중력파 파형 & 모델

Hee-Suk Cho (Pusan National University)

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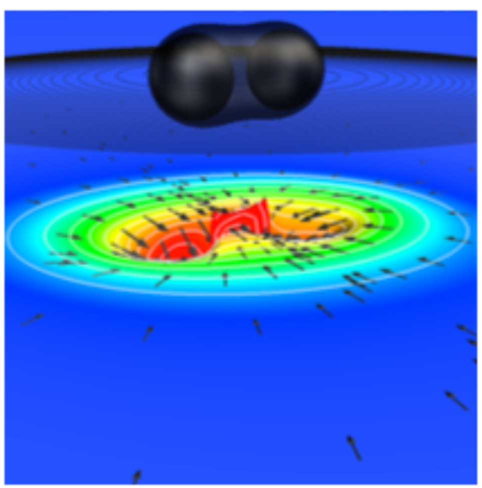
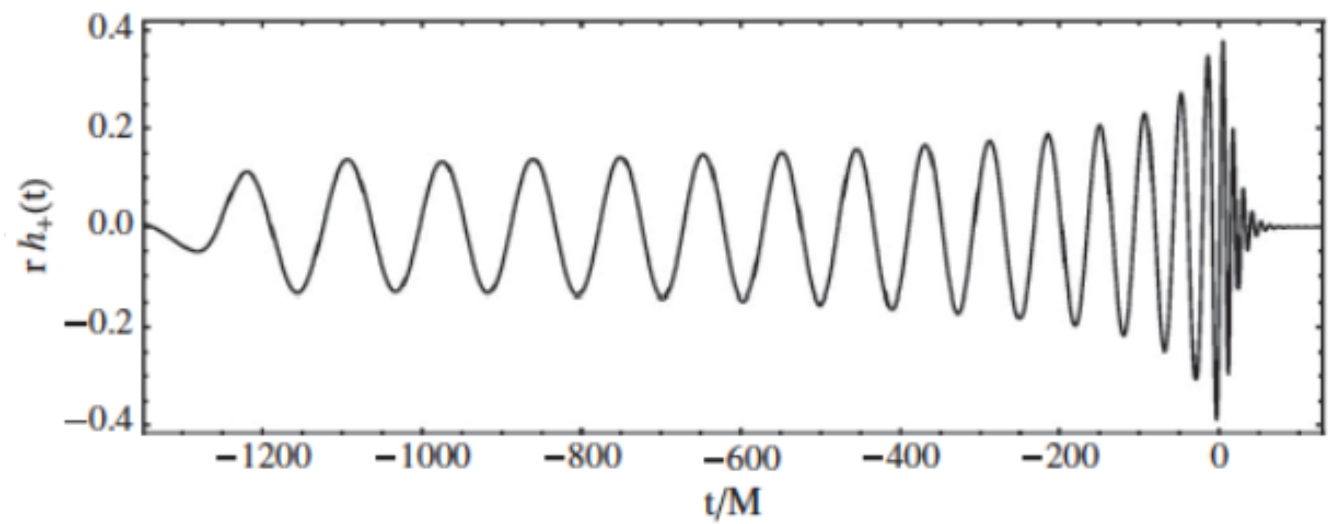




Outline

- CBC waveforms
  - nonspinning
  - spinning
- CBC waveform models
  - Inspiral model (NS-NS)
  - IMR models (BH-BH)

Gravitational-Wave Sources

$$\bar{h}_{ij}(t,r) = \frac{2G}{c^4 r}\underline{\ddot{I}_{ij}}(t - \underline{r/c}),$$

I_{ij} is the mass quadrupole moment.

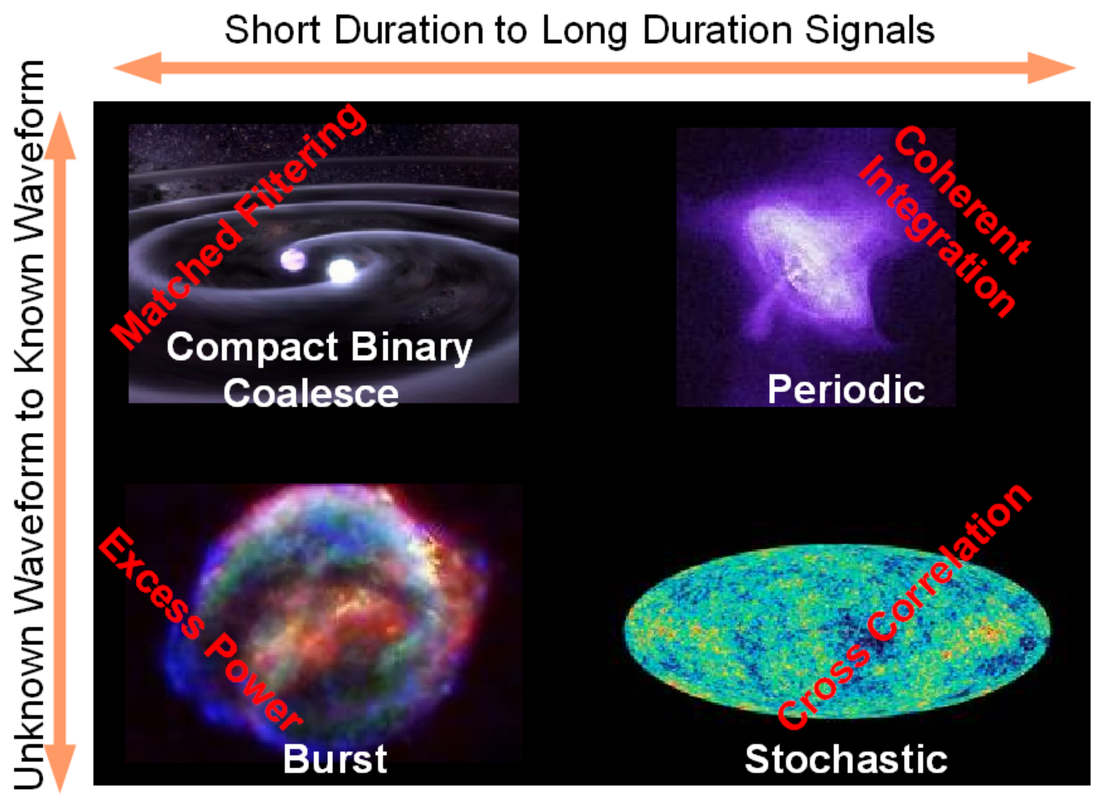
$$\boxed{\text{very weak, accelerating objects, speed of light}}$$

Compact Binary Coalescences:
Neutron Stars / Black Holes

Burst sources: Supernovae
Large uncertainties on waveforms

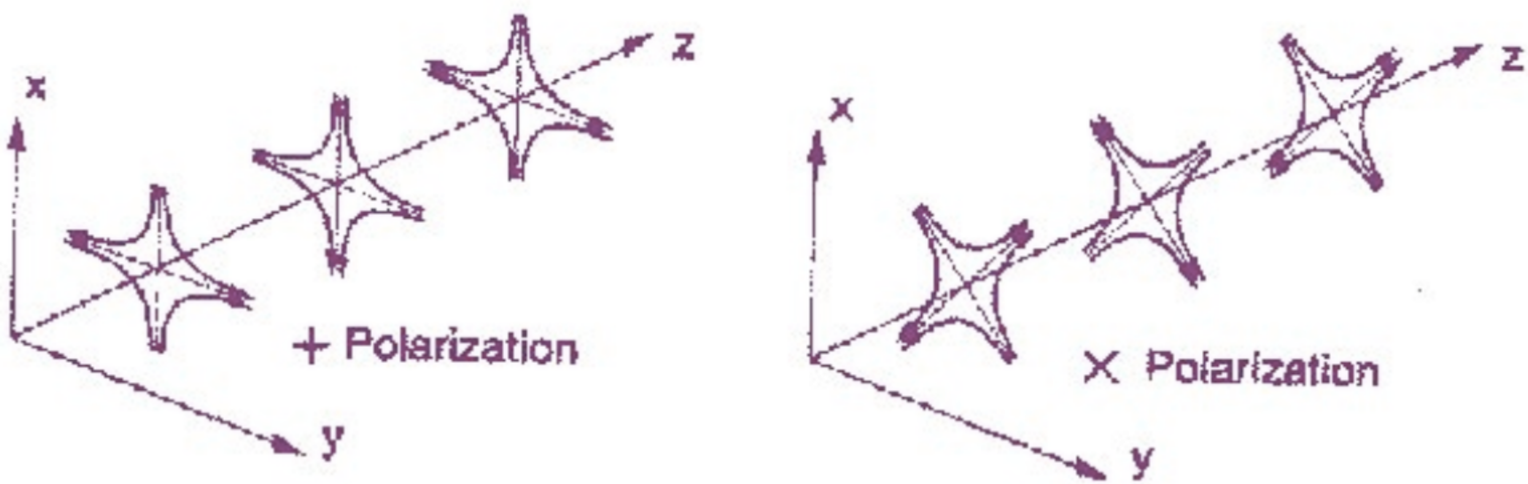
Continuous sources: Pulsars

Stochastic background
from cosmological origin



GW polarizations

$$\boxed{\begin{aligned} h_+ &= -\frac{2G\mu}{c^2 R} x(1+\cos^2\Theta)\cos 2\psi \\ h_\times &= -\frac{2G\mu}{c^2 R} x(2\cos\Theta)\sin 2\psi \end{aligned}}$$



GW polarizations: nonspinning

$$h_+ = -\frac{2G\mu}{c^2 R}\, x\,(1+\cos^2\Theta)\cos 2\psi$$

$$h_\times = -\frac{2G\mu}{c^2 R}\, x\,(2\cos\Theta)\sin 2\psi$$

Orbital phase

Inclination

PN parameter

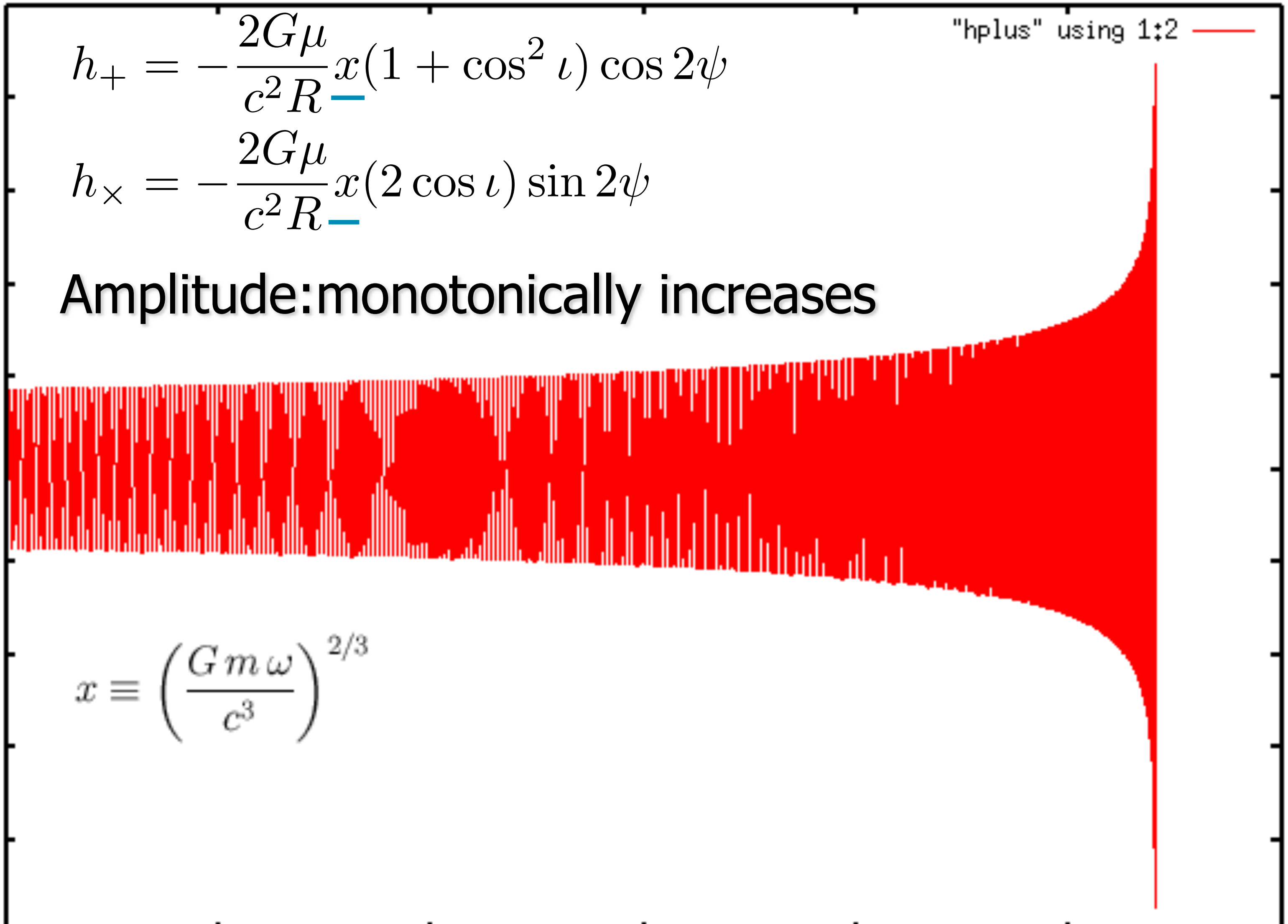
$$x \equiv \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3}$$

Waveform: nonspinning

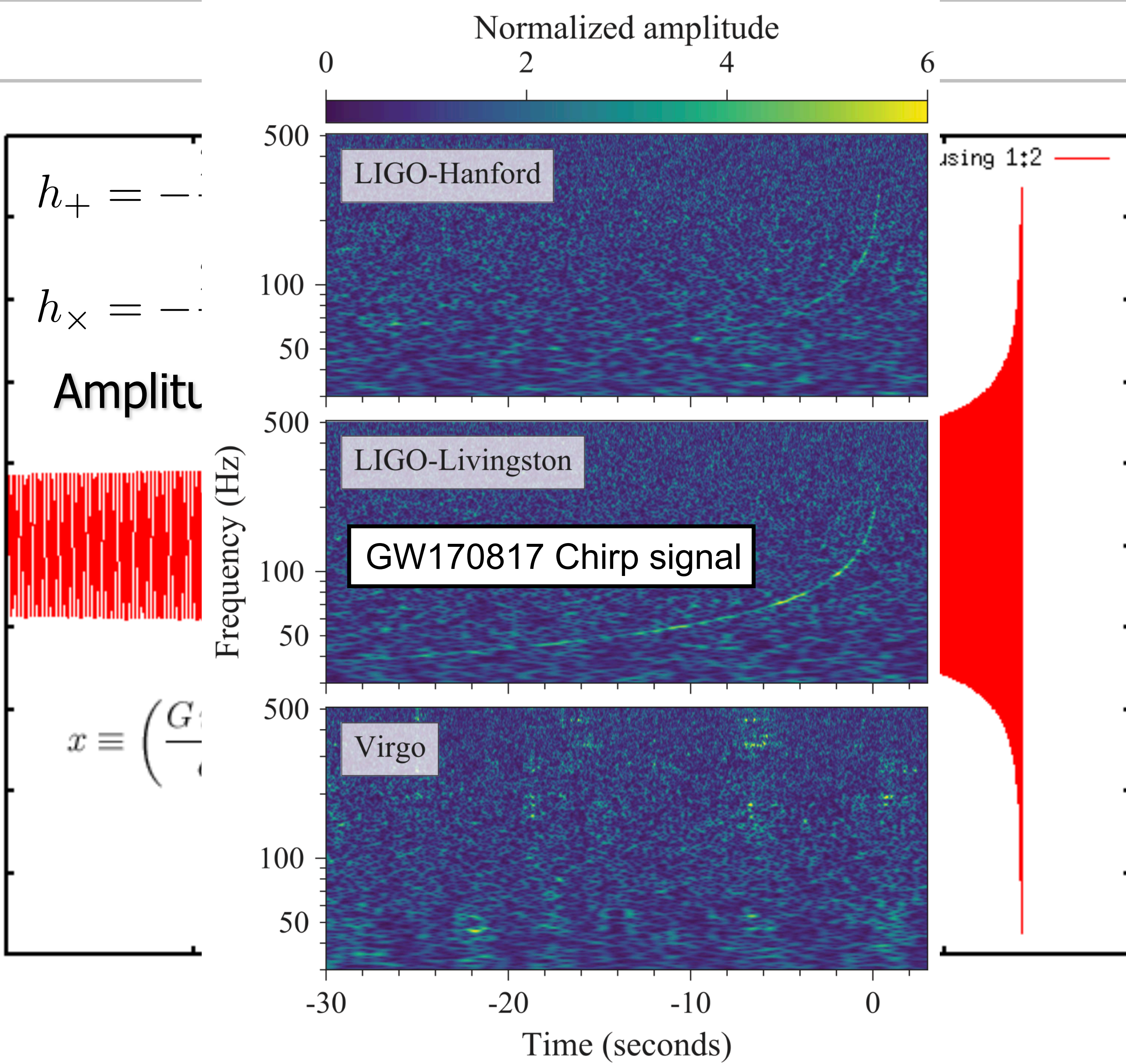
$$h_+ = -\frac{2G\mu}{c^2 R} x (1 + \cos^2 \iota) \cos 2\psi$$

