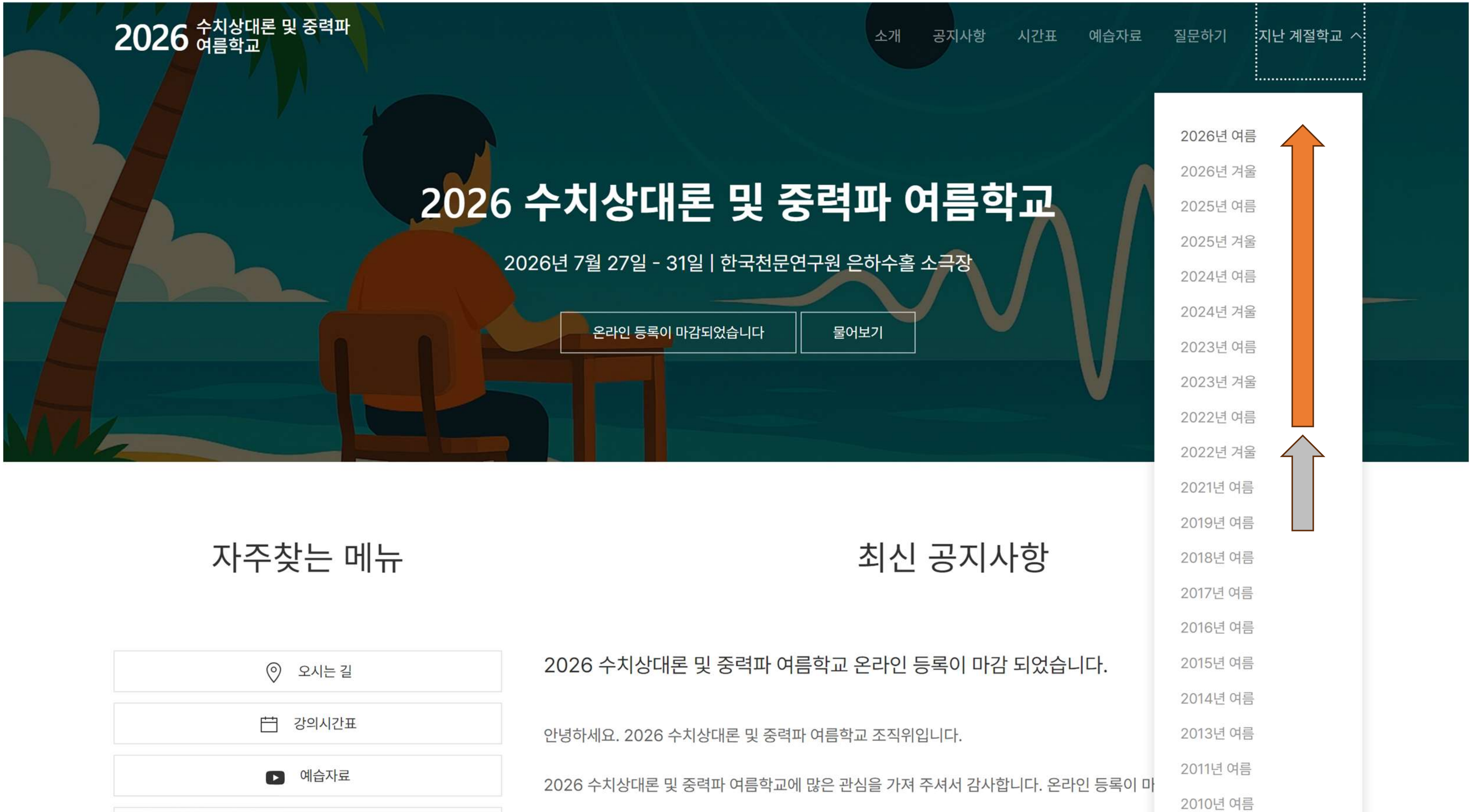
3+1 FORMALISM

Young-Hwan Hyun
2026.07.29. Wednesday
2026 GWNR Summer School
@KASI (Daejeon, Korea)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2026 수치상대론 및 중력파 여름학교
2026년 7월 27일 - 31일 | 한국천문연구원 은하수홀 소극장
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2011년 여름
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최신 공지사항
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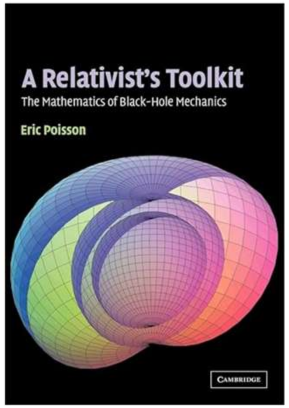
2026 수치상대론 및 중력파 여름학교 온라인 등록이 마감 되었습니다.
안녕하세요. 2026 수치상대론 및 중력파 여름학교 조직위입니다.
2026 수치상대론 및 중력파 여름학교에 많은 관심을 가져 주셔서 감사합니다. 온라인 등록이 마


A Relativist's Toolkit
The Mathematics of Black-Hole Mechanics
Eric Poisson
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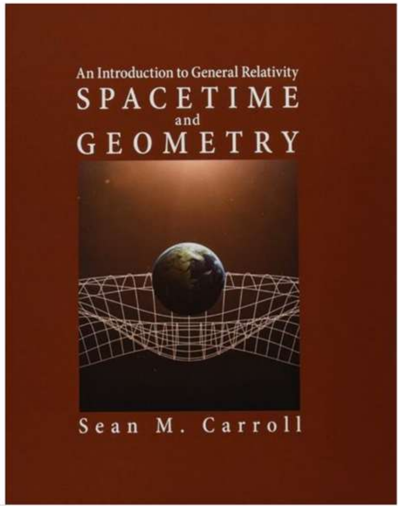
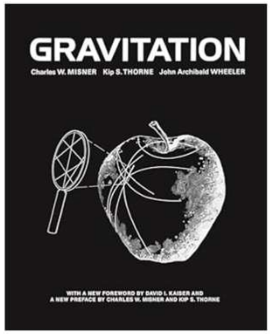
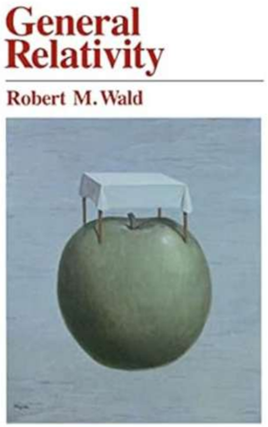
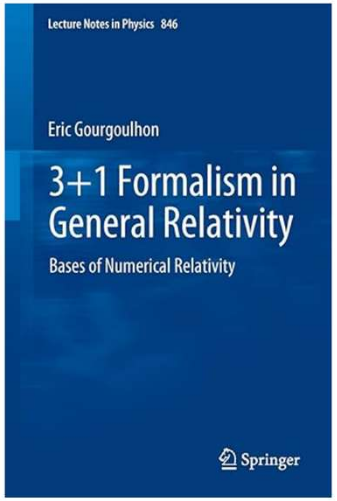

An Introduction to General Relativity
SPACETIME
and
GEOMETRY
Sean M. Carroll


GRAVITATION
Charles W. MISNER Kip S. THORNE John Archibald WHEELER

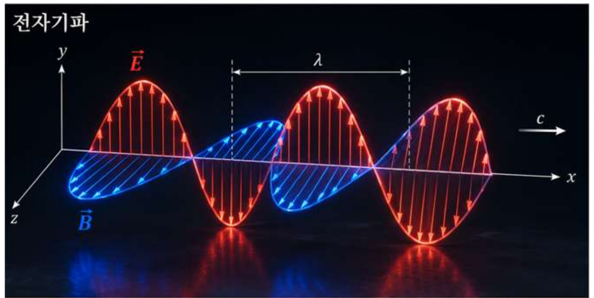

General
Relativity
Robert M. Wald


Lecture Notes in Physics 846
Eric Gourgoulhon
3+1 Formalism in
General Relativity
Bases of Numerical Relativity
Springer

References

$$\mathbf{F}_{12} = -G\,\frac{m_1 m_2}{r^2}\,\hat{\mathbf{r}}$$

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$


전자기파
y
E
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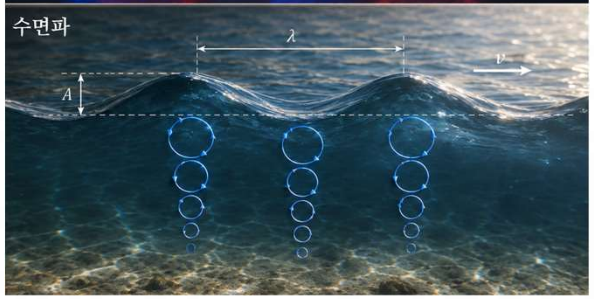
B


수면파
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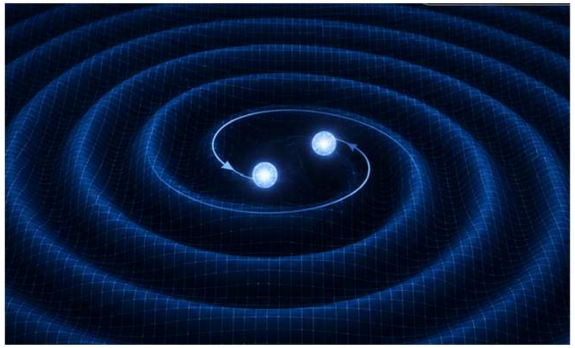

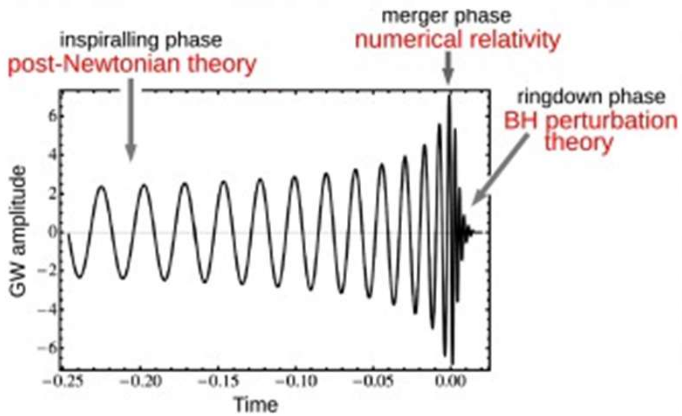
inspiralling phase
post-Newtonian theory
merger phase
numerical relativity
ringdown phase
BH perturbation theory
GW amplitude
Time
−0.25
−0.20
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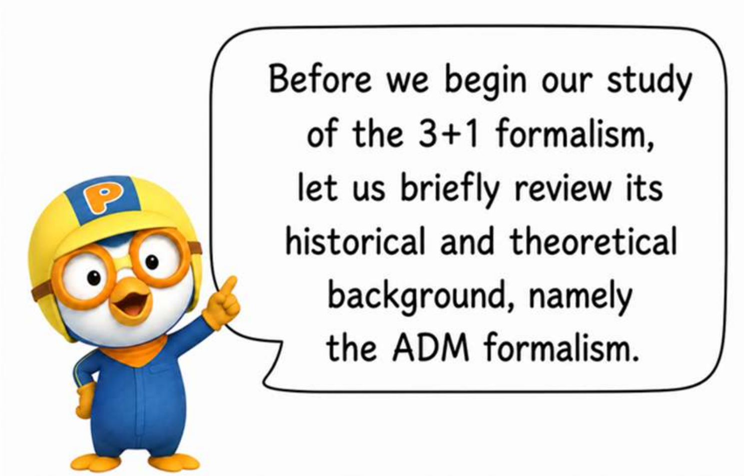
−0.05
0.00
6
4
2
0
−2
−4
−6


Before we begin our study
of the 3+1 formalism,
let us briefly review its
historical and theoretical
background, namely
the ADM formalism.

ADM formalism

PHYSICAL REVIEW VOLUME 116, NUMBER 5 DECEMBER 1, 1959

Dynamical Structure and Definition of Energy in General Relativity*†

R. Arnowitt, *Department of Physics, Syracuse University, Syracuse, New York*
S. Deser, *Department of Physics, Brandeis University, Waltham, Massachusetts*

AND

C. W. Misner‡ *Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*
(Received July 6, 1959)

The problem of the dynamical structure and definition of energy for the classical general theory of relativity is considered on a formal level. As in a previous paper, the technique used is the Schwinger action principle. Starting with the full Einstein Lagrangian in first order Palatini form, an action integral is derived in which the algebraic constraint variables have been eliminated. This action possesses a "Hamiltonian" density which, however, vanishes due to the differential constraints. If the differential constraints are then substituted into the action, the true, nonvanishing Hamiltonian of the theory emerges. From an analysis of the equations of motion and the constraint equations, the two pairs of dynamical variables which represent the two independent degrees of freedom of the gravitational field are explicitly exhibited. Four other variables remain in theory; these may be arbitrarily specified, any such specification representing a choice of coordinate frame. It is shown that it is possible to obtain truly canonical pairs of variables in terms of the dynamical and arbitrary variables. Thus a statement of the dynamics is meaningful only after a set of coordinate conditions have been chosen. In general, the true Hamiltonian will be time dependent even for an isolated gravitational field. There thus arises the notion of a preferred coordinate frame, i.e., that frame in which the Hamiltonian is conserved. In this special frame, on physical grounds, the Hamiltonian may be taken to define the energy of the field. In these respects the situation in general relativity is analogous to the parametric form of Hamilton's principle in particle mechanics.

Richard Arnowitt(-14,86), Stanley Deser(-23,92) and Charles Misner(-23,91)
at the *ADM-50: A Celebration of Current GR Innovation* conference
held in November 2009 to honor the 50th anniversary of their paper.

ADM formalism – Einstein-Hilbert Action

$$
\begin{aligned}
\delta S_{\mathrm{H}} &= \delta\left(\int d^n x \sqrt{-g}\,\frac{1}{16\pi G}R\right) \\
&= \int d^n x \frac{1}{16\pi G}\left[(\delta\sqrt{-g})R + \underbrace{\sqrt{-g}(\delta R_{\mu\nu})g^{\mu\nu}}_{\text{will be vanished by a surface integral}} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}\right] \\
&= \int d^n x \sqrt{-g}\frac{1}{16\pi G}\left(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\right)\delta g^{\mu\nu} + \int d^n x \sqrt{-g}\frac{1}{16\pi G}(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu} \\
&= \int d^n x \sqrt{-g}\frac{1}{16\pi G}G_{\mu\nu}\delta g^{\mu\nu} + \underbrace{\int d^n x \sqrt{-g}\frac{1}{16\pi G}(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu}}_{\to\text{boundary term}} \\
&= \delta S_{\text{Ei.eq.}} + \delta S_{\text{v.b.}}
\end{aligned}
$$

ς when the variation boundary term is canceled by an additional term,

$$
\to \int d^n x \sqrt{-g}\frac{1}{16\pi G}G_{\mu\nu}\delta g^{\mu\nu}
$$

ADM formalism – Gibbons-Hawking-York term

$$\boxed{S_{\mathrm{H}} = \int d^n x \sqrt{-g}\, \frac{1}{16\pi G} R}$$

$$\rightarrow \boxed{S_{\mathrm{G}} \equiv S_{\mathrm{H}} + S_{\text{G-H}} = \int_{\mathcal{M}} d^n x \sqrt{-g}\, \frac{1}{16\pi G} R + \int_{\partial\mathcal{M}} d^{n-1} \sqrt{\gamma} \frac{1}{16\pi G} \sigma_{\mathcal{K}} 2\mathcal{K}}$$

To have a finite action even in the flat case

$$\hookrightarrow \boxed{S^{\text{reg.}}_{\mathrm{G}} \equiv S_{\mathrm{H}} + S^{\text{regularized}}_{\text{G-H}} = \int_{\mathcal{M}} d^n x \sqrt{-g}\, \frac{1}{16\pi G} R + \int_{\partial\mathcal{M}} d^{n-1} \sqrt{\gamma} \frac{1}{16\pi G} \sigma_{\mathcal{K}} 2(\mathcal{K} - \mathcal{K}_0)}$$

ADM formalism – Action Decomposition

$$S_{\mathrm{H}} = \int_{\mathcal{V}} R\sqrt{-g}\, d^4x,\ \mathcal{V} := \cup_{t=t_1}^{t_2} \Sigma_t$$

$$\varsigma\ \ R = \widehat{R} + [-K^2 + K_{\mu\nu}K^{\mu\nu}] + 2\nabla_\mu(Kn^\mu) - \frac{2}{N}(\widehat{\nabla}_\mu\widehat{\nabla}^\mu N)]$$

$$\varsigma\ \ \sqrt{-g} = N\sqrt{\gamma}$$

$$\Rightarrow \boxed{\begin{aligned} S_{\mathrm{H}} &= S_{\mathrm{H,vol.}} + S_{\mathrm{H,surf.}} \\ &= \int_{t_1}^{t_2} \left\{ \int_{\Sigma_t} N(\widehat{R} - K^2 + K_{ij}K^{ij})\sqrt{\gamma}\, d^3x \right\} dt \\ &\quad - \int_{\mathcal{V}} 2(\widehat{\nabla}_i\widehat{\nabla}^i N)\sqrt{\gamma}\, d^4x + \int_{\mathcal{V}} 2\nabla_\mu(Kn^\mu)\sqrt{-g}\, d^4x \end{aligned}}$$

The gravitational Lagrangian density for $q = (\gamma_{ij}, N, \beta^i)$ is given by

$$\boxed{\mathcal{L}_{\mathrm{H,vol.}}(q,\dot{q}) = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) = N\sqrt{\gamma}[\widehat{R} + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{ij}K_{kl}]}$$

ADM formalism -Identifying Dynamical Variables

The gravitational Lagrangian density for $q = (\gamma_{ij}, N, \beta^i)$ is given by

$$\boxed{\mathcal{L}_{\mathrm{H,vol.}}(q,\dot{q}) = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) = N\sqrt{\gamma}[\widehat{R} + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{ij}K_{kl}]}$$

$$\begin{aligned}
K_{ij} &= \frac{1}{2}\mathcal{L}_{\boldsymbol{n}}P_{ij} = \frac{1}{2N}\mathcal{L}_{\boldsymbol{m}}P_{ij} = \frac{1}{2N}\mathcal{L}_{\boldsymbol{\partial_t}-\boldsymbol{\beta}}P_{ij} \\
&= \frac{1}{2N}(\mathcal{L}_{\boldsymbol{\partial_t}}P_{ij} - \mathcal{L}_{\boldsymbol{\beta}}P_{ij}) \\
&\varsigma\ \ \mathcal{L}_{\boldsymbol{\partial_t}}P_{ij} = (\partial_t)^\alpha\partial_\alpha P_{ij} + P_{\alpha j}\partial_i \cancel{\underbrace{(\partial_t)^\alpha}_{=\delta_t^\alpha}} + K_{i\alpha}\partial_j \cancel{\underbrace{(\partial_t)^\alpha}_{=\delta_t^\alpha}} = \frac{\partial P_{ij}}{\partial t} = \frac{\partial \gamma_{ij}}{\partial t} \\
&\varsigma\ \ \mathcal{L}_{\boldsymbol{\beta}}P_{ij} = \mathcal{L}_{\boldsymbol{\beta}}\gamma_{ij} = \beta^k\cancel{\widehat{\nabla}_k\gamma_{ij}} + \gamma_{kj}\widehat{\nabla}_i\beta^k + \gamma_{ik}\widehat{\nabla}_j\beta^k \\
&= \frac{1}{2N}(\dot{\gamma}_{ij} - \gamma_{kj}\widehat{\nabla}_i\beta^k - \gamma_{ik}\widehat{\nabla}_j\beta^k)
\end{aligned}$$

$$\boxed{\pi_N = \frac{\partial\mathcal{L}}{\partial\dot{N}} = 0, \qquad \pi_i = \frac{\partial\mathcal{L}}{\partial\dot{\beta}^i} = 0.}$$

ADM formalism – Conjugate momentum

The gravitational Lagrangian density for $q = (\gamma_{ij}, N, \beta^i)$ is given by

$$\boxed{\mathcal{L}_{\mathrm{H,vol.}}(q,\dot{q}) = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) = N\sqrt{\gamma}[\widehat{R} + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{ij}K_{kl}]}$$

$$\pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = \frac{\partial}{\partial \dot{\gamma}_{ij}}[N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij})]$$

$$K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - \gamma_{kj}\widehat{\nabla}_i\beta^k - \gamma_{ik}\widehat{\nabla}_j\beta^k)$$

$$= -2N\sqrt{\gamma}\,K\,\underbrace{\frac{\partial K}{\partial \dot{\gamma}_{ij}}}_{=\gamma^{ij}\frac{1}{2N}} + 2N\sqrt{\gamma}\,\underbrace{K^{kl}\frac{\partial K_{kl}}{\partial \dot{\gamma}_{ij}}}_{K^{ij}\frac{1}{2N}}$$

$$= \sqrt{\gamma}(-K\gamma^{ij} + K^{ij})$$

$$= \sqrt{\gamma}(K^{ij} - K\gamma^{ij})$$

ADM formalism – Hamiltonian construction

$$
\begin{aligned}
\mathcal{H}_{\text{H,vol.}} &= \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}_{\text{H,vol.}} \\
&= \sqrt{\gamma}(K^{ij} - K\gamma^{ij})\dot{\gamma}_{ij} - N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij})
\end{aligned}
$$

$\varsigma\ K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - \gamma_{kj}\widehat{\nabla}_i\beta^k - \gamma_{ik}\widehat{\nabla}_j\beta^k)$

$\quad \Lsh\ \dot{\gamma}_{ij} = 2NK_{ij} + \gamma_{kj}\widehat{\nabla}_i\beta^k + \gamma_{ik}\widehat{\nabla}_j\beta^k$

$$
\begin{aligned}
&= \sqrt{\gamma}\underbrace{(K^{ij} - K\gamma^{ij})(2NK_{ij}}_{\to 2N(K_{ij}K^{ij}-K^2)} + \gamma_{kj}\widehat{\nabla}_i\beta^k + \gamma_{ik}\widehat{\nabla}_j\beta^k) - N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) \\
&= -\sqrt{\gamma}[N(\widehat{R} + K^2 - K_{ij}K^{ij}) - 2(K^i{}_j - K\gamma^i{}_j)\widehat{\nabla}_i\beta^j] \\
&= -\sqrt{\gamma}[N(\widehat{R} + K^2 - K_{ij}K^{ij}) + 2\beta^j(\widehat{\nabla}_iK^i{}_j - \nabla_jK) - 2\widehat{\nabla}_i(K^i{}_j\beta^j - K\beta^i)] \\
&= -\sqrt{\gamma}[N\underbrace{(\widehat{R} + K^2 - K_{ij}K^{ij})}_{=C_0} - 2\beta^j\underbrace{(\widehat{\nabla}_jK - \nabla_iK^i{}_j)}_{=C_j}] + 2\sqrt{\gamma}\,\widehat{\nabla}_i(K^i{}_j\beta^j - K\beta^i)
\end{aligned}
$$

$$\Rightarrow \boxed{\mathcal{L}_{\text{H,vol.}} = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij})}$$

$$\Lsh\ \boxed{\mathcal{H}_{\text{H,vol.}} = -\sqrt{\gamma}(NC_0 - 2\beta^iC_i) + 2\sqrt{\gamma}\,\widehat{\nabla}_i(K^i{}_j\beta^j - K\beta^i)}$$

$$
\begin{array}{|rcl|}
\hline
\delta H/\delta N = 0 & \Longleftrightarrow & G_{nn} = 0, \\
\delta H/\delta\beta^i = 0 & \Longleftrightarrow & G_{n\hat{i}} = 0, \\
\dot{\gamma}_{ij} = \delta H/\delta\pi^{ij} & \Longleftrightarrow & K_{ij}\text{의 정의}, \\
\dot{\pi}^{ij} = -\delta H/\delta\gamma_{ij} & \Longleftrightarrow & G_{\hat{i}\hat{j}} = 0. \\
\hline
\end{array}
$$

ADM formalism – ADM Energy/Momentum

$$
\begin{aligned}
H_{\mathrm{G}}^{\mathrm{reg.}} &= H_{\mathrm{H,vol.}} + H_{\mathrm{G,surf.}}^{\mathrm{reg.}} \\
&= -\frac{1}{16\pi G}\int_{\Sigma_t} d^3x\sqrt{\gamma}\,(NC_0 - 2\beta^i C_i) \\
&\quad + \frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\;\cdot 2[r_i(K^i{}_j\beta^j - K\beta^i) - N(\kappa-\kappa_0)|_{S_t}]
\end{aligned}
$$

$$
\varsigma \quad H \xrightarrow[C_0\,=\,0\,=\,C_i]{\text{solution}} H_{\text{solution}}
$$

$$
\boxed{H_{\text{solution}} = \frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\;\cdot 2[r_i(K^i{}_j\beta^j - K\beta^i) - N(\kappa-\kappa_0)|_{S_t}]}
$$

where K_{ij} : extrinsic curvature on Σ_t,

κ : extrinsic curvature scalar on $\Sigma_t \cap S$

$$
\boxed{M_{\mathrm{ADM}} = -\frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\;\cdot 2(\kappa-\kappa_0)}
$$

Introduction to 3+1 Formalism

From Relativity to (3+1)

Special Relativity

- No absolute space or absolute time
- Space and time depend on the observer
- Simultaneity is frame-dependent
- The spacetime interval and causal structure are invariant

General Relativity

- Spacetime is dynamical and curved
- Gravity is encoded in geometry
- Matter shapes geometry; geometry governs motion

The Next Questions

Q1. How do we define physical quantities on a spatial slice?

