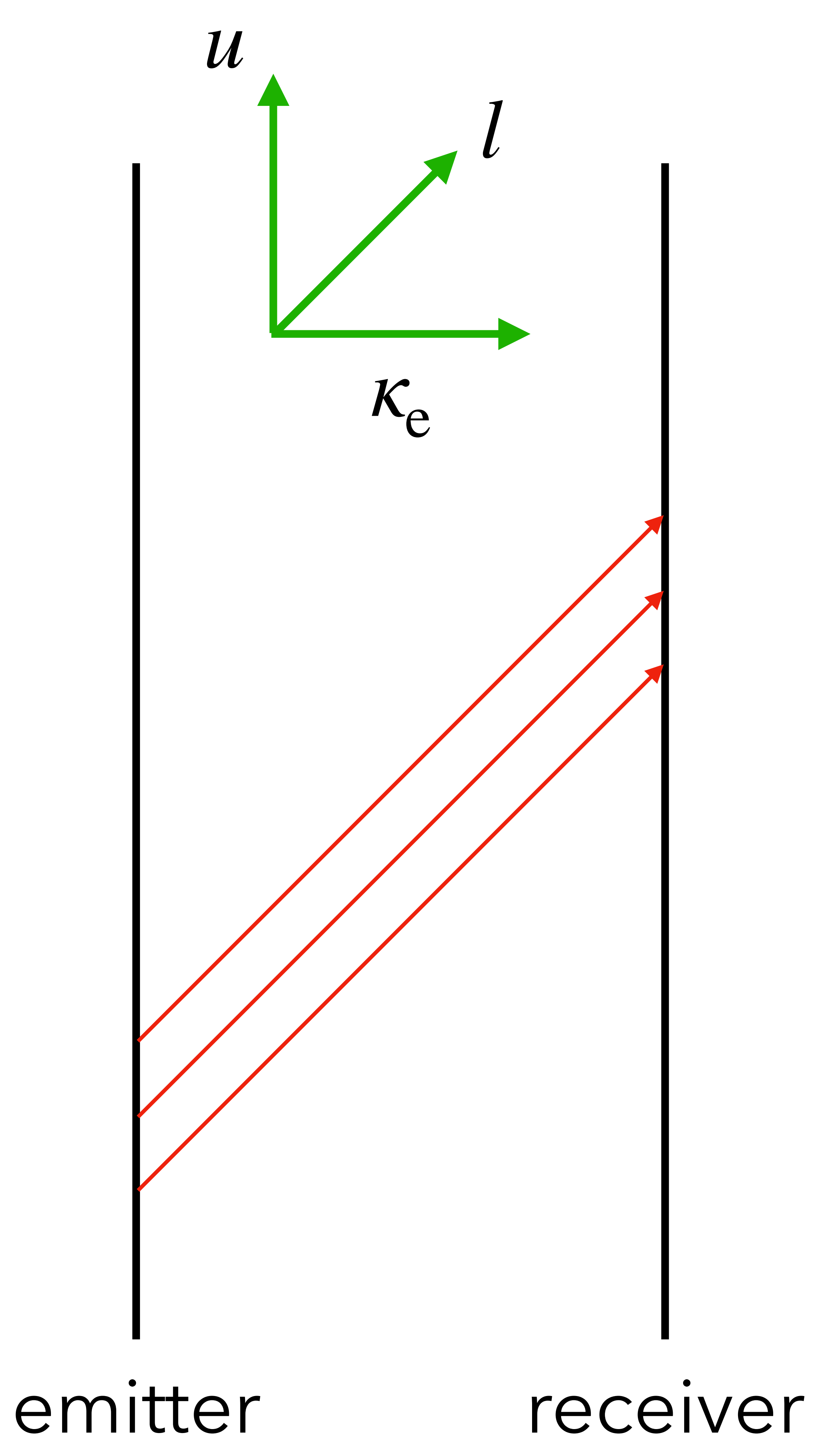
Monochromatic EMW in Flat Spacetime

- Monochromatic Wave Form
  - $A_a\left(t,\vec{x}\right) = 2\Re\left[\tilde{A}_a e^{iQ\left(t,\vec{x}\right)}\right]$
  - $\tilde{A}_a$: constant amplitude
  - $Q\left(t,\vec{x}\right) = \omega_{\mathrm{e}}\left(-t+\kappa_{\mathrm{e}}\cdot\vec{x}\right)$: real phase
  - $\omega_{\mathrm{e}} = -u^a\nabla_a Q$: frequency observed by u