$$h_\times = -\frac{2G\mu}{c^2 R} x (2 \cos \iota) \sin 2\psi$$

Amplitude:monotonically increases



$$x \equiv \left(\frac{G m \omega}{c^3}\right)^{2/3}$$

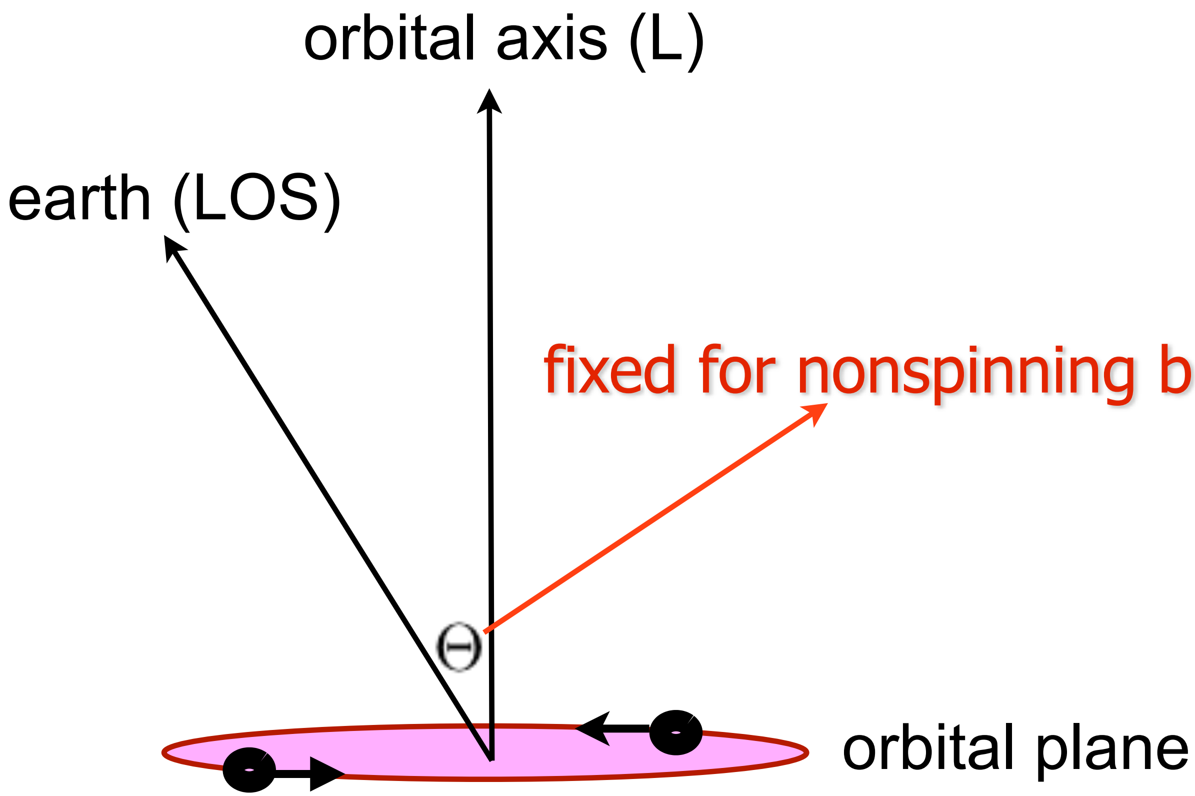


Normalized amplitude
0
2
4
6
500
100
50
LIGO-Hanford
LIGO-Livingston
GW170817 Chirp signal
Virgo
Frequency (Hz)
-30
-20
-10
0
Time (seconds)

Inclination (Θ)

$$h_+ = -\frac{2G\mu}{c^2 R} x(1 + \cos^2 \underline{\Theta}) \cos 2\psi$$

$$h_\times = -\frac{2G\mu}{c^2 R} x(2 \cos \underline{\Theta}) \sin 2\psi$$

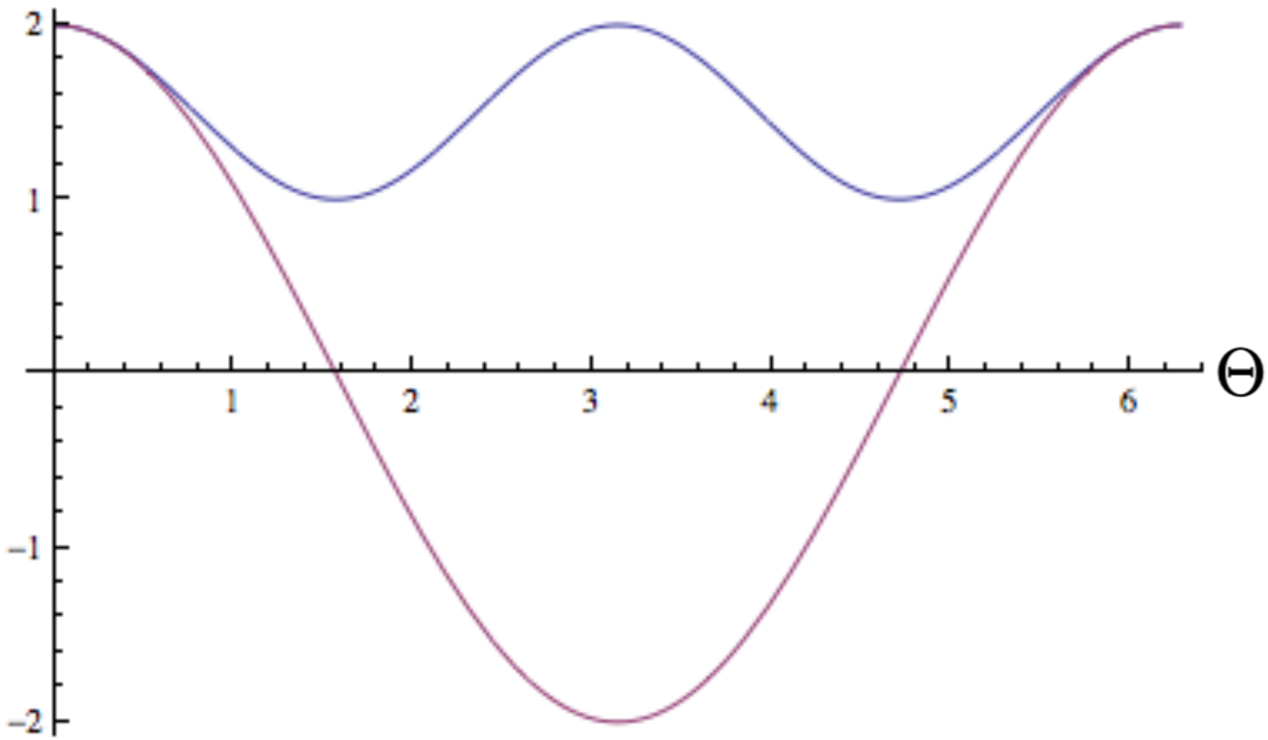


Dependence on inclination

$$h_+ = -\frac{2G\mu}{c^2 R} x \underline{(1 + \cos^2\Theta)} \cos 2\psi$$

$$h_\times = -\frac{2G\mu}{c^2 R} x \underline{(2\cos\Theta)} \sin 2\psi$$

$h_+ \propto (1 + \cos^2\Theta)$



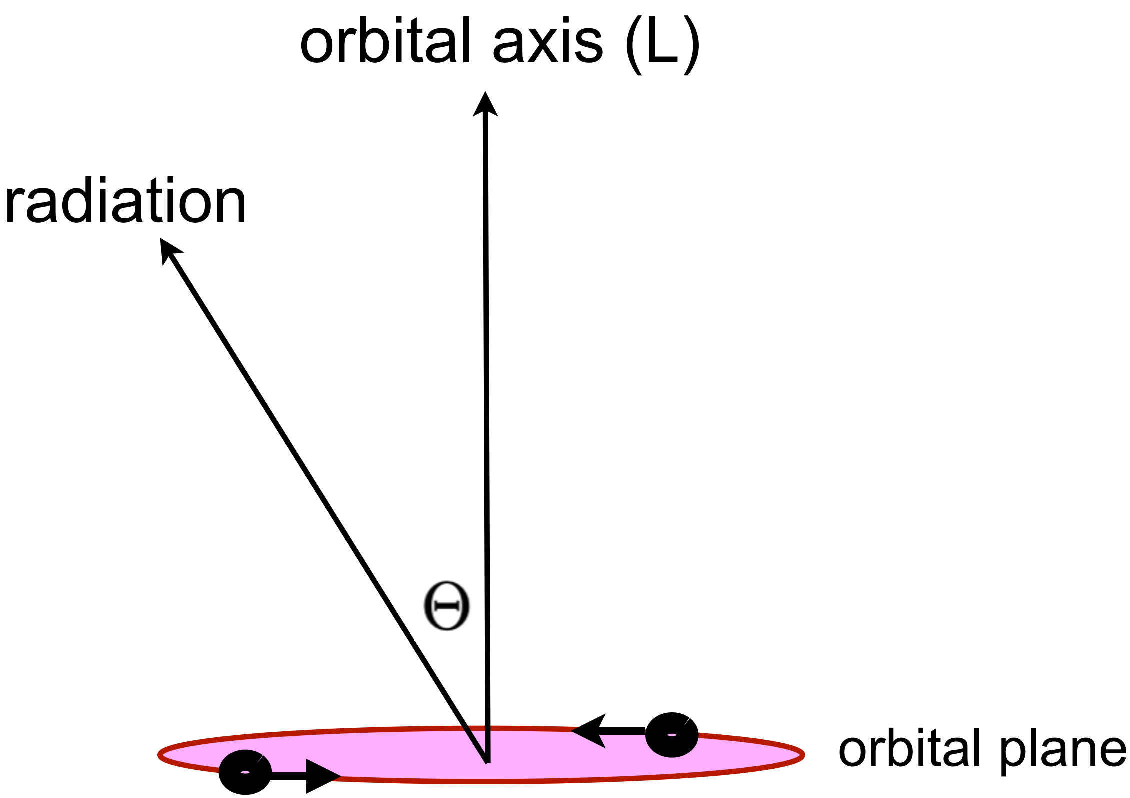
$h_\times \propto (2\cos\Theta)$

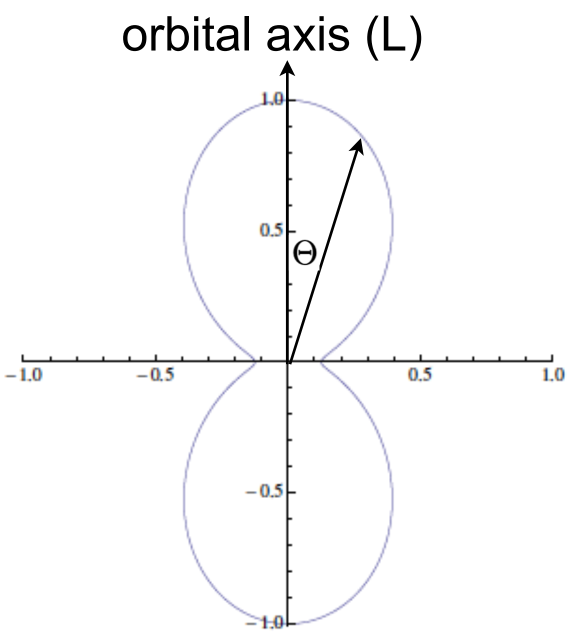
Radiation power (source frame)

$$h_+ = -\frac{2G\mu}{c^2 R} x(1+\cos^2\Theta)\cos 2\psi$$

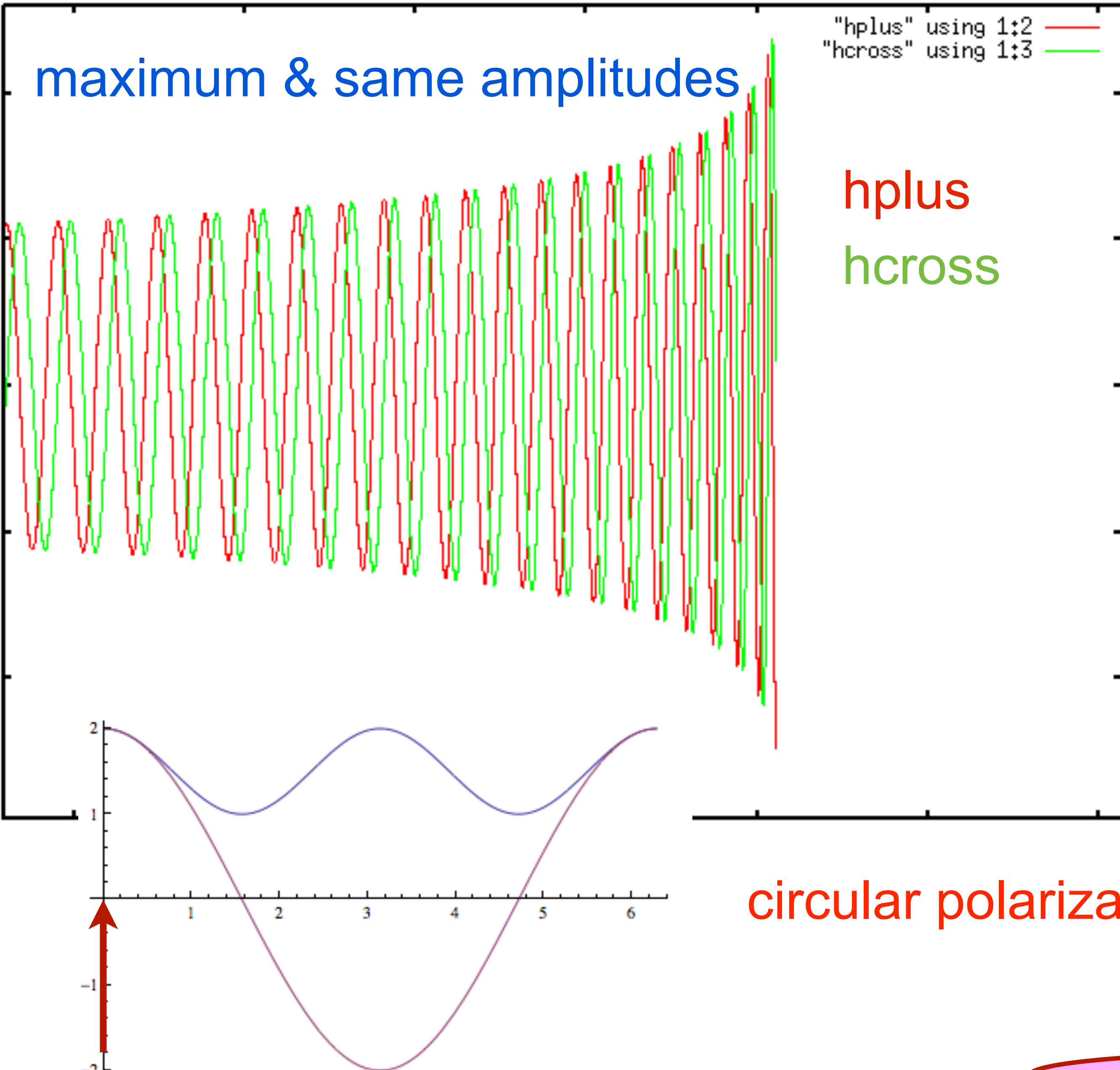
$$h_\times = -\frac{2G\mu}{c^2 R} x(2\cos\Theta)\sin 2\psi$$

$$\left(\frac{dP}{d\Omega}\right) = \frac{2G\mu^2 a^4 \omega^6}{\pi c^5} g(\Theta),$$