Observers, geometry, mass, energy, and momentum

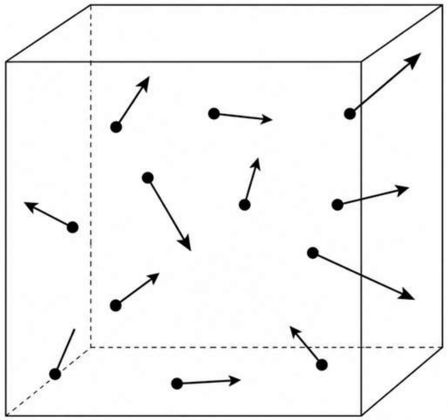
Q2. How does spatial geometry evolve, and can we predict its future?

Initial-value formulation and numerical simulation

4D spacetime → 3D spatial data + 1D time evolution

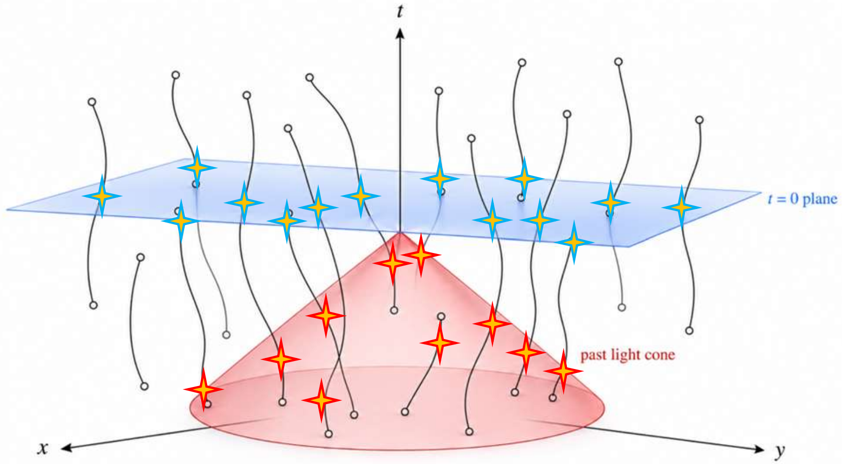
Q. How do we define physical quantities on a spatial slice?

Observers, geometry, mass, energy, and momentum



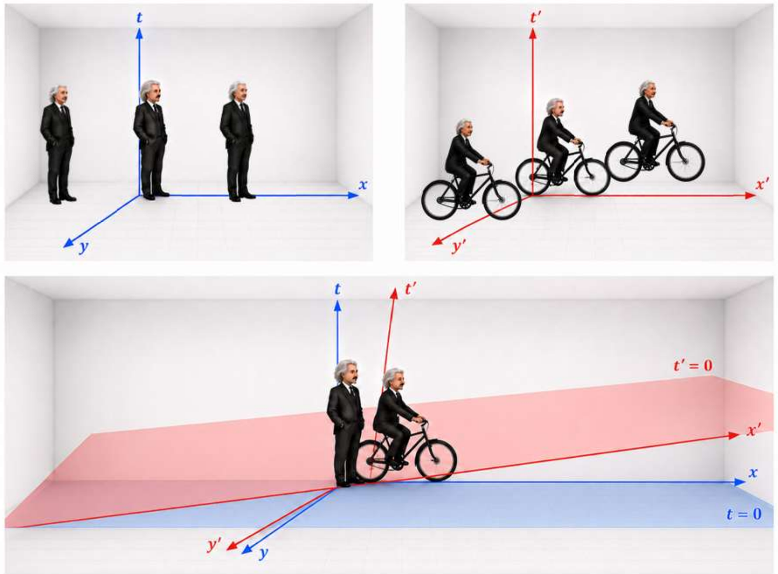
$$N_{total} = ?$$

$$P_{total} = ?$$

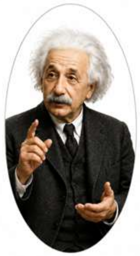


"There's only one space of 'now'."

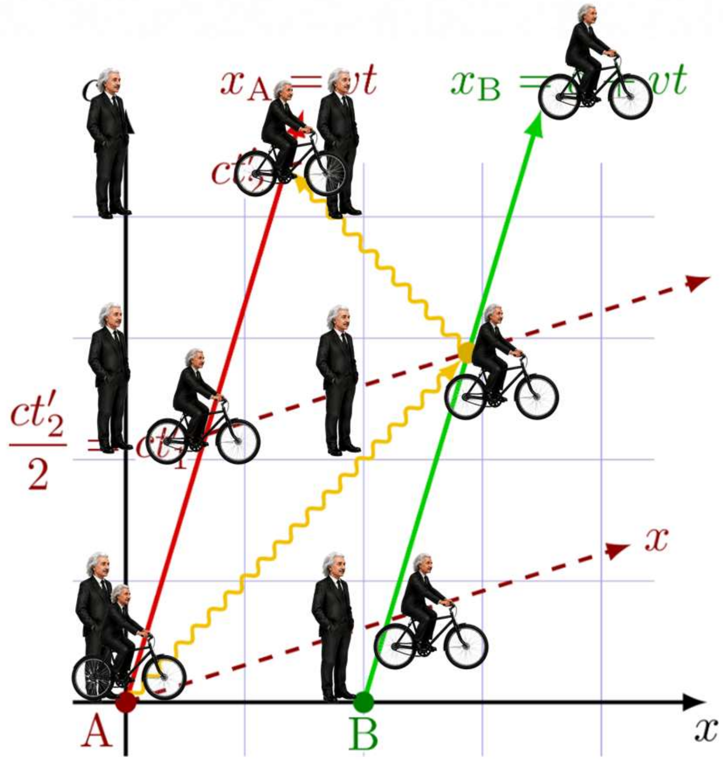
"Your space depends on how you move."



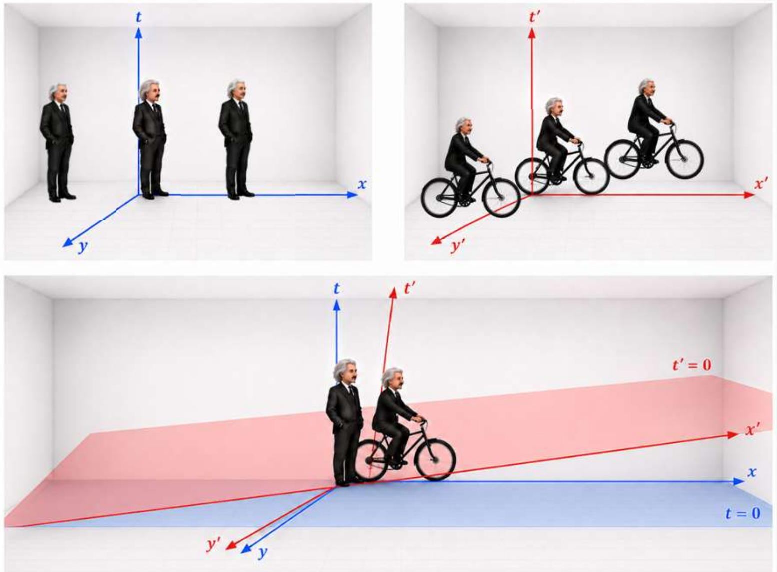
"Your space depends on how you move."



"In curved spacetime,
my 'now' is only local
—I need observers everywhere."



"Your space depends on how you move."



"In curved spacetime,
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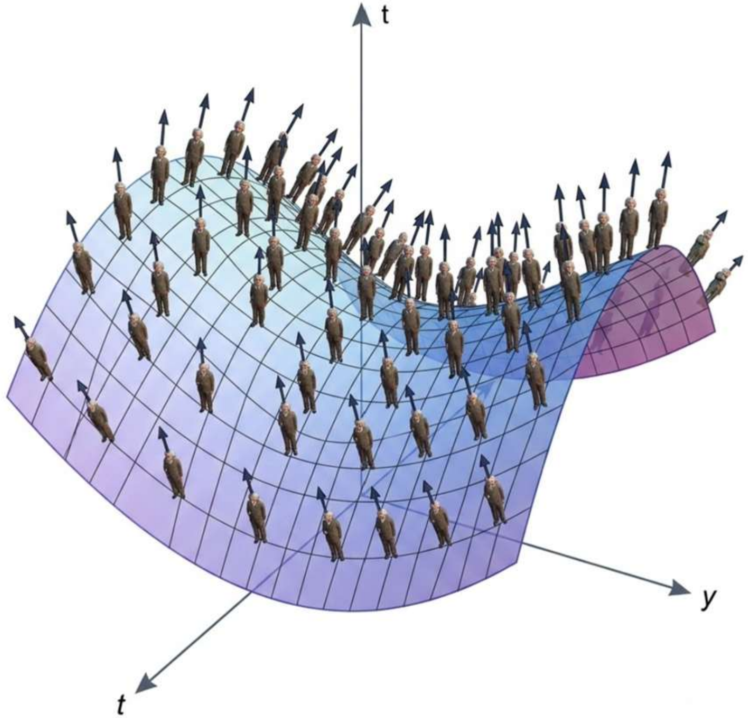
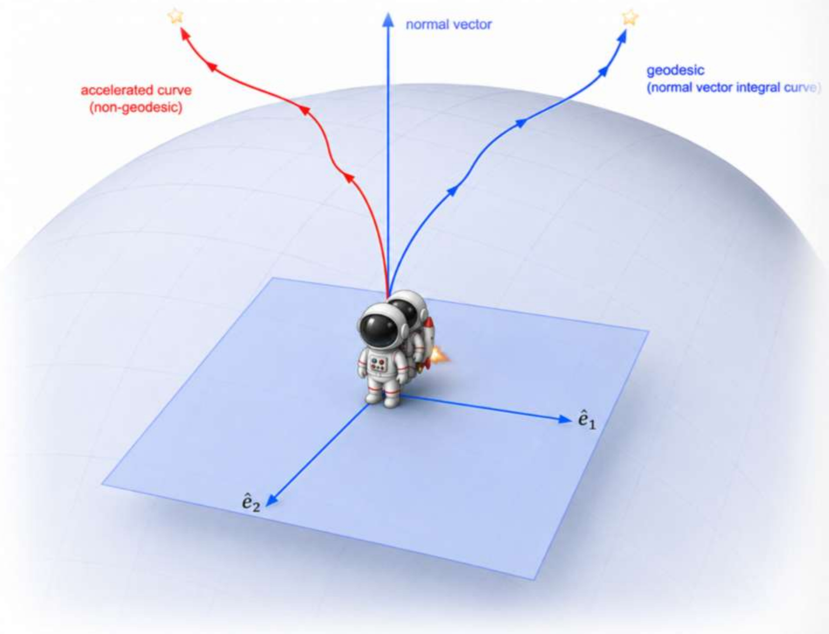


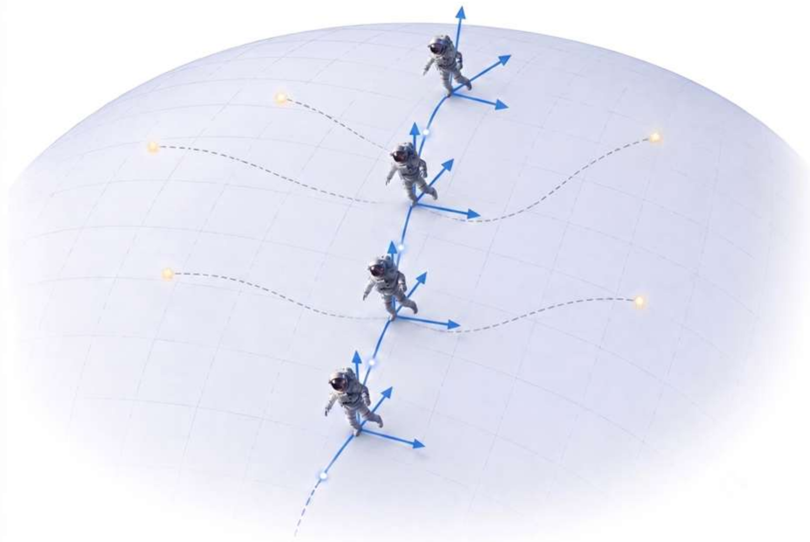
Riemann-Normal Coordinates



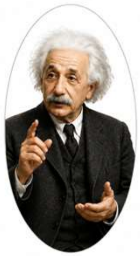
$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \quad \partial_\rho g_{\mu\nu}(p) = 0, \quad \Gamma^{\mu}{}_{\alpha\beta}(p) = 0$$

$$g_{\mu\nu}(X) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}(p) X^\alpha X^\beta + O(X^3)$$

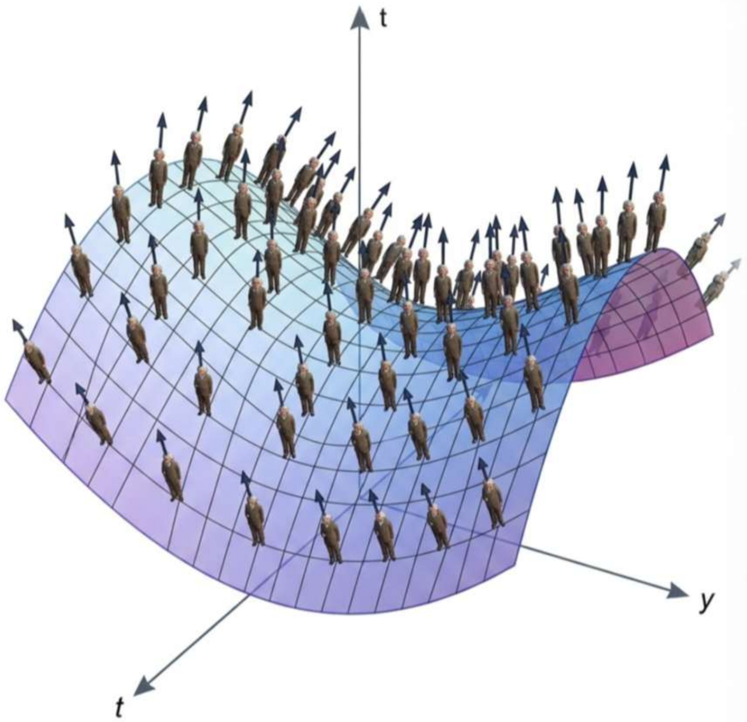
Fermi-Normal Coordinates



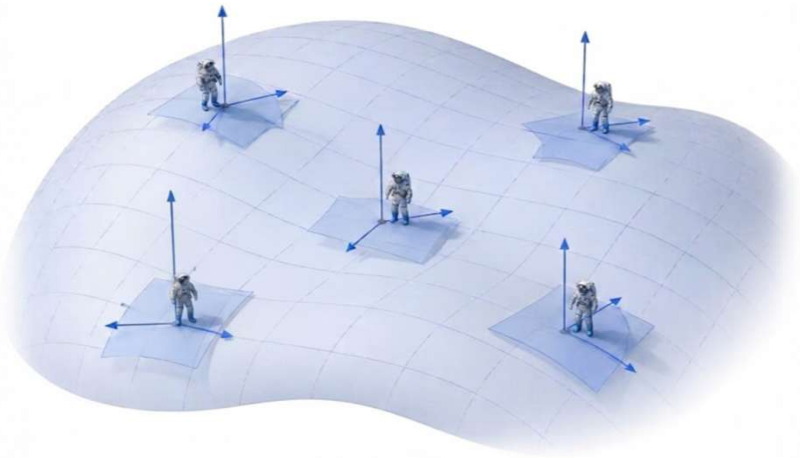
$$\begin{aligned}
g_{00} &= -(1 + a_i X^i)^2 - R_{0i0j} X^i X^j + O(|X|^3) \\
g_{0i} &= -\tfrac{2}{3} R_{0jik} X^j X^k + O(|X|^3) \\
g_{ij} &= \delta_{ij} - \tfrac{1}{3} R_{ikjl} X^k X^l + O(|X|^3)
\end{aligned}$$



*"In curved spacetime,
my 'now' is only local
—I need free-falling(?)
observers everywhere."*



Gaussian-Normal Coordinates



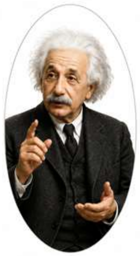
$$g_{0i}(0, x^k) = 0, \quad g_{00}(0, x^k) = -1, \quad g_{ij}(0, x^k) = h_{ij}(x^k)$$

where h_{ij} is the induced 3-metric on Σ.

$$ds^2 = -d\tau^2 + h_{ij}(\tau, x^k)\, dx^i dx^j$$

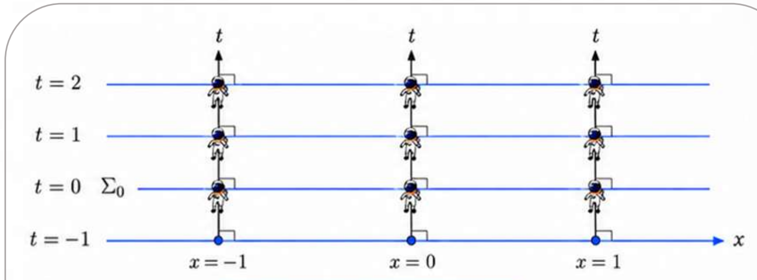
where $h_{ij}(\tau, x^k)$ is the induced 3-metric on Σ_τ.

Gaussian normal coordinates are generally well-defined only in a neighborhood of the initial hypersurface.

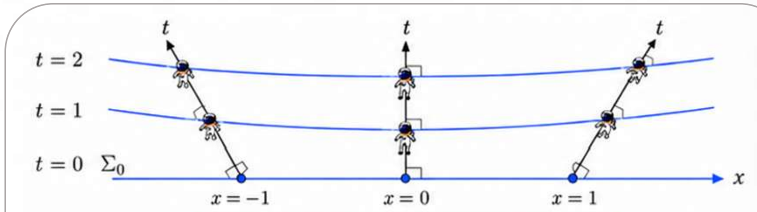


"OK. Now!
Let us see a *flat* spacetime
with geodesic observers!"

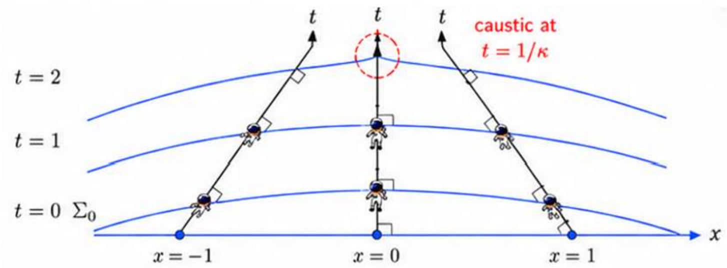
Coordinates with geodesic observers (lapse=1, shift=0)



$$ds^2 = -dt^2 + dx^2$$



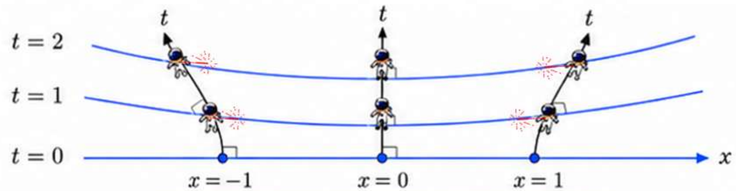
$$ds^2 = -dt^2 + (1+\kappa t)^2\, dx^2$$



$$ds^2 = -dt^2 + (1-\kappa t)^2\, dx^2, \quad 0 \le t < 1/\kappa$$

Geodesic normals can form caustics.
A spatially varying lapse allows accelerated normals.

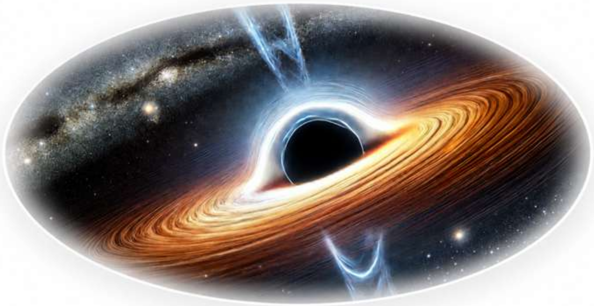
Geodesic observer → **Eulerian observers**



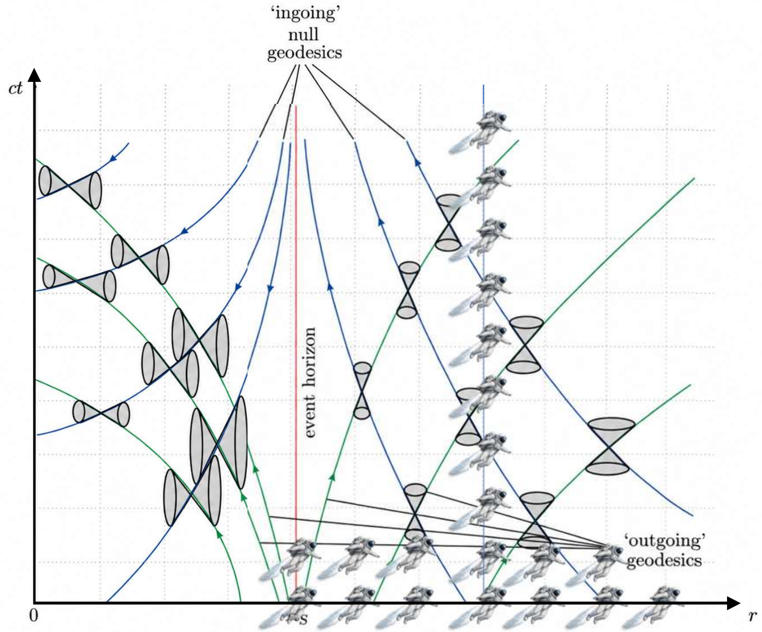
$$ds^2 = -(1+\kappa x)^2\, dt^2 + dx^2$$
$$R = 0, \quad a_x = \partial_x \ln N \neq 0$$

Eulerian observers allow us to define the next spatial slices safely, avoiding any coordinate problems.

"OK. Now!
Let us see a spacetime
with Eulerian observers!"

