“Black hole Quasinormal modes"

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"2026 Summer school for Numerical Relativity and Gravitational waves"
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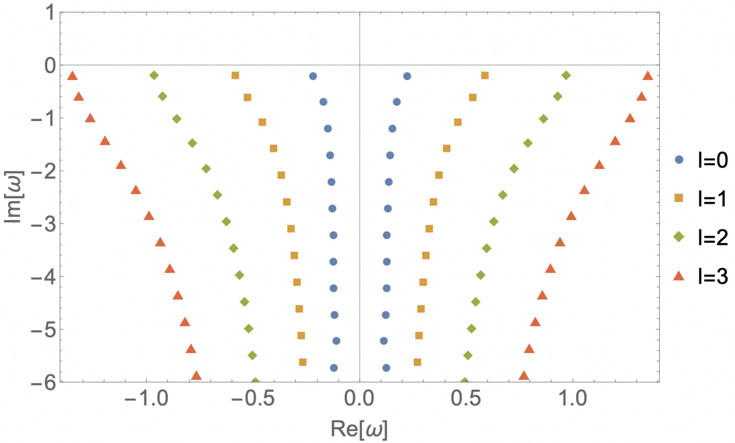
Content

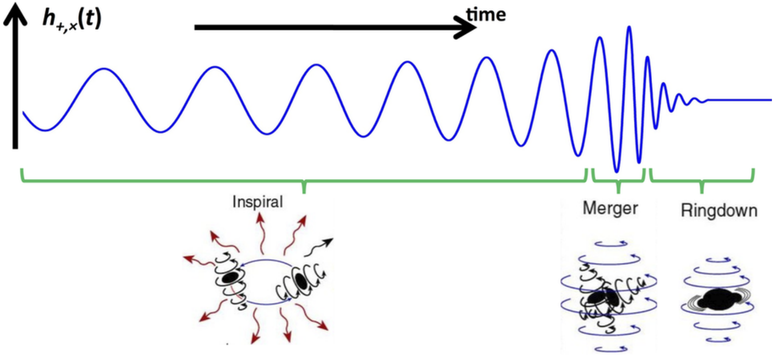
Aim : to understand quasinormal modes

Quasinormal Modes

QNM with $l = 0, 1, 2, 3$ for scalar field in Schawarzschil black hole by using pseudo spectral method



gravitational wave and quasi-normal modes



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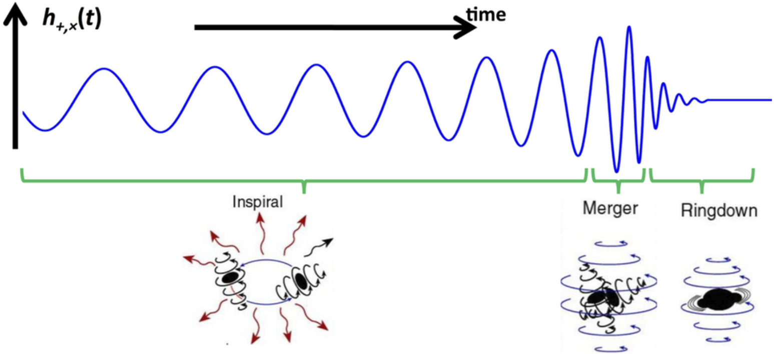

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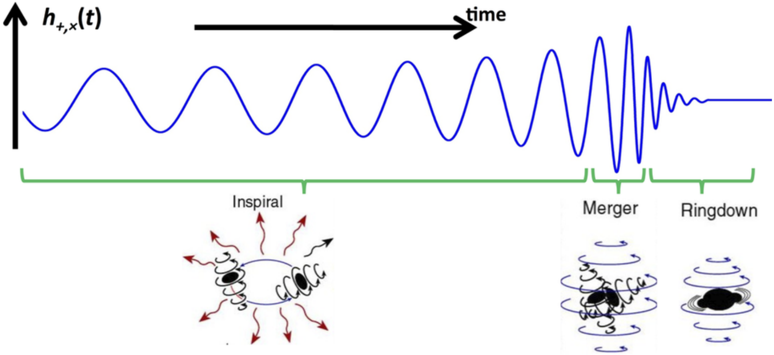

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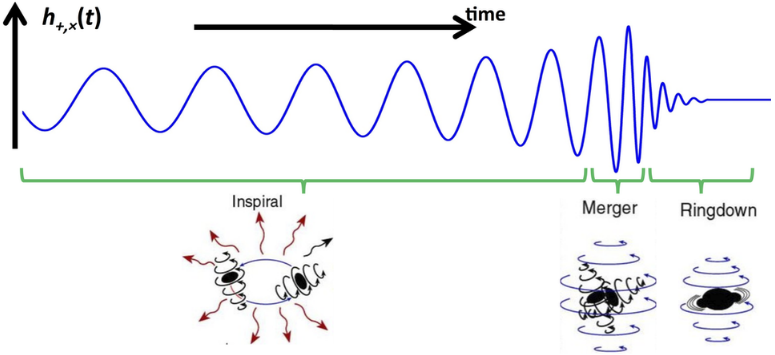

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- Inspiral Phase : Post-Newtonian (PN) Approximation
- Merger Phase : Numerical Relativity Simulation
- Ringdown Phase : Quasinormal Modes (QNMs)

$$h(t) = \sum_{lmn} A_{lmn} e^{-i\omega_{lmn}(t-t_0)}$$

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- LISA is expected to achieve significantly higher signal-to-noise ratios, making the inclusion of higher angular modes, overtones, nonlinear modes, and late-time tails essential.
- Now, it is opening up for **"black hole spectroscopy"** era.

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- test No-hair theorem or modified gravity : An isolated, stationary black hole is completely characterized by its (M, Q and J).
- event horizon structure : The presence of microstructures, quantum membranes, or physical surfaces at or near the horizon ($r = r_H + \epsilon$) changes the boundary condition from perfect absorption to partial or total reflection ($R \neq 0$).

normal modes vs quasi-normal modes

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The purpose of this lecture is to understand

normal modes vs **quasi-normal** modes

In this lecture, we will understand them by studying

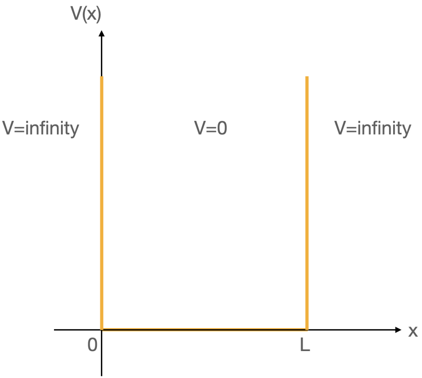
- **normal** modes : a particle in a box
- **quasi-normal** modes : a particle in a black hole

quasi-

1. cf. apparently but not really, seemingly
2. cf. being partly or almost

normal modes vs quasi-normal modes

A particle in a box



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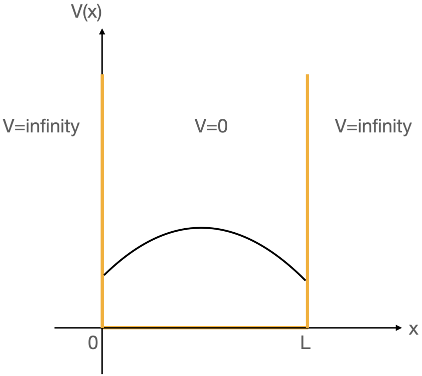

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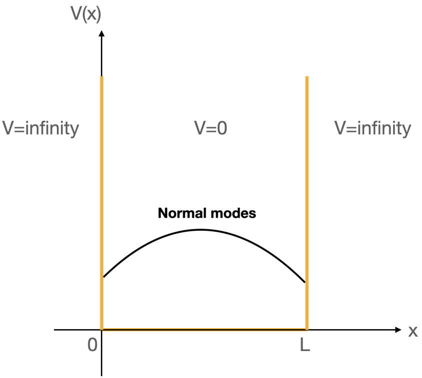

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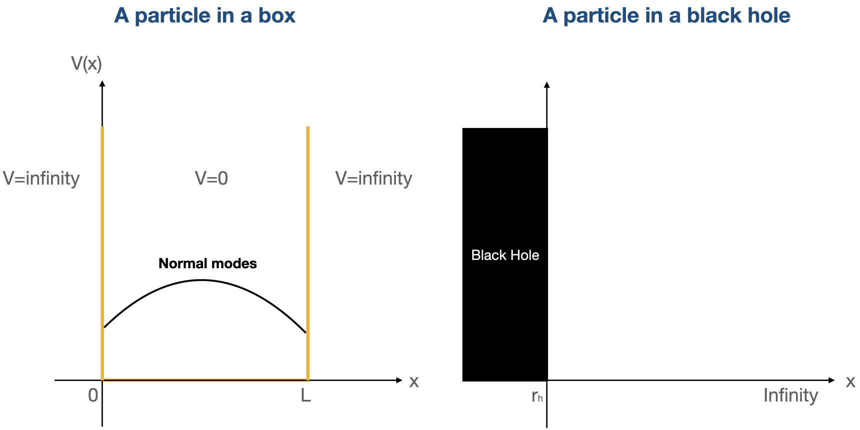