  - κ_{e}: unit spatial propagation direction
  - $l^a \equiv \nabla^a Q = \omega_{\mathrm{e}}\left(u^a+\kappa_{\mathrm{e}}^a\right)$: wave vector
- From the Maxwell's equations
  - $0 = g_{ab}l^a l^b$ (null condition)
  - $0 = l^b\nabla_b l^a$ (null geodesic)



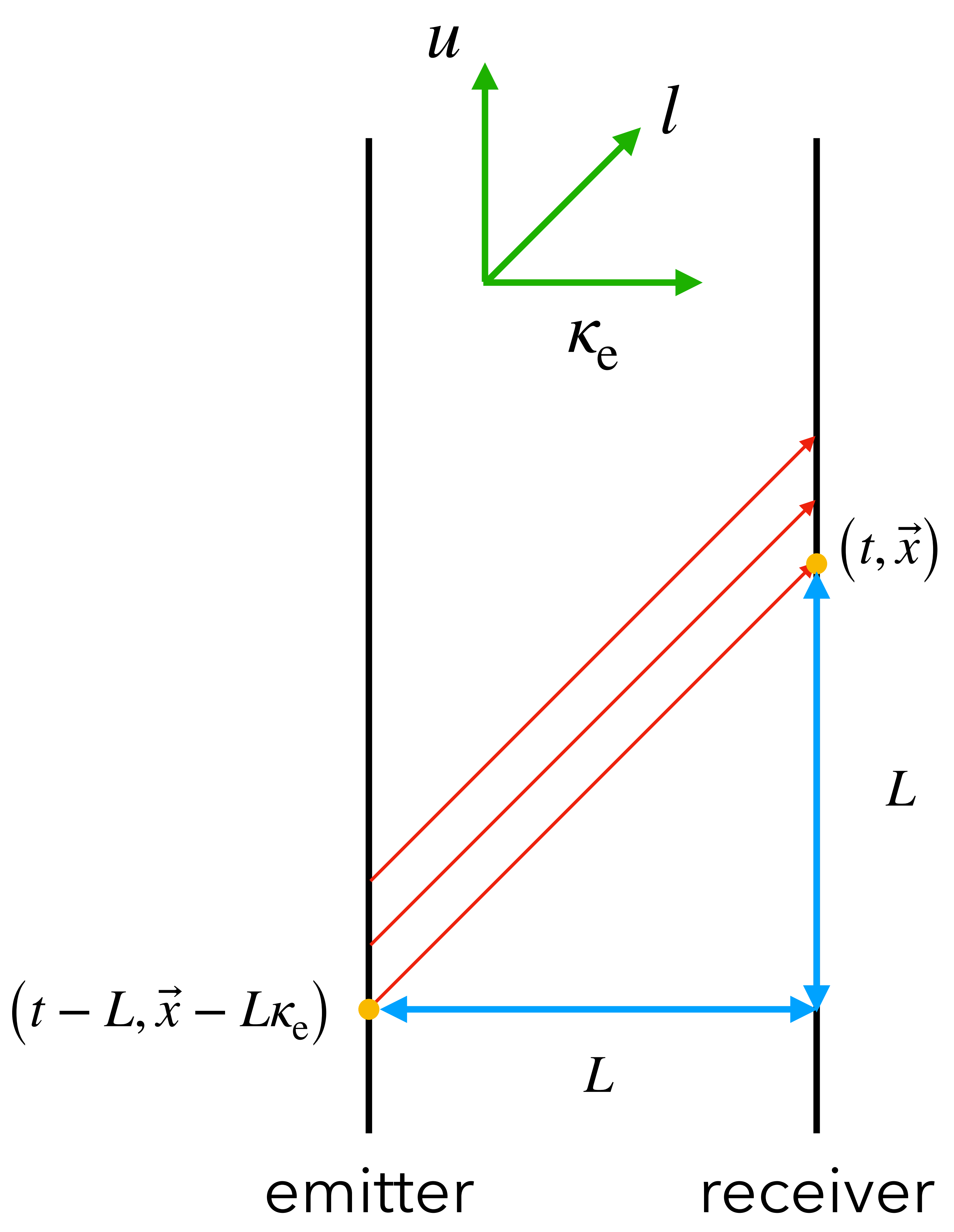
Linear Perturbation of Geometrical Optics