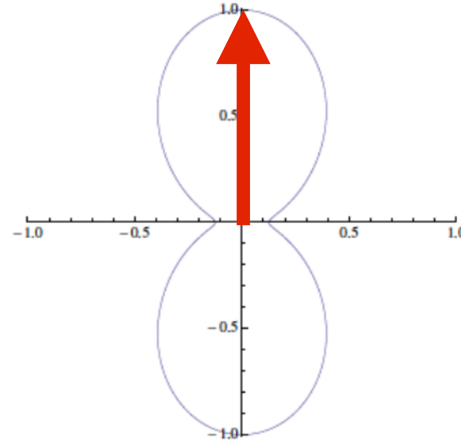




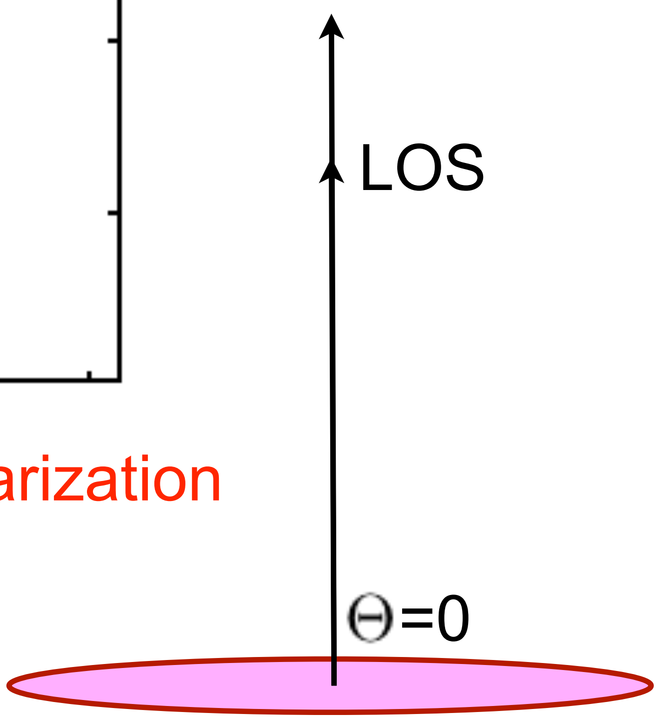
Non spinning GW polarizations (theta=0)



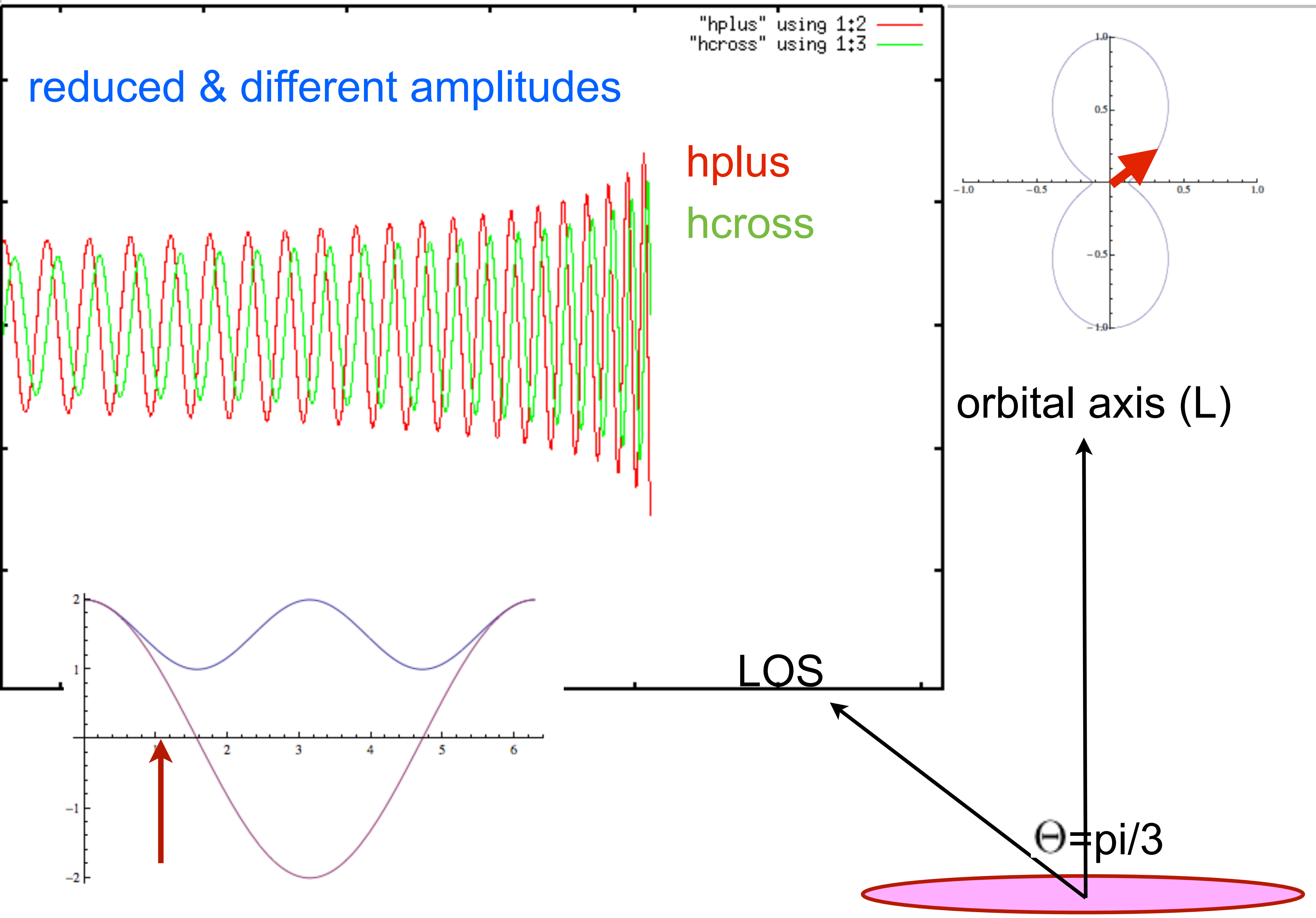
circular polarization



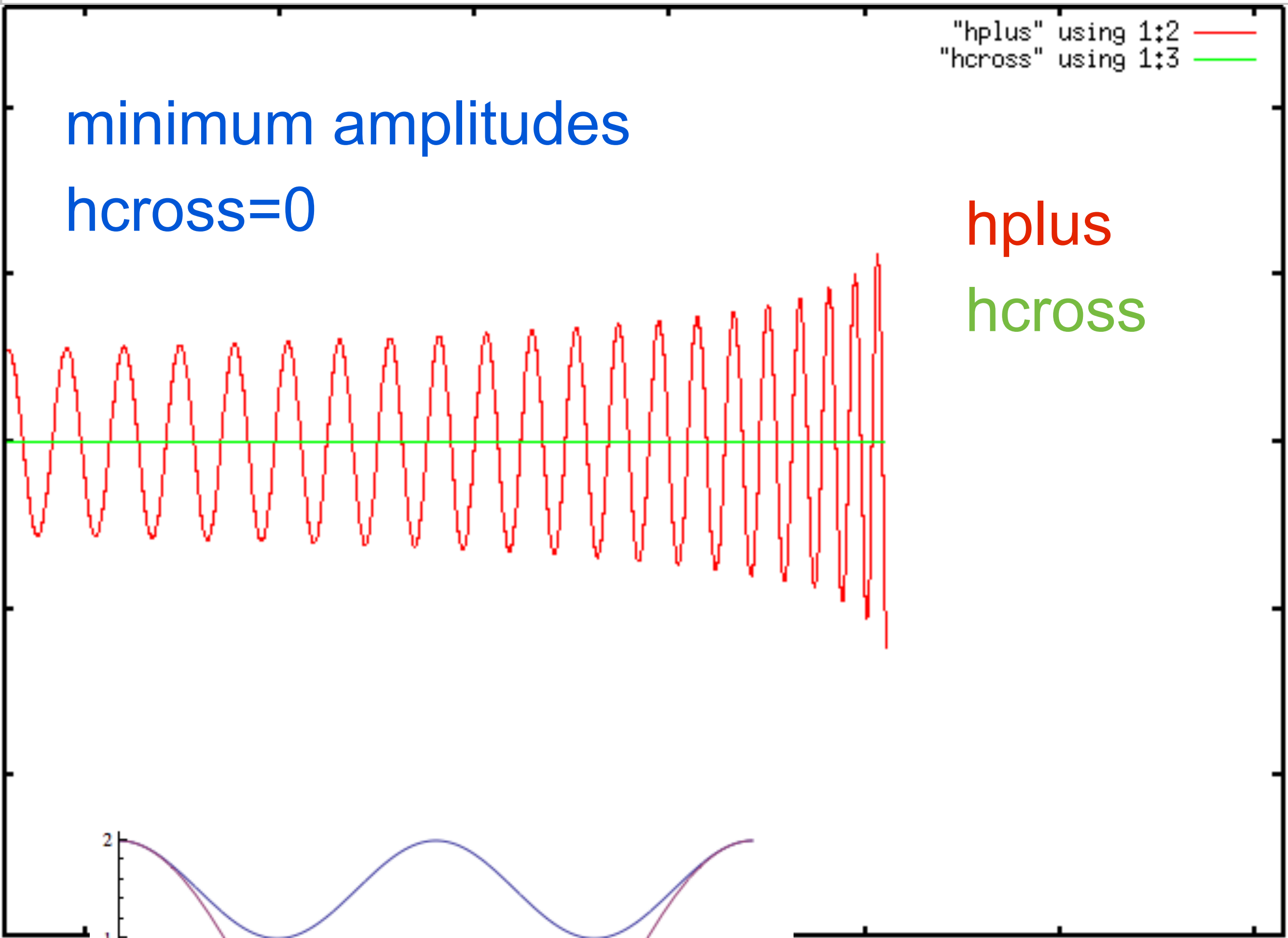
orbital axis (L)



Non spinning polarizations (theta=pi/3)



Non spinning polarizations (theta=pi/2)

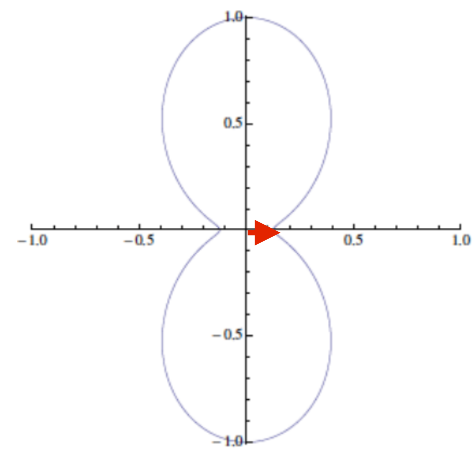


minimum amplitudes

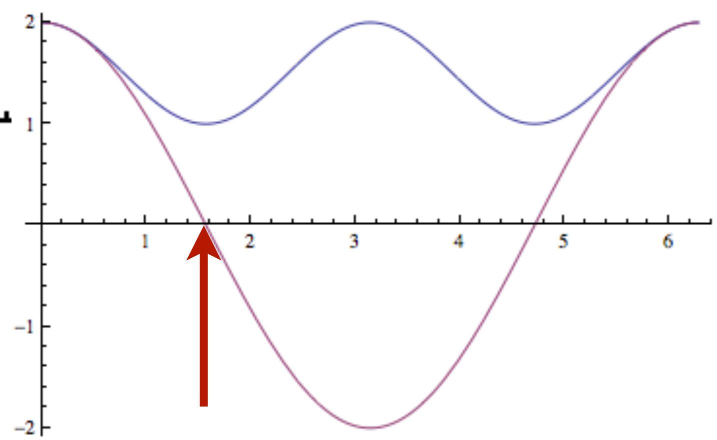
hcross=0

hplus

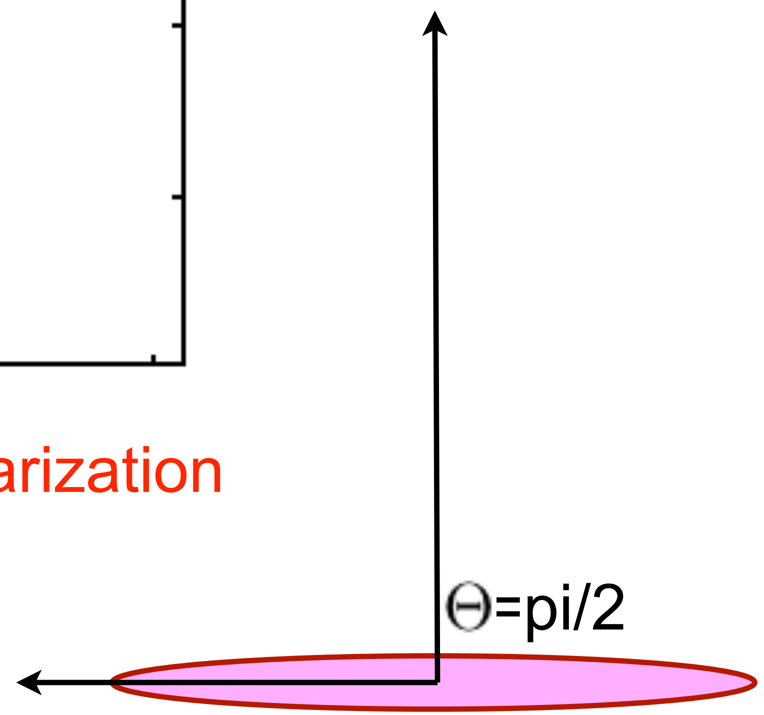
hcross



orbital axis (L)



linear polarization



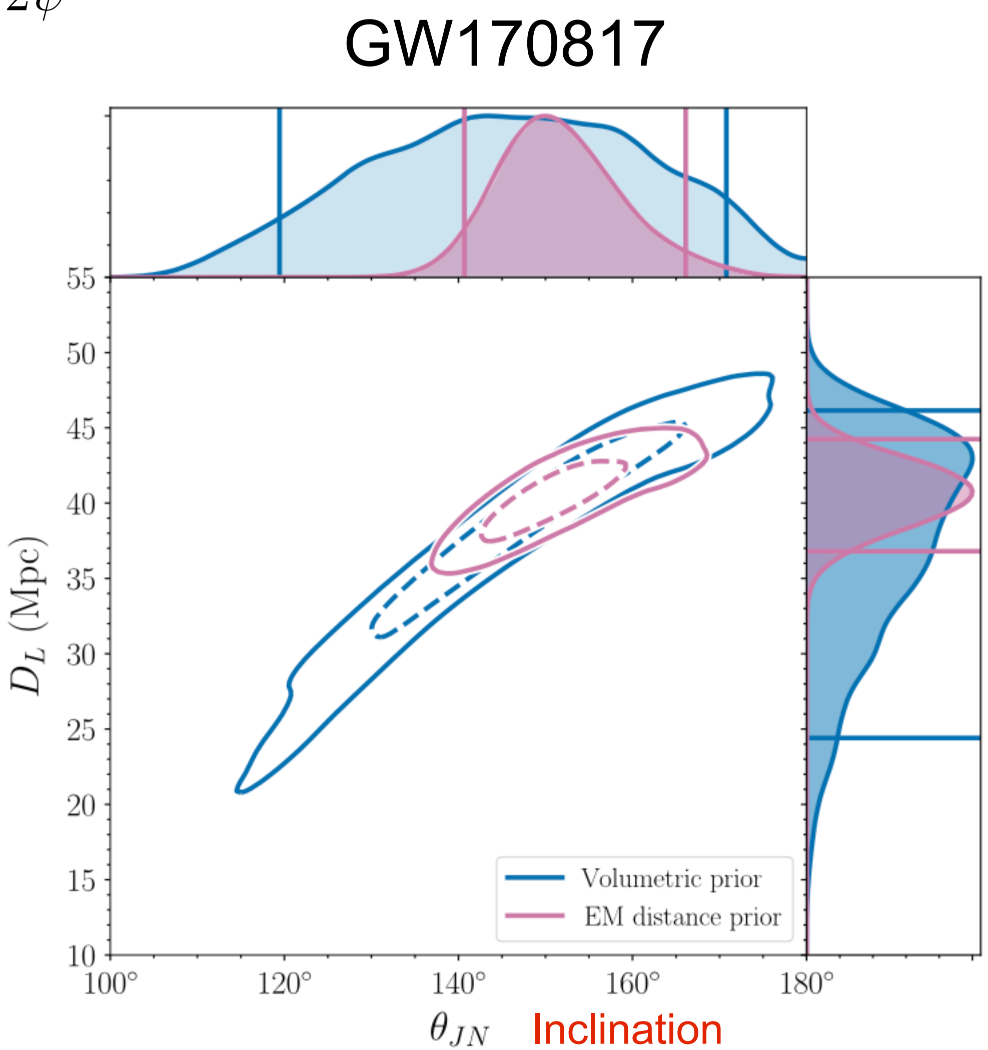
distance-inclination degeneracy

$$h_+ = -\frac{2G\mu}{c^2 R} x(1+\cos^2\Theta)\cos 2\psi$$

$$h_\times = -\frac{2G\mu}{c^2 R} x(2\cos\Theta)\sin 2\psi$$

Inclination

Distance



Precession of a spinning binary

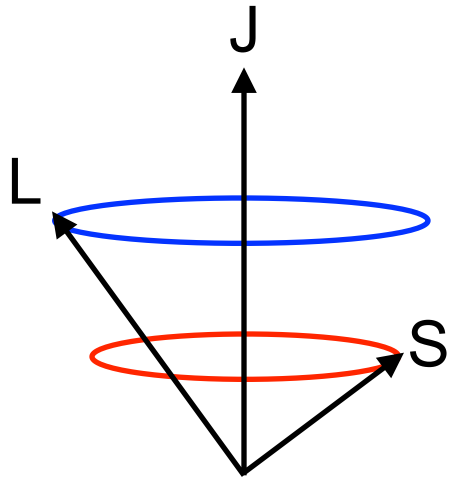
Spin-Orbit coupling, Spin-Spin coupling

$$\dot{\mathbf{S}}_1 = \frac{(M\omega)^2}{2M}\left\{\eta\,(M\omega)^{-1/3}\left(4+3\frac{m_2}{m_1}\right)\hat{\mathbf{L}}_N + \frac{1}{M^2}\left[\mathbf{S}_2 - 3(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)\hat{\mathbf{L}}_N\right]\right\}\times\mathbf{S}_1$$