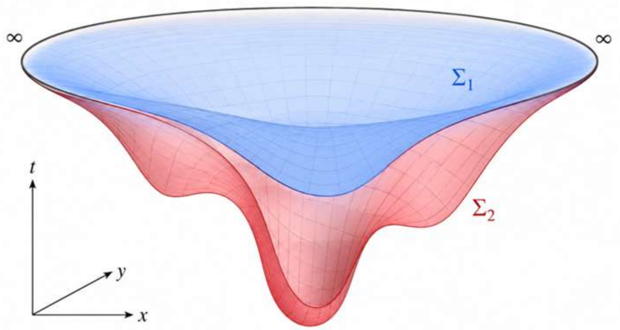


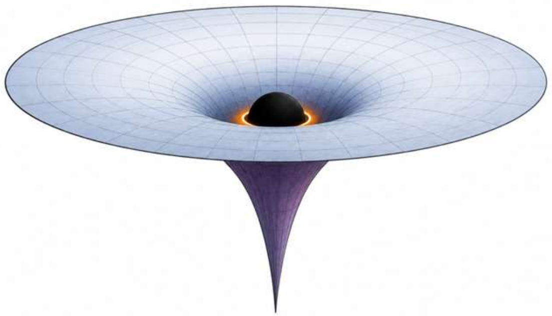
$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$



ADM Mass

$$\boxed{M_{\mathrm{ADM}} = -\frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\ \cdot 2(\kappa-\kappa_0)}$$





$$(1)\ ds^2 = -f(r)\mathrm{d}t^2 + \underbrace{\frac{1}{f(r)}\mathrm{d}r^2 + \overbrace{r^2 d\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2}^{S_t\ \to\ \sigma_{ab}}}_{\Sigma\ \to\ \gamma_{ij}},\ f(r) = 1-\frac{2GM}{r}$$

$$(2)\ r_i = \frac{1}{\sqrt{\gamma^{rr}}}\delta_i^r = \sqrt{\gamma_{rr}}\delta_i^r = (f^{-1/2},0,0),$$

$$r^i = \gamma^{ij}r_j = \left(\sqrt{1-\frac{2GM}{r}},0,0\right) = (f^{1/2},0,0)$$

$$(3)\ M_{\mathrm{ADM}} = -\frac{1}{16\pi G}\int_{S_t,r\to\infty} d^2\theta\sqrt{\sigma}\ \cdot 2(\kappa-\kappa_0)$$

$$(4)\ \kappa = \nabla_i r^i = \partial_i r^i + \Gamma^i_{ij}r^j = \partial_i r^i + \sum_i \frac{\gamma_{ii,j}}{2\gamma_{ii}}r^j = \partial_r r^r + \sum_i \frac{\gamma_{ii,r}}{2\gamma_{ii}}r^r$$

$$\varsigma\ r^r = \sqrt{f}$$

$$= \partial_r\sqrt{f} + \frac{1}{2}\left(\frac{\gamma_{rr,r}}{\gamma_{rr}} + \frac{\gamma_{\theta\theta,r}}{\gamma_{\theta\theta}} + \frac{\gamma_{\phi\phi,r}}{\gamma_{\phi\phi}}\right)\sqrt{f}$$

$$\varsigma\ (\gamma_{rr},\gamma_{\theta\theta},\gamma_{\phi\phi}) = (f^{-1},r^2,r^2\sin^2\theta)$$

$$= \frac{f_{,\cancel{r}}}{\cancel{2\sqrt{f}}} + \frac{1}{2}\left[\frac{\cancel{f^{-1}_{,r}}}{\cancel{f^{-1}}} + \underbrace{\frac{(r^2)_{,r}}{r^2} + \frac{(r^2\sin^2\theta)_{,r}}{r^2\sin^2\theta}}_{=\frac{4}{r}}\right]\sqrt{f}$$

$$= \frac{2}{r}\sqrt{f} = \frac{2}{r}\sqrt{1-\frac{2GM}{r}}$$

$$\xrightarrow{\sqrt{1-2\epsilon}\ \approx\ 1-\epsilon} \frac{2}{r}\left(1-\frac{GM}{r}+\mathcal{O}(r^{-2})\right) = \frac{2}{r}-\frac{2GM}{r^2}+\mathcal{O}(r^{-3})$$

$$(5)\ \kappa-\kappa_0 = -\frac{2GM}{r^2}+\mathcal{O}(r^{-3})$$

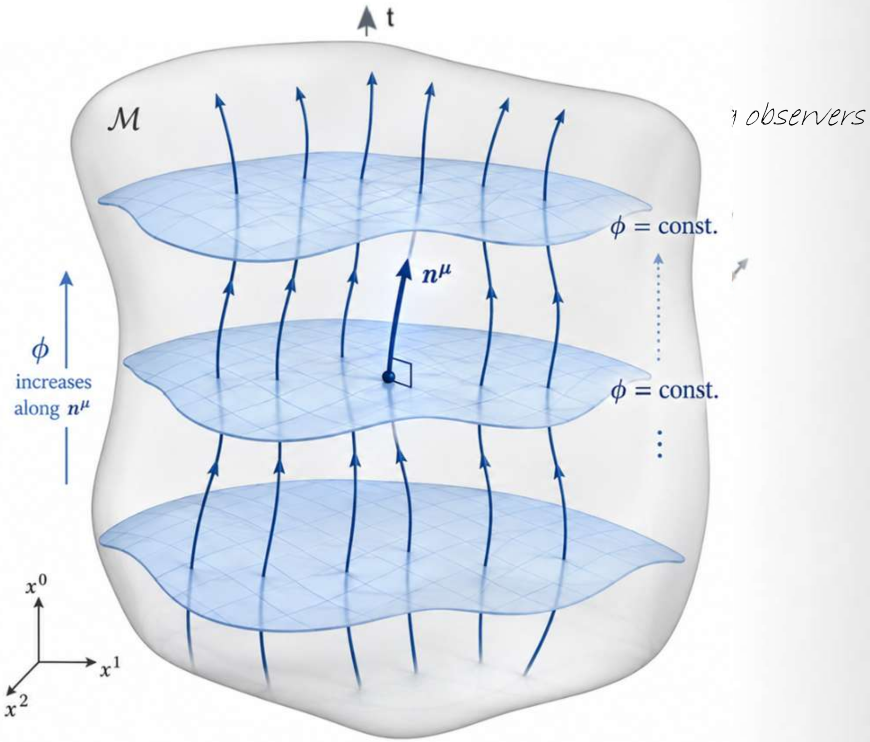
$$(6)\ M_{\mathrm{ADM}} = -\frac{1}{16\pi G}\int_{S_t,r\to\infty} d^2\theta\sqrt{\sigma}\ \cdot 2(\kappa-\kappa_0)$$

$$= -\frac{1}{16\pi G}\int_{S_t,r\to\infty} d^2\theta\sqrt{\sigma}\left(-\frac{4GM}{r^2}+\mathcal{O}(r^{-3})\right)$$

$$= \cancel{-}\frac{1}{\cancel{16\pi G}}\lim_{r\to\infty}\cancel{4\pi r^2}\cdot\left(\cancel{-}\frac{\cancel{4G}M}{\cancel{r^2}} + \cancel{\mathcal{O}(r^{-3})}\right)$$

$$= M$$

"OK. Now!
Let us split a spacetime
with Eulerian observers!"



Frobenius Theorem

Hypersurface orthogonality and foliation

- A foliation can be defined by the level sets of a scalar field $\phi(x^\mu) = \text{const.}$
- If a vector field is hypersurface-orthogonal, then locally $n_\mu \propto \nabla_\mu \phi$.
- Its integral curves form a non-crossing congruence normal to the slices.

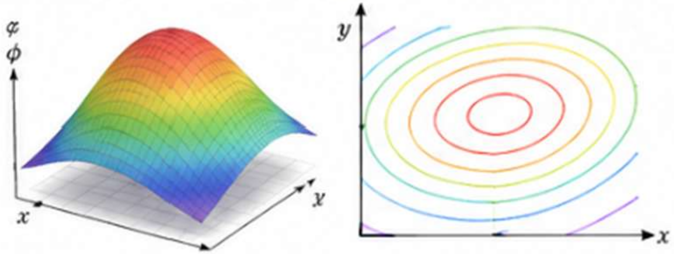
$$n_{[\mu} \nabla_\nu n_{\rho]} = 0$$

$$n_\mu = -N \, \nabla_\mu \phi$$

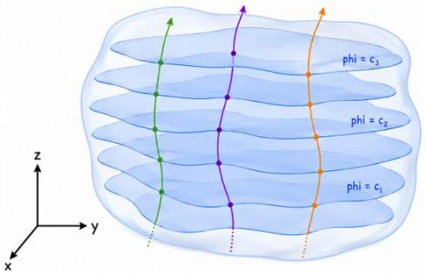
Frobenius condition $\Leftrightarrow$ zero twist / hypersurface orthogonality.

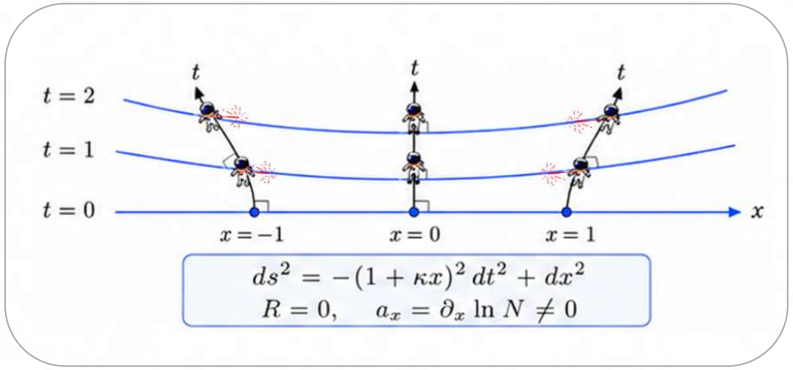
Hypersurface and folations

Foliation of a 2D Space



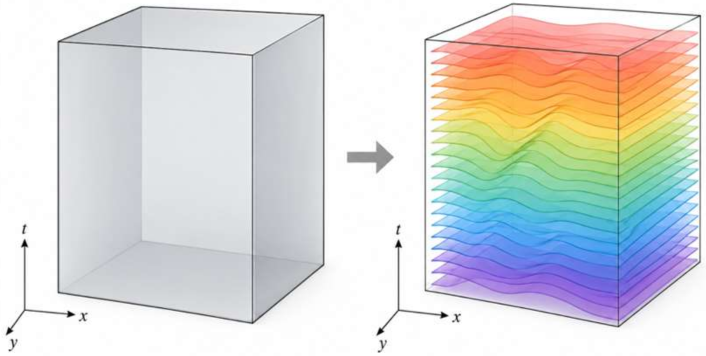
Foliation of a 3D Space



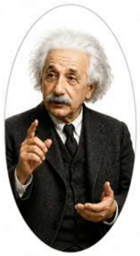


Foliation of a 4D Spacetime

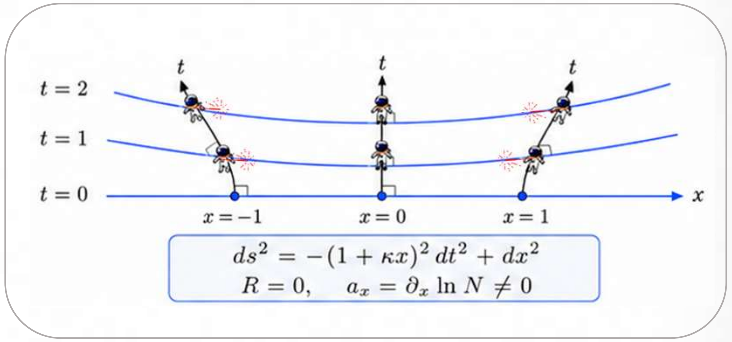
$$\mathcal{M} \simeq \mathbb{R} \times \Sigma$$



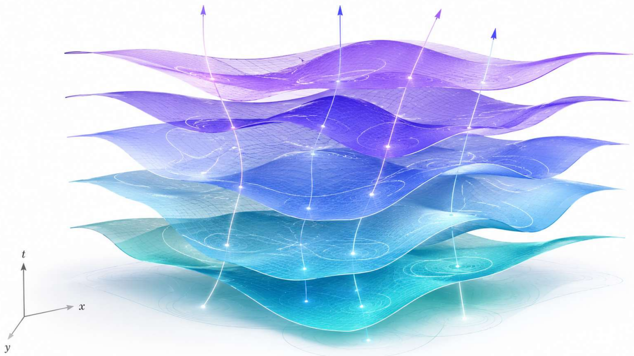
$$dt \neq 0, \qquad \nabla_\mu t\, \nabla^\mu t < 0.$$

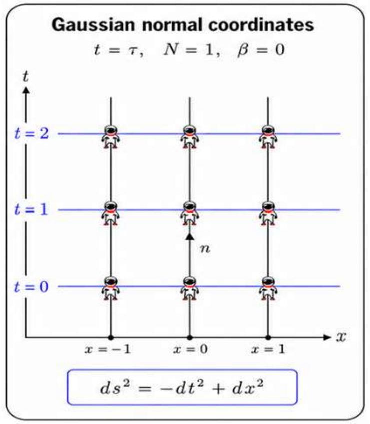


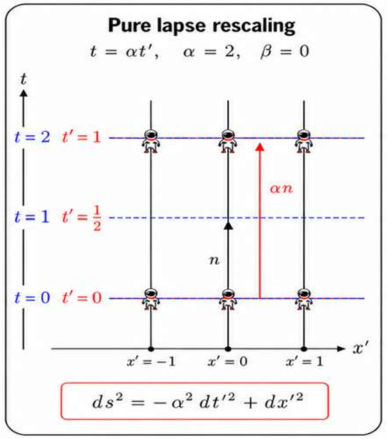
"All right! The hypersurfaces orthogonal to our Eulerian observers now foliate spacetime. What else do we need to complete the coordinate system?"

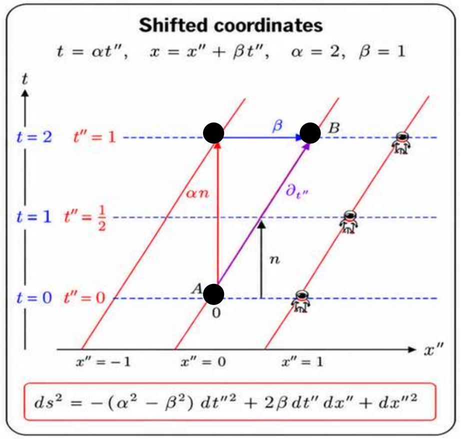


Eulerian observers allow us to define the next spatial slices safely, avoiding any coordinate problems.









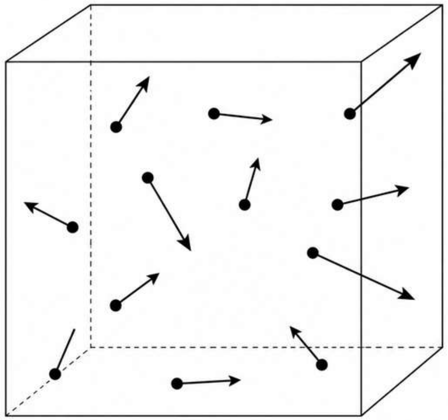
After fixing the foliation, the geometry of the slices is unchanged.
What remains is only time relabeling and threading.
The lapse changes the clock labeling,
and the shift chooses which observers are coordinate-stationary.

Q. How do we define physical quantities on a spatial slice?

Observers, geometry, mass, energy, and momentum

total

local?

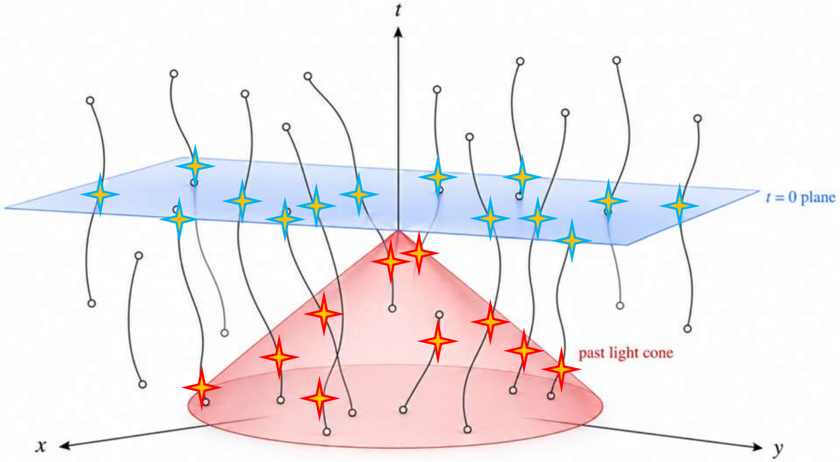


$$N_{total} = ?$$

$$P_{total} = ?$$

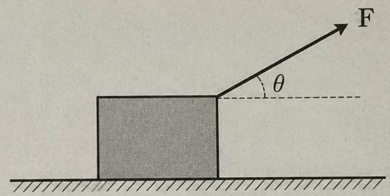
M_{ADM}

$T_{\mu\nu}U^{\mu}U^{\nu} = \rho_E$



Projection of Tensors

16. 그림과 같이 물체에 크기가 F 인 힘을 수평면과 θ 의 각을 이루며 위쪽 방향으로 작용하였다. 이 힘 F 를 수평 방향과 연직 방향 성분으로 분해할 때, 옳은 설명만을 <보기>에서 있는 대로 고른 것은?



<보기>

ㄱ. 수평 방향 성분의 크기는 $F\cos\theta$ 이다.

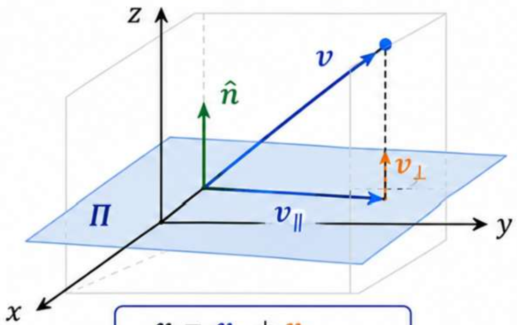
ㄴ. 연직 방향 성분의 크기는 $F\sin\theta$ 이다.

ㄷ. θ가 $0°$ 일 때, 연직 방향 성분의 크기는 0 이다.

① ㄱ ② ㄴ ③ ㄱ, ㄷ ④ ㄴ, ㄷ ⑤ ㄱ, ㄴ, ㄷ

Projection of Tensors

3D Euclidean space

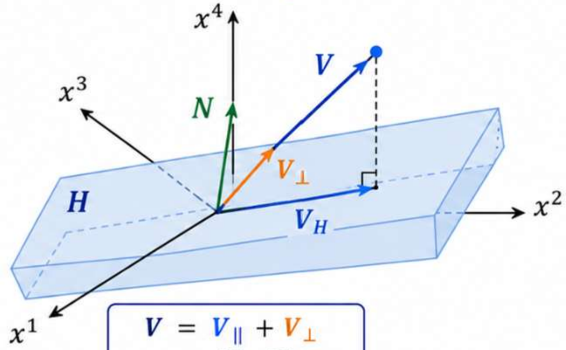


$$v = v_\parallel + v_\perp$$
$$v_\perp = \frac{(v\cdot\hat{n})}{(\hat{n}\cdot\hat{n})}\cdot\hat{n}$$
$$v_\parallel = v - v_\perp$$

$$P^i{}_j = \delta^i{}_j - \hat{n}^i\hat{n}_j$$
$$(v_\parallel)^i = P^i{}_j\, v^j$$
$$(v_\perp)^i = (\hat{n}_j\, v^j)\,\hat{n}^i$$

P projects onto the plane Π.

4D vector space



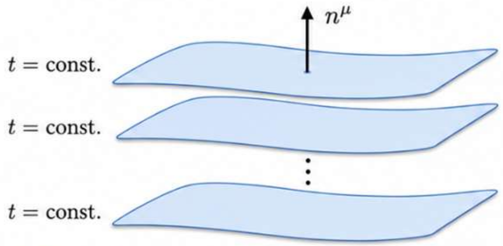
$$V = V_\parallel + V_\perp$$
$$V_\perp = \frac{(V\cdot N)}{(N\cdot N)}\cdot N$$
$$V_\parallel = V - V_\perp$$

$H = \{\, X \mid N\cdot X = 0\}$

$$h^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu \qquad (n^\mu n_\mu = -1)$$
$$V^\mu_\parallel = h^\mu{}_\nu\, V^\nu$$
$$V^\mu_\perp = -\,(n_\nu\, V^\nu)\, n^\mu$$

h projects onto the hypersurface Σ.

Projection of Energy-Momentum Tensor



In Gaussian normal coordinates

$$ds^2 = -dt^2 + \gamma_{ij}\, dx^i dx^j$$

$$g_{\mu\nu} = -n_\mu n_\nu + \gamma_{\mu\nu}$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

Using Gaussian normal coordinates makes the geometry transparent, but the projection tensor $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ *is a geometric object and is valid in any coordinates.*

Projection tensor

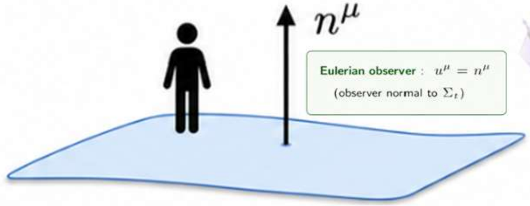
$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$$ (symmetric)

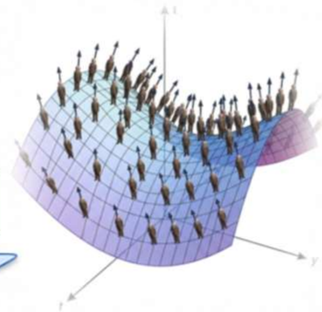
$$\gamma^\mu{}_\nu \equiv \delta^\mu{}_\nu + n^\mu n_\nu$$ (mixed)

Properties

- $\gamma^\mu{}_\nu n^\nu = 0$ (projects any vector onto Σ_t)
- $\gamma^\mu{}_\alpha \gamma^\alpha{}_\nu = \gamma^\mu{}_\nu$ (idempotent)
- $\gamma^\mu{}_\mu = 3$ (trace = spatial dimension)

Eulerian observer





Energy density

$$\boxed{\rho = T_{\mu\nu} n^\mu n^\nu}$$

(energy measured by the Eulerian observer)

Momentum density / energy flux

$$\boxed{j_\alpha \equiv - T_{\mu\nu} n^\mu P^\nu{}_\alpha}$$

(spatial energy flux seen by the Eulerian observer)

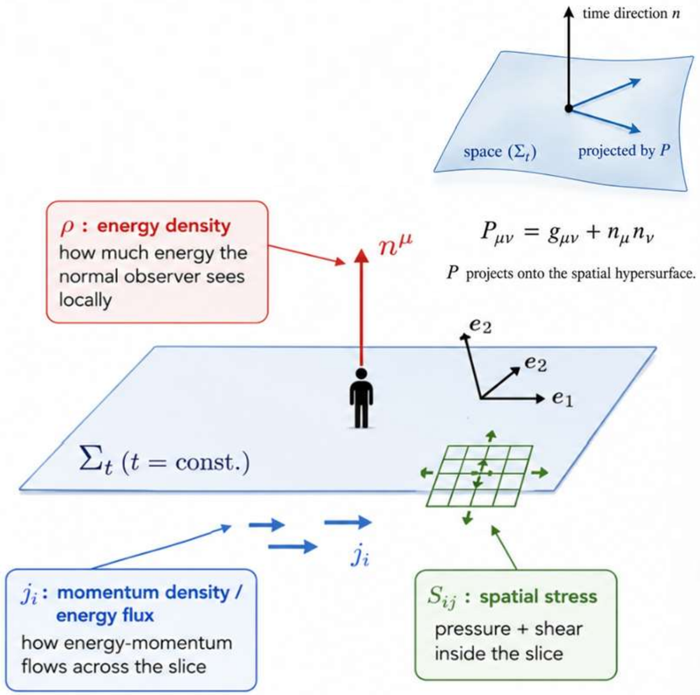
where $P^\nu{}_\alpha \equiv \gamma^\nu{}_\alpha$

Spatial projection (stress seen on Σ_t)

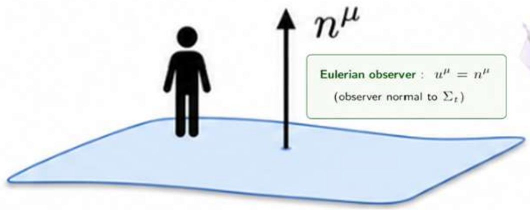
$$\boxed{S_{\mu\nu} = \gamma_\mu{}^\alpha \gamma_\nu{}^\beta T_{\alpha\beta}}$$

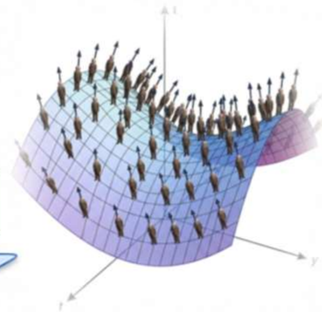
(symmetric, $S_{\mu\nu} n^\nu = 0$)

Construction of Energy-Momentum Tensor



Eulerian observer





$$
\begin{aligned}
T_{\mu\nu} &= \rho\, n_\mu n_\nu \\
&+ n_\mu j_\nu \\
&+ j_\mu n_\nu \\
&+ S_{\mu\nu}
\end{aligned}
$$

Intuition

- ρ : what energy the Eulerian observer measures
- j : how energy-momentum flows across space
- S : pressure and shear inside the slice

second index ν

first index μ	along n	projected by P
along n	$\rho = T_{\alpha\beta}n^\alpha n^\beta$ energy density	$j_\nu = -P_\nu{}^\beta T_{\alpha\beta} n^\alpha$ momentum / energy flux
projected by P	$j_\mu = -P_\mu{}^\alpha T_{\alpha\beta} n^\beta$ momentum / energy flux	$S_{\mu\nu} = P_\mu{}^\alpha P_\nu{}^\beta T_{\alpha\beta}$ spatial stress

time-time, time-space, space-time, space-space pieces

How the observer interprets $T_{\mu\nu}$

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & \rho & j_1 & j_2 & j_3 \\
1 & j_1 & S_{11} & S_{12} & S_{13} \\
2 & j_2 & S_{21} & S_{22} & S_{23} \\
3 & j_3 & S_{31} & S_{32} & S_{33}
\end{array}
$$

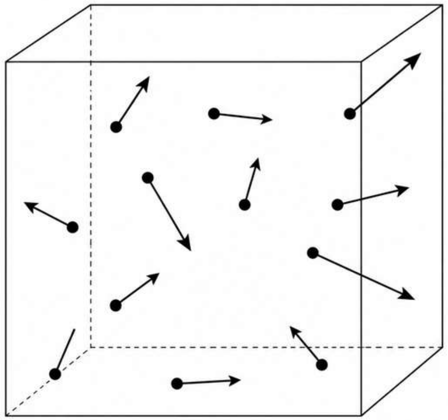
energy density

momentum density / energy flux

pressure + shear

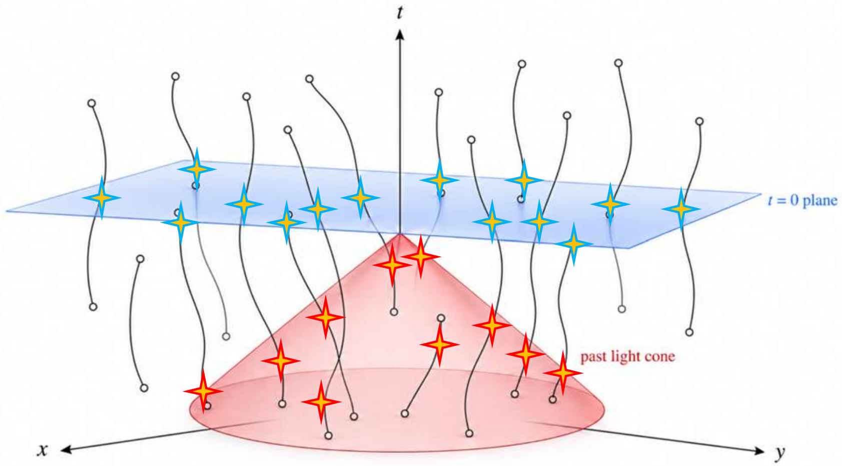
Q How do we define physical quantities on a spatial slice?