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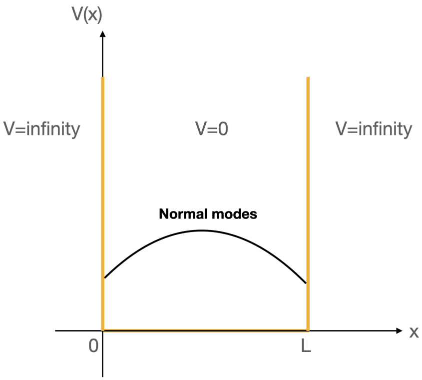
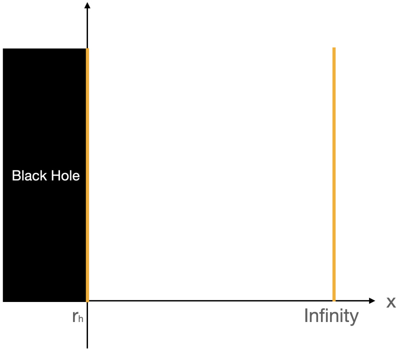


A particle in a black hole



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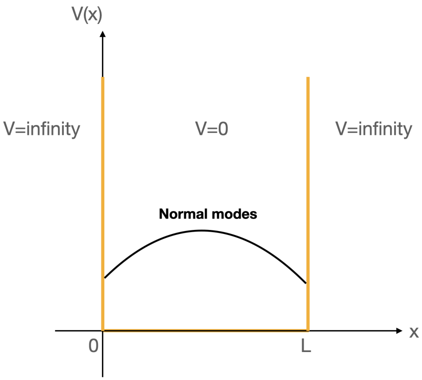

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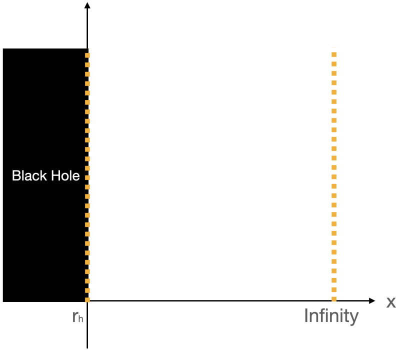

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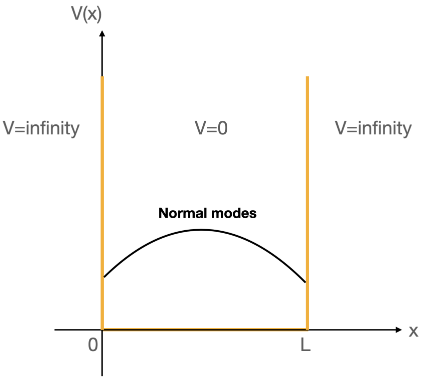

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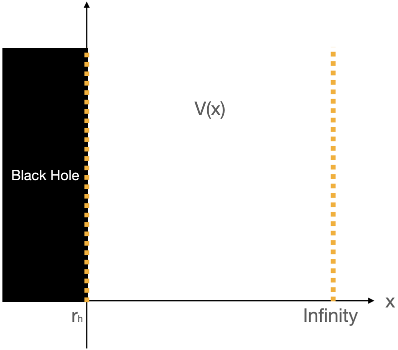

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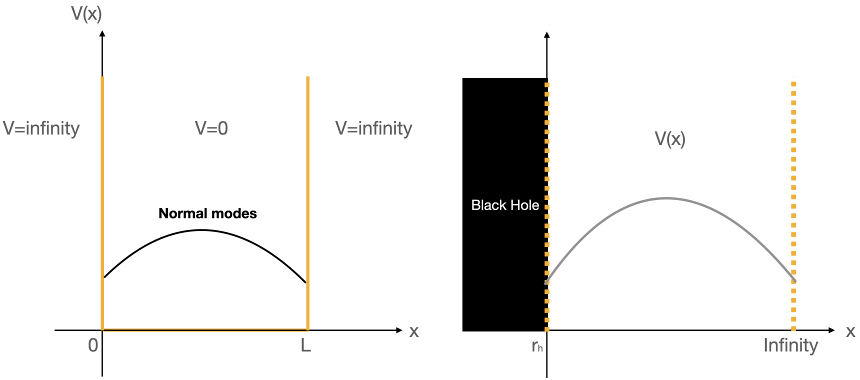

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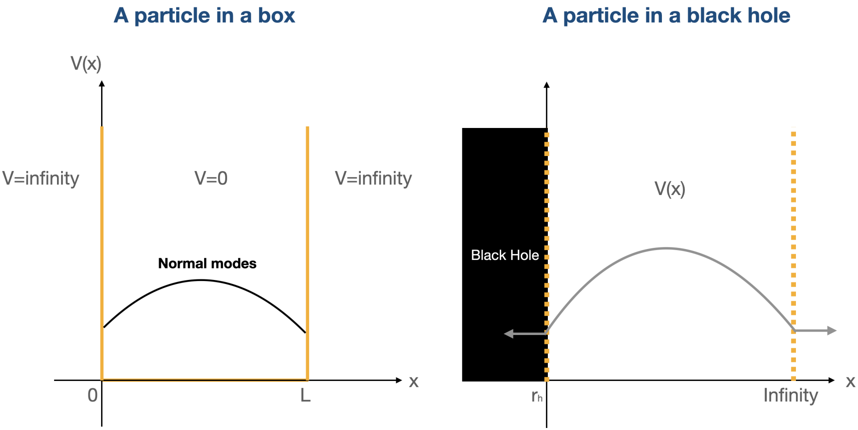

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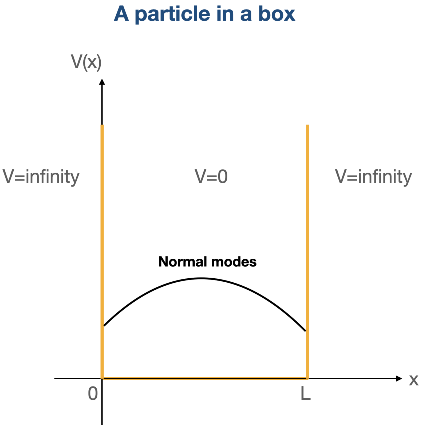

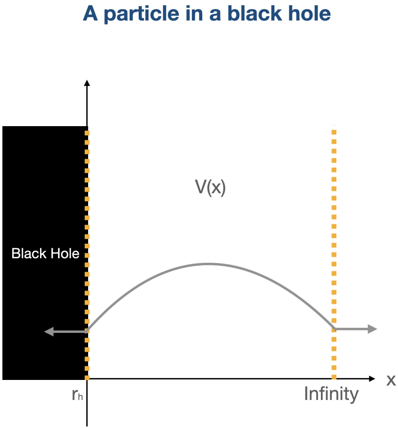

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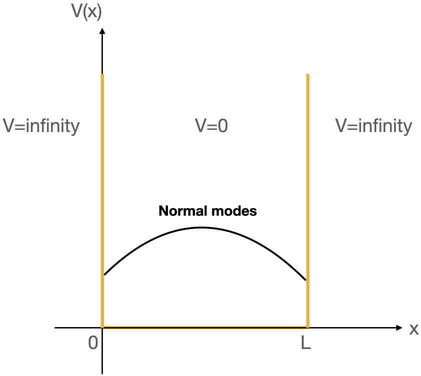

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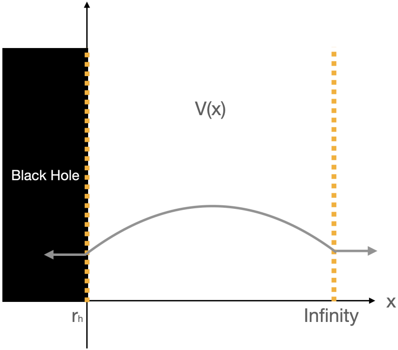

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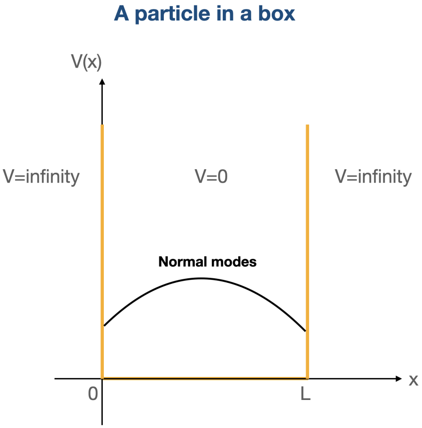

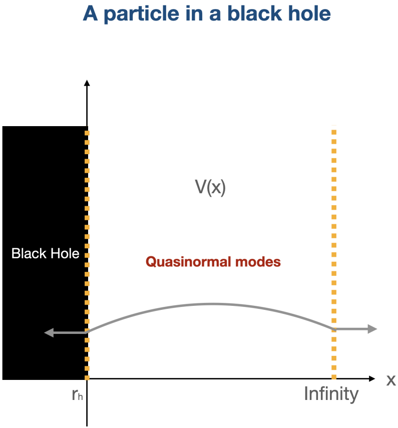

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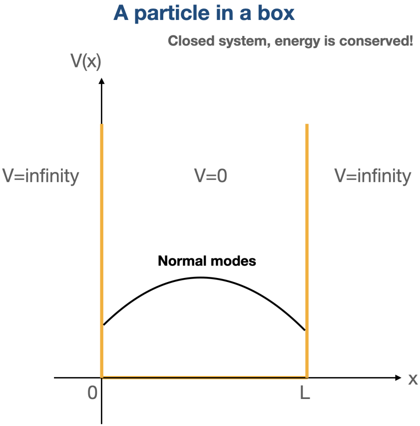