$$\dot{\mathbf{S}}_2 = \frac{(M\omega)^2}{2M}\left\{\eta\,(M\omega)^{-1/3}\left(4+3\frac{m_1}{m_2}\right)\hat{\mathbf{L}}_N + \frac{1}{M^2}\left[\mathbf{S}_1 - 3(\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)\hat{\mathbf{L}}_N\right]\right\}\times\mathbf{S}_2$$

$$\dot{\hat{\mathbf{L}}}_N = -\frac{(M\omega)^{1/3}}{\eta M^2}\dot{\mathbf{S}} = \frac{\omega^2}{2M}\left\{\left[\left(4+3\frac{m_2}{m_1}\right)\mathbf{S}_1 + \left(4+3\frac{m_1}{m_2}\right)\mathbf{S}_2\right]\times\hat{\mathbf{L}}_N - \frac{3\,\omega^{1/3}}{\eta\,M^{5/3}}\left[(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_1 + (\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_2\right]\times\hat{\mathbf{L}}_N\right\}$$

precession



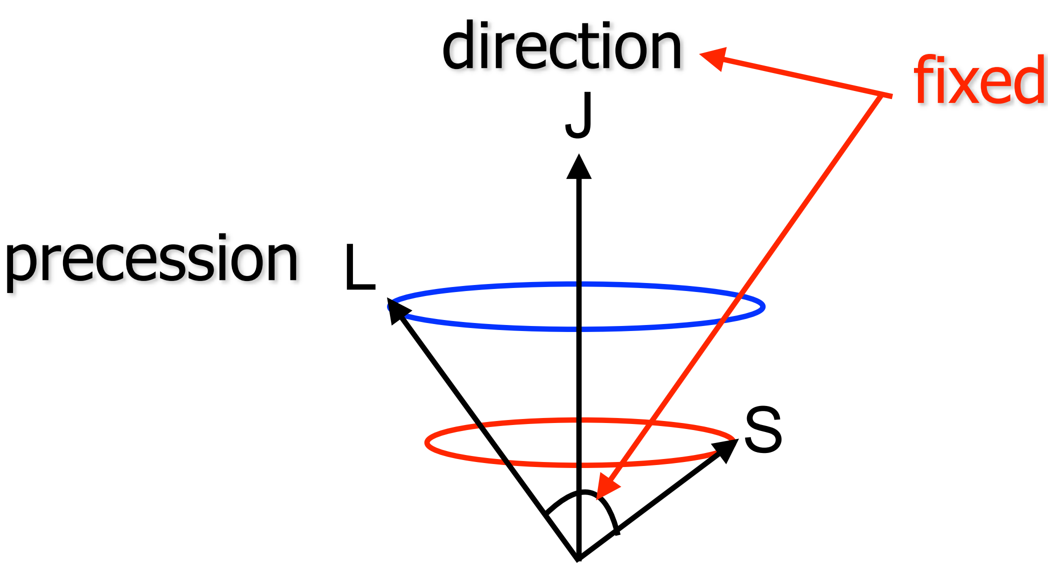
Precession of a spinning binary

Spin-Orbit coupling, Spin-Spin coupling

$$\dot{\mathbf{S}}_1 = \frac{(M\omega)^2}{2M}\left\{\eta\,(M\omega)^{-1/3}\left(4+3\frac{m_2}{m_1}\right)\hat{\mathbf{L}}_N + \frac{1}{M^2}\left[\mathbf{S}_2 - 3(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)\hat{\mathbf{L}}_N\right]\right\}\times\mathbf{S}_1$$

$$\dot{\mathbf{S}}_2 = \frac{(M\omega)^2}{2M}\left\{\eta\,(M\omega)^{-1/3}\left(4+3\frac{m_1}{m_2}\right)\hat{\mathbf{L}}_N + \frac{1}{M^2}\left[\mathbf{S}_1 - 3(\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)\hat{\mathbf{L}}_N\right]\right\}\times\mathbf{S}_2$$

$$\dot{\hat{\mathbf{L}}}_N = -\frac{(M\omega)^{1/3}}{\eta M^2}\dot{\mathbf{S}} = \frac{\omega^2}{2M}\left\{\left[\left(4+3\frac{m_2}{m_1}\right)\mathbf{S}_1 + \left(4+3\frac{m_1}{m_2}\right)\mathbf{S}_2\right]\times\hat{\mathbf{L}}_N - \frac{3\,\omega^{1/3}}{\eta\,M^{5/3}}\left[(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_1 + (\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_2\right]\times\hat{\mathbf{L}}_N\right\}$$



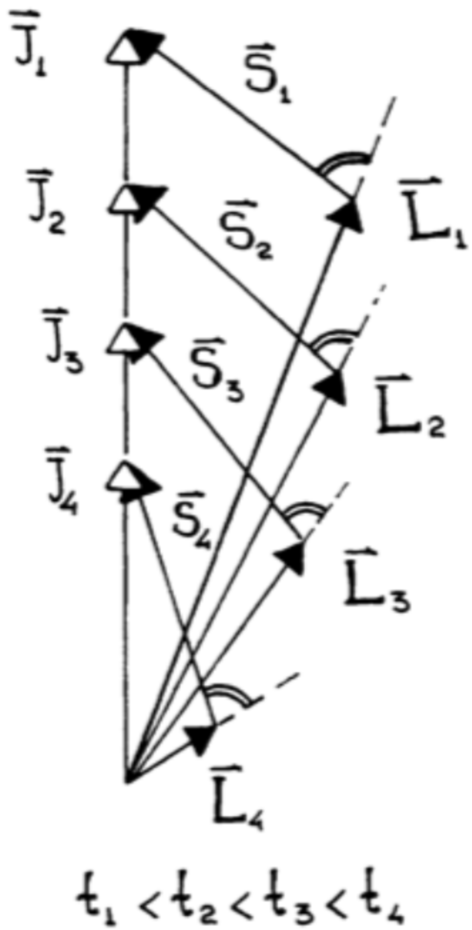
GW effects

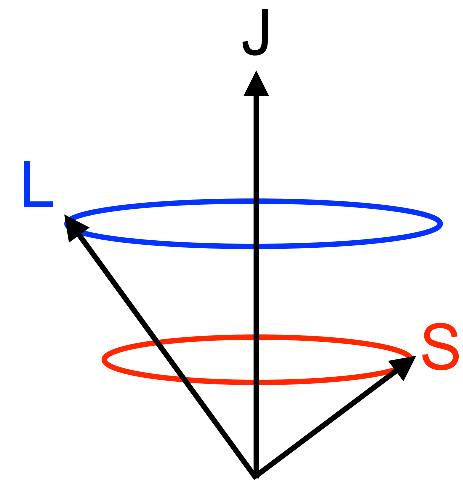
$|J|$ ↓

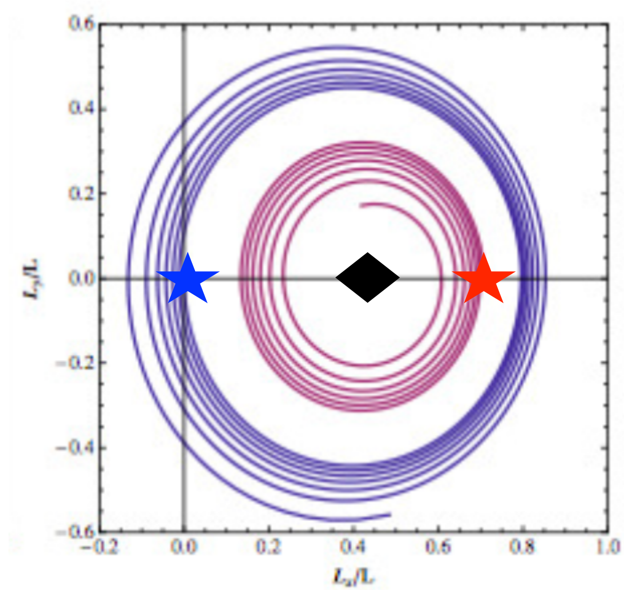
$|L|$ ↓

$|s|$ =const.

Precession of a spinning binary







Top View

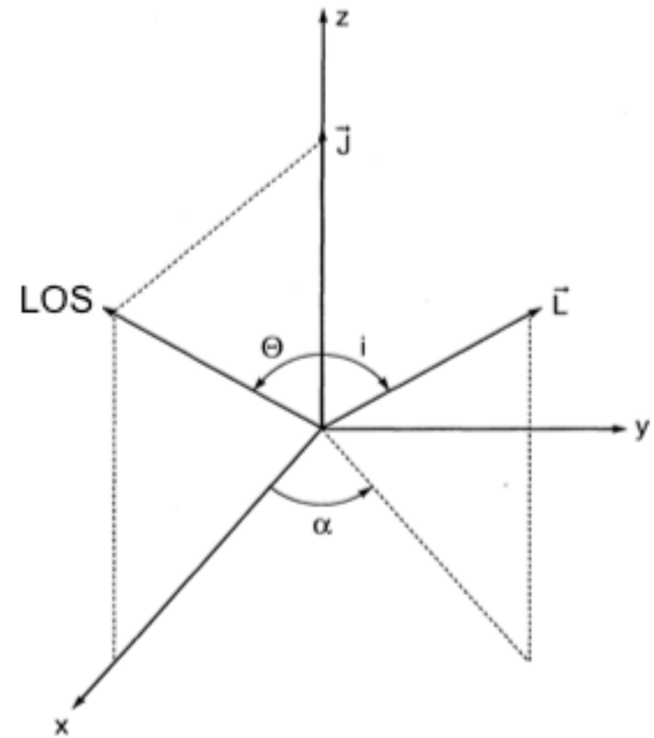
GW Polarization of spinning binary

$$h_+ = -\frac{2G\mu}{c^2 R} x[C_+ \cos 2\psi + S_+ \sin 2\psi]$$
$$h_\times = -\frac{2G\mu}{c^2 R} x[C_\times \cos 2\psi + S_\times \sin 2\psi]$$

non spinning

$$h_+ = -\frac{2G\mu}{c^2 R} x(1+\cos^2\Theta)\cos 2\psi$$
$$h_\times = -\frac{2G\mu}{c^2 R} x(2\cos\Theta)\sin 2\psi$$

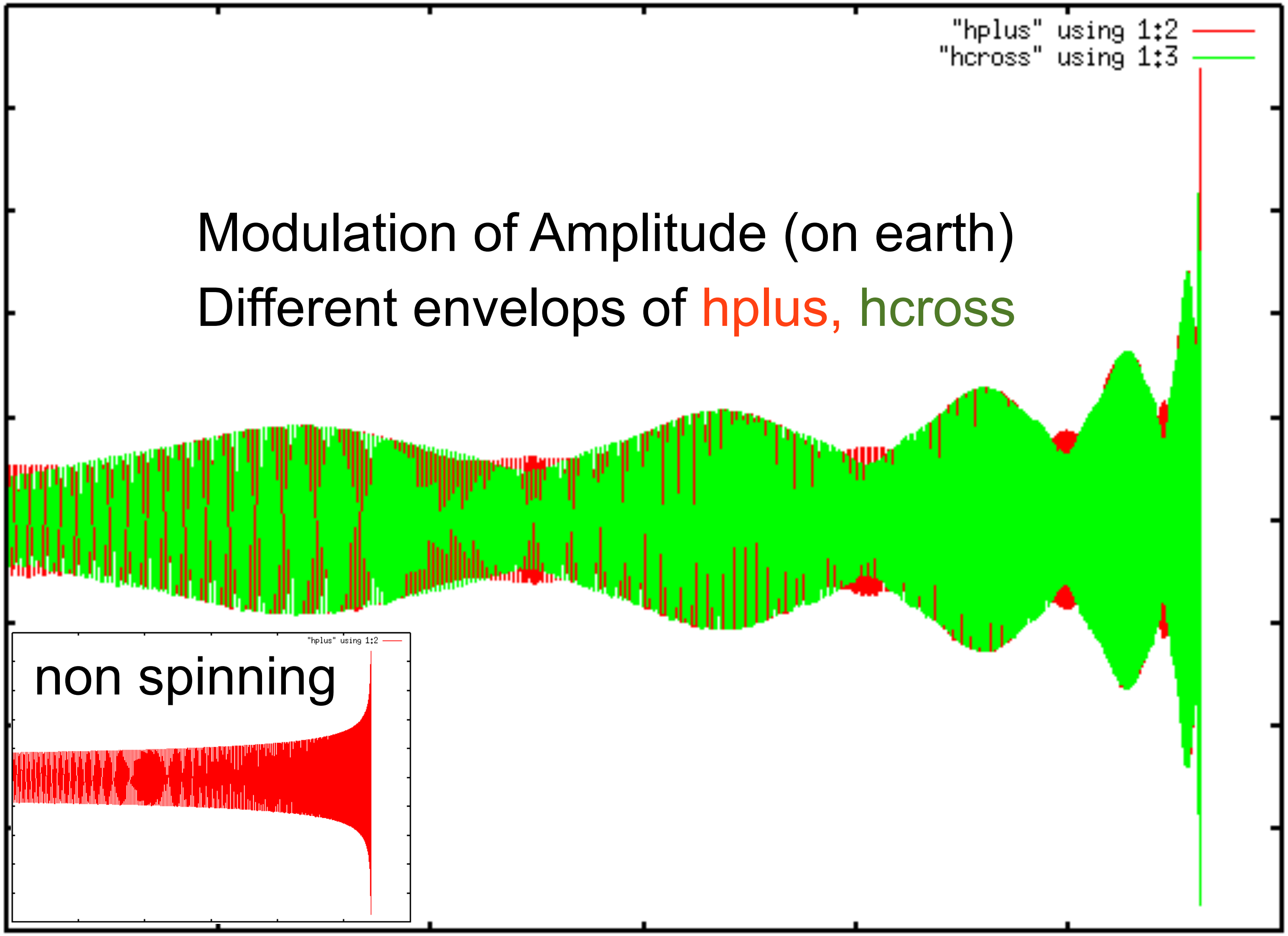
$$C_+ = \tfrac{1}{2}\cos^2\Theta\left(\sin^2\alpha - \cos^2 i\cos^2\alpha\right) + \tfrac{1}{2}(\cos^2 i\sin^2\alpha - \cos^2\alpha) - \tfrac{1}{2}\sin^2\Theta\sin^2 i - \tfrac{1}{4}\sin 2\Theta\sin 2i\cos\alpha,$$
$$S_+ = \tfrac{1}{2}\left(1+\cos^2\Theta\right)\cos i\sin 2\alpha + \tfrac{1}{2}\sin 2\Theta\sin i\sin\alpha,$$
$$C_\times = -\tfrac{1}{2}\cos\Theta\sin 2\alpha\left(1+\cos^2 i\right) - \tfrac{1}{2}\sin\Theta\sin 2i\sin\alpha,$$
$$S_\times = -\cos\Theta\cos i\cos 2\alpha - \sin\Theta\sin i\cos\alpha,$$