Observers, geometry, mass, energy, and momentum



$$N_{total} = ?$$

$$P_{total} = ?$$



Q: What physical concepts can be defined after performing a foliation?

A:

1. **Eulerian observers** — n^μ
2. **Geometric quantities** — $\alpha,\ \beta^i,\ \gamma_{ij}$
3. **Measured quantities** — $E = -p_\mu n^\mu$, $p_i = \gamma_i{}^\mu\, p_\mu$
4. **Global charges** — $E_{\text{ADM}},\ P_i^{\text{ADM}}$ (with suitable boundaries)

From Relativity to (3+1)

Special Relativity

- No absolute space or absolute time
- Space and time depend on the observer
- Simultaneity is frame-dependent
- The spacetime interval and causal structure are invariant

General Relativity

- Spacetime is dynamical and curved
- Gravity is encoded in geometry
- Matter shapes geometry; geometry governs motion

The Next Questions

Q1. How do we define physical quantities on a spatial slice?

Observers, geometry, mass, energy, and momentum

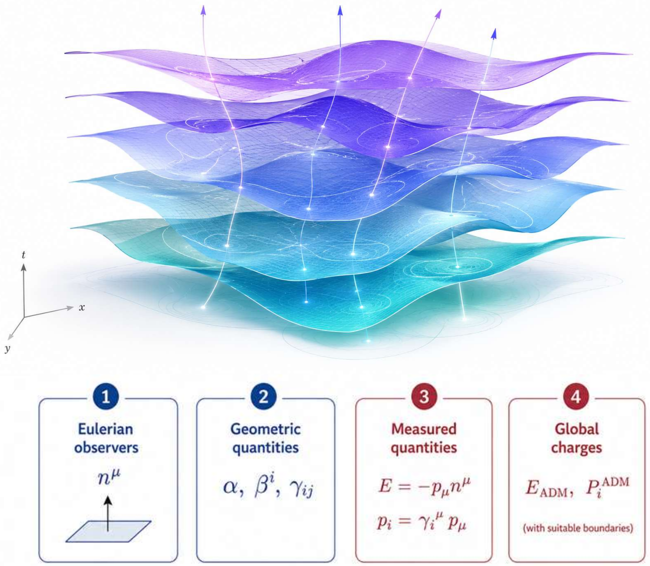
Q2. How does spatial geometry evolve, and can we predict its future?

Initial-value formulation and numerical simulation

4D spacetime → 3D spatial data + time evolution

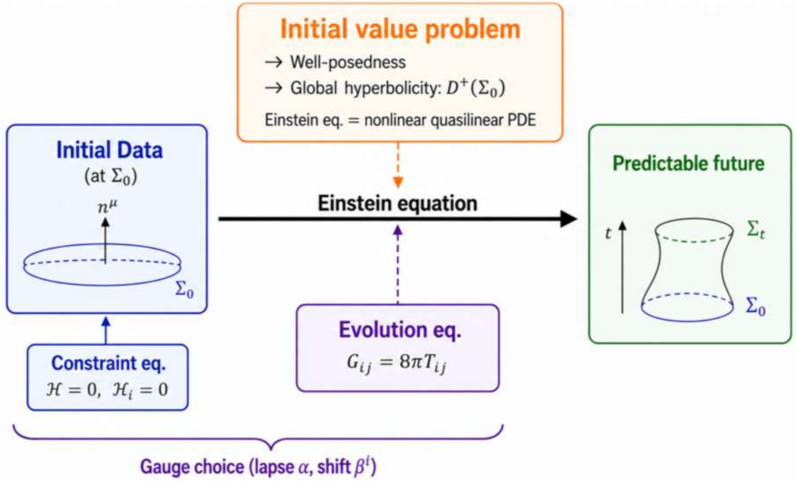
Q. How does spatial geometry evolve, and can we predict its future?

Initial-value formulation and numerical simulation



- Can any initial data be chosen?
- How do the initial data evolve in time?
- Does this evolution lead to a predictable future?

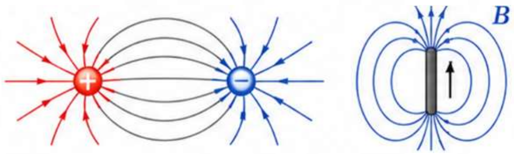
- Equation of what?
- What kind of differential eq.?



- Physical Degree of freedom?
- How many equations?

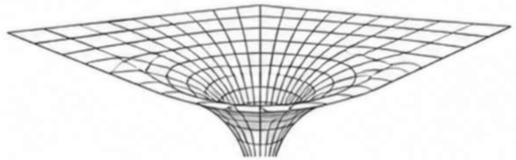
Einstein Equation

$$m\frac{d^2x^i}{dt^2} = F^i$$



$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$



$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$g_{\mu\nu}(x)$$

Einstein Equation: Nonlinear PDE

```
EinsteinCD[-μ, -ν] // ToRicci // RiemannToChristoffel // NoScalar // ChristoffelToGradMetric[#, g] & // Expand //
ToCanonical
```

$$
\begin{aligned}
&-\frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\gamma\delta} - \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\alpha g_{\mu\nu} + \boxed{\frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\mu g_{\nu\alpha}} + \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\nu\delta} \partial_\gamma g_{\mu\alpha} + \\
&\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\delta g_{\beta\gamma} + \frac{1}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta_1\delta_2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_\delta g_{\delta_1\delta_2} + \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\gamma g_{\mu\alpha} \partial_\delta g_{\nu\beta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta \partial_\beta g_{\alpha\gamma} + \\
&\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta \partial_\gamma g_{\alpha\beta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} g^{\delta_1\delta_2} g_{\mu\nu} \partial_\delta g_{\beta\delta_2} \partial_{\delta_1} g_{\alpha\gamma} - \frac{3}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta_1\delta_2} g_{\mu\nu} \partial_{\delta_1} g_{\alpha\gamma} \partial_{\delta_2} g_{\beta\delta} + \\
&\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta_1\delta_2} g_{\mu\nu} \partial_\beta g_{\alpha\gamma} \partial_{\delta_2} g_{\delta\delta_1} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta_1\delta_2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_{\delta_2} g_{\delta\delta_1} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\mu g_{\nu\alpha} - \\
&\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\mu g_{\nu\alpha} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\mu g_{\alpha\gamma} \partial_\nu g_{\beta\delta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} \partial_\nu \partial_\mu g_{\alpha\beta}
\end{aligned}
$$

$$
= \frac{8\pi G}{c^4} T_{\mu\nu}.
$$

$$
\begin{aligned}
\frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\alpha\partial_\mu g_{\beta\nu} &= \frac{1}{2}g^{00}\partial_0\partial_\mu g_{0\nu} + \frac{1}{2}g^{01}\partial_0\partial_\mu g_{1\nu} + \frac{1}{2}g^{02}\partial_0\partial_\mu g_{2\nu} + \frac{1}{2}g^{03}\partial_0\partial_\mu g_{3\nu} \\
&+ \frac{1}{2}g^{10}\partial_1\partial_\mu g_{0\nu} + \frac{1}{2}g^{11}\partial_1\partial_\mu g_{1\nu} + \frac{1}{2}g^{12}\partial_1\partial_\mu g_{2\nu} + \frac{1}{2}g^{13}\partial_1\partial_\mu g_{3\nu} + \frac{1}{2}g^{20}\partial_2\partial_\mu g_{0\nu} + \frac{1}{2}g^{21}\partial_2\partial_\mu g_{1\nu} \\
&+ \frac{1}{2}g^{22}\partial_2\partial_\mu g_{2\nu} + \frac{1}{2}g^{23}\partial_2\partial_\mu g_{3\nu} + \frac{1}{2}g^{30}\partial_3\partial_\mu g_{0\nu} + \frac{1}{2}g^{31}\partial_3\partial_\mu g_{1\nu} + \frac{1}{2}g^{32}\partial_3\partial_\mu g_{2\nu} + \frac{1}{2}g^{33}\partial_3\partial_\mu g_{3\nu}
\end{aligned}
$$

$$
g^{-1}\partial^2 g + g^{-2}(\partial g)^2
$$

Einstein Equation: Nonlinear, Quasilinear, and Hyperbolic

1. Schematic structure

$$R_{\mu\nu} \sim \boxed{g^{-1}\,\partial^2 g} + \boxed{g^{-2}(\partial g)(\partial g)}$$

- $g^{-2}(\partial g)(\partial g)$: nonlinear term (quadratic in first derivatives)
- $g^{-1}\,\partial^2 g$: contains the highest derivatives

2. Why quasilinear?

Highest derivatives appear linearly:

$$g^{\alpha\beta}(g)\,\partial_\alpha\partial_\beta g_{\mu\nu}$$

The coefficient $g^{\alpha\beta}$ depends on g, so the equation is nonlinear, but the second derivatives $\partial^2 g$ appear only linearly.

$\Rightarrow$ **quasilinear PDE**

3. Harmonic gauge

$$\boxed{\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0} \quad \text{and} \quad \boxed{\Box_g x^\mu = 0}$$

$$R_{\mu\nu} = -\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\partial_\beta g_{\mu\nu} + \boxed{\partial_{(\mu}\Gamma_{\nu)}} + Q_{\mu\nu}(g,\partial g)$$

In harmonic gauge, the gauge-dependent second-derivative term $\partial_{(\mu}\Gamma_{\nu)}$ is removed from the principal part.

4. Reduced equation and hyperbolicity

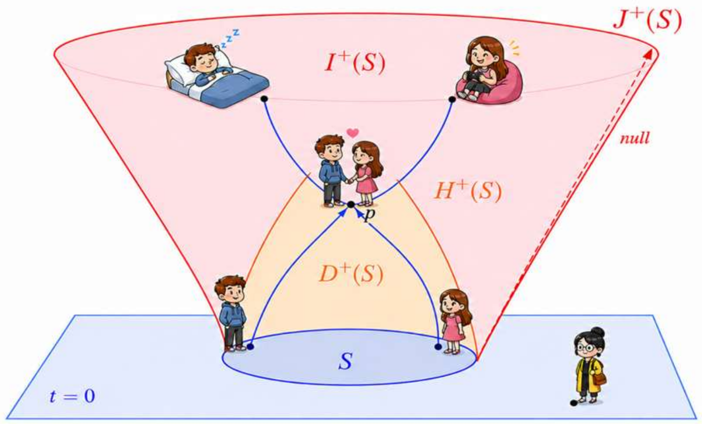
$$\boxed{g^{\alpha\beta}\,\partial_\alpha\partial_\beta g_{\mu\nu} = \mathcal{N}_{\mu\nu}(g,\partial g)}$$

- $\mathcal{N}_{\mu\nu}(g,\partial g)$ contains no second derivatives of g.
- The principal part is the wave operator with Lorentzian metric $g^{\alpha\beta}$. Its principal part has a wave-like form, e.g. $\partial_t^2 - \partial_x^2$, so the reduced system is hyperbolic.

Finite propagation speed

- Disturbances in the metric do not affect every point instantly.
- They propagate causally at a finite speed determined by the local light cones (no instantaneous action at a distance).

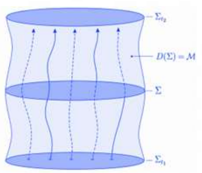
Causal Structure



Cauchy surface

A spacelike hypersurface that intersects
every inextendible causal curve exactly once.

Ex) $D(S)=N$, $D(\Sigma) = \mathcal{M}$



Globally hyperbolic spacetime

A spacetime that admits a Cauchy surface.

$N, \mathcal{M} \simeq \Sigma \times \mathbb{R}$ (topologically)

$D^{\pm}(S)$ — Domain of dependence:
Spacetime region uniquely
determined by data on S

$H^{\pm}(S)$ — Cauchy horizon:
Boundary of $D^{\pm}(S)$

$I^{\pm}(S)$ — Chronological future/past:
Spacetime region reachable
from S by timelike curves

$J^{\pm}(S)$ — Causal future/past:
Spacetime region reachable
from S by timelike or null curves

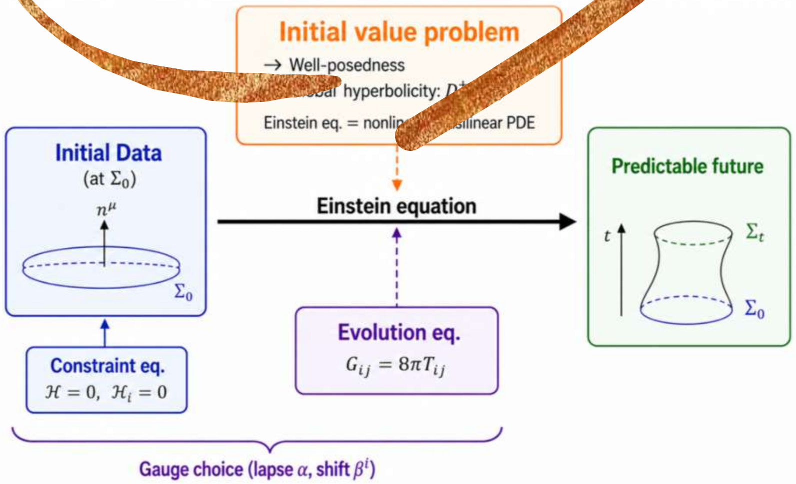
timelike

null

Initial value problem

- Einstein's equations are nonlinear, second-order PDEs—more precisely, quasilinear PDEs.
- After a suitable gauge choice, they can be written as a hyperbolic evolution system.
- Constraint-satisfying data on a spacelike Cauchy surface determine a locally existing solution that is unique up to diffeomorphism and depends continuously on the initial data.
- In a globally hyperbolic region, these data determine the evolution throughout the region, as long as the solution remains regular.
- This provides the mathematical basis for predictability and numerical simulation in general relativity.

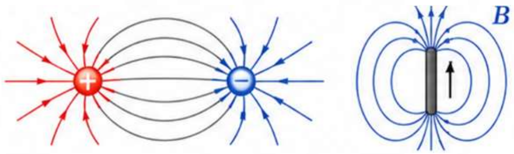
- Equation of what?
- What kind of differential eq.?



- Physical Degree of freedom?
- How many equations?

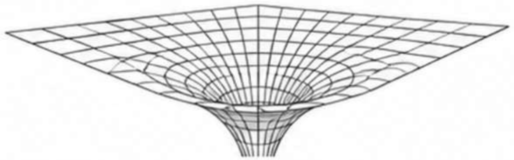
Einstein Equation

$$m\,\frac{d^2x^i}{dt^2} = F^i$$



$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$



$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$g_{\mu\nu}(x)$$

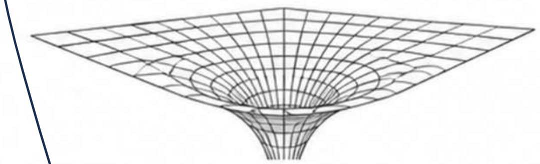
Einstein Equation

→ Field Eq. for Metric Tensor

$$\begin{pmatrix} G_{00} & G_{01} & G_{02} & G_{03} \\ G_{01} & G_{11} & G_{12} & G_{13} \\ G_{02} & G_{12} & G_{22} & G_{23} \\ G_{03} & G_{13} & G_{23} & G_{33} \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}$$

$$\boxed{\begin{aligned} G_{00} &= \frac{8\pi G}{c^4} T_{00} \\ G_{01} &= \frac{8\pi G}{c^4} T_{01} \\ G_{11} &= \frac{8\pi G}{c^4} T_{11} \end{aligned}}$$



$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$G_{\mu\nu} = G_{\nu\mu}$ **and** $g_{\mu\nu} = g_{\nu\mu}$

$\Rightarrow$ **10** equations for **10** metric components

Einstein Equation

→ Constraints and Evolution eq's

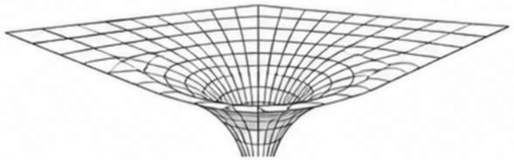
$$\nabla_\mu G^\mu{}_\nu = 0$$

$$\Downarrow$$

$$\partial_0 G^0{}_\nu = -\partial_i G^i{}_\nu + \Gamma G$$

$$\boxed{G^0{}_\nu \text{ contains no } \partial_0^2 g_{\alpha\beta}}$$

$$G^0{}_\nu = 0 \quad \Rightarrow \quad \text{4 constraints,}$$
$$G^i{}_j = 0 \quad \Rightarrow \quad \text{6 evolution equations.}$$



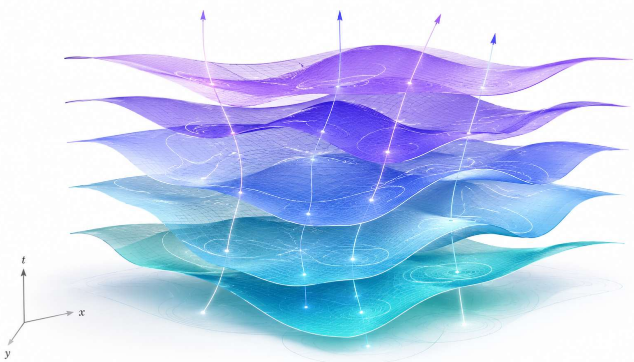
$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

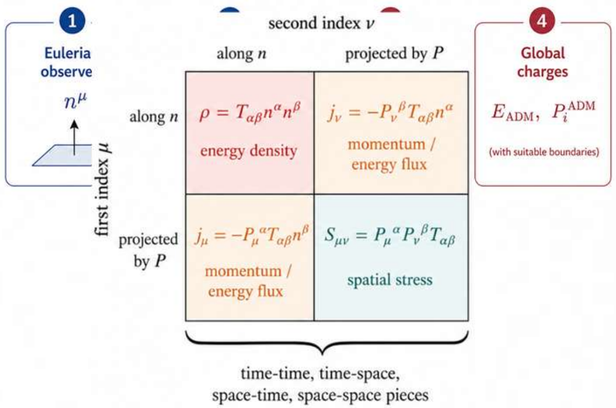
$$g_{\mu\nu}(x)$$

$$10 - 4_{\,gauge} - 4_{\,constraints} = 2_{\,propagating}$$

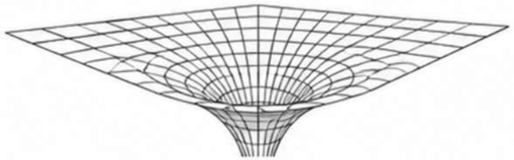
Einstein Equation

→ Need for 3+1 Decomposed Eq's





$$G^0{}_\nu = 0 \quad \Rightarrow \quad \text{4 constraints,}$$
$$G^i{}_j = 0 \quad \Rightarrow \quad \text{6 evolution equations.}$$



⋮

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

	along n	projected by P
along n	$G_{\alpha\beta} n^\alpha n^\beta$ Hamiltonian constraint	$-P^\mu{}_\nu G_{\mu\beta} n^\beta$ momentum piece
G projected by P	$-P^\nu{}_\mu G_{\alpha\nu} n^\alpha$ momentum piece	$P^\alpha{}_\mu P^\beta{}_\nu G_{\alpha\beta}$ spatial / evolution piece

vt

$+ v^2 G_{11}$

Einstein Equation

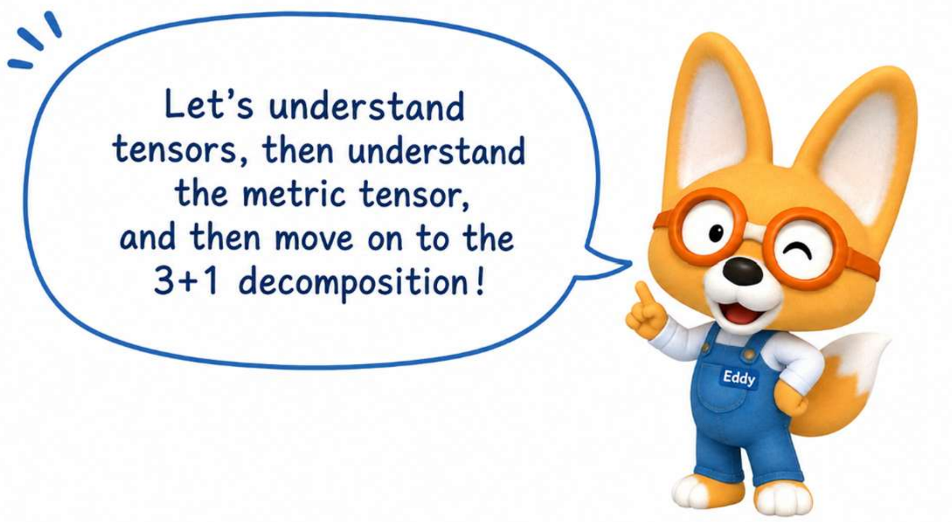
→ Result of 3+1 Decomposed Eq's

$$\alpha = N : \text{ lapse,}$$
$$\boldsymbol{\beta} = \beta^i \partial_i = N^i \partial_i : \text{ shift vector,}$$
$$X_{\hat{\mu}} \equiv P_\mu{}^\nu X_\nu : \text{ projection onto } \Sigma_t,$$
$$\nabla_{\hat{i}} X^{\hat{j}} \equiv D_i X^j \equiv P_i{}^\mu P^j{}_\nu \nabla_\mu X^\nu : \text{ spatial covariant derivative.}$$

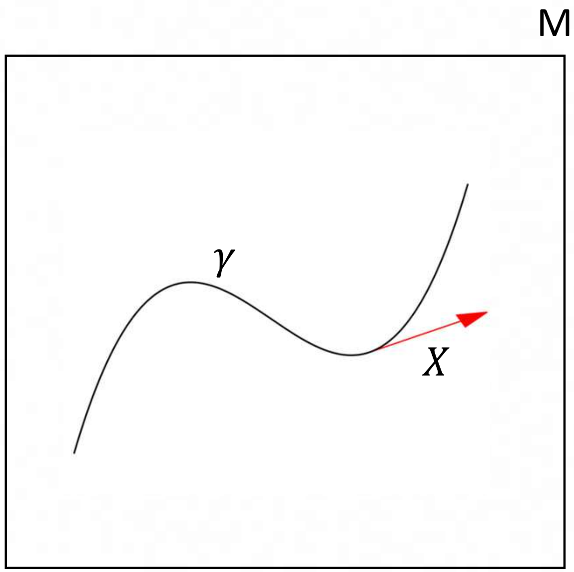
$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

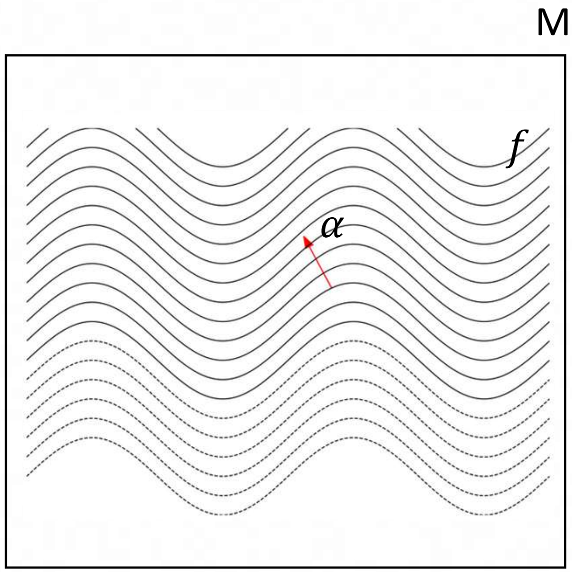
$$\hookrightarrow \begin{cases} (1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ ^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \ \rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\ \qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ \qquad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$

3+1 Decomposition of the metric tensor and Einstein's Equation


Let's understand tensors, then understand the metric tensor, and then move on to the 3+1 decomposition!
Eddy