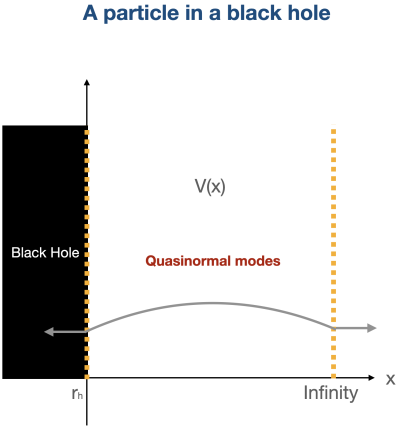

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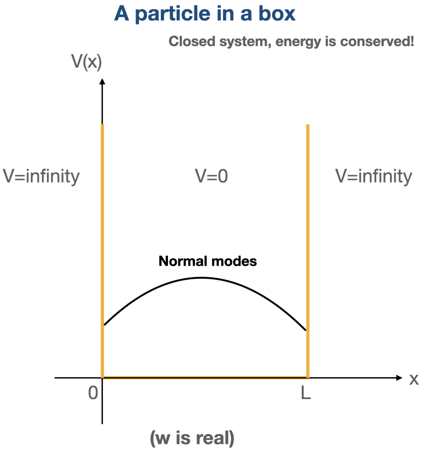

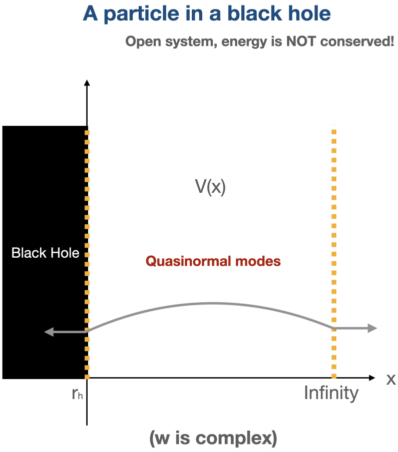

Quantum mechanics

Schrödinger equation

- the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \equiv \hat{H}\psi(x,t),$$

  where $\psi(x,t)$: wave function

- If V has no explicit time dependence, we can make separation of variables

$$\psi(x,t) = T(t)u(x),$$

$$i\hbar\frac{1}{T(t)}\frac{\mathrm{d}T(t)}{\mathrm{d}t} = \frac{1}{u(x)}\left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + V(x)u(x)\right\},$$

$$i\hbar\frac{\mathrm{d}T(t)}{\mathrm{d}t} = ET(t), \qquad T(t) = Ce^{-iEt/\hbar},$$

- the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + V(x)u(x) = \hat{H}u(x) = Eu(x)$$

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B. for operators that represent observables

$$\langle \psi, \hat{O}\psi \rangle = \langle \hat{O}\psi, \psi \rangle \qquad \Big(\langle g, f \rangle = \langle f, g \rangle^*, \ \psi(x) \in \mathcal{H} \Big) \tag{2}$$

$$\text{Proof :} \qquad \hat{O}\psi = E\psi, \tag{3}$$

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- self-adjoint :

$$\hat{\mathcal{O}} = \hat{\mathcal{O}}^\dagger \qquad (5)$$

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$$\psi(x,t) = \sum_n C_n u_n(x) e^{-iE_n t/\hbar} + \int \mathrm{d}E C(E) u_E(x) e^{-iEt/\hbar} \tag{8}$$

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- the set of all square-integrable functions, on a specific interval,

$$u(x) \ \text{ such that } \int_a^b |u(x)|^2\mathrm{d}x<\infty \tag{9}$$

$Hu_E(x) = Eu_E(x)$: Hilbert space

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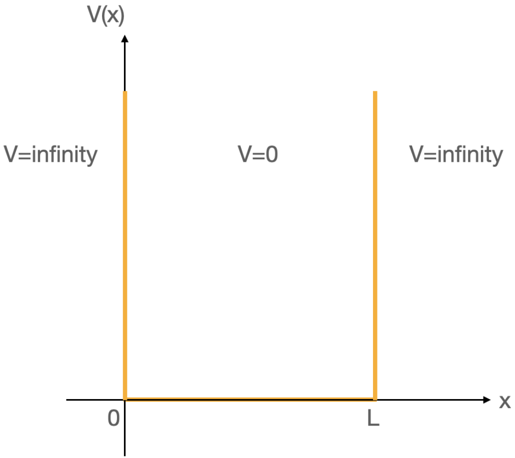
- which satisfies completeness

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

A particle in a box (an infinite potential well)



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$$V(x) = \begin{cases} 0 & \text{for } \ 0 \leq x \leq L, \\ \infty & \text{otherwise} \end{cases} \qquad -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x)$$

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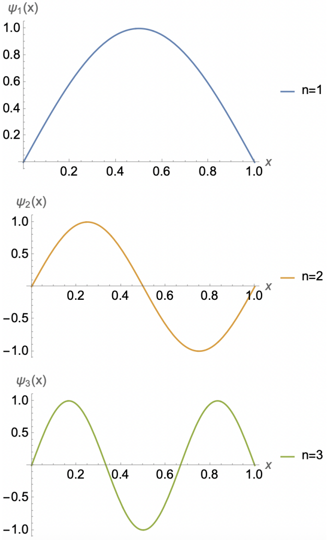
- the corresponding energy is

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{n^2 h^2}{8mL^2}$$

A particle in a box : A solution



A particle in a box

- Eigenvalues and Eigenfunctions

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{n^2 h^2}{8mL^2} = \text{real values}, \quad \psi_n(x) = \sqrt{\frac{2}{L}} \sin(\frac{n\pi x}{L}) \quad (10)$$

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Hermitian operator (self-adjoint)

$$\hat{\mathcal{O}} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) = \hat{\mathcal{O}}^\dagger \quad (13)$$

Key Concept Recap

In QM,

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- solve Schrödinger equation

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- eigenfunctions are othornormal
- eigenfunctions form a complete set
- Hilbert space (inner product is well defined and square-integrable)

A particle in a black hole

Our background spacetime

Schwarzschild black hole

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \quad (14)$$

Our background spacetime

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- asymptotically flat in 4-dimensional spacetime

Our background spacetime

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- asymptotically flat in 4-dimensional spacetime
- a static solution

Our background spacetime

Schwarzschild black hole

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- asymptotically flat in 4-dimensional spacetime
- a static solution
- a vacuum solution $T_{\mu\nu} = 0$

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi T_{\mu\nu} \quad (15)$$

from Schrödinger equation to Klein-Gordon equation

- Schrödinger eq. : the non-relativistic energy-momentum relation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \quad \left(\psi = e^{-i(Et-Px)/\hbar}\right) \quad \rightarrow \quad E = \frac{P^2}{2m} \qquad (16)$$

- the relativistic energy-momentum relation :

$$E^2 = P^2c^2 + m^2c^4 \qquad (17)$$

requires

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \hat{P} = \frac{\hbar}{i}\frac{\partial}{\partial\vec{x}} \quad \rightarrow \quad -\hbar^2\frac{\partial^2}{\partial t^2}\psi = -\hbar^2c^2\frac{\partial^2}{\partial\vec{x}^2}\psi + m^2c^4\psi \qquad (18)$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \frac{\partial^2}{\partial\vec{x}^2}\psi + \frac{m^2c^2}{\hbar^2}\psi = 0 \quad \rightarrow \quad \nabla^2\psi + m^2\psi = 0 \qquad (19)$$

$$\text{where} \qquad \nabla_\mu = \frac{\partial}{\partial x^\mu}, \quad x^\mu = \{ct, x, y, z\}, \qquad (20)$$

$$\nabla^2 = \eta^{\mu\nu}\nabla_\mu\nabla_\nu, \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta^{\mu\nu} \qquad (21)$$

Klein-Gordon equation

Quantum mechanics + Relativity → Quantum field theory

$$\text{Klein-Gordon equation:} \qquad \nabla^2\psi + m^2\psi = 0 \tag{22}$$

- ψ : fields (not a wave function)
- $\langle\psi|\psi\rangle = \int \mathrm{d}^3x|\psi(x)|^2$: not Lorentz invariant
- Klein-Gordon inner product :

$$\langle\psi|\psi\rangle_{\mathrm{KG}} = \frac{i}{2}\int_\Sigma \Big[(D_\mu\psi_n)^*\psi_m - \psi_n^*(D_\mu\psi_m)\Big]\mathrm{d}\Sigma^\mu \tag{23}$$

which is invariant under Lorentz transformation and ensures conservation under time evolution

Klein-Gordon equation on curved spacetime (I)