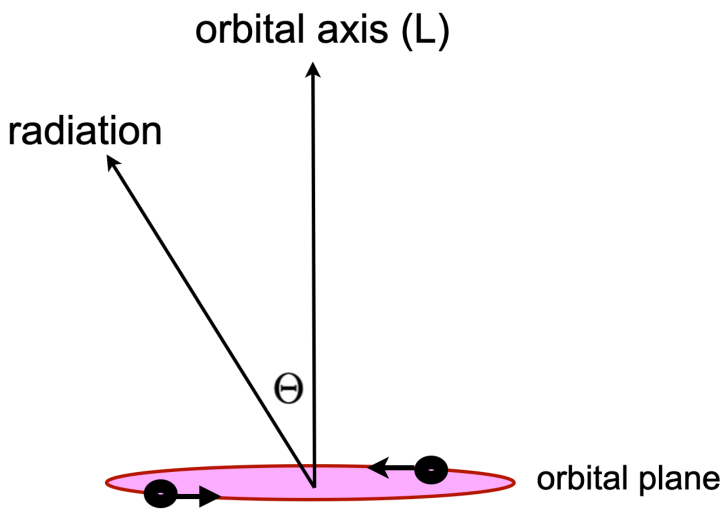
ι, α, Θ : vary in time

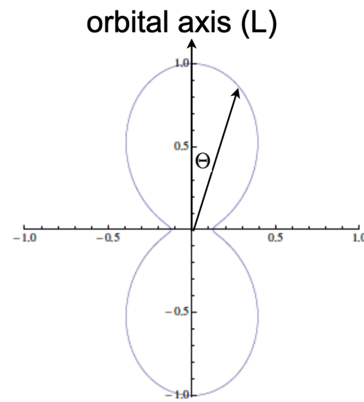
--> precessional motion

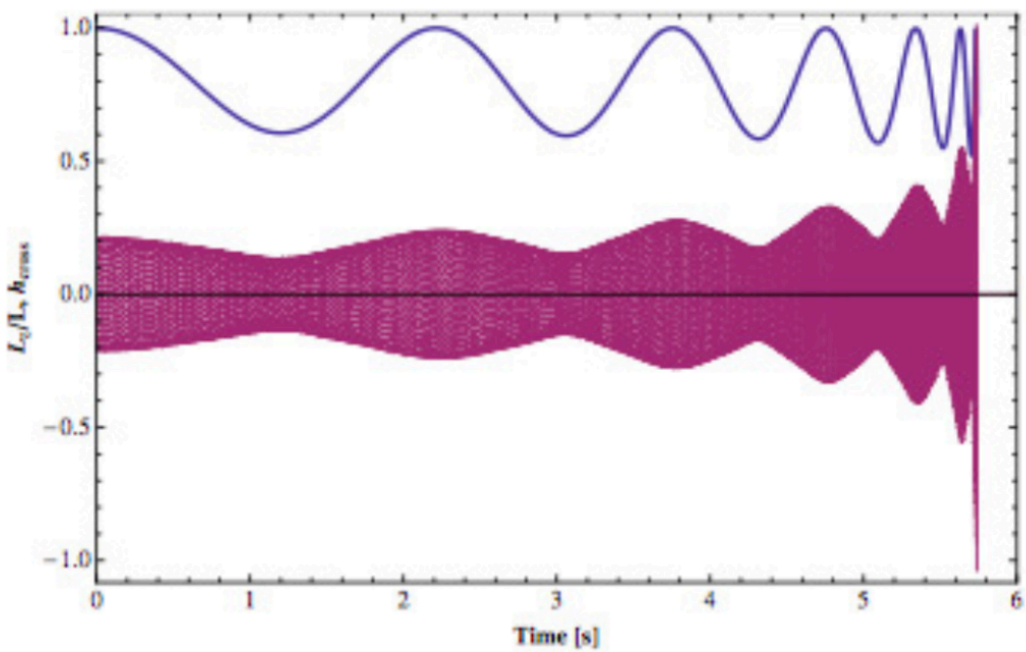
Waveform of spinning binary

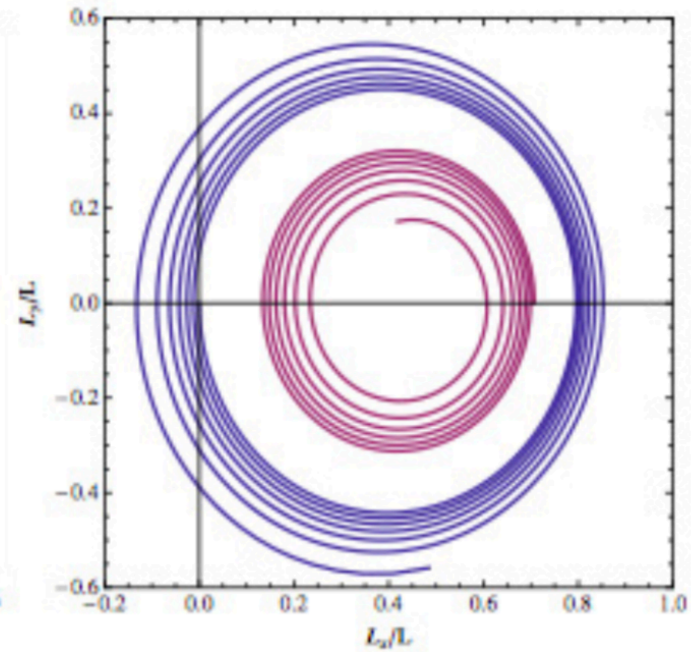


Radiation power (source frame)









Modulation magnitude

Variation of L_N depends on S, S dot L_N, S cross L_N

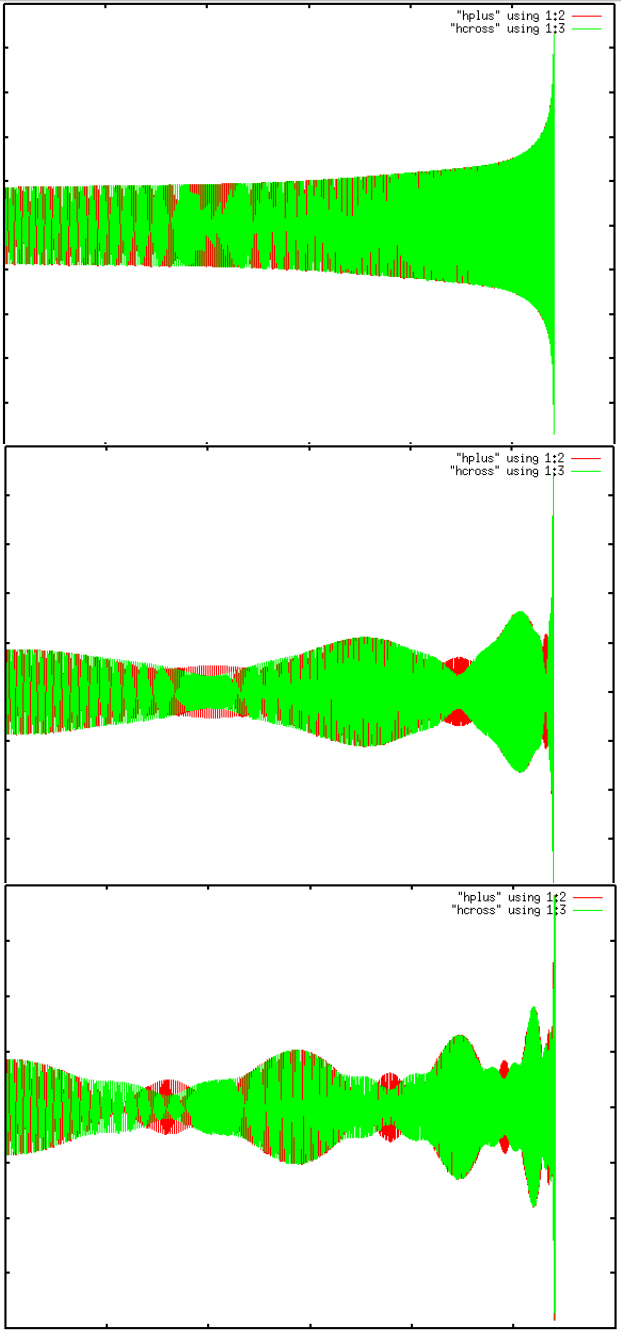
---> Precession effect depends on

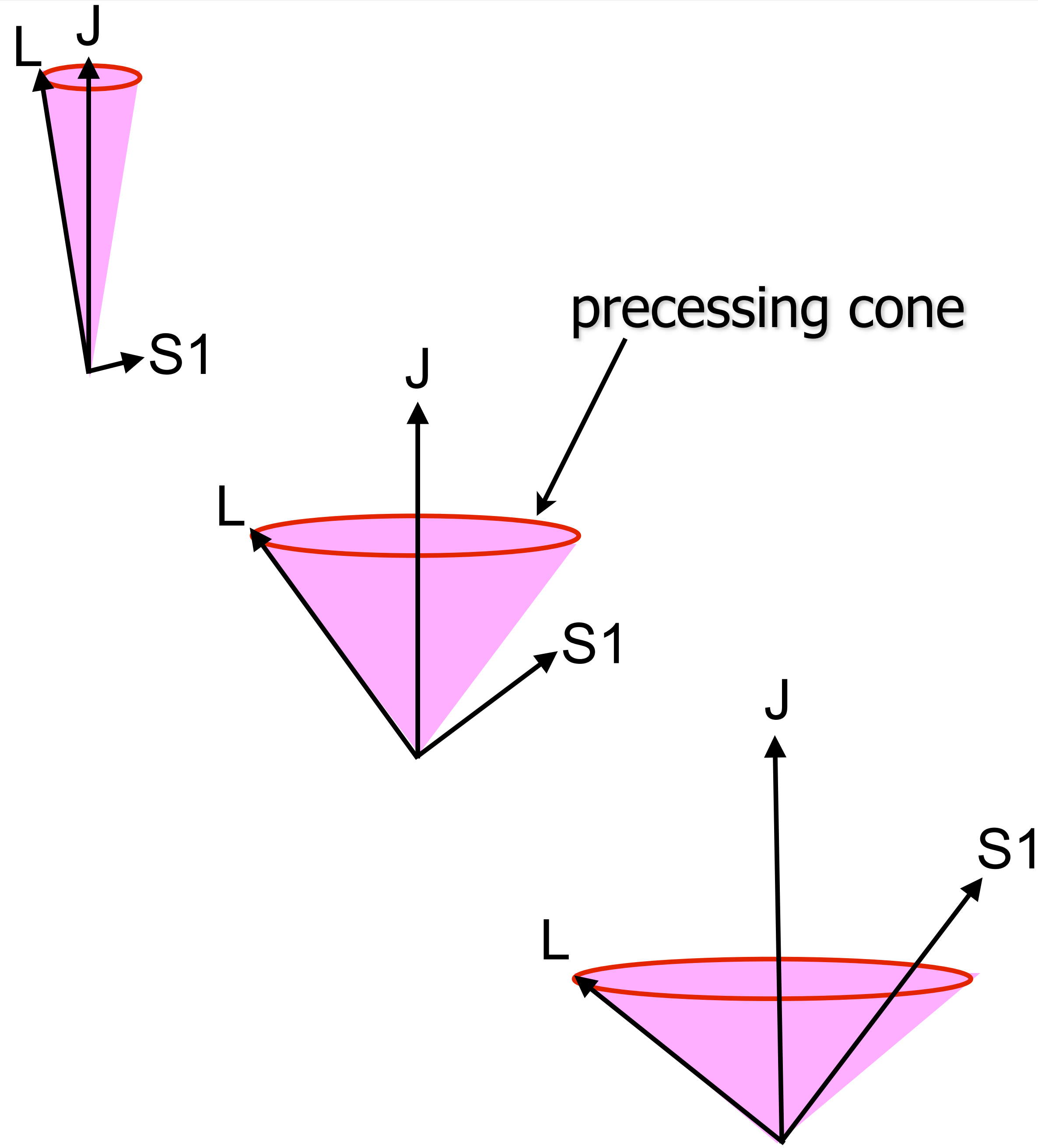
1) Spin magnitude

2) Angle between L_N and S

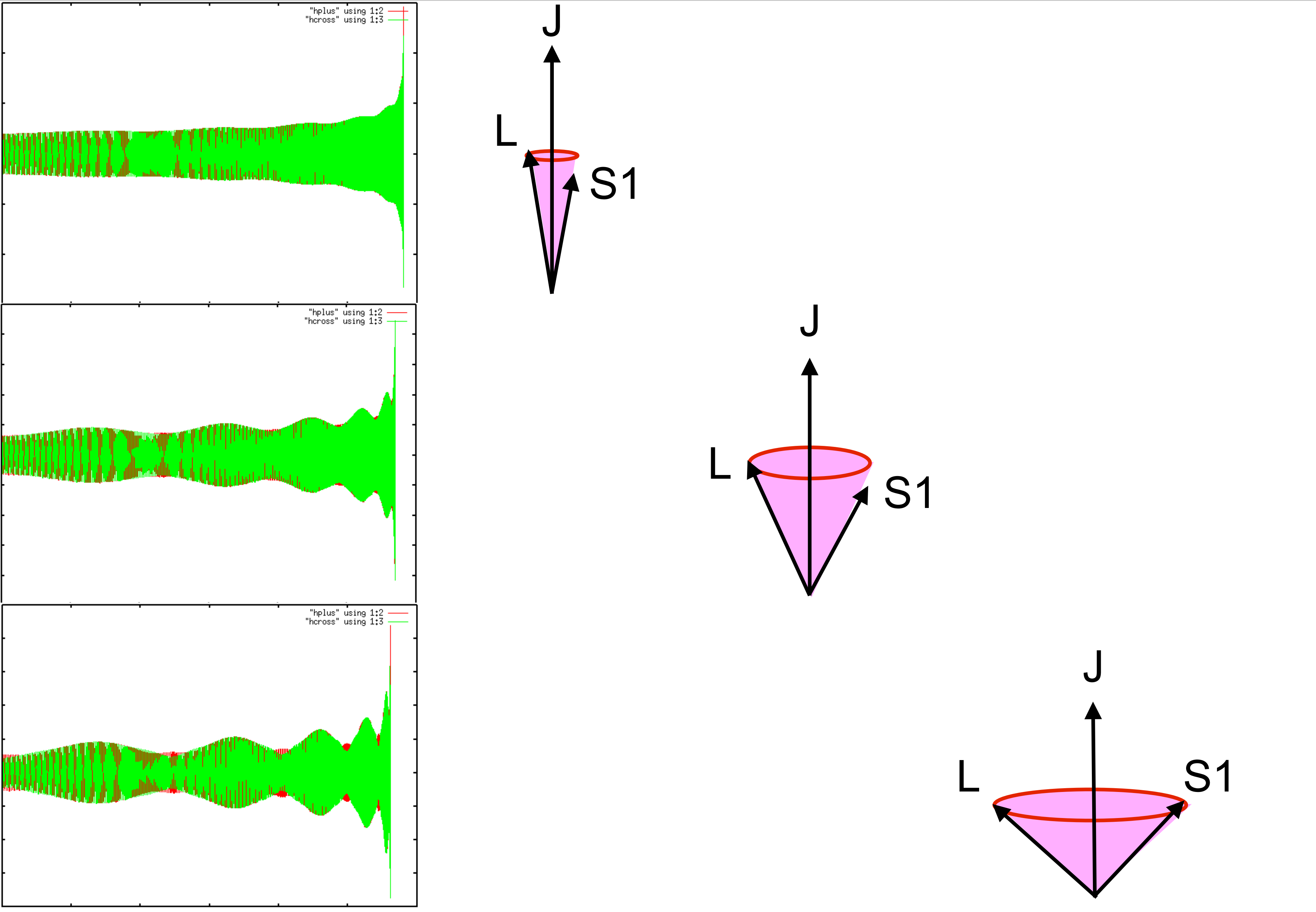
$$
\begin{aligned}
\dot{\hat{\mathbf{L}}}_N = -\frac{(M\omega)^{1/3}}{\eta M^2}\,\dot{\mathbf{S}} = \frac{\omega^2}{2M}\Bigg\{ & \left[\left(4+3\frac{m_2}{m_1}\right)\underline{\mathbf{S}_1} + \left(4+3\frac{m_1}{m_2}\right)\underline{\mathbf{S}_2}\right] \times \underline{\hat{\mathbf{L}}_N} \\
& -\frac{3\,\omega^{1/3}}{\eta\, M^{5/3}}\left[\underline{(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)}\,\underline{\mathbf{S}_1} + \underline{(\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)}\,\underline{\mathbf{S}_2}\right] \times \underline{\hat{\mathbf{L}}_N} \Bigg\}
\end{aligned}
$$