Tangent /dual vector



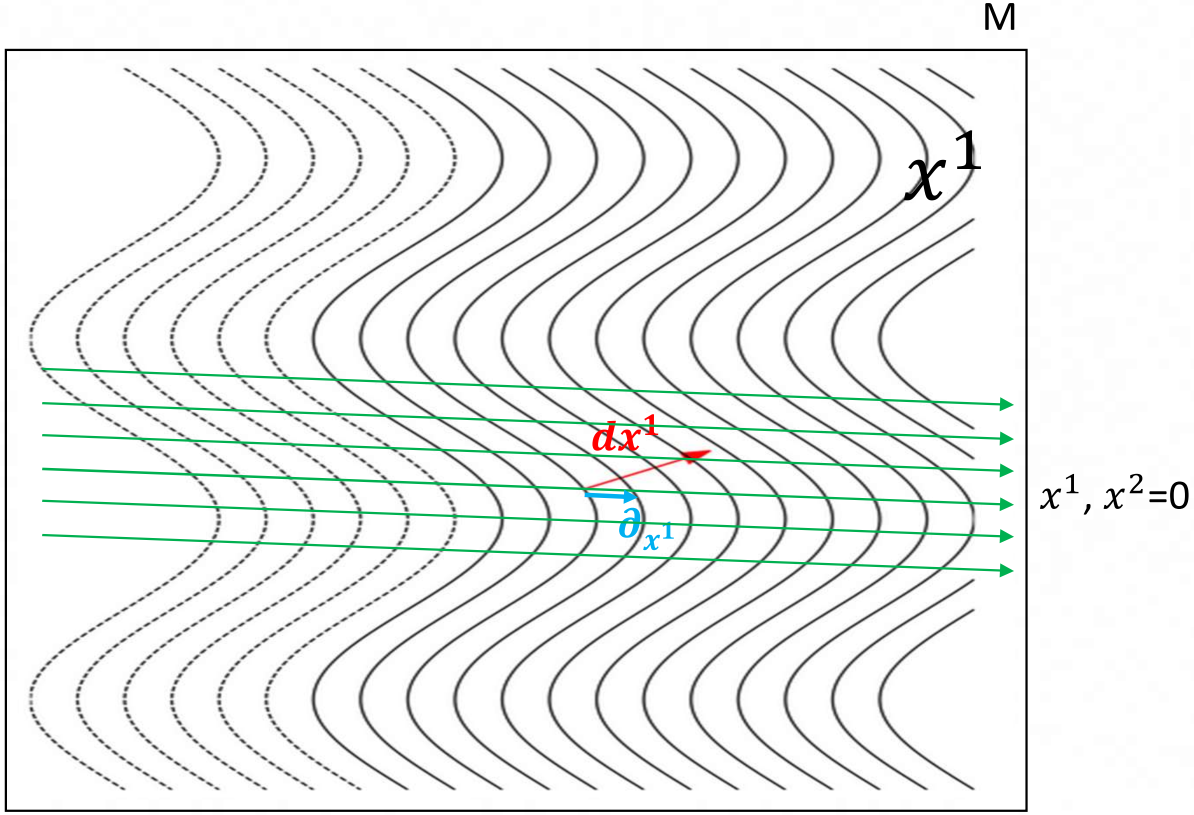


$$\mathbf{X}_p : C^\infty(M) \to \mathbb{R}, \qquad f \mapsto \mathbf{X}_p[f] = \frac{\mathrm{d}}{\mathrm{d}\lambda} f(\gamma(\lambda))\bigg|_{\lambda=0}$$

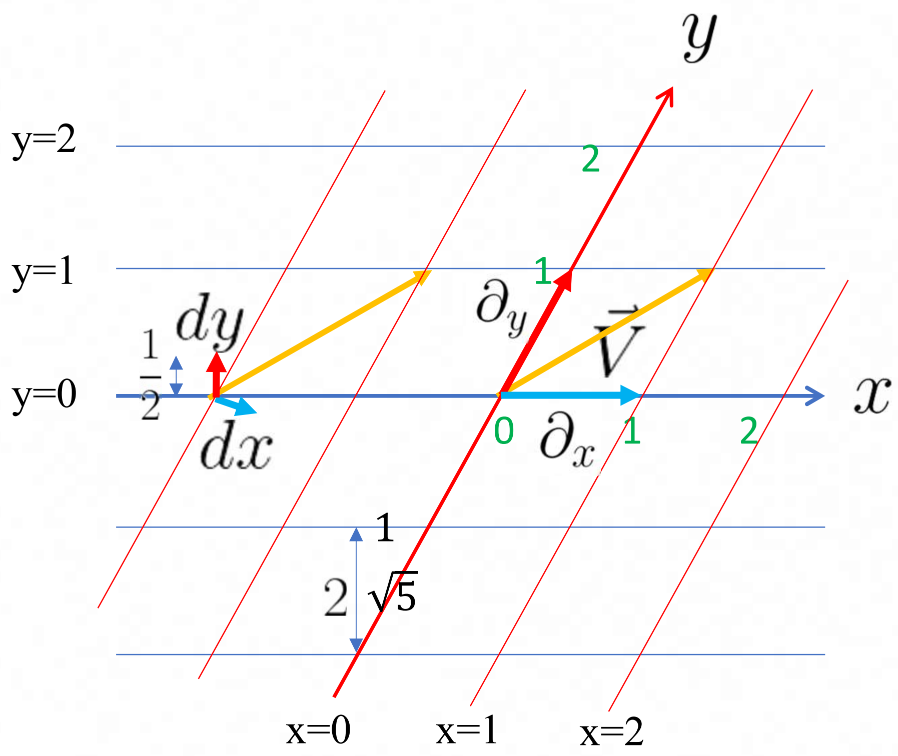
$$\alpha : T_pM \to \mathbb{R}, \quad v \mapsto \langle \alpha, v \rangle$$

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} \equiv \mathrm{d}f\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)$$

Tangent /dual vector basis from coordinate scalar fields



Tangent /dual vector components



$$\frac{\mathrm{d}}{\mathrm{d}x} = \boldsymbol{\partial}_x \text{ (tangent basis vector} \rightarrow \text{vector: covariant basis)}$$

$$\vec{V} = \boldsymbol{V} = V_x \ \boldsymbol{\partial_x} + V_y \ \boldsymbol{\partial_y}$$

$$\boldsymbol{\nabla} x = \mathrm{d}x \ (\underbrace{\text{one-form}}_{\text{(normal)}} \text{ basis vector} \rightarrow \text{dual-vector: contravariant basis})$$

$$\boldsymbol{\partial_x} \cdot \boldsymbol{dx} = \frac{\partial x}{\partial x} = 1, \ \boldsymbol{\partial_y} \cdot \boldsymbol{dy} = \frac{\partial y}{\partial y} = 1,$$

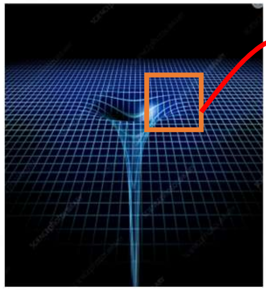
$$\boldsymbol{\partial_x} \cdot \boldsymbol{dy} = \frac{\partial y}{\partial x} = 0, \ \boldsymbol{\partial_y} \cdot \boldsymbol{dx} = \frac{\partial x}{\partial y} = 0$$

$$\vec{V} = (\vec{V} \cdot \partial_x)\mathrm{d}x + (\vec{V} \cdot \partial_y)\mathrm{d}y = V_x \mathrm{d}x + V_y \mathrm{d}y = V_i \mathrm{d}x^i$$

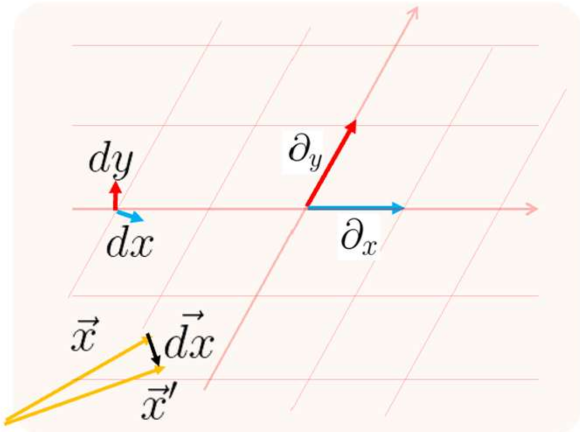
$$\vec{V} = (\vec{V} \cdot \mathrm{d}x)\partial_x + (\vec{V} \cdot \mathrm{d}y)\partial_y = V^x \partial_x + V^y \partial_y = V^i \partial_i$$

(covariant/contravariant)-(component/basis)

Tangent /dual vector basis to understand the metric tensor



$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$



- Metric tensor components from coordinate bases
  - Information about the lengths and angles of the basis vectors

$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu = \boldsymbol{g}(\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu) = (g_{\mu'\nu'}\mathbf{d}x^{\mu'} \otimes \mathbf{d}x^{\nu'})(\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu)$$
$$g^{\mu\nu} = \mathbf{d}x^\mu \cdot \mathbf{d}x^\nu = \boldsymbol{g}(\mathbf{d}x^\mu, \mathbf{d}x^\nu) = (g^{\mu'\nu'}\boldsymbol{\partial}_{\mu'} \otimes \boldsymbol{\partial}_{\nu'})(\mathbf{d}x^\mu, \mathbf{d}x^\nu)$$

$$\left.\begin{cases} g_{xx} = \boldsymbol{\partial}_x \cdot \boldsymbol{\partial}_x = |\boldsymbol{\partial}_x|^2 & \rightarrow \text{ length}^2 \\ g_{xy} = \boldsymbol{\partial}_x \cdot \boldsymbol{\partial}_y = |\boldsymbol{\partial}_x||\boldsymbol{\partial}_y|\cos\theta & \rightarrow \text{ angle info.} \end{cases}\right. \rightarrow \text{grid info.}$$
$$ds^2 = \boldsymbol{g}(\mathbf{d}x, \mathbf{d}x) = \boldsymbol{g}(\mathrm{d}x^\mu\boldsymbol{\partial}_\mu, \mathrm{d}x^\nu\boldsymbol{\partial}_\nu) = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \rightarrow \text{invariant length}$$

Tangent /dual vector basis to understand foliations

- Scalar field > Submanifold Σ_t
- Gradient of scalar field > normal vector (one-form basis)

$$dt = \nabla t\,, \qquad \langle V \text{ on } \Sigma_t, \nabla t \rangle = 0$$

- A curve intersecting the hypersurface > tangent basis vector

$$\gamma(t) \text{ by } \Sigma_t : \quad t = \partial_t$$

- Tangent basis and one-form

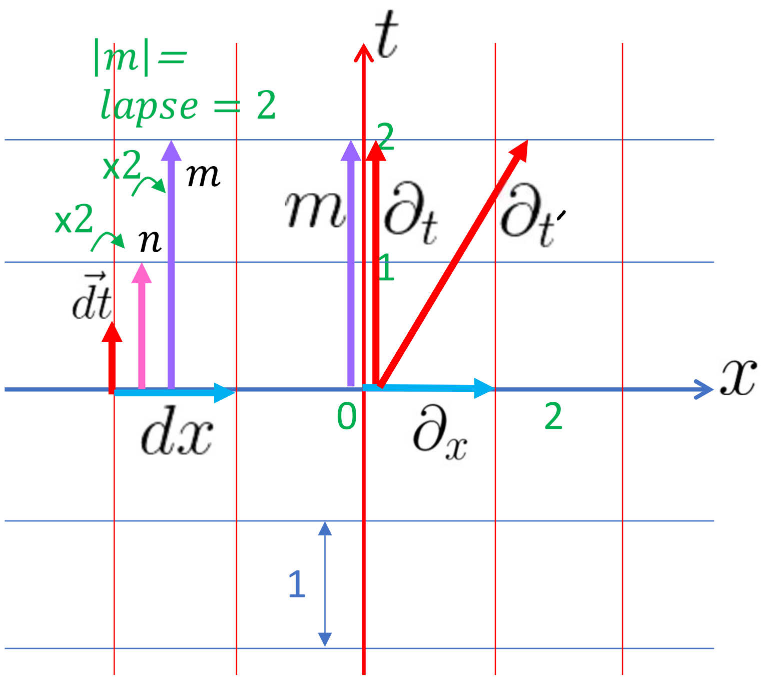
$$\gamma(t) \text{ by } \Sigma_t : \quad \langle t, \nabla t \rangle = t^\alpha \nabla_\alpha t = t^\alpha \partial_\alpha t = \partial_t t = 1$$

- A congruence of curves > coordinates > vector components

$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} \equiv (\partial_t)^\alpha \qquad \text{when } x^\alpha \equiv (t, y^a) \quad \rightarrow \quad \delta^\alpha_t = (1,0,0,0)$$

$$(y_a)^\alpha = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t \equiv (\partial_{y^a})^\alpha \qquad \text{when } x^\alpha \equiv (t, y^a) \quad \rightarrow \quad \delta^\alpha_a = (0,1,0,0)$$

$$\nabla_\alpha t = \left(\frac{\partial t}{\partial x^\alpha}\right) \equiv (\mathrm{d}t)_\alpha \qquad \text{when } x^\alpha \equiv (t, y^a) \quad \rightarrow \quad \delta^t_\alpha = (1,0,0,0)$$



Tangent /dual vector basis to understand foliations

1 Foliation of spacetime

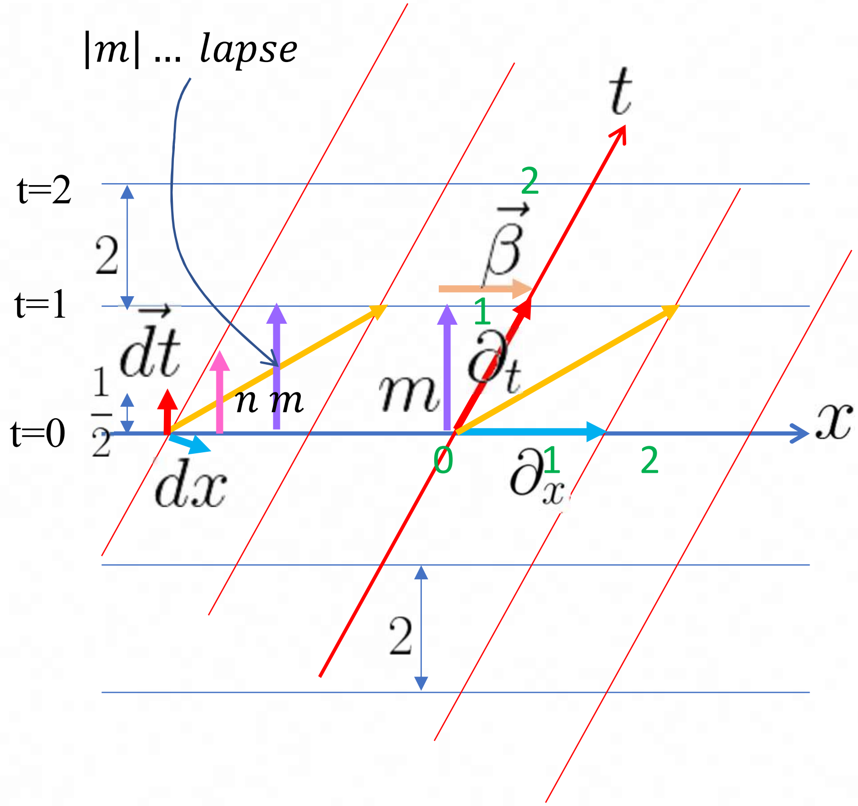
- Constant-time hypersurfaces: $t = \text{const.}$
- The normal one-form is dt.
- Because $\langle dt, \partial_t \rangle = 1$, the norm of dt is $|dt| = \frac{1}{N}$.
- The unit normal is $n = -N\,dt$.
- The normal evolution vector to the next slice is $m = Nn$, so $|m| = N$ (lapse).

2 Coordinate time flow

- Along the curve $x = \text{const}$, the tangent basis vector is ∂_t.
- Decompose the time-flow vector as $\partial_t = m + \beta$.
- Therefore the shift vector is $\beta = \partial_t - m$.

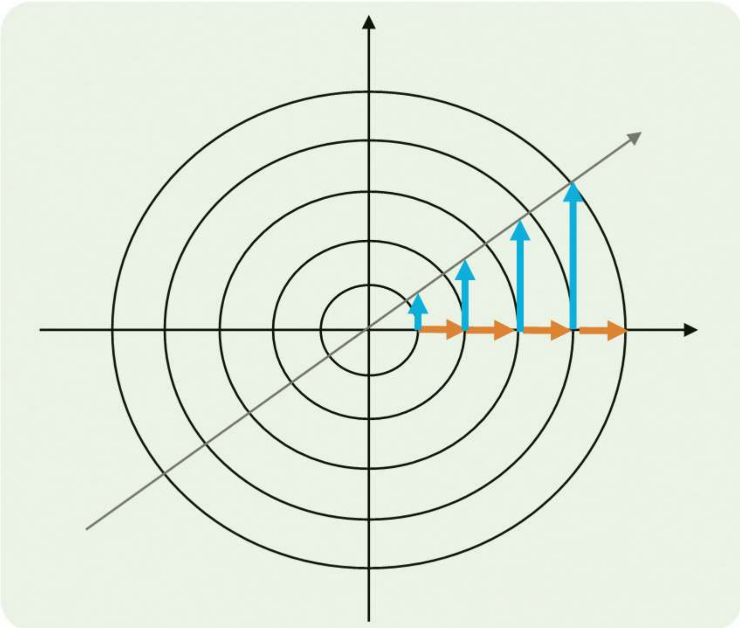
3 Geometry on each hypersurface

- Spatial basis vectors lie within the hypersurface.
- The induced spatial metric is γ_{ij}.



Metric tensor Examples

- 2-D Polar coordinates



$$
\begin{aligned}
ds^2 &= g_{rr}dr^2 + g_{\theta\theta}d\theta^2 \\
&= (\boldsymbol{\partial_r} \cdot \boldsymbol{\partial_r})dr^2 + (\boldsymbol{\partial_\theta} \cdot \boldsymbol{\partial_\theta})d\theta^2 \\
&= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2
\end{aligned}
$$

$$
g_{ij} = \begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}
$$

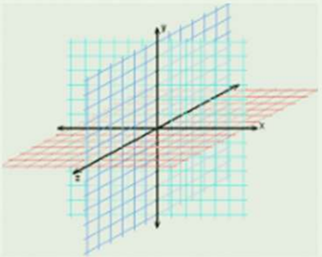
- Alternative coordinate system of flat 2-D space
- Orthogonal coordinates
- Symmetric tensor > metric tensor property
- Diagonal components indicates lengths of the basis

Metric tensor Examples

3-D flat space

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

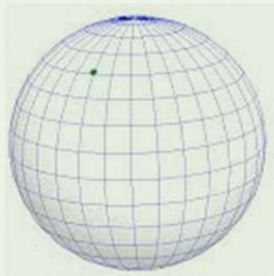
$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



2-D sphere surface

$$ds^2 = r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$$

$$g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix}$$

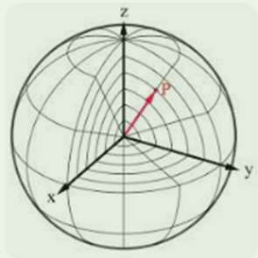


3-D spherical coordinates

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$$

$$(= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

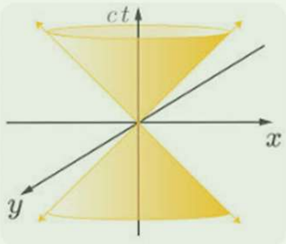


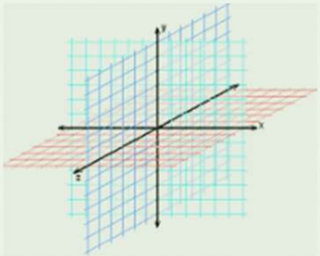
Metric tensor Examples (Pseudo-Riemannian, Flat)

4-D flat Minkowski spacetime (Cartesian)

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

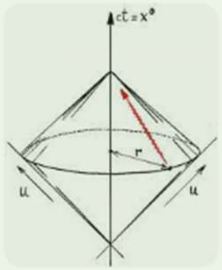


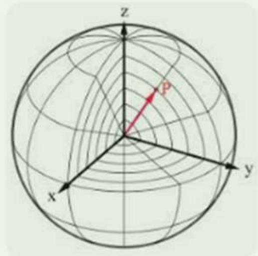


4-D flat Minkowski spacetime (Spherical)

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$



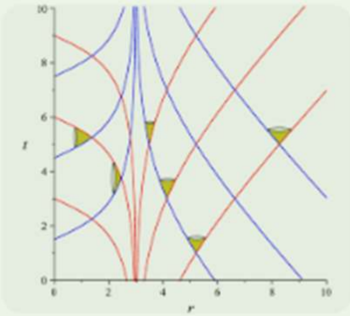


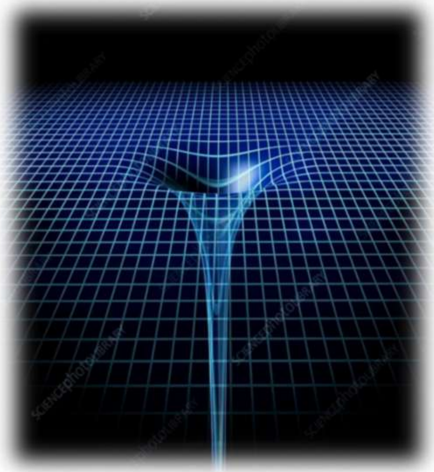
Metric tensor Examples (Pseudo-Riemannian, Curved)

4-D curved spacetime

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{2GM}{r}}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$





Metric tensor 3+1 Decomposition

$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu,\ g^{\mu\nu} = (\mathbf{d}x^\mu)\cdot(\mathbf{d}x^\nu)$$

$$\downarrow \begin{cases} \boldsymbol{t} = \boldsymbol{m} + \boldsymbol{\beta} & \\ N \equiv \alpha & \text{(lapse function)} \\ N^\alpha \equiv \beta^\alpha = (0, \vec{\beta}) & \text{(shift vector)} \\ m^\alpha = N n^\alpha = (1, -\vec{\beta}) & \text{(evolution vector)} \end{cases}$$

$$\begin{aligned} ds^2 &= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -(N^2 - \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ &= -N^2\mathrm{d}t^2 + \gamma_{ij} \underbrace{(\mathrm{d}x^i + \beta^i\mathrm{d}t)}_{\mathrm{d}\hat{x}^i} \underbrace{(\mathrm{d}x^j + \beta^j\mathrm{d}t)}_{\mathrm{d}\hat{x}^j} \end{aligned}$$

Note that $\mathrm{d}x^i$ is not on Σ_t when $\vec{\beta} \neq 0$, but $\mathrm{d}\hat{x}^i$ is on Σ_t.

$$\varsigma\ \ \mathbf{d}t = -N^{-1}\boldsymbol{n} \qquad -\boldsymbol{n} = -n_\mu(\mathbf{d}x^\mu)$$

$$= -n_\mu n_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu + \gamma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

$$= (-n_\mu n_\nu + \gamma_{\mu\nu})\mathrm{d}x^\mu\mathrm{d}x^\nu$$

$$\begin{aligned} \therefore\ \gamma_{\mu\nu} &= g_{\mu\nu} - \sigma n_\mu n_\nu \\ &= P_{\mu\nu} \quad \text{(projection tensor)} \end{aligned}$$

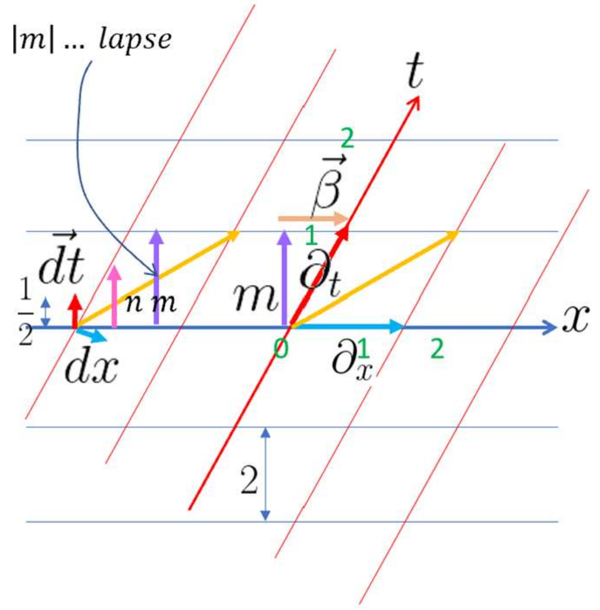
> Projection tensor = metric on hypersurface

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{\partial_t}\cdot\boldsymbol{\partial_t} & \boldsymbol{\partial_t}\cdot\boldsymbol{\partial_x} & \cdots & \cdots \\ \cdots & & & \\ \cdots & & \gamma_{ij} & \\ \cdots & & & \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{m}\cdot\boldsymbol{m} + \boldsymbol{\beta}\cdot\boldsymbol{\beta} & \boldsymbol{\beta}\cdot\boldsymbol{\partial_x} & \cdots & \cdots \\ \cdots & & & \\ \cdots & & \gamma_{ij} & \\ \cdots & & & \end{pmatrix}$$

$$= \begin{pmatrix} -N^2 + \beta_k\beta^k & \cdots & \beta_i & \cdots \\ \cdots & & & \\ \beta_i & & \gamma_{ij} & \\ \cdots & & & \end{pmatrix}$$



Metric tensor 3+1 Decomposition (projection tensor)

- Definition: $\boxed{P_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu}$

- Projected vectors are tangent to the hypersurface.