- Klein-Gordon equation on curved spacetime ($m^2 = 0$)

$$\nabla^2\psi = \nabla^\mu\nabla_\mu\psi = \nabla_\mu g^{\mu\nu}\nabla_\nu\psi = 0, \quad (\nabla_\mu g_{\alpha\beta} = 0) \tag{24}$$

$$g^{\mu\nu}\nabla_\mu(\partial_\nu\psi) = g^{\mu\nu}\left(\partial_\mu(\partial_\nu\psi) - \Gamma^\gamma_{\mu\nu}(\partial_\gamma\psi)\right) = 0 \tag{25}$$

where $\nabla_\mu\psi = \partial_\mu\psi$ and $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\gamma_{\mu\nu}A_\gamma$

and

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{1}{A(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \tag{26}$$

Klein-Gordon equation on curved spacetime (I)

Klein-Gordon equation on curved spacetime ($m^2 = 0$)

$$\nabla^2\psi = g^{\mu\nu}\left(\partial_\mu(\partial_\nu\psi) - \Gamma^\gamma_{\mu\nu}(\partial_\gamma\psi)\right) = 0 \tag{27}$$

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$$g_{\mu\nu} = \begin{pmatrix} -A(r) & 0 & 0 & 0 \\ 0 & \frac{1}{A(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad g_{\mu\nu}g^{\nu\gamma} = \delta^\gamma_\mu \tag{29}$$

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2}g^{\gamma\delta}\left[\frac{\partial g_{\mu\delta}}{\partial x^\nu} + \frac{\partial g_{\nu\delta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\delta}\right], \tag{30}$$

$$\Gamma^t_{tt} = \frac{1}{2}g^{t\delta}\left[\frac{\partial g_{t\delta}}{\partial x^t} + \frac{\partial g_{t\delta}}{\partial x^t} - \frac{\partial g_{tt}}{\partial x^\delta}\right] = \frac{1}{2}g^{tt}\left[\frac{\partial g_{tt}}{\partial x^t} + \frac{\partial g_{tt}}{\partial x^t} - \frac{\partial g_{tt}}{\partial x^t}\right] = 0$$

$$\Gamma^t_{tr} = \frac{1}{2}g^{t\delta}\left[\frac{\partial g_{t\delta}}{\partial x^r} + \frac{\partial g_{r\delta}}{\partial x^t} - \frac{\partial g_{tr}}{\partial x^\delta}\right] = \frac{1}{2}g^{tt}\left[\frac{\partial g_{tt}}{\partial x^r} + \frac{\partial g_{tt}}{\partial x^t} - \frac{\partial g_{tr}}{\partial x^t}\right]$$

$$= \frac{1}{2}\left(-\frac{1}{A(r)}\right)\left[\frac{\partial(-A(r))}{\partial x^r} + \frac{\partial(-A(r))}{\partial x^t} - 0\right] = \frac{A'(r)}{2A(r)} \tag{31}$$

Klein-Gordon equation on curved spacetime (I)

$$\Gamma^t_{rt} = \frac{A'(r)}{2A(r)}, \quad \Gamma^r_{tt} = \frac{1}{2}A(r)A'(r), \quad \Gamma^r_{rr} = -\frac{A'(r)}{2A(r)}, \tag{27}$$

$$\Gamma^r_{\theta\theta} = -rA(r), \quad \Gamma^r_{\phi\phi} = -rA(r)\sin^2(\theta), \quad \Gamma^\theta_{\theta r} = \frac{1}{r}, \tag{28}$$

$$\Gamma^\theta_{\phi\phi} = -\cos(\theta)\sin(\theta), \quad \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\phi\theta} = \cot(\theta) \tag{29}$$

$$\begin{aligned}
\nabla^2\psi &= g^{\mu\nu}\Big(\partial_\mu(\partial_\nu\psi) - \Gamma^\gamma_{\mu\nu}(\partial_\gamma\psi)\Big) \\
&= g^{tt}\Big(\partial_t(\partial_t\psi) - \Gamma^\gamma_{tt}(\partial_\gamma\psi)\Big) + g^{rr}\Big(\partial_r(\partial_r\psi) - \Gamma^\gamma_{rr}(\partial_\gamma\psi)\Big) \\
&+ g^{\theta\theta}\Big(\partial_\theta(\partial_\theta\psi) - \Gamma^\gamma_{\theta\theta}(\partial_\gamma\psi)\Big) + g^{\phi\phi}\Big(\partial_\phi(\partial_\phi\psi) - \Gamma^\gamma_{\phi\phi}(\partial_\gamma\psi)\Big) \\
&= g^{tt}\Big(\partial_t(\partial_t\psi) - \Gamma^r_{tt}(\partial_r\psi)\Big) + g^{rr}\Big(\partial_r(\partial_r\psi) - \Gamma^r_{rr}(\partial_r\psi)\Big) \\
&+ g^{\theta\theta}\Big(\partial_\theta(\partial_\theta\psi) - \Gamma^r_{\theta\theta}(\partial_r\psi)\Big) + g^{\phi\phi}\Big(\partial_\phi(\partial_\phi\psi) - \Gamma^r_{\phi\phi}(\partial_r\psi) - \Gamma^\theta_{\phi\phi}(\partial_\theta\psi)\Big)
\end{aligned}$$

Klein-Gordon equation on curved spacetime (I)

$$\psi(t,r,\theta,\phi) = e^{-i\omega t}\frac{\Phi(r)}{r}Y^l_m(\theta,\phi), \tag{30}$$

$$\begin{aligned}\nabla^2\psi = g^{tt}\left[\partial_t(\partial_t\psi) - \frac{A(r)A'(r)}{2}(\partial_r\psi)\right] \\ + g^{rr}\left[\partial_r(\partial_r\psi) - \left(-\frac{A'(r)}{2A(r)}\right)(\partial_r\psi)\right] + g^{\theta\theta}\left[\partial_\theta(\partial_\theta\psi) - (-rA(r))(\partial_r\psi)\right] \\ + g^{\phi\phi}\left[\partial_\phi(\partial_\phi\psi) - (-rA(r)\sin^2\theta)(\partial_r\psi) - (-\cos\theta\sin\theta)(\partial_\theta\psi)\right]\end{aligned} \tag{31}$$

using

$$\frac{1}{Y^l_m}\frac{\partial^2 Y^l_m}{\partial\phi^2} = -m^2, \quad l(l+1)\sin^2\theta + \frac{\sin\theta}{Y^l_m}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y^l_m}{\partial\theta}) = m^2 \tag{32}$$

then

$$\Phi''(r) + \frac{A'(r)}{A(r)}\Phi'(r) - \frac{1}{A(r)}\left(\frac{l(l+1)}{r^2} + \frac{A'(r)}{r} - \frac{\omega^2}{A(r)}\right)\Phi(r) = 0 \tag{33}$$

Klein-Gordon equation on curved spacetime (II)

$$\nabla^2\psi(t,r,\theta,\phi) = \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi) = 0, \tag{34}$$

$$g_{\mu\nu} = \begin{pmatrix} -A(r) & 0 & 0 & 0 \\ 0 & \frac{1}{A(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix}, \quad \sqrt{-g} = r^2\sin(\theta) \tag{35}$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{A(r)} & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2(\theta)} \end{pmatrix} \tag{36}$$

$$\psi(t,r,\theta,\phi) = e^{-i\omega t}\frac{\Phi(r)}{r}Y_m^l(\theta,\phi) \tag{37}$$

$$\nabla^2\psi(t,r,\theta,\phi) = \frac{1}{r^2\sin(\theta)}\left[\partial_t\left(g^{tt}r^2\sin(\theta)\partial_t\psi\right) + \partial_r\left(g^{rr}r^2\sin(\theta)\partial_r\psi\right)\right.$$
$$\left. + \partial_\theta\left(g^{\theta\theta}r^2\sin(\theta)\partial_\theta\psi\right) + \partial_\phi\left(g^{\phi\phi}r^2\sin(\theta)\partial_\phi\psi\right)\right] = 0 \tag{38}$$

Klein-Gordon equation on curved spacetime (II)

$$\nabla^2\psi(t,r,\theta,\phi) = \partial_t\Big(-\frac{1}{A(r)}\partial_t\psi\Big) + \frac{1}{r^2}\partial_r\Big(A(r)r^2\partial_r\psi\Big)$$
$$+ \frac{1}{r^2\sin(\theta)}\partial_\theta\Big(\sin(\theta)\partial_\theta\psi\Big) + \frac{1}{r^2\sin^2(\theta)}\,\partial_\phi\Big(\partial_\phi\psi\Big) \tag{39}$$

$$\text{Let us insert :}\quad \psi(t,r,\theta,\phi) = e^{-i\omega t}\frac{\Phi(r)}{r}Y^l_m(\theta,\phi)$$

$$= \partial_t\Big(-\frac{1}{A(r)}\partial_t e^{-i\omega t}\Big)\frac{\Phi(r)}{r}Y^l_m + \frac{1}{r^2}\partial_r\Big(A(r)r^2\partial_r\frac{\Phi(r)}{r}\Big)e^{-i\omega t}Y^l_m$$