Amplitude modulation with Spin (Angle=pi/2, S2=0)





Amplitude modulation with Angle (S1=0.9, S2=0)



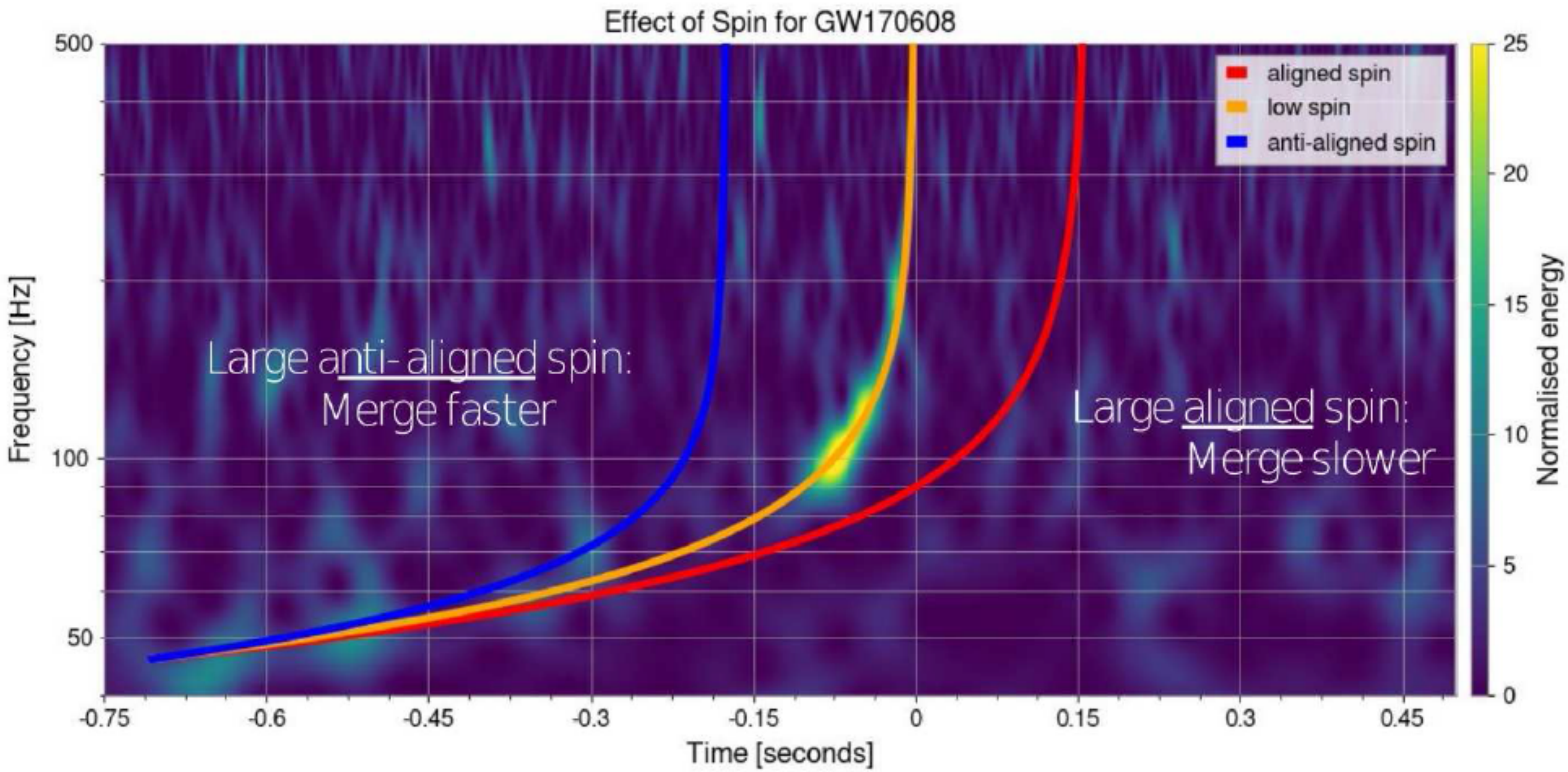
aligned-spin binaries

$$\dot{\hat{\mathbf{L}}}_N = -\frac{(M\omega)^{1/3}}{\eta M^2}\,\dot{\mathbf{S}} = \frac{\omega^2}{2M}\Bigg\{\left[\left(4+3\frac{m_2}{m_1}\right)\mathbf{S}_1 + \left(4+3\frac{m_1}{m_2}\right)\mathbf{S}_2\right]\underline{\times\,\hat{\mathbf{L}}_N}$$

$$-\frac{3\,\omega^{1/3}}{\eta\,M^{5/3}}\left[(\mathbf{S}_2\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_1 + (\mathbf{S}_1\cdot\hat{\mathbf{L}}_N)\,\mathbf{S}_2\right]\underline{\times\,\hat{\mathbf{L}}_N}\Bigg\}$$

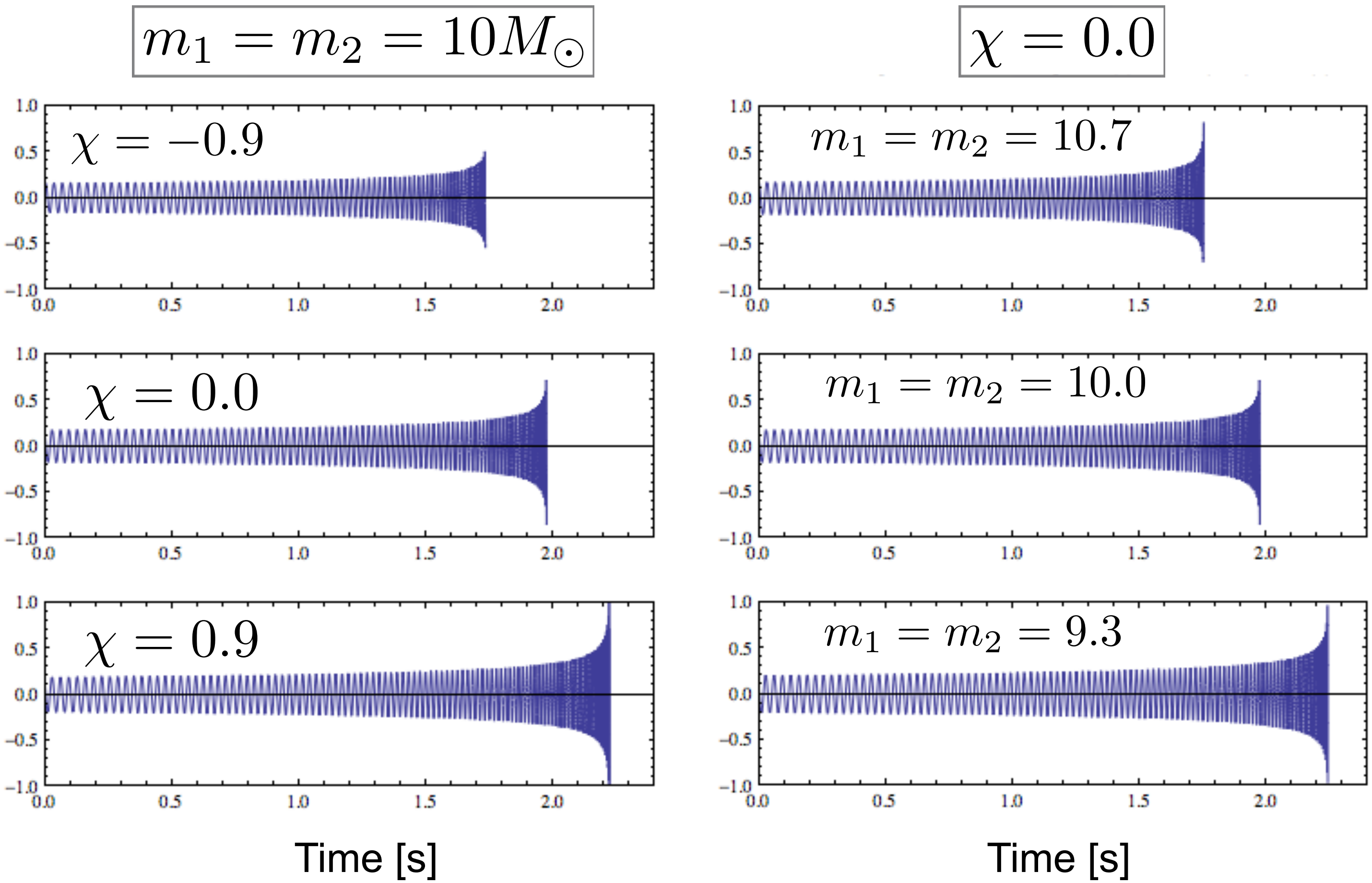
aligned-spin: no precession

aligned-spin binaries

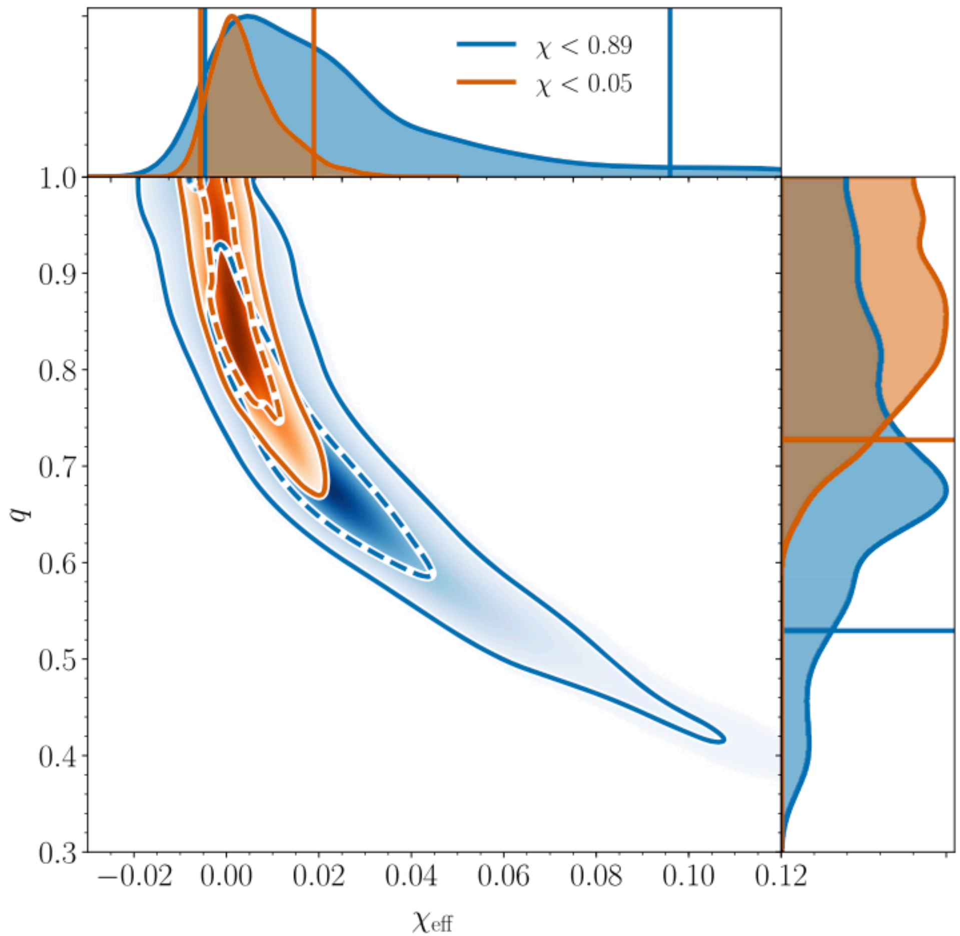


(see the `pyCBC` tutorial:
`http://pycbc.org/pycbc/latest/html/waveform.html#plotting-frequency-evolution-of-td-waveform`)

spin-mass degeneracy



GW170817



$\chi < 0.89$
$\chi < 0.05$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
q
−0.02
0.00
0.02
0.04
0.06
0.08
0.10
0.12
$\chi_{\rm eff}$

GW phase: nonspinning

$$h_+ = -\frac{2G\mu}{c^2 R} x (1 + \cos^2 \Theta) \cos 2\underline{\psi}$$
$$h_\times = -\frac{2G\mu}{c^2 R} x (2 \cos \Theta) \sin 2\underline{\psi}$$

Phase evolution from
post Newtonian (PN)

Post Newtonian Energy & Flux

$$E = -\underline{\frac{\mu c^2 x}{2}}\left\{1 + \left(-\frac{3}{4} - \frac{1}{12}\nu\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right)x^2 + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)x^3\right\} + \mathcal{O}\left(\frac{1}{c^8}\right).$$

Newtonian binding energy of a binary

$$\mathcal{L} = \frac{32c^5}{5G}\nu^2 x^5\left\{1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{856}{105}\ln(16x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}.$$

$$x \equiv \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3} \qquad m = m_1 + m_2 \qquad \mu = m_1 m_2/m \qquad \nu \equiv \frac{\mu}{m} \equiv \frac{m_1 m_2}{(m_1+m_2)^2}.$$

Phase evolution

Energy balance equation : orbital binding energy loss = GW emission energy

$$\boxed{\frac{dE}{dt} = -L} \qquad \frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} \qquad \frac{dx}{dt} = -\frac{L}{(dE/dx)}$$

expand with x , $x \equiv \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3}$

$$\frac{\dot\omega}{\omega^2} = \frac{96}{5}\eta(M\omega)^{5/3}\Bigg(1 - \frac{743+924\eta}{336}(M\omega)^{2/3} + \left(\frac{34\,103}{18\,144} + \frac{13\,661}{2016}\eta + \frac{59}{18}\eta^2\right)(M\omega)^{4/3} - \frac{1}{672}(4159+14\,532\eta)\pi(M\omega)^{5/3}\eta$$
$$+\left[\left(\frac{16\,447\,322\,263}{139\,708\,800} - \frac{1712}{105}\gamma_E + \frac{16}{3}\pi^2\right) + \left(-\frac{273\,811\,877}{1\,088\,640} + \frac{451}{48}\pi^2 - \frac{88}{3}\hat\theta\right)\eta + \frac{541}{896}\eta^2 - \frac{5605}{2592}\eta^3\right.$$
$$\left. - \frac{856}{105}\log[16(M\omega)^{2/3}]\right](M\omega)^2 + \left(-\frac{4\,415}{4\,032} + \frac{661\,775}{12\,096}\eta + \frac{149\,789}{3\,024}\eta^2\right)\pi(M\omega)^{7/3}\Bigg),$$

$$w(t) = \int_0^t \dot w(w)dt, \qquad \psi(t) = \int_0^t w(t)dt$$ ---> Numerical integration

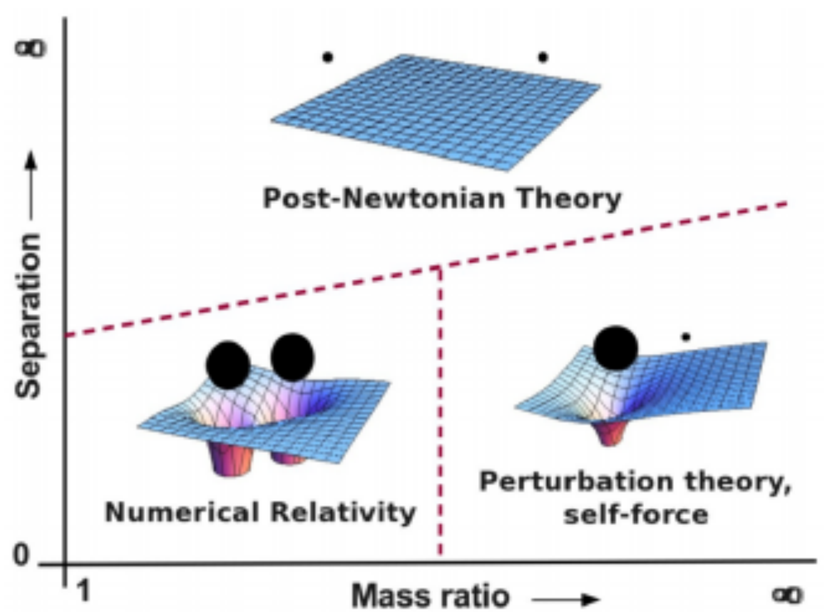
TaylorT4, T1,2,3,.. : time domain models