- Perturbed EMWs
  - $\phi_\epsilon^* \hat{Q} = Q + \epsilon \delta Q + O\left(\epsilon^2\right)$
  - $\phi_\epsilon^* \hat{\omega}_{\mathrm{e}} = \omega_{\mathrm{e}} + \epsilon \delta \omega_{\mathrm{e}} + O\left(\epsilon^2\right)$
  - Note that $\delta\omega_{\mathrm{e}}$ is gauge invariant because ω_{e} is constant scalar.
- Perturbation of Geometrical-Optical Equations
  - $l^a \nabla_a \delta Q = \dfrac{1}{2} h_{ab} l^a l^b$
  - $\delta\omega_{\mathrm{e}} = -u^a \nabla_a \delta Q + (\delta u)^a l_a = -\partial_t \delta Q$ in TT gauge.



Linear Perturbation of Geometrical Optics

- Boundary Condition
  - $\delta\omega_{\rm e} = -\partial_t \delta Q = 0$ at the emitter.
  - For convenience, we just set $\delta Q = 0$ at the emitter.
- Solution
  - $\delta Q = \int d^2\kappa_{\rm g} \int_{-\infty}^{\infty} \frac{d\omega_{\rm g}}{2\pi} \frac{1}{2i\left(k \cdot l\right)} \tilde{h}_{ab}\left(\omega_{\rm g}, \kappa_{\rm g}\right) l^a l^b \left(e^{iP} - e^{iP'}\right)$
- where
  - $P\left(t, \vec{x}\right) = \omega_{\rm g}\left(-t + \kappa_{\rm g} \cdot \vec{x}\right)$: GW phase
  - $P'\left(t, \vec{x}\right) = P\left(t - L, \vec{x} - L\kappa_{\rm e}\right)$: retarded GW phase