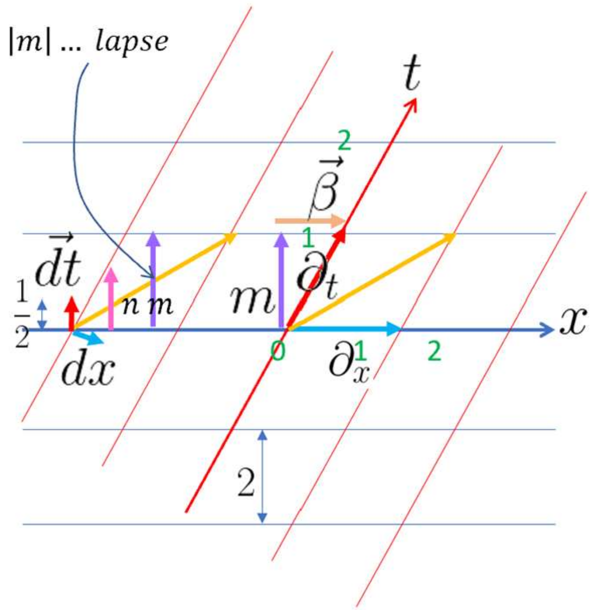
$$(P_{\mu\nu}V^\mu)n^\nu = g_{\mu\nu}V^\mu n^\nu - \sigma n_\mu \overbrace{n_\nu V^\mu n^\nu}^{=\sigma V^\mu,\ \sigma^2=1}$$
$$= 0$$

- Act like the metric for tangent vectors

$$P_{\mu\nu}V^\mu W^\nu = g_{\mu\nu}V^\mu W^\nu - \cancelto{0}{\sigma n_\mu n_\nu V^\mu W^\nu}$$
$$= g_{\mu\nu}V^\mu W^\nu$$

- Idempotent f(f(x))=f(x)

$$P^\mu{}_\lambda P^\lambda{}_\nu = \ldots = P^\mu{}_\nu$$



Metric tensor 3+1 Decomposition (inverse metric)

$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu,\ g^{\mu\nu} = (\mathbf{d}x^\mu)\cdot(\mathbf{d}x^\nu)$$

$$\downarrow \begin{cases} \boldsymbol{t} = \boldsymbol{m} + \boldsymbol{\beta} & \\ N \equiv \alpha & \text{(lapse function)} \\ N^\alpha \equiv \beta^\alpha = (0, \vec{\beta}) & \text{(shift vector)} \\ m^\alpha = N n^\alpha = (1, -\vec{\beta}) & \text{(evolution vector)} \end{cases}$$

$$g^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{N^2} & \dfrac{\beta^j}{N^2} \\ \dfrac{\beta^i}{N^2} & \gamma^{ij} - \dfrac{\beta^i\beta^j}{N^2} \end{pmatrix}$$

Note that $g^{ij} \neq \gamma^{ij}$

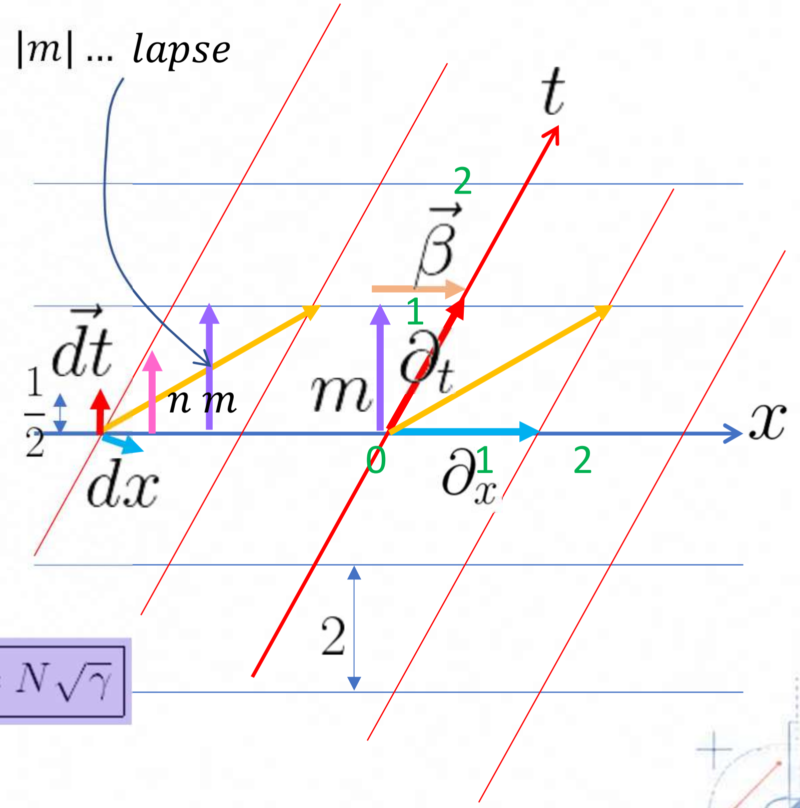
$$g^{00} = (\mathbf{d}t)\cdot(\mathbf{d}t) = \left(-\frac{1}{N}n^\mu\right)\left(-\frac{1}{N}n_\mu\right) = \sigma\frac{1}{N^2} = -\frac{1}{N^2}$$

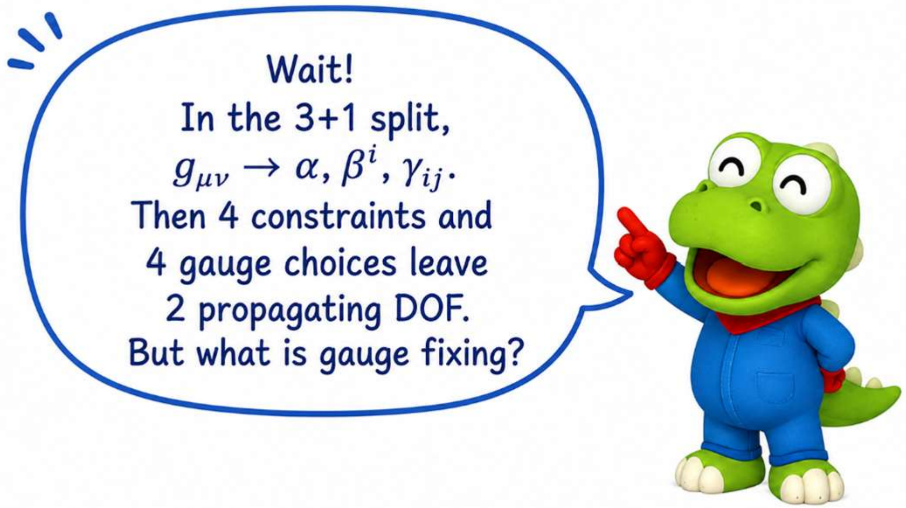
$$g^{0\mu} = (\mathbf{d}t)\cdot(\mathbf{d}x^\mu) = \delta^\beta_{\ \alpha}\left(-\frac{1}{N}n^\alpha\right)\underbrace{(\mathbf{d}x^\mu)_\beta}_{=\delta^\mu_{\ \beta}} = -\frac{1}{N}n^\mu = \frac{1}{N^2}(-1, \vec{\beta})$$

$$g^{ij} = (\mathbf{d}x^i)\cdot(\mathbf{d}x^j) = g^{\mu\nu}(\mathbf{d}x^i)_\mu(\mathbf{d}x^j)_\nu = (\sigma n^\mu n^\nu + P^{\mu\nu})\underbrace{(\mathbf{d}x^i)_\mu}_{=\delta^i_\beta}\underbrace{(\mathbf{d}x^j)_\nu}_{=\delta^j_\nu}$$

$$= \sigma n^i n^j + \gamma^{ij} = -\frac{\beta^i\beta^j}{N^2} + \gamma^{ij}$$

$$\Longrightarrow \boxed{\sqrt{-g} = N\sqrt{\gamma}}$$




Wait!
In the 3+1 split,
$g_{\mu\nu} \to \alpha, \beta^i, \gamma_{ij}$.
Then 4 constraints and
4 gauge choices leave
2 propagating DOF.
But what is gauge fixing?

GR: Diffeomorphism Invariant Theory

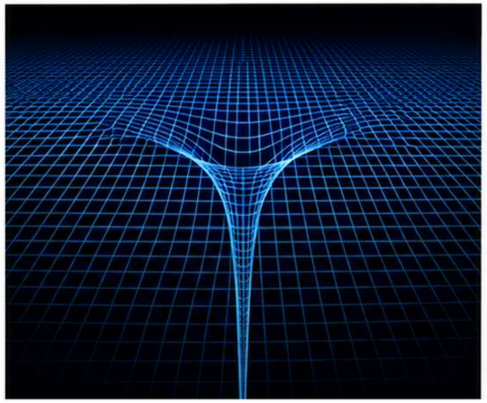
Diffeomorphism invariance in general relativity

In general relativity, a diffeomorphism deforms spacetime together with the metric and all physical fields, so the same physics is preserved.

GR: Diffeomorphism Invariant Theory

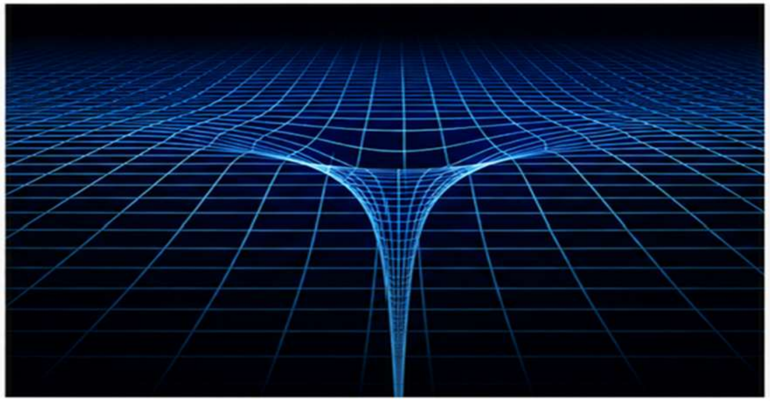
→ Freedom to choose coordinates

Diffeomorphism invariance in general relativity means that different coordinate descriptions can represent the same physical spacetime.



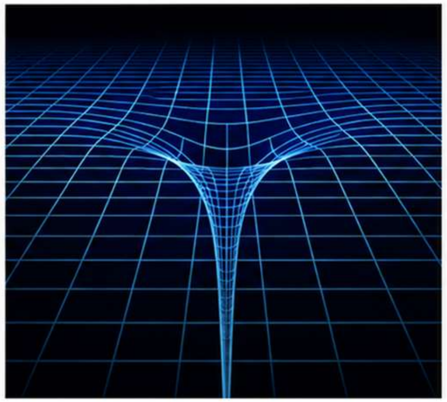
Original geometry

$$G_{\mu\nu}[g] = \frac{8\pi G}{c^4} T_{\mu\nu}$$



Diffeomorphism

$$G_{\mu\nu}[\phi^* g] = \frac{8\pi G}{c^4} (\phi^* T)_{\mu\nu}$$



Coordinate transformation

$$G_{\mu'\nu'}[g'] = \frac{8\pi G}{c^4} T'_{\mu'\nu'}$$

Physical dofs of metric tensor

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Metric tensor

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}$$

10 independent components

$$g_{\mu\nu} = g_{\nu\mu}$$

→ coordinate freedom / gauge choice

Coordinate freedom

$$x'^{\mu} = f^{\mu}(x)$$

$$g'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x)$$

Same physics, different coordinates

4 arbitrary coordinate functions
$\Rightarrow$ choose 4 gauge conditions

10 → 6

→ Einstein constraints on the initial slice

Constraints

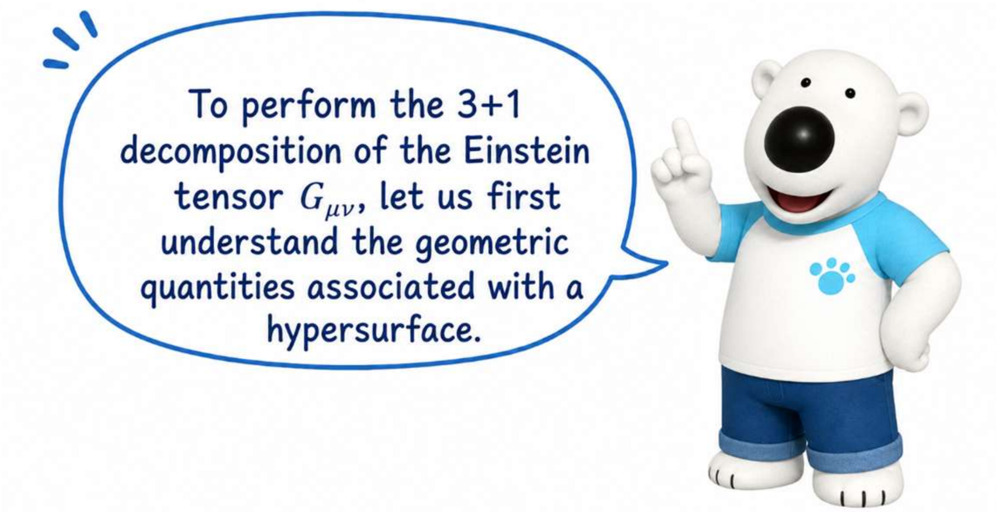
$$G_{00} = 8\pi T_{00},$$
$$G_{0i} = 8\pi T_{0i} \quad (i = 1, 2, 3)$$

Hamiltonian + momentum constraints

These 4 equations restrict the remaining 6 unknowns, so not all 6 are freely specifiable.

6 → 2

Remaining 2 propagating DOF = gravitational-wave polarizations h_+ and $h_\times$


To perform the 3+1 decomposition of the Einstein tensor $G_{\mu\nu}$, let us first understand the geometric quantities associated with a hypersurface.

3+1 Decomposition of Einstein equations

$$\begin{aligned} \alpha = N &: \text{ lapse}, \\ \boldsymbol{\beta} = \beta^i \partial_i = N^i \partial_i &: \text{ shift vector}, \\ X_{\hat{\mu}} \equiv P_\mu{}^\nu X_\nu &: \text{ projection onto } \Sigma_t, \\ \nabla_{\hat{i}} X^j \equiv D_i X^j \equiv P_i{}^\mu P^j{}_\nu \nabla_\mu X^\nu &: \text{ spatial covariant derivative.} \end{aligned}$$

$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ {}^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ {}^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \ \rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\ \qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ \qquad - D_i D_j \alpha - 8\pi G \alpha[S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$

$$G_{\mu\nu} \ \rightarrow\ \left\{ \begin{matrix} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu}\widehat{\nu}} \end{matrix} \right. \quad \frac{^{(4)}R \to {}^{(3)}R}{\begin{matrix} ^{(4)}\nabla \to {}^{(3)}\nabla \\ g_{\mu\nu} \to \gamma_{ij} \end{matrix}} \ (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)$$

① Geometry and evolution are described by fields on each Σ_t,

γ_{ij}, K_{ij}, $^{(3)}R$, N, β^i

② D_i is intrinsic to Σ_t, defining spatial derivatives, geodesics, and parallel transport,

$\nabla_{\hat{i}} X^j = D_i X^j$

③ We have already learned that matter variables are projected onto Σ_t,

E, p_i, S_{ij}

3+1 Decomposition of Einstein equations

$$\alpha = N: \text{ lapse},$$
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$$X_{\hat{\mu}} \equiv P_\mu{}^\nu X_\nu: \text{ projection onto } \Sigma_t,$$
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$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\left\{\begin{array}{ll} (1)\ {}^{(4)}G_{nn} = 8\pi G T_{nn} & \to R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ {}^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} & \to D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} & \to \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) \\ & \quad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ & \quad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} & \to \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{array}\right.$$

1

$$G_{\mu\nu} \to \left\{\begin{array}{l} G_{\boldsymbol{nn}} \\ G_{\hat{\mu}\hat{\nu}} \end{array}\right. \xrightarrow[\substack{^{(4)}\nabla \to {}^{(3)}\nabla \\ g_{\mu\nu} \to \gamma_{ij}}]{^{(4)}R \to {}^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)$$

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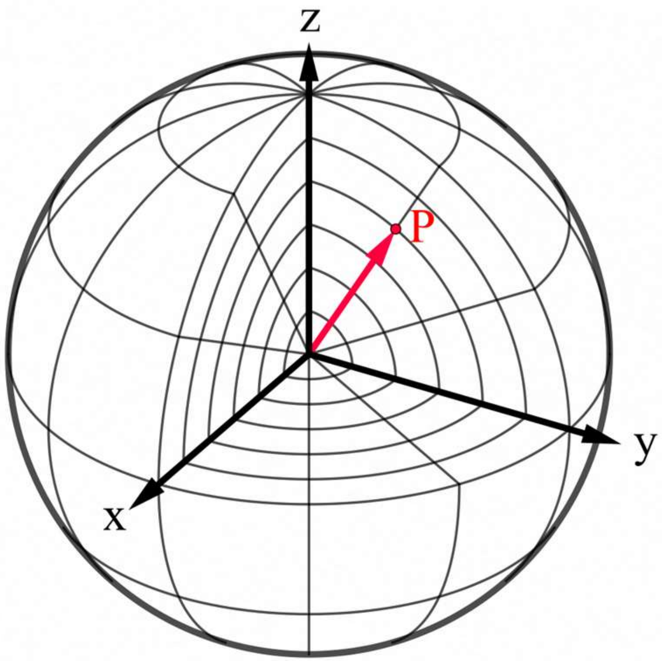
3+1 Quantities: Intrinsic curvature

$$G_{\mu\nu} \rightarrow \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu\nu}} \end{cases} \xrightarrow[\substack{{}^{(4)}\nabla \,\to\, {}^{(3)}\nabla \\ g_{\mu\nu} \,\to\, \gamma_{ij}}]{{}^{(4)}R \,\to\, {}^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)$$



$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$\Lsh\ x^i = (r, \theta, \phi) \ \rightarrow\ \gamma_{ij} \ \rightarrow\ {}^{(3)}\nabla \ \rightarrow\ {}^{(3)}R$$

$$ds^2 = r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$\Lsh\ x^a = (\theta, \phi) \ \rightarrow\ \gamma_{ab} \ \rightarrow\ {}^{(2)}\nabla \ \rightarrow\ {}^{(2)}R$$

We need to understand the quantities of the hypersurface.

3+1 Decomposition of Einstein equations

$$\alpha = N: \text{ lapse},$$
$$\boldsymbol{\beta} = \beta^i \partial_i = N^i \partial_i: \text{ shift vector},$$
$$X_{\hat{\mu}} \equiv P_\mu{}^\nu X_\nu: \text{ projection onto } \Sigma_t,$$
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$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow \ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow \ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ ^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \ \rightarrow \ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) \\ \qquad\qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ \qquad\qquad - D_i D_j \alpha - 8\pi G \alpha[S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow \ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

2

$$G_{\mu\nu} \rightarrow \begin{cases} G_{nn} \\ G_{\widehat{\mu\nu}} \end{cases} \xrightarrow[\substack{^{(4)}\nabla \,\rightarrow\, ^{(3)}\nabla \\ g_{\mu\nu} \,\rightarrow\, \gamma_{ij}}]{^{(4)}R \,\rightarrow\, ^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)$$

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③ We have already learned that matter variables are projected onto Σ_t,

E, p_i, S_{ij}

3+1 Quantities: Extrinsic curvature

$$G_{\mu\nu} \rightarrow \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu}\widehat{\nu}} \end{cases} \xrightarrow[\substack{^{(4)}\nabla \rightarrow {}^{(3)}\nabla \\ g_{\mu\nu} \rightarrow \gamma_{ij}}]{^{(4)}R \rightarrow {}^{(3)}R} (K_{ij}, \gamma_{ij}, N, \beta^i, \partial_t)$$

- 1st fundamental form of the hypersurface:

  Projection tensor > project all tensors on to the hyper surface

  $$\boxed{P_{\mu\nu}} = g_{\mu\nu} - \sigma n_\mu n_\nu$$

- 2nd fundamental form of the hypersurface:

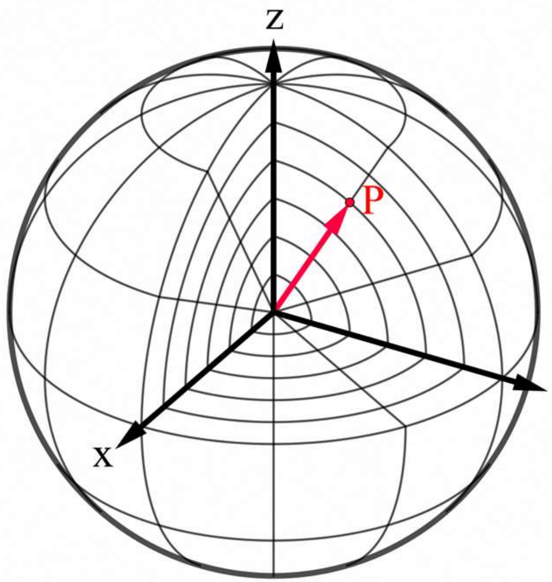
  Change of projection tensor along the normal direction

  > bended hypersurface

  >> Extrinsic curvature

  $$\boxed{K_{\mu\nu}} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2}P^\alpha{}_\mu P^\beta{}_\nu \mathcal{L}_n g_{\alpha\beta} = \nabla_\mu n_\nu - \sigma n_\mu a_\nu$$

$$ds^2 = \gamma_{ij} d\widehat{x}^i d\widehat{x}^j = \widehat{r}^2 d\widehat{\theta}^2 + \widehat{r}^2 \sin^2\widehat{\theta} d\widehat{\phi}^2, \ \widehat{x}^i = (\widehat{\theta}, \widehat{\phi})$$



Example of Kμν .

(1) 3-dimensional manifold M

(2) foliation of M with $\Sigma_r' s$

(3) coordinates: $x^i = (r, \theta, \phi)$ (r along normal dir., (θ, ϕ) on Σ_r)

(4) metric: $ds^2 = g_{ij} \mathrm{d}x^i \mathrm{d}x^j = \mathrm{d}r^2 + \underbrace{r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2}_{=\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b}$

(5) Christoffel symbol: $\Gamma^r_{\theta\theta} = \dfrac{g_{\theta\theta,r}}{-2g_{rr}} = \dfrac{2r}{-2} = -r,\ \Gamma^\theta_{\theta r} = \cdots,\ \cdots$

(6) curvature: $M \rightarrow {}^{(3)}R = 0, \quad \Sigma_r \rightarrow {}^{(2)}R = \dfrac{2}{r^2}$

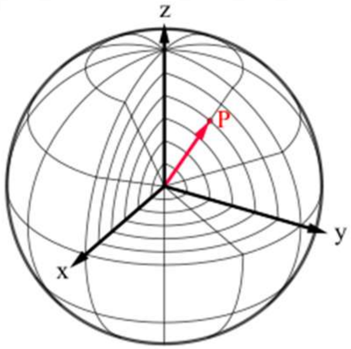
(7) normal vector: $(\mathbf{d}r)_i = \nabla_i r = \partial_i r = \delta^r_i = (1, 0, 0) \equiv r_i$

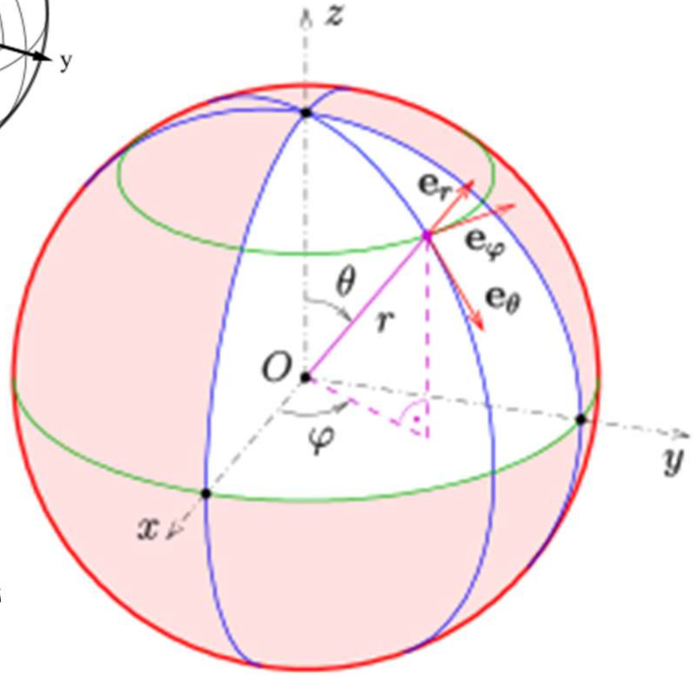
projection tensor: $\gamma_{ij} = g_{ij} - r_i r_j$

(8) extrinsic curvature: $K_{\theta\theta} = \begin{cases} = \dfrac{1}{2}\mathcal{L}_{\boldsymbol{r}}\gamma_{\theta\theta} = \dfrac{1}{2}(\underbrace{r^i\partial_i}_{\partial_r} \underbrace{\gamma_{\theta\theta}}_{=r^2} + \underbrace{\cancel{2(\partial_\theta r^i)\gamma_{i\theta}}}_{=2\partial_\theta r^\theta \gamma_{\theta\theta}=0}) = r \\ = \nabla_\theta r_\theta - \sigma_r r_\theta \cancel{a_\theta} = \nabla_\theta r_\theta = \nabla_\theta \delta^r_\theta = -\Gamma^i_{\theta\theta}\delta^r_i = -\Gamma^r_{\theta\theta} = \end{cases}$

$$K_{\phi\phi} = -\Gamma^r_{\phi\phi} = -\frac{g_{\phi\phi,r}}{-2g_{rr}} = r\sin^2\theta$$

$$K = K^\theta{}_\theta + K^\phi{}_\phi = g^{\theta\theta}K_{\theta\theta} + g^{\phi\phi}K_{\phi\phi} = \frac{1}{r^2}\cdot r + \frac{1}{r^2\sin^2\theta}\cdot r\sin^2\theta = \frac{2}{r}$$