Waveform modeling

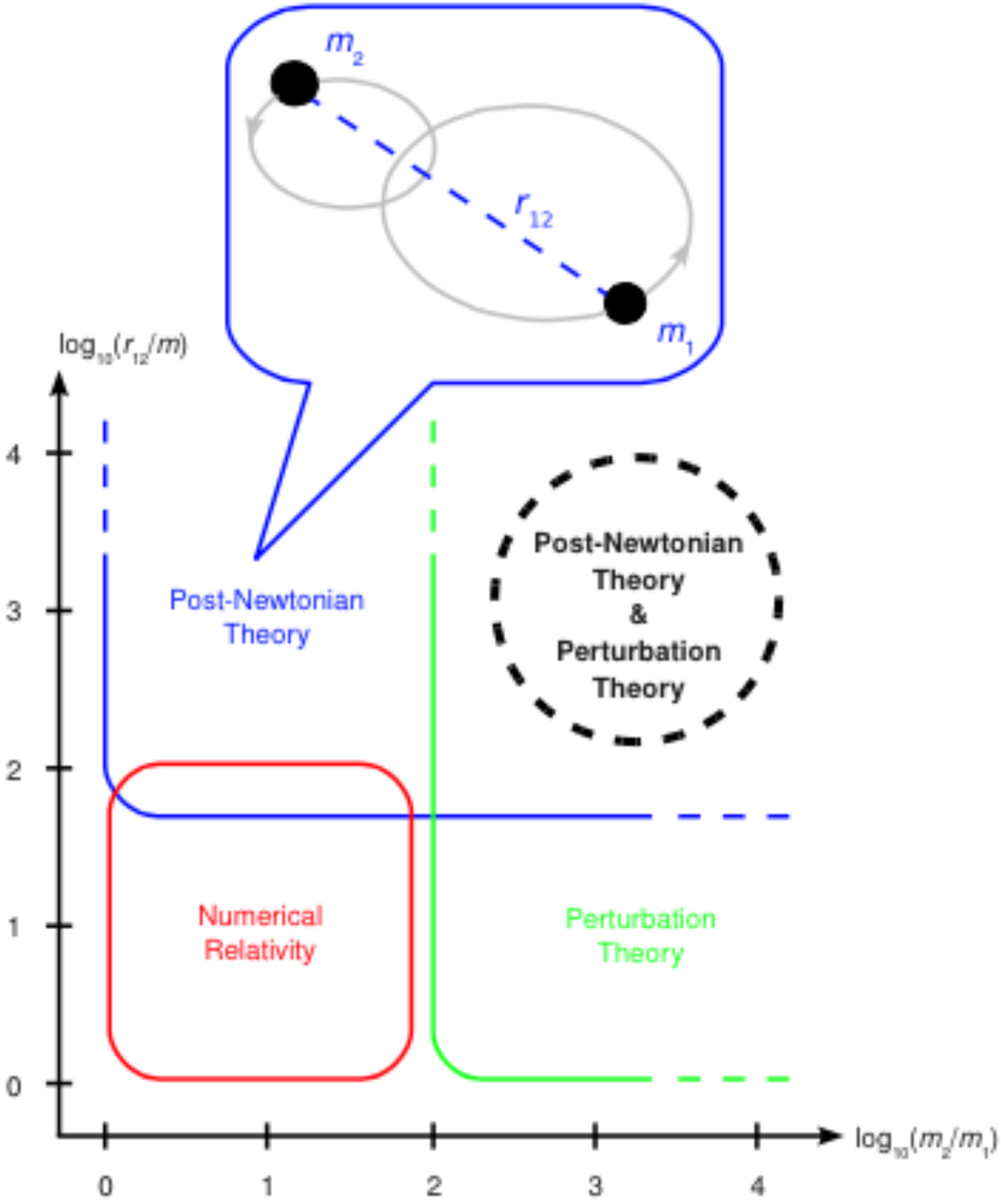
- **Post-Newtonian (PN) approximation**: slow motion approximation $(v/c << 1)$
- **Perturbative theory**: small mass ratio $(m/M \ll 1)$
- **Effective-One-Body approach**: combination of PN, Perturbative approach and NR

time-domain model !!

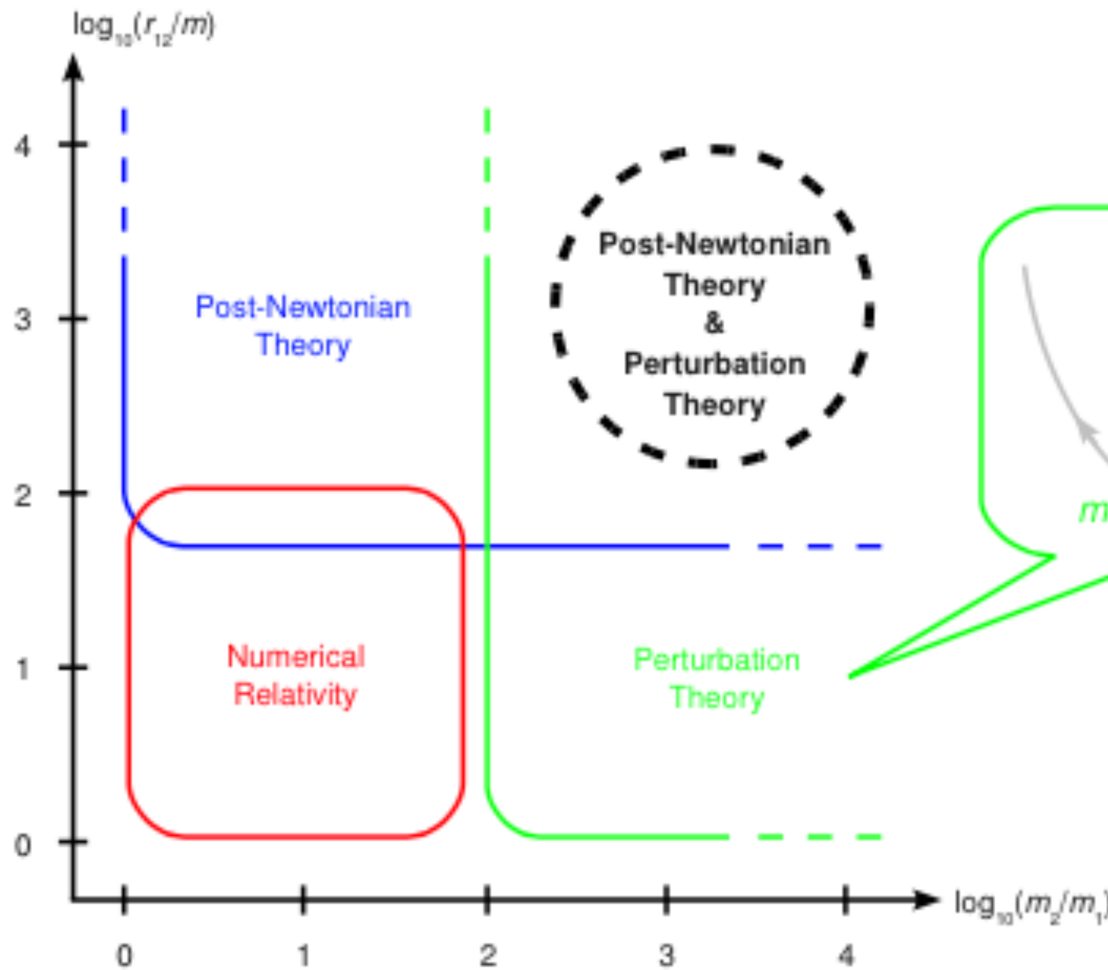
Binary parameter space



[Leor Barack]



m_2
r_{12}
m_1
$\log_{10}(r_{12}/m)$
4
3
2
1
0
Post-Newtonian Theory
Post-Newtonian Theory & Perturbation Theory
Numerical Relativity
Perturbation Theory
0
1
2
3
4
$\log_{10}(m_2/m_1)$



$\log_{10}(r_{12}/m)$
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Post-Newtonian Theory
Post-Newtonian Theory & Perturbation Theory
m_2
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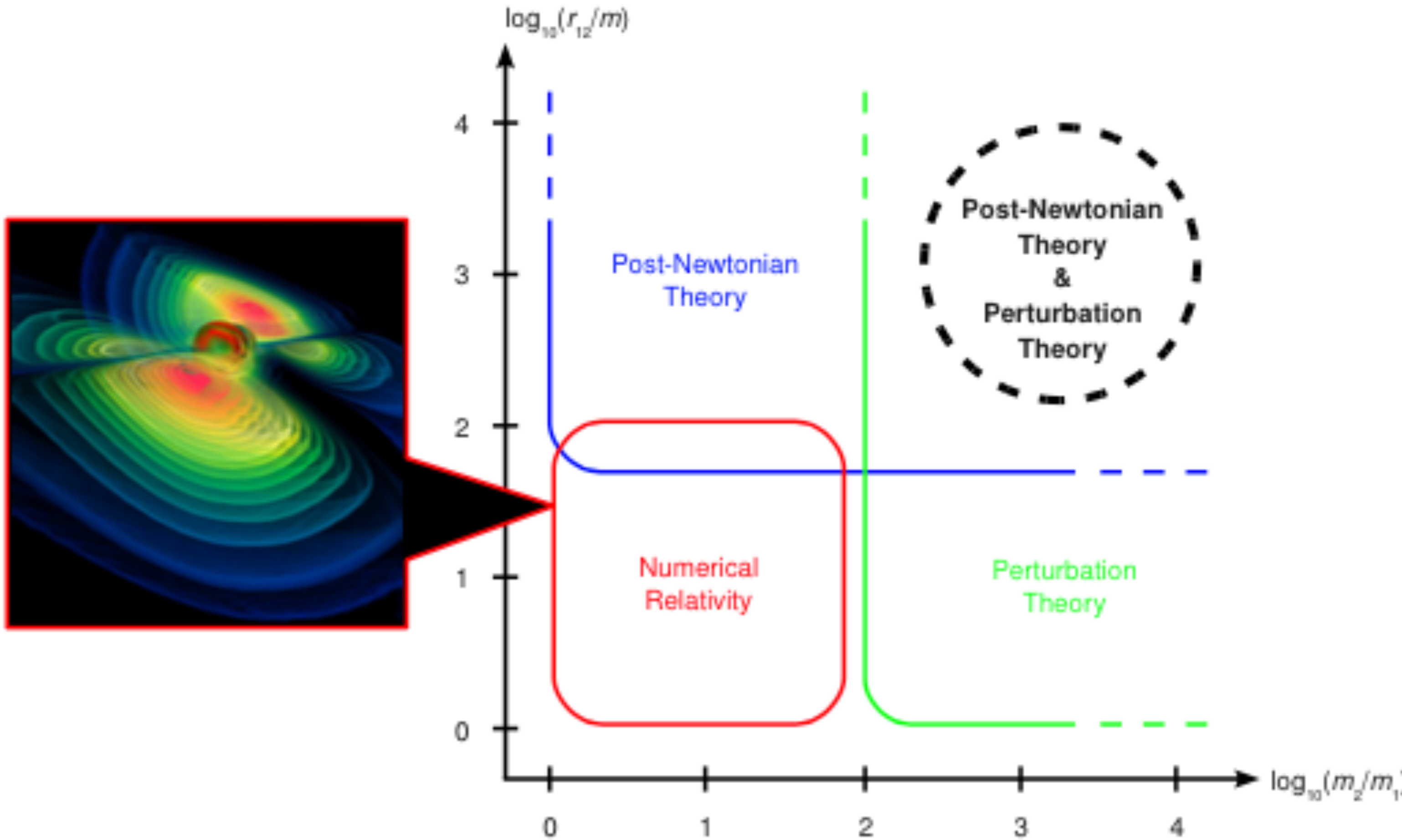
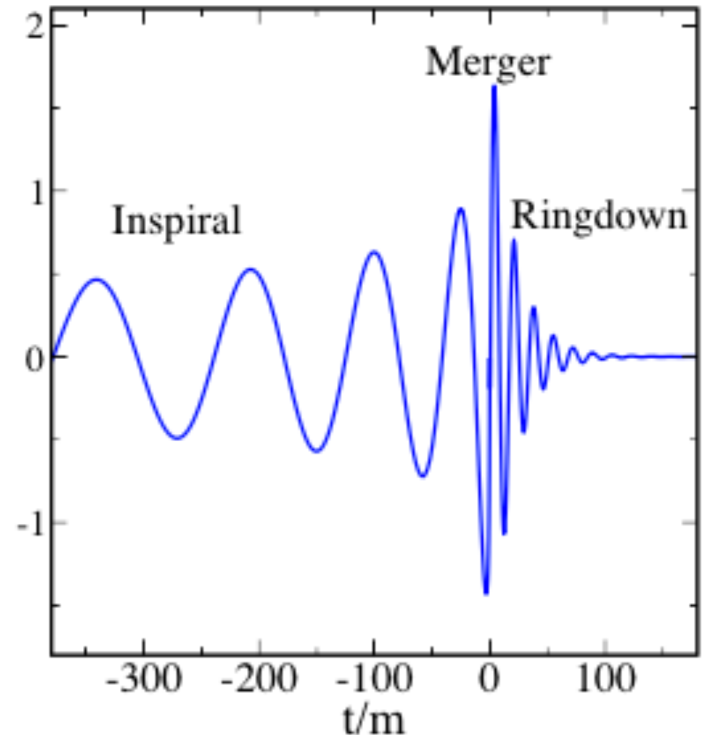
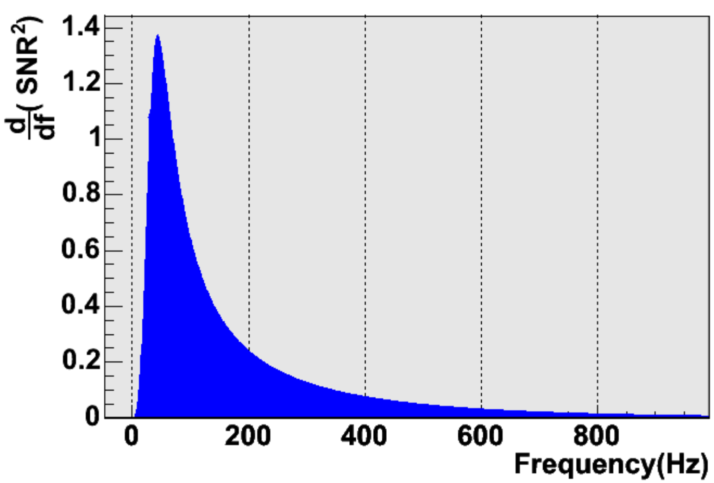


$\log_{10}(r_{12}/m)$
4
3
2
1
0
Post-Newtonian Theory
Post-Newtonian Theory & Perturbation Theory
Numerical Relativity
Perturbation Theory
0
1
2
3
4
$\log_{10}(m_2/m_1)$

Tools for computing the waveform

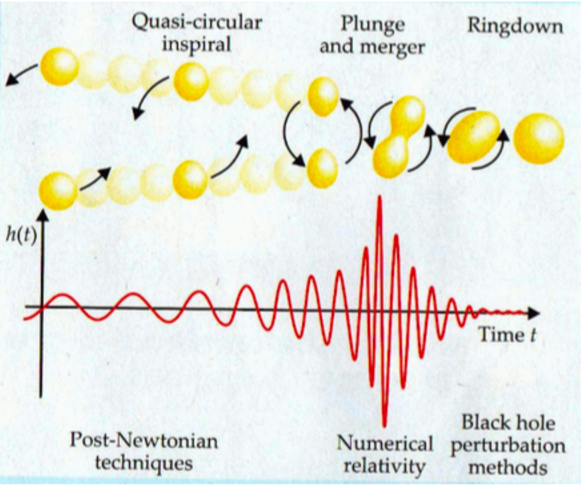
- Inspiral
  - $v \ll c$: perturbative expansion in v/c (post-Newtonian expansion)
  - v/c large: Numerical relativity
- Merger
  - Numerical relativity
- Ringdown
  - BH perturbation theory
  - Numerical relativity
- Tasks for Numerical relativity:
  - simulate "late" inspiral and merger
  - determine what "**late**" means