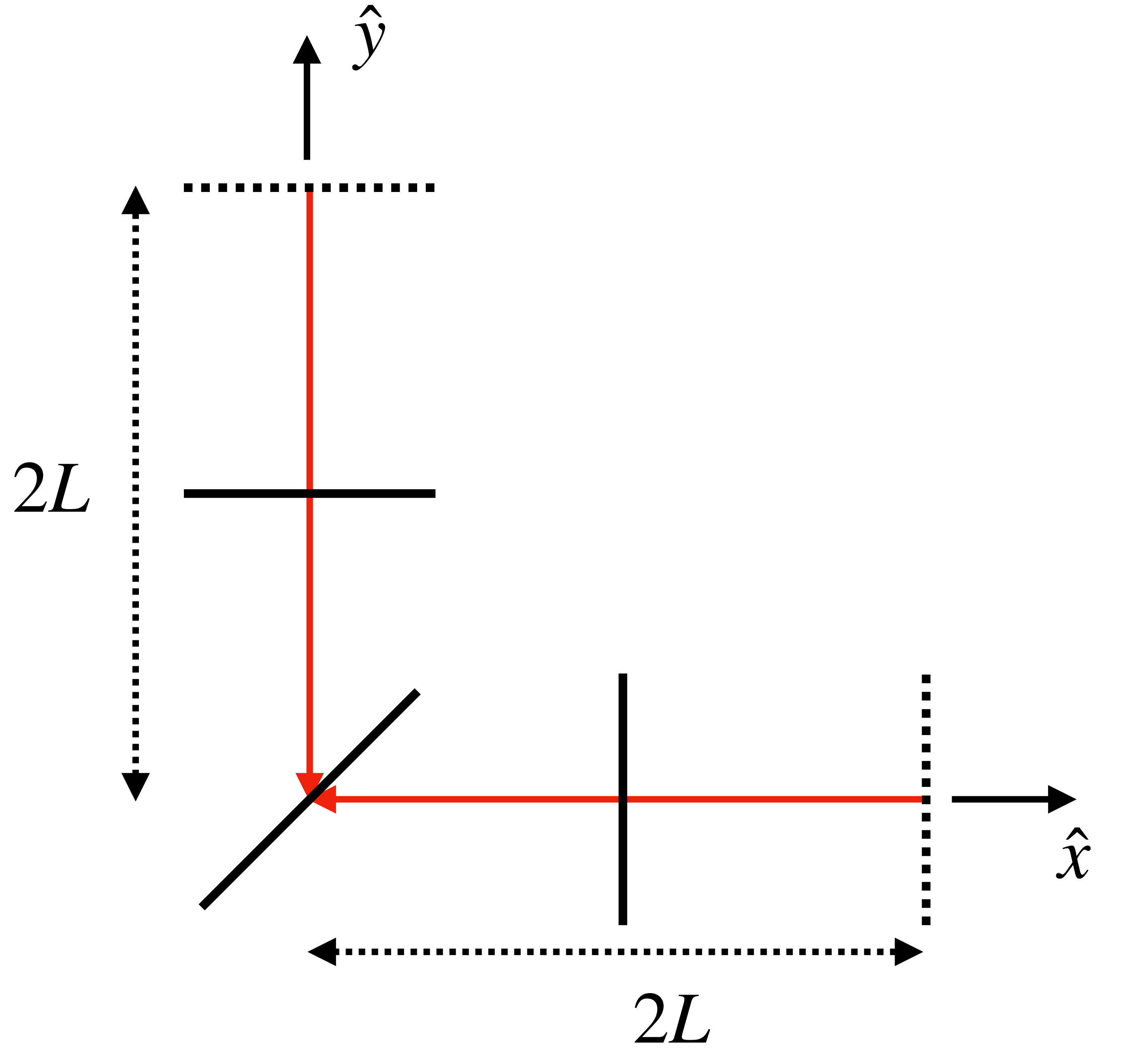
Interferometric GW Detector

- Phase Difference in Dark Port at $\vec{x} = 0$

$$\delta Q^{(x)} - \delta Q^{(y)} = -\frac{1}{2}\omega_{\mathrm{e}} \int d^2\kappa_{\mathrm{g}} \int_{-\infty}^{\infty} \frac{d\omega_{\mathrm{g}}}{2\pi} \frac{1}{i\omega_{\mathrm{g}}} \left[\frac{\tilde{h}_{xx}}{1+\kappa_x} \left(e^{iP\left(t,-2L\hat{x}\right)} - e^{iP(t-2L,0)} \right) - \frac{\tilde{h}_{yy}}{1+\kappa_y} \left(e^{iP\left(t,-2L\hat{y}\right)} - e^{iP(t-2L,0)} \right) \right]$$