Properties of P$\mu\nu$, K$\mu\nu$

$$G_{\mu\nu} \;\rightarrow\; \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu\nu}} \end{cases} \xrightarrow[\substack{{}^{(4)}\nabla \,\to\, {}^{(3)}\nabla \\ g_{\mu\nu} \,\to\, \gamma_{ij}}]{{}^{(4)}R \,\to\, {}^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)$$

$$\boxed{P_{\mu\nu}} = g_{\mu\nu} - \sigma n_\mu n_\nu \qquad \begin{cases} n^\mu P_{\mu\nu} = 0,\ n^\nu P_{\mu\nu} = 0 \\ P_{[\mu\nu]} = 0 \\ P_{\mu\nu} V^\mu W^\nu = g_{\mu\nu} V^\mu W^\nu \end{cases} \rightarrow \gamma_{ij}$$

$$\begin{aligned} \boxed{K_{\mu\nu}} &= \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \\ &= \frac{1}{2} P^\alpha{}_\mu P^\beta{}_\nu \mathcal{L}_n g_{\alpha\beta} \\ &= \nabla_\mu n_\nu - \sigma n_\mu a_\nu \end{aligned} \qquad \begin{cases} n^\mu K_{\mu\nu} = 0,\ n^\nu K_{\mu\nu} = 0 \\ K_{[\mu\nu]} = 0 \end{cases} \rightarrow K_{ij}$$

3+1 Decomposition of Einstein equations

$$\begin{aligned} \alpha = N &: \text{ lapse,} \\ \boldsymbol{\beta} = \beta^i \partial_i = N^i \partial_i &: \text{ shift vector,} \\ X_{\hat{\mu}} \equiv P_\mu{}^\nu X_\nu &: \text{ projection onto } \Sigma_t, \\ \nabla_{\hat{i}} X^j \equiv D_i X^j \equiv P_i{}^\mu P^j{}_\nu \nabla_\mu X^\nu &: \text{ spatial covariant derivative.} \end{aligned}$$

$${}^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\begin{cases} (1)\ {}^{(4)}G_{nn} = 8\pi G T_{nn} & \rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ {}^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} & \rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} & \rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) \\ & \qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ & \qquad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \tfrac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \tfrac{1}{2}\mathcal{L}_n P_{\mu\nu} & \rightarrow\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$

$$G_{\mu\nu} \rightarrow \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu\nu}} \end{cases} \xrightarrow[\substack{{}^{(4)}\nabla \,\rightarrow\, {}^{(3)}\nabla \\ g_{\mu\nu} \,\rightarrow\, \gamma_{ij}}]{{}^{(4)}R \,\rightarrow\, {}^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)$$

① Geometry and evolution are described by fields on each Σ_t,

γ_{ij}, K_{ij},
${}^{(3)}R$, N, β^i

② D_i is intrinsic to Σ_t, defining spatial derivatives, geodesics, and parallel transport,

$\nabla_{\hat{i}} X^j = D_i X^j$

③ We have already learned that matter variables are projected onto Σ_t,

E, p_i, S_{ij}

3+1 Quantities: Intrinsic covariant derivative

$$G_{\mu\nu} \;\rightarrow\; \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu}\widehat{\nu}} \end{cases} \xrightarrow[\substack{^{(4)}\nabla \,\rightarrow\, ^{(3)}\nabla \\ g_{\mu\nu} \,\rightarrow\, \gamma_{ij}}]{^{(4)}R \,\rightarrow\, ^{(3)}R} (K_{ij},\; \gamma_{ij},\; N,\; \beta^i,\; \partial_t)$$

$$\boldsymbol{M}:\; g_{\mu\nu},\; \nabla_\mu[\Gamma(g)] \;\rightarrow\; [\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda{}_{\rho\mu\nu}V^\rho \;\rightarrow R$$

$$\Lsh\; \nabla_\mu g_{\nu\rho} = 0$$

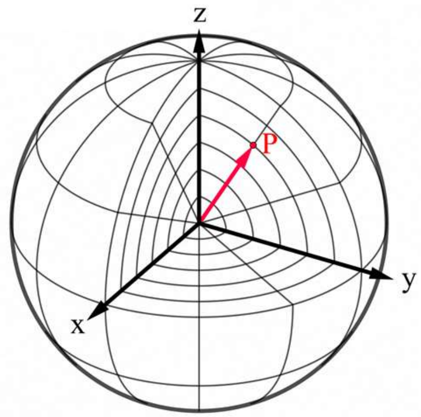
$$\begin{aligned} \varsigma\;\; \nabla_\mu P_{\nu\rho} &= \nabla_\mu(g_{\nu\rho} + n_\nu n_\rho) \\ &= \nabla_\mu g_{\nu\rho} + (\nabla_\mu n_\nu)n_\rho + n_\nu(\nabla_\mu n_\rho) \\ &= K_{\mu\nu}n_\rho + n_\nu K_{\mu\rho} + n_\mu a_\nu n_\rho + n_\nu a_\mu n_\rho \\ &\neq 0 \end{aligned}$$

$$\varsigma\;\; \widehat{\nabla}_\sigma X^{\mu\cdots}_{\nu\cdots} = P^\alpha{}_\sigma P^\mu{}_\beta \cdots \nabla_\alpha X^{\beta\cdots}_{\gamma\cdots}$$

$$\widehat{\nabla}_\mu \widehat{\nabla}_\nu X^\rho = P_\mu{}^{\mu'} P_\nu{}^{\nu''} P^{\rho'}_{\rho''} \nabla_{\mu'}(P_{\rho'}{}^{\rho''} P^{\nu'}_{\nu''} \nabla_{\nu'} X^{\rho'})$$

$$\boldsymbol{\Sigma}_t:\; \gamma_{\mu\nu},\; \widehat{\nabla}_\mu[\Gamma(\gamma)] \;\rightarrow\; [\widehat{\nabla}_\mu, \widehat{\nabla}_\nu]V^\lambda = \widehat{R}^\lambda{}_{\rho\mu\nu}V^\rho \;\rightarrow \widehat{R}$$

$$\Lsh\; \widehat{\nabla}_\mu \gamma_{\nu\rho} = 0$$



$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$\Lsh\; x^i = (r, \theta, \phi) \;\rightarrow\; \gamma_{ij} \;\rightarrow\; ^{(3)}\nabla \;\rightarrow\; ^{(3)}R$$

$$ds^2 = r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$\Lsh\; x^a = (\theta, \phi) \;\rightarrow\; \gamma_{ab} \;\rightarrow\; ^{(2)}\nabla \;\rightarrow\; ^{(2)}R$$

3+1 Decomposition of Einstein equations

$$
\begin{aligned}
\alpha = N &: \text{lapse},\\
\boldsymbol{\beta} = \beta^i\partial_i = N^i\partial_i &: \text{shift vector},\\
X_{\hat{\mu}} \equiv P_\mu{}^\nu X_\nu &: \text{projection onto } \Sigma_t,\\
\nabla_{\hat{i}}X^j \equiv D_i X^j \equiv P_i{}^\mu P^j{}_\nu \nabla_\mu X^\nu &: \text{spatial covariant derivative.}
\end{aligned}
$$

$$
{}^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}
$$

$$
\hookrightarrow
\begin{cases}
(1)\ {}^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E\\
(2)\ {}^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i\\
(3)\ {}^{(4)}G_{\widehat{\mu\nu}} = 8\pi G T_{\widehat{\mu\nu}} \ \rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij})\\
\qquad\qquad + (\beta^k\partial_k K_{ij} + \partial_i\beta^k K_{kj} + \partial_j \beta^k K_{ik})\\
\qquad\qquad - D_i D_j \alpha - 8\pi G\alpha[S_{ij} - \frac{1}{2}\gamma_{ij}(S-E)]\\
K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow\ \partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i
\end{cases}
$$

$$
G_{\mu\nu} \rightarrow \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu\nu}} \end{cases} \xrightarrow[\substack{{}^{(4)}\nabla \,\rightarrow\, {}^{(3)}\nabla \\ g_{\mu\nu} \,\rightarrow\, \gamma_{ij}}]{{}^{(4)}R \,\rightarrow\, {}^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)
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① Geometry and evolution are described by fields on each Σ_t,

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E, p_i, S_{ij}

3+1 Quantities: intrinsic curvature derivation (Gauss eq.)

$$\boxed{\widehat{\nabla}_\mu \widehat{\nabla}_\nu V^\rho} = P^\alpha{}_\mu P^\beta{}_\nu P^\rho{}_\gamma \underbrace{\nabla_\alpha (P^\delta{}_\beta P^\gamma{}_\lambda \nabla_\delta V^\lambda)}_{= -\sigma n^\delta (\nabla_\alpha n_\beta) P^\gamma{}_\lambda \nabla_\delta V^\lambda - \sigma P^\delta{}_\beta (\nabla_\alpha n^\gamma) n_\lambda \nabla_\delta V^\lambda + P^\delta{}_\beta P^\gamma{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda}$$

(using $\nabla_\mu P^\alpha{}_\beta = \nabla_\mu(\delta^\alpha{}_\beta - \sigma n^\alpha n_\beta) = -\sigma \nabla_\mu (n^\alpha n_\beta)$
and $P^\alpha{}_\beta n_\alpha = 0$)

$$= P^\alpha{}_\mu P^\beta{}_\nu P^\rho{}_\gamma (-\sigma n^\delta (\nabla_\alpha n_\beta) P^\gamma{}_\lambda \nabla_\delta V^\lambda - \sigma (\nabla_\alpha n^\gamma) P^\delta{}_\beta \underbrace{n_\lambda \nabla_\delta V^\lambda}_{= -V^\lambda \nabla_\delta n_\lambda \ \because\ \nabla_\delta (n_\lambda V^\lambda) = 0} + P^\delta{}_\beta P^\gamma{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda)$$

$$= -\sigma \underbrace{P^\alpha{}_\mu P^\beta{}_\nu (\nabla_\alpha n_\beta)}_{K_{\mu\nu}} P^\rho{}_\gamma P^\gamma{}_\lambda n^\delta \nabla_\delta V^\lambda + \sigma \overbrace{P^\alpha{}_\mu P^\rho{}_\gamma (\nabla_\alpha n^\gamma)}^{K^\rho{}_\mu} \underbrace{P^\beta{}_\nu P^\delta{}_\beta (\nabla_\delta n_\lambda) V^\lambda}_{P^\delta{}_\nu (\nabla_\delta n_\lambda) \delta^\lambda{}_\kappa V^\kappa = P^\delta{}_\nu (\nabla_\delta n_\lambda) P^\lambda{}_\eta P^\eta{}_\kappa V^\kappa = K_{\nu\eta} P^\eta{}_\kappa V^\kappa = K_{\nu\lambda} V^\lambda}$$

$$+ P^\alpha{}_\mu \underbrace{P^\beta{}_\nu P^\delta{}_\beta}_{P^\delta{}_\nu} \underbrace{P^\rho{}_\gamma P^\gamma{}_\lambda}_{P^\rho{}_\lambda} \nabla_\alpha \nabla_\delta V^\lambda$$

$$= -\sigma K_{\mu\nu} P^\rho{}_\lambda n^\delta \nabla_\delta V^\lambda + \sigma K^\rho{}_\mu K_{\nu\lambda} V^\lambda + P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda$$

$$\boxed{\equiv \nabla_{\widehat{\mu}} \nabla_{\widehat{\nu}} V^{\widehat{\rho}} - \sigma K_{\mu\nu} \nabla_n V^{\widehat{\rho}} + \sigma K^\rho{}_\mu K_{\nu\lambda} V^\lambda}$$

(where we defined $\nabla_{\widehat{\mu}} \nabla_{\widehat{\nu}} V^{\widehat{\rho}} \equiv P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda$)

3+1 Quantities: Intrinsic curvature derivation (Gauss eq.)

$$\widehat{\nabla}_\mu \widehat{\nabla}_\nu V^\rho = \nabla_{\widehat{\mu}} \nabla_{\widehat{\nu}} V^{\widehat{\rho}} - \sigma K_{\mu\nu} \nabla_n V^{\widehat{\rho}} + \sigma K^\rho{}_\mu K_{\nu\lambda} V^\lambda$$

$$(\text{where } \nabla_{\widehat{\mu}} \nabla_{\widehat{\nu}} V^{\widehat{\rho}} \equiv P^\alpha_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda)$$

$$\Rightarrow 2\widehat{\nabla}_{[\mu} \widehat{\nabla}_{\nu]} V^\rho \equiv \widehat{R}^\rho{}_{\sigma\mu\nu} V^\sigma$$

$$= 2(-\sigma \cancelto{0}{K_{[\mu\nu]}} P^\rho{}_\gamma n^\delta \nabla_\delta V^\lambda + \sigma K^\rho{}_{[\mu} K_{\nu]\lambda} V^\lambda + \underbrace{P^\alpha_{[\mu} P^\delta{}_{\nu]} P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda}_{= P^\alpha_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_{[\alpha} \nabla_{\delta]} V^\lambda = \frac{1}{2} P^\alpha_\mu P^\delta{}_\nu P^\rho{}_\lambda R^\lambda{}_{\sigma\alpha\delta} V^\sigma})$$

$$= 2(\sigma K^\rho{}_{[\mu} K_{\nu]\sigma} + \frac{1}{2} P^\alpha_\mu P^\delta{}_\nu P^\rho{}_\lambda R^\lambda{}_{\sigma\alpha\delta}) V^\sigma$$

$$\Rightarrow \widehat{R}^\rho{}_{\sigma\mu\nu} = P^\rho{}_\alpha P^\beta_\sigma P^\gamma{}_\mu P^\delta{}_\nu R^\alpha{}_{\beta\gamma\delta} + \sigma(K^\rho{}_\mu K_{\sigma\nu} - K^\rho{}_\nu K_{\sigma\mu})$$

$$\Rightarrow \boxed{\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma}} \quad \textbf{(Gauss eq.)}$$

(intrinsic)=(projected R)+(bending)

3+1 Quantities: Intrinsic curvature derivation (Gauss eq.)

Contracted Gauss' equation :

$$P^{\mu\rho}(\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2K_{\rho[\mu}K_{\nu]\sigma})$$

$$\rightarrow \quad \widehat{R}_{\nu\sigma} = \underbrace{P^{\mu\rho}R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}}}_{=P^{\mu\rho}P^{\alpha}{}_{\mu}P^{\beta}{}_{\nu}P^{\gamma}{}_{\rho}P^{\delta}{}_{\sigma}R_{\alpha\beta\gamma\delta}=P^{\alpha\gamma}P^{\beta}{}_{\nu}P^{\delta}{}_{\sigma}R_{\alpha\beta\gamma\delta}=(g^{\alpha\gamma}-\sigma n^{\alpha}n^{\gamma})P^{\beta}{}_{\nu}P^{\delta}{}_{\sigma}R_{\alpha\beta\gamma\delta}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma})$$

$$\rightarrow \quad \boxed{\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}}$$

3+1 Quantities: Intrinsic curvature derivation(Gauss eq.)

Scalar Gauss relation :

$$P^{\nu\sigma}(\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma})$$

$$\rightarrow \quad \widehat{R} = \underbrace{P^{\nu\sigma}P^{\alpha}_{\ \nu}P^{\beta}_{\ \sigma}}_{=P^{\alpha\beta}=g^{\alpha\beta}-\sigma n^{\alpha}n^{\beta}}(R_{\alpha\beta} - \sigma R_{n\alpha n\beta}) + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]}^{\ \nu}$$

$$= R - \sigma R_{nn} - \sigma R_{nn} + \sigma^2 \cancel{R_{nnnn}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]}^{\ \nu}$$

$$= R - \sigma 2R_{nn} + \sigma(K^2 - K^{\mu\nu}K_{\mu\nu})$$

$$\Rightarrow \boxed{\widehat{R} = R - \sigma 2R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})}$$

$$\Rightarrow \text{when } \sigma = -1,\ \widehat{R} = R + 2R_{nn} - (K^2 - K_{\mu\nu}K^{\mu\nu})$$

Gauss relations (tangential projections)

$\hat{R}_{\mu\nu\rho\sigma},\ \hat{R}_{\mu\nu},\ \hat{R}$: intrinsic curvature of the hypersurface Σ_t.

1. $$2\,\hat{\nabla}_{[\mu}\,\hat{\nabla}_{\nu]}\,V^{\rho} \equiv \hat{R}^{\rho}{}_{\sigma\mu\nu}\,V^{\sigma}$$

2. $$\hat{R}_{\mu\nu\rho\sigma} = R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \sigma\,2\,K_{\rho[\mu}\,K_{\nu]\sigma}$$

contract $P^{\mu\rho}$

3. $$\hat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma\,R_{n\hat{\nu}n\hat{\sigma}} + \sigma\,2\,K^{\mu}{}_{[\mu}\,K_{\nu]\sigma}$$

contract $P^{\nu\sigma}$

4. $$\hat{R} = R - 2\sigma\,R_{nn} + \sigma\,(K^2 - K_{\mu\nu}\,K^{\mu\nu})$$

Codazzi relations (one normal projection)

1. $$2\,\hat{\nabla}_{[\hat{\mu}}\,\hat{\nabla}_{\hat{\nu}]}\,n^{\hat{\rho}} = \hat{R}^{\hat{\rho}}{}_{\lambda\hat{\mu}\hat{\nu}}\,n^{\lambda}$$

2. $$2\,\hat{\nabla}_{[\mu}\,K_{\nu]}{}^{\rho} = R^{\rho}{}_{n\hat{\mu}\hat{\nu}}$$

contract $P^{\mu\rho}$

3. $$2\,\hat{\nabla}_{[\mu}\,K_{\nu]}{}^{\mu} = R_{n\hat{\nu}}$$

Notation

$\hat{\mu}$: projected index,

$X_{\hat{\mu}} \equiv P_{\mu}{}^{\nu}\,X_{\nu}$,

$X_n \equiv n^{\nu}\,X_{\nu}$.

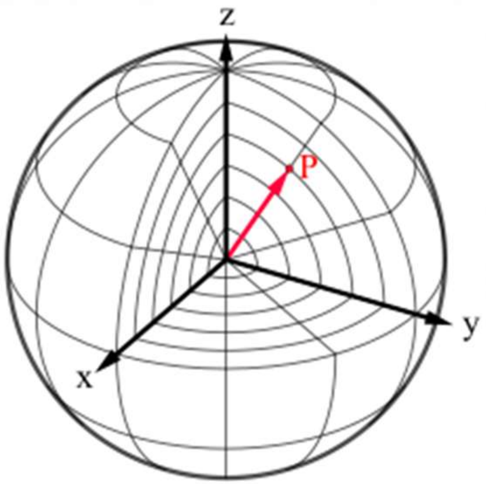
$\hat{\nabla}_{\mu} \equiv \nabla_{\hat{\mu}}$.

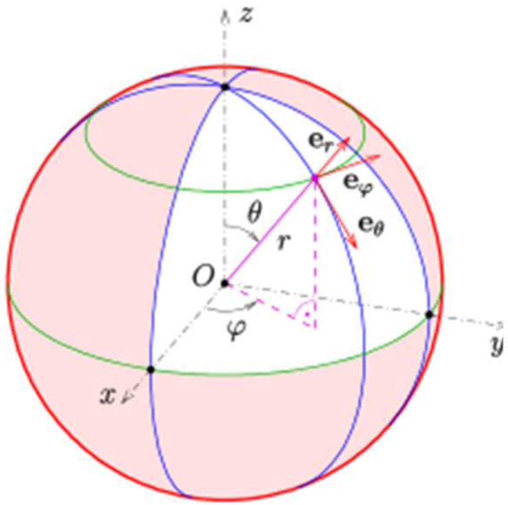
Example: Curvature to intrinsic/extrinsic curvatures of hypersurface

For a sphere of radius r in the flat space, we obtained,

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$$

$$\widehat{R} = \frac{2}{r^2}, \ K_{\theta\theta} = r, \ K_{\phi\phi} = r\sin^2\theta, \ K = \frac{2}{r}$$





The contracted Gauss equation in the flat space:

$$\boxed{\widehat{R} = \cancel{R} - \sigma 2 \cancel{R_{nn}} + \overbrace{\sigma}^{=1}(K^2 - K_{\mu\nu}K^{\mu\nu})}$$

$$\widehat{R} = K^2 - K_{\mu\nu}K^{\mu\nu} = K^2 - K_{\theta\theta}K^{\theta\theta} - K_{\phi\phi}K^{\phi\phi}$$

$$K^{\theta\theta} = g^{\theta\theta}g^{\theta\theta}K_{\theta\theta}, \ K^{\phi\phi} = g^{\phi\phi}g^{\phi\phi}K_{\phi\phi}$$

$$= K^2 - (g^{\theta\theta})^2(K_{\theta\theta})^2 - (g^{\phi\phi})^2(K_{\phi\phi})^2$$

$$= \left(\frac{2}{r}\right)^2 - \frac{1}{r^4}r^2 - \frac{1}{r^4\sin^4\theta}r^2\sin^4\theta$$

$$= \frac{4}{r^2} - \frac{1}{r^2} - \frac{1}{r^2}$$

$$= \frac{2}{r^2}$$

Dimension check: K and R

Example: Curvature to intrinsic/extrinsic curvatures of hypersurface

$$x^\mu = (r, \phi, z),$$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}z^2.$$

$${}^{(3)}R^\mu{}_{\nu\rho\sigma} = 0.$$

$${}^{(3)}R_{\mu\nu} = 0, \qquad {}^{(3)}R = 0.$$

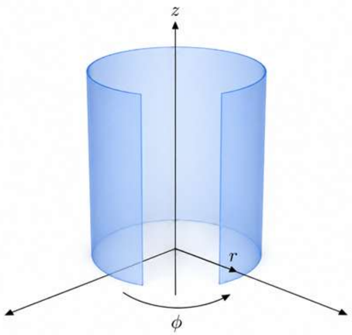
$$K_{ab} = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}.$$

$$K^{\phi\phi} = h^{\phi\phi}h^{\phi\phi}K_{\phi\phi} = \frac{1}{R^2}\frac{1}{R^2}R = \frac{1}{R^3}$$

$$\mathrm{d}\ell^2 = R^2\mathrm{d}\phi^2 + \mathrm{d}z^2.$$

$${}^{(2)}R^a{}_{bcd} = 0.$$

$${}^{(2)}R_{ab} = 0, \qquad {}^{(2)}R = 0.$$



$$\boxed{{}^{(2)}R = 0 = {}^{(3)}R_{\|} + K^2 - K_{ab}K^{ab} = 0 + \frac{1}{R^2} - \frac{1}{R^2}}.$$

3+1 Decomposition of Einstein eq.

$$\begin{cases} \text{(Gauss scalar eq.)} & \hat{R} = R - \sigma 2 R_{nn} + \sigma (K^2 - K_{\mu\nu}K^{\mu\nu}) \\ \text{(energy density)} & \rho_e = T_{nn} \end{cases}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\Lsh\ G_{nn} = 8\pi G T_{nn} \cdots (1)$$

$$\Rightarrow \overbrace{R_{nn}}^{=\frac{1}{2}(\hat{R} - \cancel{R} + K^2 - K_{ij}K^{ij})} - \frac{1}{2} R \underbrace{g_{nn}}_{= g_{\mu\nu}n^\mu n^\nu = n_\mu n^\mu = \sigma = -1} = 8\pi G \overbrace{T_{nn}}^{E}$$

$$\Rightarrow \boxed{\hat{R} + K^2 - K_{ij}K^{ij} = 16\pi G E}$$

3+1 Decomposition of Einstein equations

$$
\begin{aligned}
\alpha = N &: \text{ lapse,}\\
\boldsymbol{\beta} = \beta^i\partial_i = N^i\partial_i &: \text{ shift vector,}\\
X_{\widehat{\mu}} \equiv P_\mu{}^\nu X_\nu &: \text{ projection onto } \Sigma_t,\\
\nabla_{\hat{i}} X^j \equiv D_i X^j \equiv P_i{}^\mu P^j{}_\nu \nabla_\mu X^\nu &: \text{ spatial covariant derivative.}
\end{aligned}
$$

$$
{}^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}
$$

$$
\hookrightarrow \begin{cases}
(1)\ {}^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow \ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\
(2)\ {}^{(4)}G_{n\widehat{\mu}} = 8\pi G T_{n\widehat{\mu}} \ \rightarrow \ D_i K - D_j K^j{}_i = -8\pi G p_i \\
(3)\ {}^{(4)}G_{\widehat{\mu}\widehat{\nu}} = 8\pi G T_{\widehat{\mu}\widehat{\nu}} \ \rightarrow \ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\
\qquad\qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\
\qquad\qquad - D_i D_j \alpha - 8\pi G\alpha[S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\
K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow \ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i
\end{cases}
$$

$$
G_{\mu\nu} \ \rightarrow \ \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu}\widehat{\nu}} \end{cases} \xrightarrow[{}^{(4)}\nabla \to {}^{(3)}\nabla \atop g_{\mu\nu} \to \gamma_{ij}]{{}^{(4)}R \to {}^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)
$$

① Geometry and evolution are described by fields on each Σ_t,

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${}^{(3)}R$, N, β^i

② D_i is intrinsic to Σ_t, defining spatial derivatives, geodesics, and parallel transport,

$\nabla_{\hat{i}} X^j = D_i X^j$

③ We have already learned that matter variables are projected onto Σ_t,

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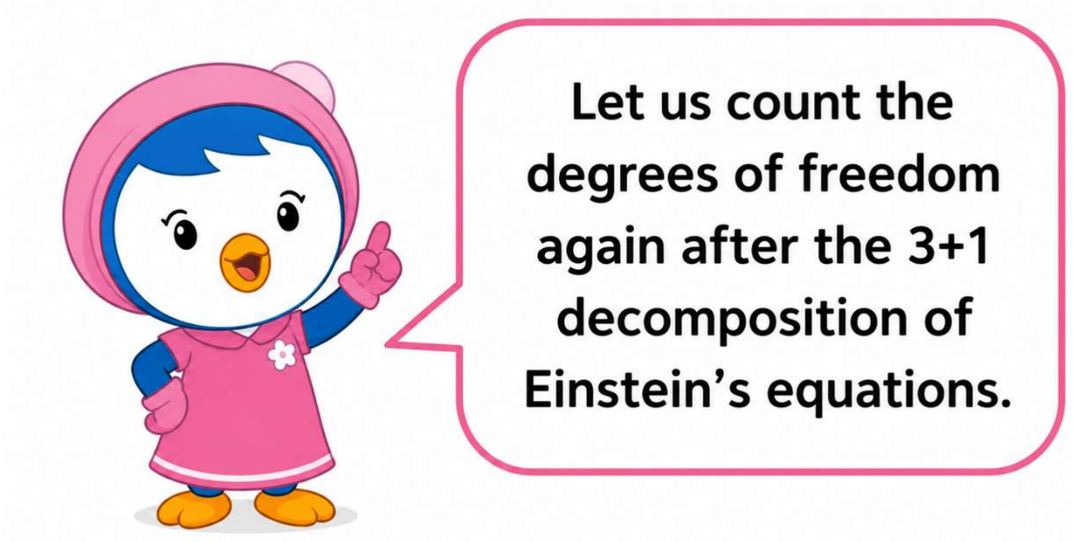

Let us count the degrees of freedom again after the 3+1 decomposition of Einstein's equations.