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$$+\frac{1}{r^2}\Big[\frac{1}{\sin(\theta)}\partial_\theta\Big(\sin(\theta)\partial_\theta Y^l_m\Big) + \frac{1}{\sin^2(\theta)}\ \partial_\phi\Big(\partial_\phi Y^l_m\Big)\Big]e^{-i\omega t}\frac{\Phi(r)}{r} \qquad (40)$$
$$\Big(\frac{1}{\sin(\theta)}\partial_\theta\Big(\sin(\theta)\partial_\theta Y^l_m\Big) + \frac{1}{\sin^2(\theta)}\ \partial_\phi\Big(\partial_\phi Y^l_m\Big) = -l(l+1)Y^l_m\Big) \qquad (41)$$

Klein-Gordon equation on curved spacetime (II)

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$$\rightarrow \quad \Phi''(r) + \frac{A'(r)}{A(r)}\Phi'(r) - \frac{1}{A(r)}\left(\frac{l(l+1)}{r^2} + \frac{A'(r)}{r} - \frac{\omega^2}{A(r)}\right)\Phi(r) = 0$$

Klein-Gordon equation on curved spacetime

A scalar field equation on curved spacetime

$$\Phi''(r) + \frac{A'(r)}{A(r)}\Phi'(r) - \frac{1}{A(r)}\left(\frac{l(l+1)}{r^2} + \frac{A'(r)}{r} - \frac{\omega^2}{A(r)}\right)\Phi(r) = 0 \quad (41)$$

Let us employ a tortoise coordinate

$$\mathrm{d}r_* = \frac{1}{A(r)}\mathrm{d}r \qquad \left(\mathrm{d}s^2 = -A(r)(\mathrm{d}t^2 + \mathrm{d}r_*^2) + r^2\mathrm{d}\Omega_2\right) \quad (42)$$

it becomes Schrodinger equation like

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}\Phi''(r) - V_{\text{eff}}\,\Phi(r) = -\omega^2\Phi(r) \quad (43)$$

where

$$V_{\text{eff}} = A(r)\left(\frac{l(l+1)}{r^2} + \frac{A'(r)}{r}\right) \quad (44)$$

We will solve this equation classically (not quantize).

Quasi-Normal Modes (QNM)

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}\Phi''(r) + V_{\mathrm{eff}}\,\Phi(r) = -\omega^2\Phi(r) \tag{45}$$

Boundary conditions

- near the horizon : incoming, $e^{-i(t+r)\omega}$
- at infinity : outgoing, $e^{-i(t-r)\omega}$

which indicates *"open system"*

- non-Hermitian : $\mathcal{L} \neq \mathcal{L}^\dagger$
- no conservation in time evolution
- the Klein-Gordon inner product at infinity diverge
- ω is not real but complex (discrete and infinity number of set) $\rightarrow$ Quasi-Normal Modes (QNM)

How to impose the Boundary Conditions?

$$\frac{d^2}{dr_*^2}\Phi''(r) + V_{\text{eff}}\,\Phi(r) = -\omega^2\Phi(r) \tag{46}$$

- (B.C.) near the horizon : incoming, $e^{-i(t+r_*)\omega}$
- (B.C.) at infinity : outgoing, $e^{-i(t-r_*)\omega}$

$$V_{\text{eff}} = 0: \quad \Phi(r) = c_1\cos(r_*\omega) + c_2\sin(r_*\omega) \tag{47}$$
$$= \frac{1}{2}c_1 e^{-ir_*\omega} + \frac{1}{2}c_1 e^{ir_*\omega} - \frac{1}{2}ic_2 e^{ir_*\omega} + \frac{1}{2}ic_2 e^{-ir_*\omega}$$

- incoming boundary condition : $c_1 = ic_2$

$$\Phi(r) = ic_2 e^{-ir_*\omega}, \qquad e^{-i\omega t}\Phi(r) = -ic_2 e^{-i(t+r_*)\omega} \tag{48}$$

- outgoing boundary condition : $c_1 = -ic_2$

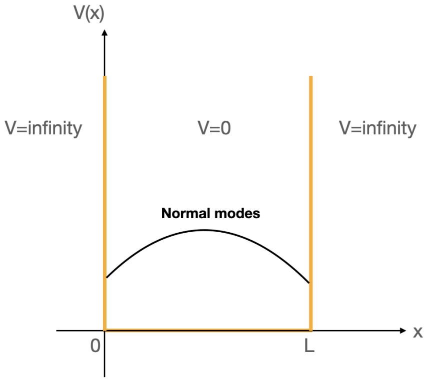
$$\Phi(r) = -ic_2 e^{ir_*\omega}, \qquad e^{-i\omega t}\Phi(r) = -ic_2 e^{-i(t-r_*)\omega} \tag{49}$$

- previously (a particle in a box)

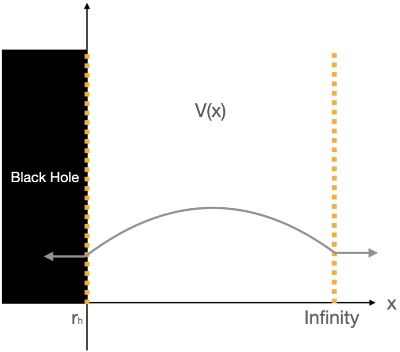
$$(\text{B.C.}) \quad V_{\text{eff}} = 0 \ (0 \le x \le L), \qquad \psi(0) = \psi(L) = 0,$$

$$\psi(x) = A\sin(kx) + B\cos(kx) \quad \to \quad \psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

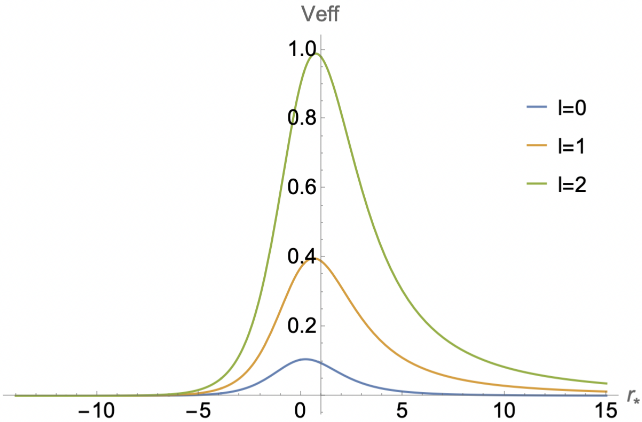
A particle in a box



A particle in a black hole



$V_{\rm eff}$: effective potential for KG field



Let us solve KG equation: Green's function method

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - V(x)\right]\psi(t,x) = 0, \qquad x = \frac{r_*}{c}, \tag{50}$$

$$\text{Laplace transformation}: \quad \hat{\psi}(s,x) = \int_0^\infty \mathrm{d}t\, e^{-st}\psi(t,x), \tag{51}$$

$$\left[\frac{\partial^2}{\partial x^2} - s^2 - V(x)\right]\hat{\psi}(s,x) = \mathcal{J}(s,x), \qquad (s = -i\omega) \tag{52}$$

$$\text{where} \quad \mathcal{J}(s,x) = -\partial_t\psi(t,x) - s\psi(t,x)\Big|_{t=0} \tag{53}$$

$$\left[\frac{\partial^2}{\partial x^2} - s^2 - V(x)\right]\hat{G}(s,x,x') = \delta(x - x') \tag{54}$$

Then the corresponding solution is given by

$$\hat{\psi}(s,x) = \int_\infty^\infty \mathrm{d}x'\hat{G}(s,x,x')\mathcal{J}(s,x') \tag{55}$$

If the initial data $\mathcal{J}(s,x')$ is given the solution is uniquely determined.

Let us solve KG equation: Green's function method

$$\left[\frac{\partial^2}{\partial x^2} - s^2 - V(x)\right]\hat{\psi}_\pm(s,x) = 0,$$

where $\hat{\psi}_\pm$ are any two linearly independent sol's of homogenous eq.

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$$\hat{G} = \frac{1}{\mathcal{W}(s)}\left[\Theta(x-x')\hat{\psi}_-(s,x')\hat{\psi}_+(s,x) + \Theta(x'-x)\hat{\psi}_-(s,x)\hat{\psi}_+(s,x')\right],$$

where $\mathcal{W}(s) = \hat{\psi}_-(s,x)\partial_x\hat{\psi}_+(s,x) - \hat{\psi}_+(s,x)\partial_x\hat{\psi}_-(s,x),$

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$$\text{If } x > x_R \ : \hat{\psi}(s,x) = \frac{c_-(s)}{\mathcal{W}(s)}\hat{\psi}_+(s,x), \quad c_-(s) = \int_{x_L}^{x_R}\mathrm{d}x'\hat{\psi}_-(s,x')\mathcal{J}(s,x')$$

$$\text{If } x < x_L \ : \hat{\psi}(s,x) = \frac{c_+(s)}{\mathcal{W}(s)}\hat{\psi}_-(s,x), \quad c_+(s) = \int_{x_L}^{x_R}\mathrm{d}x'\hat{\psi}_+(s,x')\mathcal{J}(s,x')$$

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which is kwon as *Bromwich integral* or *Mellin's inverse formula*.