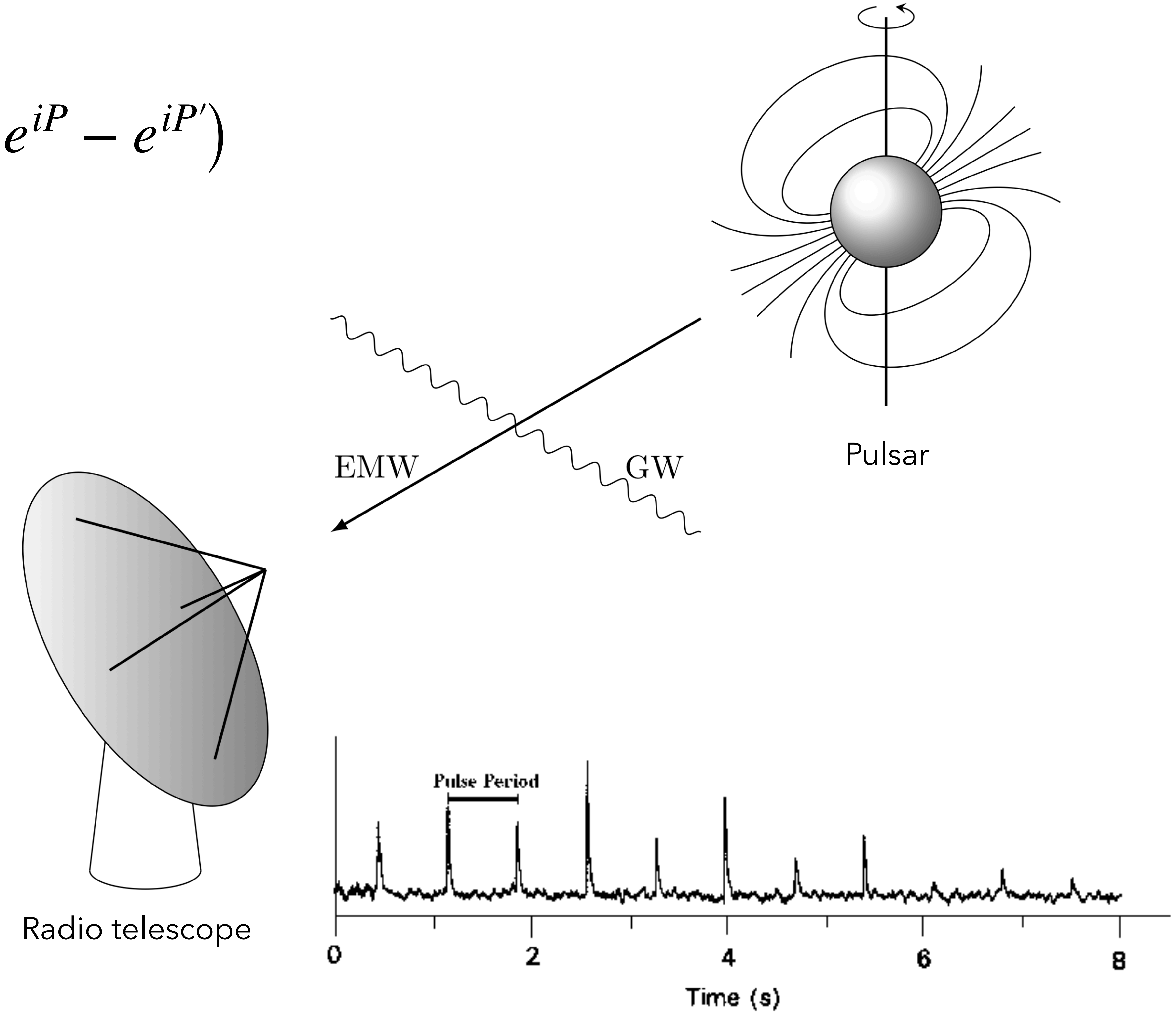
- $$\simeq L\omega_{\mathrm{e}} \left(\hat{x}^a \hat{x}^b - \hat{y}^a \hat{y}^b \right) h_{ab}(t,0) \qquad \text{for} \qquad \left(\omega_{\mathrm{g}} L \ll 1 \right)$$

- For LIGO,
  - $\omega_g \sim 10^2\,\mathrm{Hz}$
  - $L \sim 10^3\,\mathrm{m}$
  - $\omega_g L \sim 10^{-3} \ll 1$