Physical dofs of metric tensor

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Metric tensor

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}$$

10 independent components

$$g_{\mu\nu} = g_{\nu\mu}$$

coordinate freedom / gauge choice

Coordinate freedom

$$x'^{\mu} = f^{\mu}(x)$$

$$g'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x)$$

Same physics, different coordinates

4 arbitrary coordinate functions
$\Rightarrow$ choose 4 gauge conditions

10 → 6

Einstein constraints on the initial slice

Constraints

$$G_{00} = 8\pi T_{00},$$

$$G_{0i} = 8\pi T_{0i} \quad (i = 1, 2, 3)$$

Hamiltonian + momentum constraints

These 4 equations restrict the remaining 6 unknowns, so not all 6 are freely specifiable.

6 → 2

Remaining 2 propagating DOF = gravitational-wave polarizations h_+ and $h_\times$

Physical dofs after 3+1 decomposition

$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} & \to \quad R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} & \to \quad D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ ^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} & \to \quad \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) \\ & \qquad\qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ & \qquad\qquad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} & \to \quad \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$

$$\boxed{\begin{aligned} (\gamma_{ij}, K_{ij}) :\ & \text{12 phase-space variables,} \\ \text{4 constraints} \implies & 12 - 4 = 8, \\ \text{4 coordinate freedoms} \implies & 8 - 4 = 4. \end{aligned}}$$

$$\boxed{12 - 4_{\,\text{constraints}} - 4_{\,\text{gauge}} = 4_{\,\text{phase space}} = 2_{\,\text{configuration}}}$$


Let's go back to our formalism~

ADM formalism – Action Decomposition

$$S_{\rm H} = \int_{\mathcal{V}} R\sqrt{-g}\, d^4x,\ \ \mathcal{V} := \cup_{t=t_1}^{t_2} \Sigma_t$$

$$\varsigma\ \ R = \widehat{R} + [-K^2 + K_{\mu\nu}K^{\mu\nu}] + 2\nabla_\mu(Kn^\mu) - \frac{2}{N}(\widehat{\nabla}_\mu \widehat{\nabla}^\mu N)]$$

$$\varsigma\ \ \sqrt{-g} = N\sqrt{\gamma}$$

$$\Rightarrow \boxed{\begin{aligned} S_{\rm H} &= S_{\rm H,vol.} + S_{\rm H,surf.} \\ &= \int_{t_1}^{t_2} \left\{ \int_{\Sigma_t} N(\widehat{R} - K^2 + K_{ij}K^{ij})\sqrt{\gamma}\, d^3x \right\} dt \\ &\quad - \int_{\mathcal{V}} 2(\widehat{\nabla}_i \widehat{\nabla}^i N)\sqrt{\gamma}\, d^4x + \int_{\mathcal{V}} 2\nabla_\mu(Kn^\mu)\sqrt{-g}\, d^4x \end{aligned}}$$

The gravitational Lagrangian density for $q = (\gamma_{ij}, N, \beta^i)$ is given by

$$\boxed{\mathcal{L}_{\rm H,vol.}(q,\dot{q}) = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) = N\sqrt{\gamma}[\widehat{R} + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{ij}K_{kl}]}$$

ADM formalism –Identifying Dynamical Variables

The gravitational Lagrangian density for $q = (\gamma_{ij}, N, \beta^i)$ is given by

$$\boxed{\mathcal{L}_{\mathrm{H,vol.}}(q,\dot{q}) = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) = N\sqrt{\gamma}[\widehat{R} + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{ij}K_{kl}]}$$

$$\begin{aligned}
K_{ij} &= \frac{1}{2}\mathcal{L}_{\boldsymbol{n}}P_{ij} = \frac{1}{2N}\mathcal{L}_{\boldsymbol{m}}P_{ij} = \frac{1}{2N}\mathcal{L}_{\boldsymbol{\partial_t}-\boldsymbol{\beta}}P_{ij} \\
&= \frac{1}{2N}(\mathcal{L}_{\boldsymbol{\partial_t}}P_{ij} - \mathcal{L}_{\boldsymbol{\beta}}P_{ij})
\end{aligned}$$

$$\varsigma \quad \mathcal{L}_{\boldsymbol{\partial_t}}P_{ij} = (\partial_t)^\alpha\partial_\alpha P_{ij} + P_{\alpha j}\partial_i \underbrace{\cancel{(\partial_t)^\alpha}}_{=\delta_t^\alpha} + K_{i\alpha}\partial_j \underbrace{\cancel{(\partial_t)^\alpha}}_{=\delta_t^\alpha} = \frac{\partial P_{ij}}{\partial t} = \frac{\partial \gamma_{ij}}{\partial t}$$

$$\varsigma \quad \mathcal{L}_{\boldsymbol{\beta}}P_{ij} = \mathcal{L}_{\boldsymbol{\beta}}\gamma_{ij} = \beta^k\cancel{\widehat{\nabla}_k\gamma_{ij}} + \gamma_{kj}\widehat{\nabla}_i\beta^k + \gamma_{ik}\widehat{\nabla}_j\beta^k$$

$$= \frac{1}{2N}(\dot{\gamma}_{ij} - \gamma_{kj}\widehat{\nabla}_i\beta^k - \gamma_{ik}\widehat{\nabla}_j\beta^k)$$

$$\boxed{\pi_N = \frac{\partial\mathcal{L}}{\partial\dot{N}} = 0, \qquad \pi_i = \frac{\partial\mathcal{L}}{\partial\dot{\beta}^i} = 0.}$$

ADM formalism – Conjugate momentum

The gravitational Lagrangian density for $q = (\gamma_{ij}, N, \beta^i)$ is given by

$$\boxed{\mathcal{L}_{\mathrm{H,vol.}}(q,\dot q) = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) = N\sqrt{\gamma}[\widehat{R} + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{ij}K_{kl}]}$$

$$\pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot\gamma_{ij}} = \frac{\partial}{\partial \dot\gamma_{ij}}[N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij})]$$

$$\varsigma \quad K_{ij} = \frac{1}{2N}(\dot\gamma_{ij} - \gamma_{kj}\widehat{\nabla}_i\beta^k - \gamma_{ik}\widehat{\nabla}_j\beta^k)$$

$$= -2N\sqrt{\gamma}\,K\underbrace{\frac{\partial K}{\partial \dot\gamma_{ij}}}_{=\gamma^{ij}\frac{1}{2N}} + 2N\sqrt{\gamma}\,\underbrace{K^{kl}\frac{\partial K_{kl}}{\partial \dot\gamma_{ij}}}_{K^{ij}\frac{1}{2N}}$$

$$= \sqrt{\gamma}(-K\gamma^{ij} + K^{ij})$$

$$= \sqrt{\gamma}(K^{ij} - K\gamma^{ij})$$

$$\boxed{\pi^{ij} = \sqrt{\gamma}(K^{ij} - K\gamma^{ij})}$$

ADM formalism – Hamiltonian construction

$$
\begin{aligned}
\mathcal{H}_{\mathrm{H,vol.}} &= \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}_{\mathrm{H,vol.}} \\
&= \sqrt{\gamma}(K^{ij} - K\gamma^{ij})\dot{\gamma}_{ij} - N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij})
\end{aligned}
$$

$$
K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - \gamma_{kj}\widehat{\nabla}_i\beta^k - \gamma_{ik}\widehat{\nabla}_j\beta^k)
$$

$$
\dot{\gamma}_{ij} = 2NK_{ij} + \gamma_{kj}\widehat{\nabla}_i\beta^k + \gamma_{ik}\widehat{\nabla}_j\beta^k
$$

$$
\begin{aligned}
&= \sqrt{\gamma}\underbrace{(K^{ij} - K\gamma^{ij})(2NK_{ij}}_{\to 2N(K_{ij}K^{ij}-K^2)} + \gamma_{kj}\widehat{\nabla}_i\beta^k + \gamma_{ik}\widehat{\nabla}_j\beta^k) - N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) \\
&= -\sqrt{\gamma}[N(\widehat{R} + K^2 - K_{ij}K^{ij}) - 2(K^i{}_j - K\gamma^i{}_j)\widehat{\nabla}_i\beta^j] \\
&= -\sqrt{\gamma}[N(\widehat{R} + K^2 - K_{ij}K^{ij}) + 2\beta^j(\widehat{\nabla}_i K^i{}_j - \nabla_j K) - 2\widehat{\nabla}_i(K^i{}_j\beta^j - K\beta^i) \\
&= -\sqrt{\gamma}[N\underbrace{(\widehat{R} + K^2 - K_{ij}K^{ij})}_{=C_0} - 2\beta^j\underbrace{(\widehat{\nabla}_j K - \widehat{\nabla}_i K^i{}_j)}_{=C_j}] + 2\sqrt{\gamma}\,\widehat{\nabla}_i(K^i{}_j\beta^j - K\beta^i)
\end{aligned}
$$

$$
\Rightarrow \boxed{\mathcal{L}_{\mathrm{H,vol.}} = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij})}
$$

$$
\boxed{\mathcal{H}_{\mathrm{H,vol.}} = -\sqrt{\gamma}(NC_0 - 2\beta^i C_i) + 2\sqrt{\gamma}\,\widehat{\nabla}_i(K^i{}_j\beta^j - K\beta^i)}
$$

$$
\boxed{
\begin{aligned}
\delta H/\delta N = 0 \quad &\Longleftrightarrow \quad G_{nn} = 0, \\
\delta H/\delta\beta^i = 0 \quad &\Longleftrightarrow \quad G_{n\hat{i}} = 0, \\
\dot{\gamma}_{ij} = \delta H/\delta\pi^{ij} \quad &\Longleftrightarrow \quad K_{ij}\text{의 정의}, \\
\dot{\pi}^{ij} = -\delta H/\delta\gamma_{ij} \quad &\Longleftrightarrow \quad G_{\hat{i}\hat{j}} = 0.
\end{aligned}
}
$$

ADM formalism – ADM Energy/Momentum

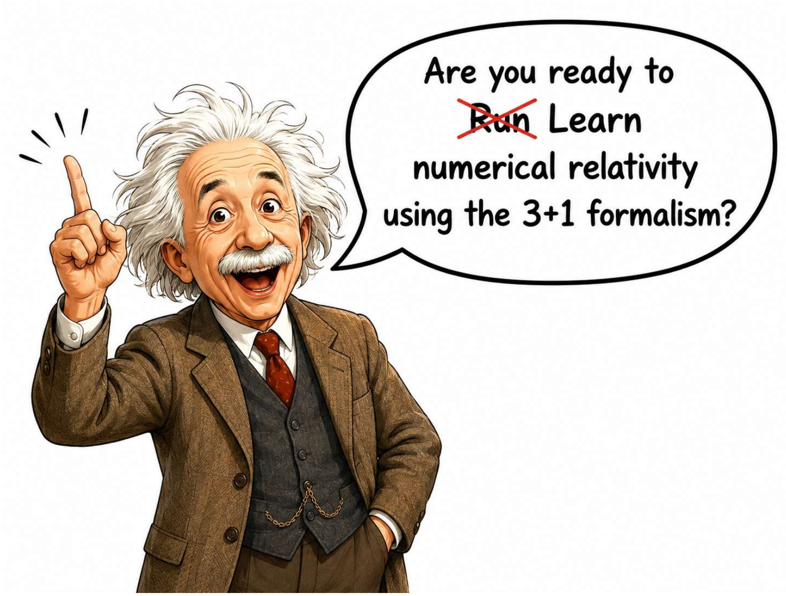
$$\begin{aligned} H_{\rm G}^{\rm reg.} &= H_{\rm H,vol.} + H_{\rm G,surf.}^{\rm reg.} \\ &= -\frac{1}{16\pi G}\int_{\Sigma_t} d^3x \sqrt{\gamma}\,(NC_0 - 2\beta^i C_i) \\ &\quad + \frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\,\cdot 2[r_i(K^i{}_j\beta^j - K\beta^i) - N(\kappa-\kappa_0)|_{S_t}] \end{aligned}$$

$$\circlearrowleft \quad H \xrightarrow[C_0\,=\,0\,=\,C_i]{\text{solution}} H_{\rm solution}$$

$$\boxed{H_{\rm solution} = \frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\,\cdot 2[r_i(K^i{}_j\beta^j - K\beta^i) - N(\kappa-\kappa_0)|_{S_t}]}$$

where K_{ij} : extrinsic curvature on Σ_t,

κ : extrinsic curvature scalar on $\Sigma_t \cap S$


Are you ready to
~~Run~~ Learn
numerical relativity
using the 3+1 formalism?

1. Initial-data methods

A Conformal decomposition (York–Lichnerowicz)

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}\,, \qquad K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K$$

$$\tilde{A}^{ij} = (\tilde{L}W)^{ij} + \tilde{A}^{ij}_{TT}$$

Solve the constraint equations as elliptic equations for conformal data.

B Conformal thin-sandwich (CTS / XCTS)

$$\tilde{A}^{ij} = \frac{1}{2\tilde{N}}\left[(\tilde{L}\beta)^{ij} - \tilde{u}^{ij}\right]$$

$$\partial_t \tilde{\gamma}_{ij} = \tilde{u}_{ij}$$

Uses lapse–shift data to build quasi-equilibrium initial data.

2. Evolution formulations

C ADM (Arnowitt–Deser–Misner)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

$$\partial_t K_{ij} = \cdots$$

Geometrically transparent, but not the most robust numerically.

D BSSN (Baumgarte–Shapiro–Shibata–Nakamura)

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}\,, \quad \tilde{A}_{ij} = e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right)$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$$

Conformal variables improve numerical stability.

E Z4 / Z4c

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

Promotes constraint violation to evolved fields and allows damping.

3. Black-hole treatment

F Puncture method / moving puncture

$$\psi = 1 + \sum_a \frac{m_a}{2r_a} + u$$

$$(\partial_t - \beta^i\partial_i)\alpha = -2\alpha K$$

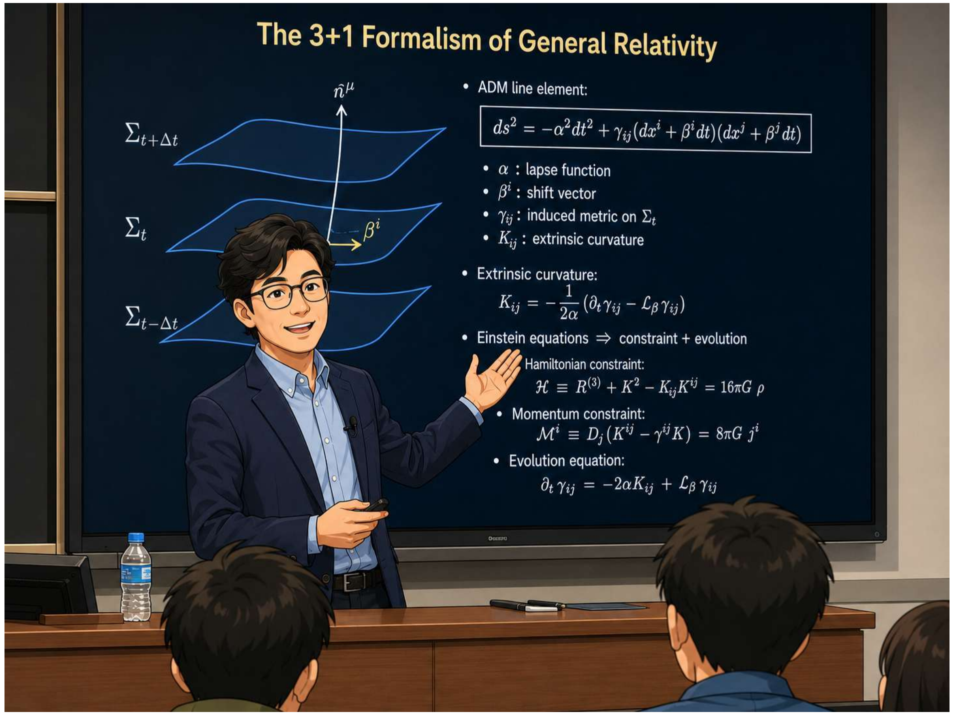
$$\partial_t \beta^i = \frac{3}{4}B^i,$$

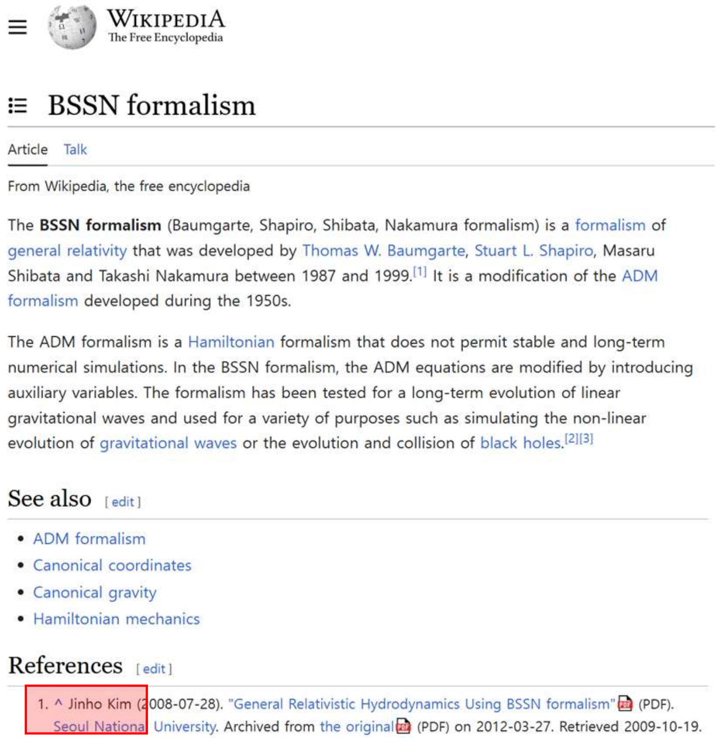
$$\partial_t B^i = \partial_t \tilde{\Gamma}^i - \eta B^i$$

Represents black holes without excision; widely used in binary black-hole evolutions.

Initial data: solve constraints $\longrightarrow$ **Evolution:** choose a stable formulation $\longrightarrow$ **Black holes:** choose a suitable treatment

More about NR applications





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BSSN formalism

Article Talk

From Wikipedia, the free encyclopedia

The **BSSN formalism** (Baumgarte, Shapiro, Shibata, Nakamura formalism) is a formalism of general relativity that was developed by Thomas W. Baumgarte, Stuart L. Shapiro, Masaru Shibata and Takashi Nakamura between 1987 and 1999.[1] It is a modification of the ADM formalism developed during the 1950s.

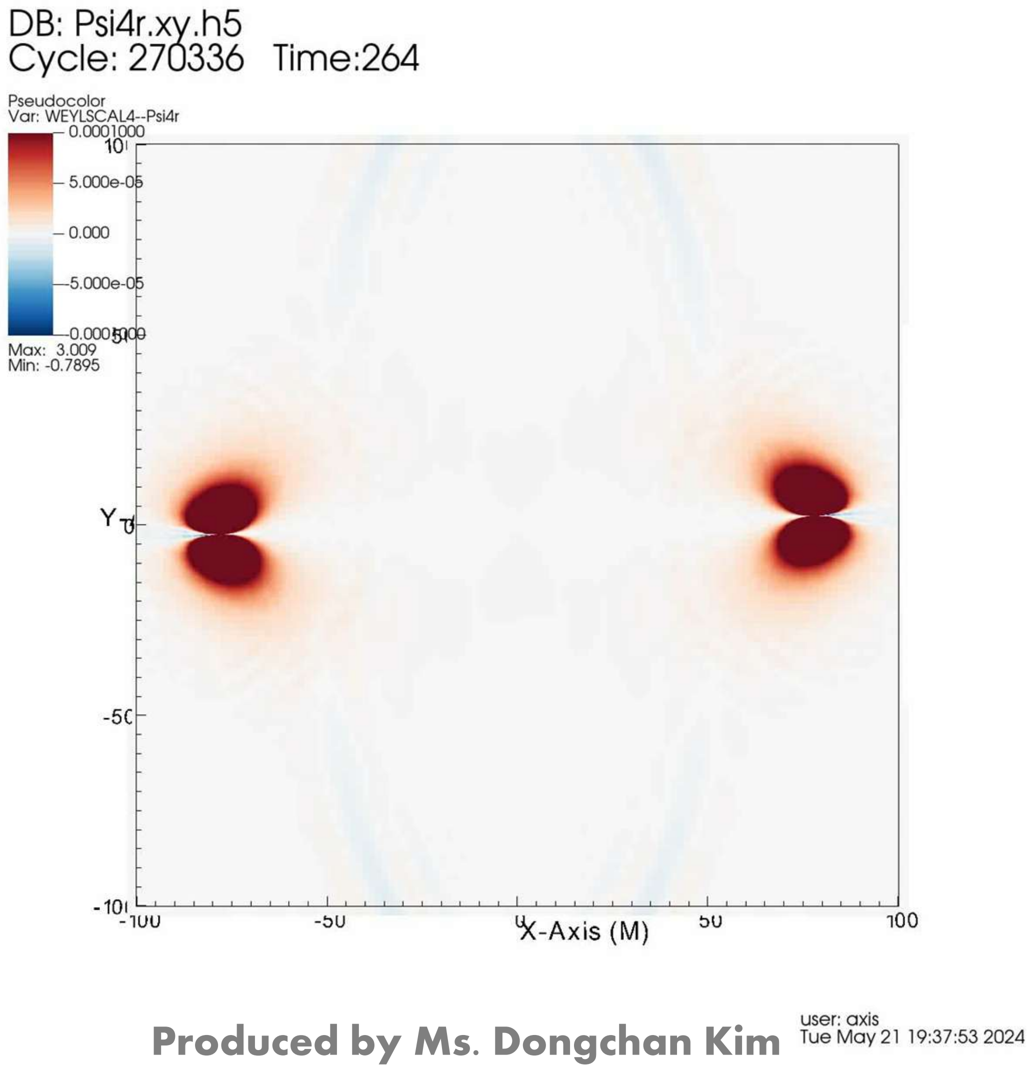
The ADM formalism is a Hamiltonian formalism that does not permit stable and long-term numerical simulations. In the BSSN formalism, the ADM equations are modified by introducing auxiliary variables. The formalism has been tested for a long-term evolution of linear gravitational waves and used for a variety of purposes such as simulating the non-linear evolution of gravitational waves or the evolution and collision of black holes.[2][3]

See also [edit]

- ADM formalism
- Canonical coordinates
- Canonical gravity
- Hamiltonian mechanics

References [edit]

1. ^ Jinho Kim (2008-07-28). "General Relativistic Hydrodynamics Using BSSN formalism" (PDF). Seoul National University. Archived from the original (PDF) on 2012-03-27. Retrieved 2009-10-19.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