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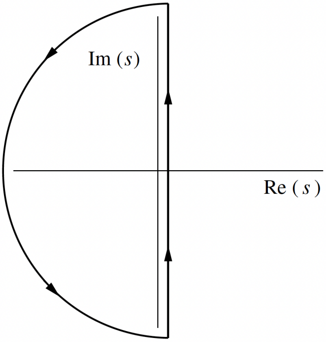

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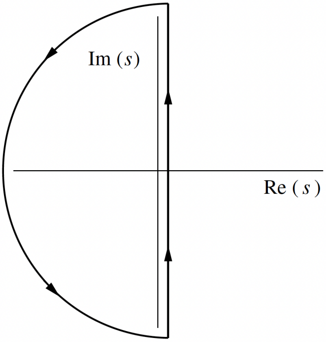

If $\hat{\psi}_+(s,x)$ are analytic in s (there are no essential singularities or branch cuts inside of the contour), the only contribution to the integral come from the poles of integrand due to zeros of $W(s)$.

$$\mathcal{W}(s) = \mathcal{W}'(s_n)(s-s_n) + \mathcal{O}(s-s_n)^2,$$

$$\text{where} \quad \mathcal{W}'(s_n) \equiv \left.\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}s}\right|_{s=s_n}$$

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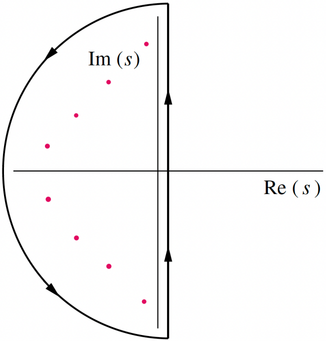

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Let us solve KG equation: Green's function method

$$\mathcal{W}(s_n) = \hat{\psi}_-(s_n,x)\partial_x\hat{\psi}_+(s_n,x) - \hat{\psi}_+(s_n,x)\partial_x\hat{\psi}_-(s_n,x) = 0 \qquad (56)$$

implies

$$\frac{\hat{\psi}'_+(s_n,x)}{\hat{\psi}_+(s_n,x)} = \frac{\hat{\psi}'_-(s_n,x)}{\hat{\psi}_-(s_n,x)} \qquad (57)$$

which integrates to

$$\hat{\psi}_+(s_n,x) = c(s_n)\hat{\psi}_-(s_n,x) \qquad (58)$$

"This means that, for $s = s_n$ the two solutions of the homogenous equation, $\hat{\phi}_+(s,x)$ and $\hat{\phi}_-(s,x)$, are no longer independent, and we have a single solution $\hat{\psi}_+(s_n,x) \propto \hat{\psi}_-(s_n,x) \equiv \hat{\psi}_*(s_n,s)$."

Let us solve KG equation: Asymptotic solution

Asymptotic solutions ($V(x) \to 0$ as $x \to \pm\infty$)

$$[\partial_x^2 - s^2]\hat{\psi}_{\pm}(s,x) = 0 \tag{59}$$

$$\hat{\psi}_{-}(s,x) \simeq \begin{cases} e^{sx} & (x \to -\infty) \\ a_1(s)e^{sx} + a_2(s)e^{-sx} & (x \to +\infty) \end{cases}, \tag{60}$$

$$\hat{\psi}_{+}(s,x) \simeq \begin{cases} b_1(s)e^{sx} + b_2(s)e^{-sx} & (x \to -\infty) \\ e^{-sx} & (x \to +\infty) \end{cases}, \tag{61}$$

$$\mathcal{W}(s) = -2sb_2(s) \;\; (x \to -\infty), \qquad \mathcal{W}(s) = -2sa_1(s) \;\; (x \to \infty) \tag{62}$$

$$\Longrightarrow a_1(s) = b_2(s) \tag{63}$$

$$\mathcal{W}(s_n) = 0 \quad \to \quad a_1(s_n) = b_2(s_n) = 0 \tag{64}$$

KG inner product

when $x \to \infty$,

$$\psi_n(x) = e^{-i\omega_n t}\Phi_n(x), \quad \psi_m(x) = e^{-i\omega_m t}\Phi_m(x)$$

where $s_n = -i\omega_n$, $\Phi_n(x) = e^{-sx}$

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$$\langle \psi_n | \psi_m \rangle_{\mathrm{KG}} = \frac{i}{2} \int_\Sigma \Big[(D_\mu \psi_n)^* \psi_m - \psi_n^* (D_\mu \psi_m) \Big] \mathrm{d}\Sigma^\mu$$

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$$= -\frac{1}{2}(\omega_n^* + \omega_m)e^{i(\omega_n^* - \omega_m)t}\int_\Sigma \Phi_n^*(x)\Phi_m \mathrm{d}\Sigma^t$$

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$$= -\frac{1}{2}(\omega_n^* + \omega_m)e^{i(\omega_n^* - \omega_m)t}\int_\Sigma \Phi_n^*(x)\Phi_m \mathrm{d}\Sigma^t$$

$$= -\frac{1}{2}(\omega_n^* + \omega_m)e^{i(\omega_n^* - \omega_m)t}\int_\Sigma e^{-i\omega_n^* x}e^{i\omega_m x}\mathrm{d}\Sigma^t \neq \delta_{nm}$$

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KG inner product

when $x \to \infty$,

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In QM,

$$\langle\psi_n|\psi_m\rangle = \delta(\omega_n - \omega_m) = \delta_{nm}$$

Analytic Structure of the Klein–Gordon Green's Function : $G = G_{\mathbf{F}} + G_{\mathbf{QNM}} + G_{\mathbf{Branch\text{-}cut}}$

Solution for KG equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}\Phi''(r) + \left(\omega^2 - V_{\text{eff}}\right)\Phi(r) = 0, \quad V_{\text{eff}} = A(r)\left(\frac{l(l+1)}{r^2} + \frac{A'(r)}{r}\right)$$

[3]Michele Maggiore - "Gravitational Waves: Volume 2: Astrophysics and Cosmology"

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- This form is known as a generalized spheroidal wave equation
- their solutions are known exactly such as [3]

$$\hat{\phi}_+(s,r) = (2is)^s e^{i\phi_+}(1-1/r)^s \sum_{L=-\infty}^{\infty} b_L[G_{L+\nu}(-is, isr) + iF_{L+\nu}(-is, isr)]$$

$$\hat{\phi}_-(s,r) = r^{-2s}(r-1)^s e^{-s(r-2)} \sum_{n=0}^{\infty} a_n(1-1/r)^n$$

where $G_{L+\nu}$ and $F_{L+\nu}$ are Coulomb wave functions. $\phi_-(s,r)$ is analytic in s, but $\phi_+(s,r)$ has a branch cut in the complex s-plane.

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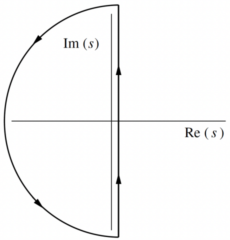

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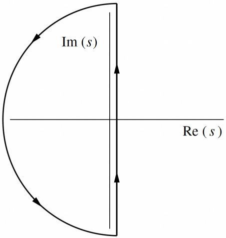

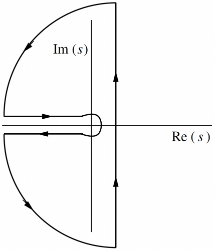

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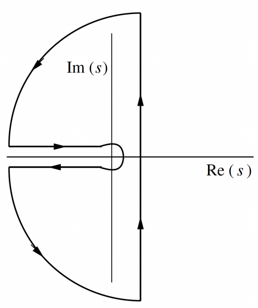
Solution for KG equation

$$\hat{\psi}(s,x) = \int_{-\infty}^{\infty} \mathrm{d}x' \hat{G}(s,x,x') \mathcal{J}(s,x), \tag{65}$$

$$G(t,x,x') = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{\mathrm{d}s}{2\pi i} e^{st} \hat{G}(s,x,x') \tag{66}$$



$$G(t,x,x') = G_{\mathrm{QNM}}(t,x,x') + G_{\mathrm{B}}(t,x,x') + G_{\mathrm{F}}(t,x,x') \tag{67}$$

Branch cut solution for small ω : late time tail

- In the limit $t \to \infty$ with x fixed :

$$G_B(t,x,x') \simeq (-1)^{l+1} \frac{2(2l+2)!}{[(2l+1)!!]^2} \frac{r_h}{c} \frac{(xx')^{l+1}}{t^{2l+3}} \tag{68}$$

  there is a power-law tail at spatial infinity, i.e. a non-radiative tail.

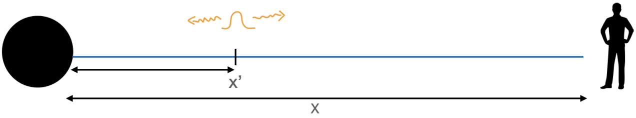
- In the limit $t \to \infty$ with t/x fixed :

$$G_B(t,x,x') \simeq (-1)^{l+1} \frac{(l+1)!}{(2l+1)!!} \frac{r_h}{c} \frac{(x')^{l+1}}{u^{l+2}}, \qquad u \equiv t - x \tag{69}$$

  there is a radiative tail at future null infinity.