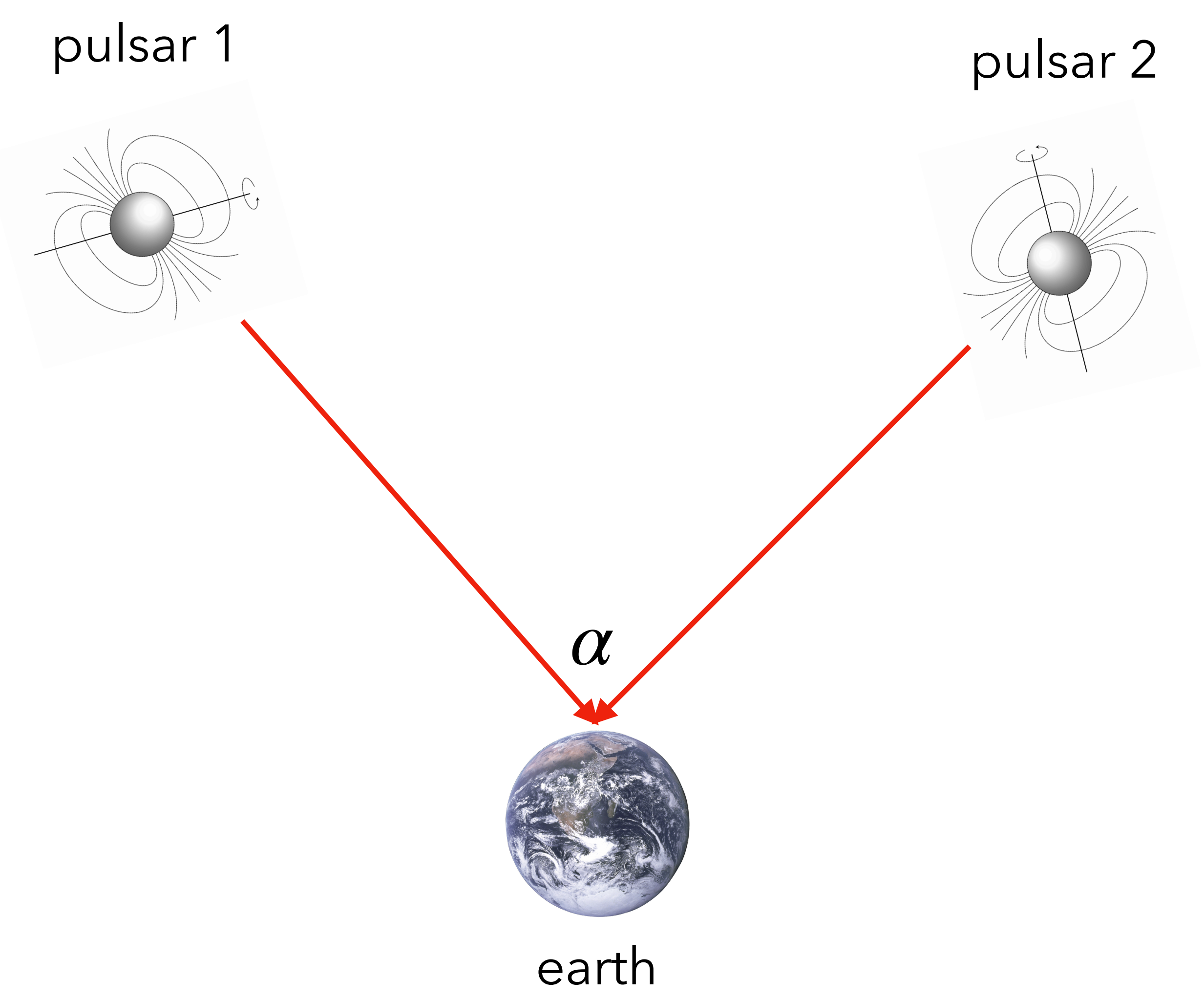
Pulsar Timing

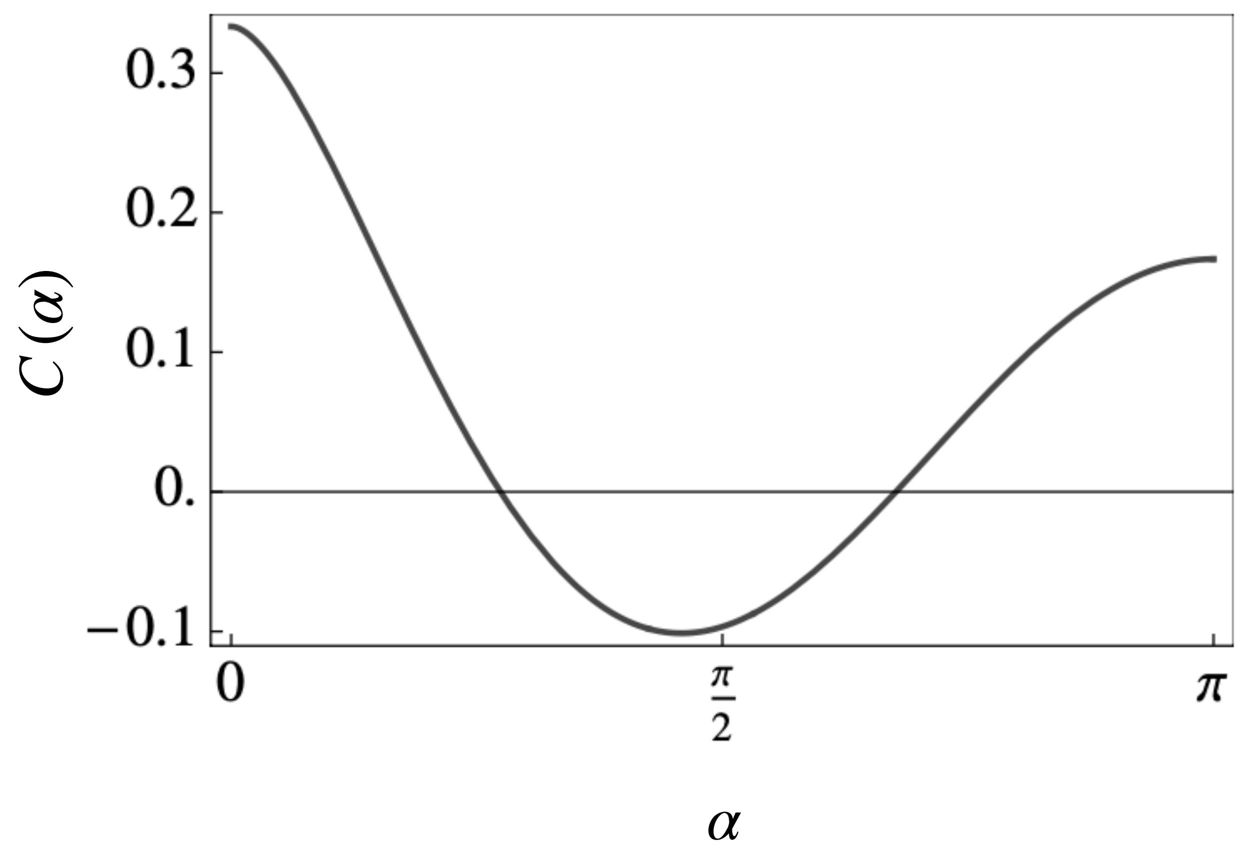
- Change of Pulse Period
  - $\dfrac{\delta T}{T} = -\dfrac{\delta \omega_{\mathrm{e}}}{\omega_e} = \displaystyle\int d^2\kappa_{\mathrm{g}} \int_{-\infty}^{\infty} \dfrac{d\omega_{\mathrm{g}}}{2\pi} \dfrac{1}{2} \dfrac{\tilde{h}_{ab}\kappa_{\mathrm{e}}^a \kappa_{\mathrm{e}}^b}{1 - \kappa_{\mathrm{g}} \cdot \kappa_{\mathrm{e}}} \left(e^{iP} - e^{iP'}\right)$
- For PTAs,
  - $\omega_{\mathrm{g}} \sim 10^{-9}\,\mathrm{Hz}$
  - $L \sim 10^3\,\mathrm{pc} \sim 10^{19}\,\mathrm{m}$
  - $\omega_g L \sim 10^2 \gg 1$



Pulsar Timing Array

- Correlation
  - $\langle h_1 h_2 \rangle = \left\langle \frac{\delta T_1}{T_1} \frac{\delta T_2}{T_2} \right\rangle = C(\alpha) \langle h_{ab} h^{ab} \rangle$
  - where
  - $C(\alpha) = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{6}\sin^2(\alpha/2) + \sin^2(\alpha/2)\ln\sin^2(\alpha/2)\right)$





Hellings and Downs / ApJ (1983)

Summary

- Perturbation gauges are mathematical redundancies in the description of perturbations, arising from the mappings between the background and perturbed spacetimes.
- The linearized Einstein's equations in vacuum imply the wave equation for GWs.
- For weak and slowly varying source, the time-varying quadrupole moment generates GWs.
- The geometrical optics is valid when the frequency of electromagnetic waves is much larger than the inverse of the characteristic spacetime length scales.
- GWs affects the phase of light rays. This effect enables the detection of GWs.
- Thank you for your attention. 🤩