Solution for KG equation: Green's function method

$$G(t,x,x') = G_{\rm QNM}(t,x,x') + G_{\rm B}(t,x,x') + G_{\rm F}(t,x,x') \tag{70}$$



- $t \simeq x - x'$: the outward propagating wave can effectively ignore the influence of this potential and travel outward reaches the observer, $G_{\rm F}$
- $t \simeq x + x'$: (quasinormal modes) the part of the low-frequency scalar perturbation traveling towards the black hole bounces off the effective potential $V_{\rm eff}(r)$ near the horizon and propagates back to the observer, $G_{\rm QNM}$
- $t \gg x + x'$: (late-time behavior) the QNMs gradually fade away, and only the decaying modes of the signal persist. The outgoing wave that traveled to a large distance and then scattered back towards the observer due to the spacetime curvature at large distances, $G_{\rm B}$

How to determine QNMs (ω)

Wronskian-based Approaches (ω)

Shooting method

- Set Asymptotic Boundary Solutions
  - Horizon ($r_* \to -\infty$): Pure ingoing wave, $\Psi_{\text{in}} \sim e^{-i\omega r_*}$
  - Infinity ($r_* \to +\infty$): Pure outgoing wave, $\Psi_{\text{out}} \sim e^{+i\omega r_*}$
- Bi-directional Numerical Integration (Shooting)
  - Integrate Ψ_{in} outward and Ψ_{out} inward
  - Meet at a matching point r_{match} (typically photon sphere, $r \approx 3M$)
- Wronskian Evaluation at r_{match} $\to$ Complex Root-Finding

$$W(\omega) = \left(\Psi_{\text{in}} \frac{d\Psi_{\text{out}}}{dr_*} - \frac{d\Psi_{\text{in}}}{dr_*} \Psi_{\text{out}} \right) \Bigg|_{r=r_{\text{match}}} = 0$$

Leaver method

- Expand radial function in a Frobenius series at the horizon :
  $\Psi(r) = \mathcal{A}(r) \sum_{n=0}^{\infty} a_n \left(1 - \frac{r_H}{r}\right)^n , \qquad \mathcal{A}(r) = P_H(r,\omega) P_\infty(r,\omega)$
- Three-Term Recurrence Relation
  - Substitute ansatz into wave equation (Teukolsky / Regge-Wheeler)
  - Derive 3-term recurrence relation for coefficients a_n:

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0$$

- Continued Fraction Condition $\to$ Complex Root-Finding

$$\beta_0 - \cfrac{\alpha_0 \gamma_1}{\beta_1 - \cfrac{\alpha_1 \gamma_2}{\beta_2 - \cfrac{\alpha_2 \gamma_3}{\cdots}}} = 0$$

  - Imposes convergence at $r \to \infty$ (a_n forms a minimal solution)

Determining QNM Frequencies (ω) : WKB method [4]

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}\Phi''(r) + V_{\mathrm{eff}}\,\Phi(r) = -\omega^2\Phi(r)$$

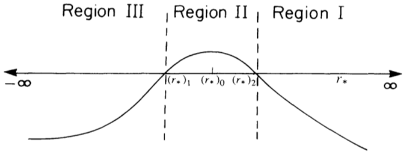


FIG. 1. The function $V(r_*) - \omega^2$.

[4] Sai Iyer, "Black-hole normal modes: A WKB approach. II. Schwarzschild black holes", Phys. Rev. D 35, 3632

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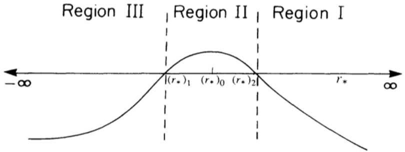

FIG. 1. The function $V(r_*)-\omega^2$.

mula for the normal-mode frequencies, given by

$$\omega^2=[V_0+(-2V_0'')^{1/2}\tilde{\Lambda}] -i(n+\tfrac{1}{2})(-2V_0'')^{1/2}(1+\tilde{\Omega})\,, \qquad (1.2)$$

$$n=\begin{cases}0,1,2,\ldots, & \mathrm{Re}(\omega)>0\,,\\ -1,-2,-3,\ldots, & \mathrm{Re}(\omega)<0\,,\end{cases}$$

where

$$\tilde{\Lambda}(n)=\frac{1}{(-2V_0'')^{1/2}}\left[\frac{1}{8}\left[\frac{V_0^{(4)}}{V_0''}\right]\left[\frac{1}{4}+\alpha^2\right]-\frac{1}{288}\left[\frac{V_0'''}{V_0''}\right]^2(7+60\alpha^2)\right]\,, \qquad (1.3a)$$

$$\tilde{\Omega}(n)=\frac{1}{(-2V_0'')}\left[\frac{5}{6912}\left[\frac{V_0'''}{V_0''}\right]^4(77+188\alpha^2)-\frac{1}{384}\left[\frac{V_0'''^2V_0^{(4)}}{V_0''^{\,3}}\right](51+100\alpha^2)+\frac{1}{2304}\left[\frac{V_0^{(4)}}{V_0''}\right]^2(67+68\alpha^2)\right.$$
$$\left.+\frac{1}{288}\left[\frac{V_0'''V_0^{(5)}}{V_0''^{\,2}}\right](19+28\alpha^2)-\frac{1}{288}\left[\frac{V_0^{(6)}}{V_0''}\right](5+4\alpha^2)\right]\,. \qquad (1.3b)$$

[4] Sai Iyer, "Black-hole normal modes: A WKB approach. II. Schwarzschild black holes", Phys. Rev. D 35, 3632

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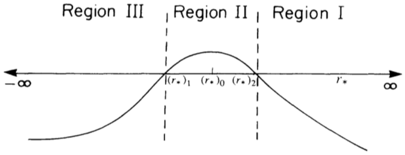

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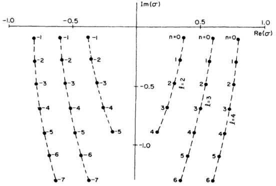

FIG. 2. Gravitational normal modes. Here $\sigma=M\omega$, where M is the mass of the black hole.

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$$n=\begin{cases}0,1,2,\ldots, & \mathrm{Re}(\omega)>0\,,\\ -1,-2,-3,\ldots, & \mathrm{Re}(\omega)<0\,,\end{cases}$$

where

$$\tilde{\Lambda}(n)=\frac{1}{(-2V_0'')^{1/2}}\left[\frac{1}{8}\left(\frac{V_0^{(4)}}{V_0''}\right)\left(\frac{1}{4}+\alpha^2\right)-\frac{1}{288}\left(\frac{V_0'''}{V_0''}\right)^2(7+60\alpha^2)\right], \qquad (1.3a)$$

$$\tilde{\Omega}(n)=\frac{1}{(-2V_0'')}\left[\frac{5}{6912}\left(\frac{V_0'''}{V_0''}\right)^4(77+188\alpha^2)-\frac{1}{384}\left(\frac{V_0'''^2V_0^{(4)}}{V_0''^{\,3}}\right)(51+100\alpha^2)+\frac{1}{2304}\left(\frac{V_0^{(4)}}{V_0''}\right)^2(67+68\alpha^2)\right.$$
$$\left.+\frac{1}{288}\left(\frac{V_0'''V_0^{(5)}}{V_0''^{\,2}}\right)(19+28\alpha^2)-\frac{1}{288}\left(\frac{V_0^{(6)}}{V_0''}\right)(5+4\alpha^2)\right]. \qquad (1.3b)$$

[4] Sai Iyer, "Black-hole normal modes: A WKB approach. II. Schwarzschild black holes", Phys. Rev. D 35, 3632

Matrix & Numerical Approach : Eigenvalues Problem

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}\Phi''(r) + V_{\text{eff}}\,\Phi(r) = -\omega^2\Phi(r) \tag{71}$$

which takes a form of

$$\mathcal{L}\,\Phi(r) = \lambda\Phi(r) \tag{72}$$

this is nothing but the eigenvalue problem.

$$\mathcal{L}u_n = \lambda_n u_n \tag{73}$$

- By using numerical methods (e.g. pseudo spectral methods or shooting method), we impose the boundary condition for QNM and can calculate λ_n and u_n.

Matrix & Numerical Approach : Eigenvalues Problem

This perturbed equation takes a form of

$$\mathcal{L}_{\omega^2}\tilde{\Phi}(z) = A_2\tilde{\Phi}''(z) + (A_1 + A_{1\omega}\omega)\tilde{\Phi}'(z) + (A_0 + A_{0\omega_1}\omega + A_{0\omega_2}\omega^2)\tilde{\Phi}(z) = 0, \tag{74}$$

and can be rearranged to

$$(\alpha + \beta_1\omega + \beta_2\omega^2)\tilde{\Phi}(z) = 0 \tag{75}$$

where

$$\alpha = A_2\frac{\mathrm{d}^2}{\mathrm{d}z^2} + A_1\frac{\mathrm{d}}{\mathrm{d}z} + A_0, \qquad \beta_1 = A_{1\omega}\frac{\mathrm{d}}{\mathrm{d}z} + A_{0\omega_1}, \qquad \beta_2 = A_{0\omega_2} \tag{76}$$

$$(\tilde{\alpha} - \tilde{\beta}\,\omega)\begin{pmatrix}\tilde{\Phi}(z)\\ \omega\tilde{\Phi}(z)\end{pmatrix} = 0 \tag{77}$$

We redefine α, β_1, and β_2 to take a form of

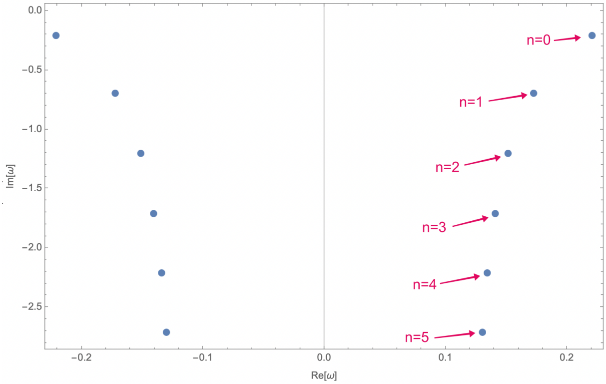
$$\tilde{\alpha} = \begin{pmatrix}\alpha & \beta_1\\ 0 & I\end{pmatrix}, \qquad \tilde{\beta} = \begin{pmatrix}0 & -\beta_2\\ I & 0\end{pmatrix}. \tag{78}$$

This can be written

$$(\tilde{\beta}^{-1}\tilde{\alpha} - \omega I_{2\times 2})\begin{pmatrix}\tilde{\Phi}(z)\\ \omega\tilde{\Phi}(z)\end{pmatrix} = 0, \qquad \rightarrow \qquad (\mathcal{L}_M - \omega_p)v_p = 0 \tag{79}$$

Matrix & Numerical Approach : Eigenvalues Problem

QNM with $l = 0$ for scalar field in Schawarzschil black hole by using pseudo spectral method

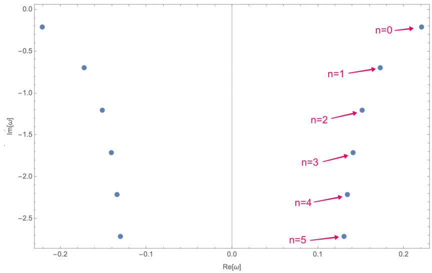


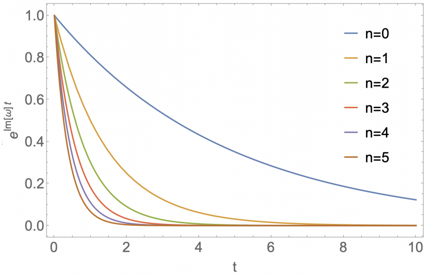
	$Re[\omega_n]$	$Im[\omega_n]$
1	±0.13024640	-2.7140490
2	±0.13444262	-2.2112523
3	±0.14078647	-1.7073040
4	±0.15148323	-1.2021605
5	±0.17223383	-0.69610489
6	±0.22090988	-0.20979143

$$e^{-i\omega t} = e^{-i(\omega_R + i\omega_I)t} = e^{-i\omega_R t} e^{\omega_I t}$$

Matrix & Numerical Approach : Eigenvalues Problem

QNM with $l = 0$ for scalar field in Schawarzschil black hole by using pseudo spectral method

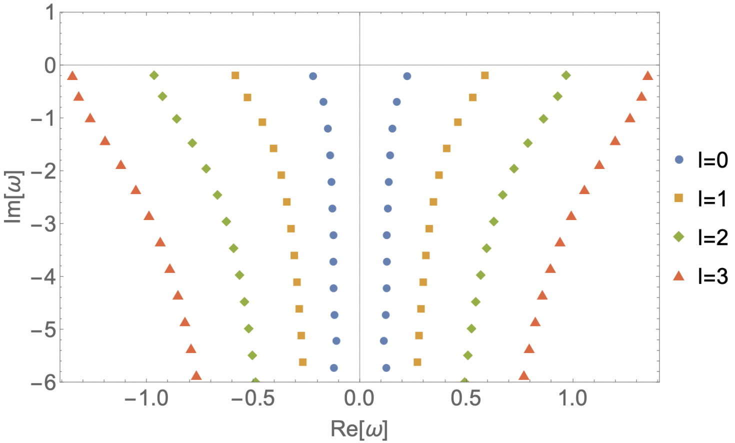




$$e^{-i\omega t} = e^{-i(\omega_R + i\omega_I)t} = e^{-i\omega_R t} e^{\omega_I t}$$

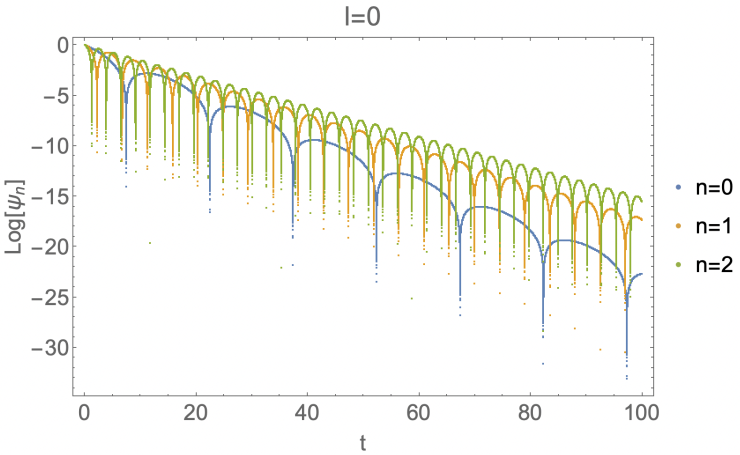
Matrix & Numerical Approach : Eigenvalues Problem

QNM with $l = 0, 1, 2, 3$ for scalar field in Schawarzschil black hole by using pseudo spectral method



Matrix & Numerical Approach : Eigenvalues Problem

QNM with $l = 0$ for scalar field in Schawarzschil black hole by using pseudo spectral method in time domain



Summary

In QM (vs in QNMs),

- solve Schrodinger equation

- Hermitian system
- energy conservation
- observable is real (e.g. $E \sim \omega$ is real)
- eigenfunctions are othornormal

- eigenfunctions form a complete set

- Hilbert space (inner product is well defined and square-integrable)

Summary

In QM (vs in QNMs),

- solve Schrodinger equation vs solve KG equation (but, Schrodinger equation like)
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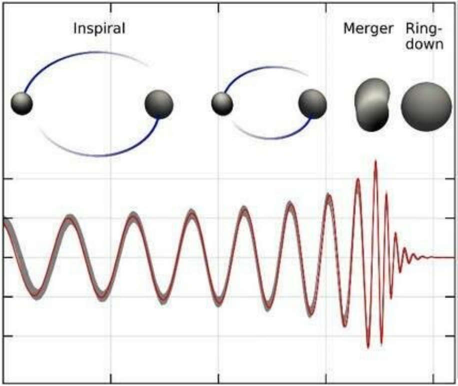
- eigenfunctions are othornormal vs eigenfunctions are not othornormal
- eigenfunctions form a complete set vs eigenfunctions do not form a complete set
- Hilbert space (inner product is well defined and square-integrable) vs no Hilbert space (KG inner product is not well defined and not square-integrable)
- Despite QNMs not being a Hermitian system themselves, numerous developments have been made to uitilize the powerful mathematical properties of Hermitian systems.

Black Hole ringdown

Ring-down phase



The newly formed, perturbed black hole relaxes to a stable state (usually a Kerr black hole characterized by its mass M and spin J) by emitting gravitational waves. These waves are described by the black hole's quasinormal modes (QNMs)

QNMs for gravitational wave

We need to consider metric perturbation

$$g_{\alpha\beta}(x) = \bar{g}_{\alpha\beta}(x) + h_{\alpha\beta}(x) \tag{80}$$

- for a Schwarzschild black hole
  - Odd-parity (axial) modes: Use Regge-Wheeler equation, related to off-diagonal metric components.
  - Even-parity (polar) modes: Use Zerilli equation, related to diagonal metric components.
- for a Kerr black hole : Teukolsky equation

They all take a form of

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}\Phi''(r) + \left(\omega^2 - V\right)\Phi(r) = 0 \tag{81}$$

Ringdown phase with QNMs for gravitational wave

However, as we have seen that *QNMs generally do not form a complete set in the usual mathematical sense due to their non-Hermitian nature* and the boundary conditions imposed. They can be part of a generalized basis when combined with other modes, such as:

$$\psi(t.x) = \sum_n c_n \psi_n(x) e^{-i\omega_n t} + \text{branch cut contribution} \tag{82}$$

[5]Kostas D. Kokkotas, Bernd G. Schmidt, "Quasi-Normal Modes of Stars and Black Holes",Volume 2, article number 2, (1999)

Ringdown phase with QNMs for gravitational wave

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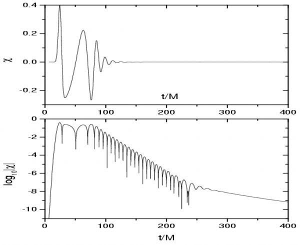

Figure 4: *The response of a Schwarzschild black hole as a Gaussian wave packet impinges upon it. The QNM signal dominates the signal after $t \approx 70M$ while at later times (after $t \approx 300M$) the signal is dominated by a power-law fall-off with time.*

5

[5] Kostas D. Kokkotas, Bernd G. Schmidt, "Quasi-Normal Modes of Stars and Black Holes",Volume 2, article number 2, (1999)

Thank you!