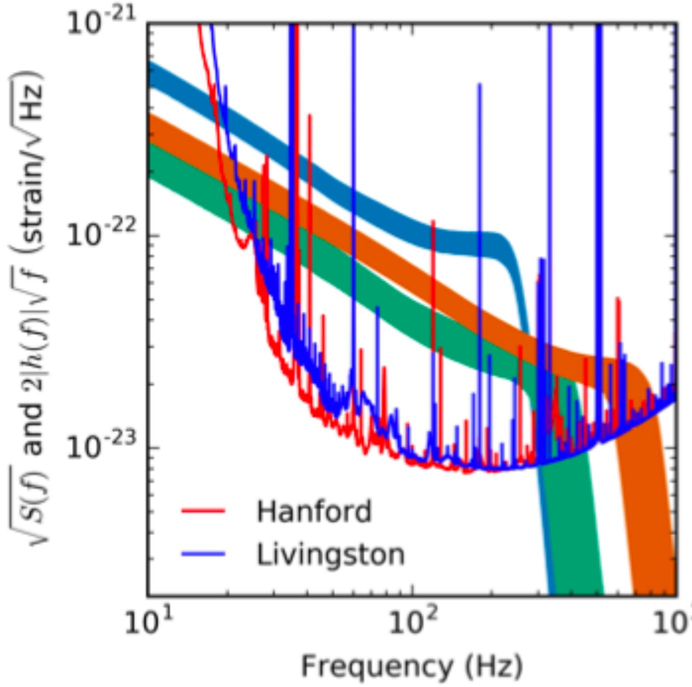


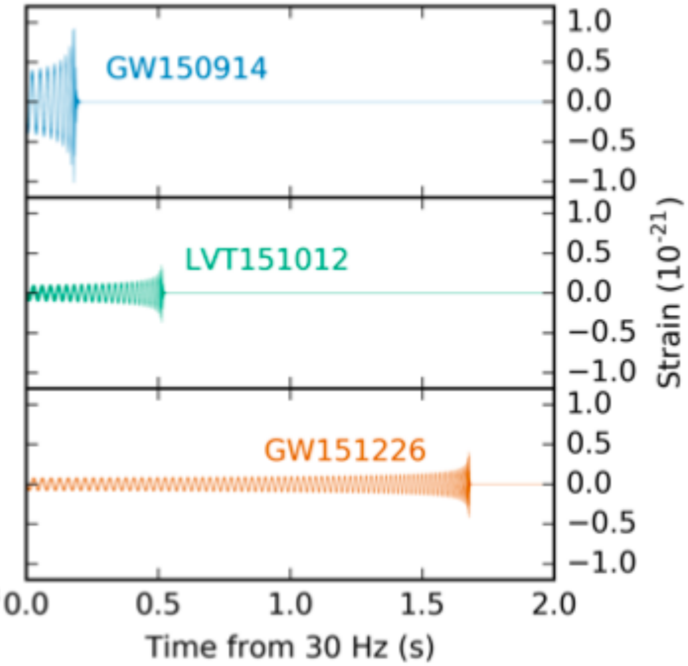


Why full IMR waveforms ?

	m_1 / m_2	f_{ISCO}
NS-NS	1.4 /1.4 $M_\odot$	1600 Hz
BH-BH	17.5/17.5 $M_\odot$	128 Hz







What is the post-Newtonian approximation?

Post-Newtonian approximation is

Weak field approximation & **Slow motion approximation**

Gravitational field is weak. Strength of gravitational field produced by matter is represented by G. Thus, in this approximation, $\boxed{\text{"}G\text{"} \ll 1}$ is assumed, and Einstein equations are solved with $\boxed{G^n}$ expansion.

Speed of matter is much smaller than the speed of light.

$$\boxed{v/c \ll 1}$$

$(v/c)^n$ expansion

G : Newton's gravitational constant
v : typical speed of the system
c : speed of light

PN equations of motion for compact binaries

$$\vec{a} = -\frac{m}{r^3}\vec{x} + 1PN + 1PN_{SO} + 1PN_{SS} + 2PN + 2.5PN$$

$+ 3PN$ B F S

$+ 3.5PN$ W B

$+ 3.5PN_{SO}$ W

$+ 3.5PN_{SS}$ W

B = Blanchet, Damour, Iyer et al
F = Futamase, Itoh
S = Schäfer, Jaranowski
W = WUGRAV

Example : nonspinning, point-mass CBC

$$
\begin{aligned}
\boldsymbol{a}_1 = &-\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} \\
&\boxed{+\frac{1}{c^2}}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\frac{Gm_2}{r_{12}^2}\left(\frac{3}{2}(n_{12}v_2)^2-v_1^2+4(v_1v_2)-2v_2^2\right)\right]\boldsymbol{n}_{12}\right. \\
&\qquad\left.+\frac{Gm_2}{r_{12}^2}\left(4(n_{12}v_1)-3(n_{12}v_2)\right)\boldsymbol{v}_{12}\right\} \\
&\boxed{+\frac{1}{c^4}}\left\{\left[-\frac{57G^3m_1^2m_2}{4r_{12}^4}-\frac{69G^3m_1m_2^2}{2r_{12}^4}-\frac{9G^3m_2^3}{r_{12}^4}\right.\right. \\
&\qquad+\frac{Gm_2}{r_{12}^2}\left(-\frac{15}{8}(n_{12}v_2)^4+\frac{3}{2}(n_{12}v_2)^2v_1^2-6(n_{12}v_2)^2(v_1v_2)-2(v_1v_2)^2+\frac{9}{2}(n_{12}v_2)^2v_2^2\right. \\
&\qquad\qquad\left.+4(v_1v_2)v_2^2-2v_2^4\right) \\
&\qquad+\frac{G^2m_1m_2}{r_{12}^3}\left(\frac{39}{2}(n_{12}v_1)^2-39(n_{12}v_1)(n_{12}v_2)+\frac{17}{2}(n_{12}v_2)^2-\frac{15}{4}v_1^2-\frac{5}{2}(v_1v_2)+\frac{5}{4}v_2^2\right) \\
&\qquad\left.+\frac{G^2m_2^2}{r_{12}^3}\left(2(n_{12}v_1)^2-4(n_{12}v_1)(n_{12}v_2)-6(n_{12}v_2)^2-8(v_1v_2)+4v_2^2\right)\right]\boldsymbol{n}_{12} \\
&\qquad+\left[\frac{G^2m_2^2}{r_{12}^3}\left(-2(n_{12}v_1)-2(n_{12}v_2)\right)+\frac{G^2m_1m_2}{r_{12}^3}\left(-\frac{63}{4}(n_{12}v_1)+\frac{55}{4}(n_{12}v_2)\right)\right. \\
&\qquad\quad+\frac{Gm_2}{r_{12}^2}\left(-6(n_{12}v_1)(n_{12}v_2)^2+\frac{9}{2}(n_{12}v_2)^3+(n_{12}v_2)v_1^2-4(n_{12}v_1)(v_1v_2)\right. \\
&\qquad\qquad\left.\left.\left.+4(n_{12}v_2)(v_1v_2)+4(n_{12}v_1)v_2^2-5(n_{12}v_2)v_2^2\right)\right]\boldsymbol{v}_{12}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{c^5}\Bigg\{\left[\frac{208G^3m_1m_2^2}{15r_{12}^4}(n_{12}v_{12})-\frac{24G^3m_1^2m_2}{5r_{12}^4}(n_{12}v_{12})+\frac{12G^2m_1m_2}{5r_{12}^3}(n_{12}v_{12})v_{12}^2\right]\boldsymbol{n}_{12}\\
&\qquad+\left[\frac{8G^3m_1^2m_2}{5r_{12}^4}-\frac{32\,G^3m_1m_2^2}{5r_{12}^4}-\frac{4G^2m_1m_2}{5r_{12}^3}v_{12}^2\right]\boldsymbol{v}_{12}\Bigg\}\\
&+\frac{1}{c^6}\Bigg\{\Bigg[\frac{Gm_2}{r_{12}^2}\bigg(\frac{35}{16}(n_{12}v_2)^6-\frac{15}{8}(n_{12}v_2)^4v_1^2+\frac{15}{2}(n_{12}v_2)^4(v_1v_2)+3(n_{12}v_2)^2(v_1v_2)^2\\
&\qquad\qquad-\frac{15}{2}(n_{12}v_2)^4v_2^2+\frac{3}{2}(n_{12}v_2)^2v_1^2v_2^2-12(n_{12}v_2)^2(v_1v_2)v_2^2-2(v_1v_2)^2v_2^2\\
&\qquad\qquad+\frac{15}{2}(n_{12}v_2)^2v_2^4+4(v_1v_2)v_2^4-2v_2^6\bigg)\\
&\qquad+\frac{G^2m_1m_2}{r_{12}^3}\bigg(-\frac{171}{8}(n_{12}v_1)^4+\frac{171}{2}(n_{12}v_1)^3(n_{12}v_2)-\frac{723}{4}(n_{12}v_1)^2(n_{12}v_2)^2\\
&\qquad\qquad+\frac{383}{2}(n_{12}v_1)(n_{12}v_2)^3-\frac{455}{8}(n_{12}v_2)^4+\frac{229}{4}(n_{12}v_1)^2v_1^2\\
&\qquad\qquad-\frac{205}{2}(n_{12}v_1)(n_{12}v_2)v_1^2+\frac{191}{4}(n_{12}v_2)^2v_1^2-\frac{91}{8}v_1^4-\frac{229}{2}(n_{12}v_1)^2(v_1v_2)\\
&\qquad\qquad+244(n_{12}v_1)(n_{12}v_2)(v_1v_2)-\frac{225}{2}(n_{12}v_2)^2(v_1v_2)+\frac{91}{2}v_1^2(v_1v_2)\\
&\qquad\qquad-\frac{177}{4}(v_1v_2)^2+\frac{229}{4}(n_{12}v_1)^2v_2^2-\frac{283}{2}(n_{12}v_1)(n_{12}v_2)v_2^2\\
&\qquad\qquad+\frac{259}{4}(n_{12}v_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2+43(v_1v_2)v_2^2-\frac{81}{8}v_2^4\bigg)\\
&\qquad+\frac{G^2m_2^2}{r_{12}^3}\bigg(-6(n_{12}v_1)^2(n_{12}v_2)^2+12(n_{12}v_1)(n_{12}v_2)^3+6(n_{12}v_2)^4\\
&\qquad\qquad+4(n_{12}v_1)(n_{12}v_2)(v_1v_2)+12(n_{12}v_2)^2(v_1v_2)+4(v_1v_2)^2\\
&\qquad\qquad-4(n_{12}v_1)(n_{12}v_2)v_2^2-12(n_{12}v_2)^2v_2^2-8(v_1v_2)v_2^2+4v_2^4\bigg)\\
&\qquad+\frac{G^3m_2^3}{r_{12}^4}\left(-(n_{12}v_1)^2+2(n_{12}v_1)(n_{12}v_2)+\frac{43}{2}(n_{12}v_2)^2+18(v_1v_2)-9v_2^2\right)\\
&\qquad+\frac{G^3m_1m_2^2}{r_{12}^4}\bigg(\frac{415}{8}(n_{12}v_1)^2-\frac{375}{4}(n_{12}v_1)(n_{12}v_2)+\frac{1113}{8}(n_{12}v_2)^2-\frac{615}{64}(n_{12}v_{12})^2\pi^2\\
&\qquad\qquad+18v_1^2+\frac{123}{64}\pi^2v_{12}^2+33(v_1v_2)-\frac{33}{2}v_2^2\bigg)\\
&\qquad+\frac{G^3m_1^2m_2}{r_{12}^4}\bigg(-\frac{45887}{168}(n_{12}v_1)^2+\frac{24025}{42}(n_{12}v_1)(n_{12}v_2)-\frac{10469}{42}(n_{12}v_2)^2+\frac{48197}{840}v_1^2\\
&\qquad\qquad-\frac{36227}{420}(v_1v_2)+\frac{36227}{840}v_2^2+110(n_{12}v_{12})^2\ln\left(\frac{r_{12}}{r'_1}\right)-22v_{12}^2\ln\left(\frac{r_{12}}{r'_1}\right)\bigg)\\
&\qquad+\frac{16G^4m_2^4}{r_{12}^5}+\frac{G^4m_1^2m_2^2}{r_{12}^5}\left(175-\frac{41}{16}\pi^2\right)+\frac{G^4m_1^3m_2}{r_{12}^5}\left(-\frac{3187}{1260}+\frac{44}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right)\\
&\qquad+\frac{G^4m_1m_2^3}{r_{12}^5}\left(\frac{110741}{630}-\frac{41}{16}\pi^2-\frac{44}{3}\ln\left(\frac{r_{12}}{r'_2}\right)\right)\Bigg]\boldsymbol{n}_{12}\\
&\qquad+\Bigg[\frac{Gm_2}{r_{12}^2}\bigg(\frac{15}{2}(n_{12}v_1)(n_{12}v_2)^4-\frac{45}{8}(n_{12}v_2)^5-\frac{3}{2}(n_{12}v_2)^3v_1^2+6(n_{12}v_1)(n_{12}v_2)^2(v_1v_2)
\end{aligned}
$$

$$
\begin{aligned}
&- 6(n_{12}v_2)^3(v_1v_2) - 2(n_{12}v_2)(v_1v_2)^2 - 12(n_{12}v_1)(n_{12}v_2)^2v_2^2 + 12(n_{12}v_2)^3v_2^2 \\
&+ (n_{12}v_2)v_1^2v_2^2 - 4(n_{12}v_1)(v_1v_2)v_2^2 + 8(n_{12}v_2)(v_1v_2)v_2^2 + 4(n_{12}v_1)v_2^4 \\
&- 7(n_{12}v_2)v_2^4\Big) \\
&+ \frac{G^2m_2^2}{r_{12}^3}\Big(- 2(n_{12}v_1)^2(n_{12}v_2) + 8(n_{12}v_1)(n_{12}v_2)^2 + 2(n_{12}v_2)^3 + 2(n_{12}v_1)(v_1v_2) \\
&\qquad + 4(n_{12}v_2)(v_1v_2) - 2(n_{12}v_1)v_2^2 - 4(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^2m_1m_2}{r_{12}^3}\Big(-\frac{243}{4}(n_{12}v_1)^3 + \frac{565}{4}(n_{12}v_1)^2(n_{12}v_2) - \frac{269}{4}(n_{12}v_1)(n_{12}v_2)^2 \\
&\qquad - \frac{95}{12}(n_{12}v_2)^3 + \frac{207}{8}(n_{12}v_1)v_1^2 - \frac{137}{8}(n_{12}v_2)v_1^2 - 36(n_{12}v_1)(v_1v_2) \\
&\qquad + \frac{27}{4}(n_{12}v_2)(v_1v_2) + \frac{81}{8}(n_{12}v_1)v_2^2 + \frac{83}{8}(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^3m_2^3}{r_{12}^4}\left(4(n_{12}v_1) + 5(n_{12}v_2)\right) \\
&+ \frac{G^3m_1m_2^2}{r_{12}^4}\left(-\frac{307}{8}(n_{12}v_1) + \frac{479}{8}(n_{12}v_2) + \frac{123}{32}(n_{12}v_{12})\pi^2\right) \\
&+ \frac{G^3m_1^2m_2}{r_{12}^4}\left(\frac{31397}{420}(n_{12}v_1) - \frac{36227}{420}(n_{12}v_2) - 44(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\right)\Bigg]\boldsymbol{v}_{12}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\boxed{+\frac{1}{c^7}}\Bigg\{\Bigg[\frac{G^4m_1^3m_2}{r_{12}^5}\left(\frac{3992}{105}(n_{12}v_1) - \frac{4328}{105}(n_{12}v_2)\right) \\
&+ \frac{G^4m_1^2m_2^2}{r_{12}^6}\left(-\frac{13576}{105}(n_{12}v_1) + \frac{2872}{21}(n_{12}v_2)\right) - \frac{3172}{21}\frac{G^4m_1m_2^3}{r_{12}^6}(n_{12}v_{12}) \\
&+ \frac{G^3m_1^2m_2}{r_{12}^4}\Big(48(n_{12}v_1)^3 - \frac{696}{5}(n_{12}v_1)^2(n_{12}v_2) + \frac{744}{5}(n_{12}v_1)(n_{12}v_2)^2 - \frac{288}{5}(n_{12}v_2)^3 \\
&\qquad - \frac{4888}{105}(n_{12}v_1)v_1^2 + \frac{5056}{105}(n_{12}v_2)v_1^2 + \frac{2056}{21}(n_{12}v_1)(v_1v_2) \\
&\qquad - \frac{2224}{21}(n_{12}v_2)(v_1v_2) - \frac{1028}{21}(n_{12}v_1)v_2^2 + \frac{5812}{105}(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^3m_1m_2^2}{r_{12}^4}\Big(-\frac{582}{5}(n_{12}v_1)^3 + \frac{1746}{5}(n_{12}v_1)^2(n_{12}v_2) - \frac{1954}{5}(n_{12}v_1)(n_{12}v_2)^2 \\
&\qquad + 158(n_{12}v_2)^3 + \frac{3568}{105}(n_{12}v_{12})v_1^2 - \frac{2864}{35}(n_{12}v_1)(v_1v_2) \\
&\qquad + \frac{10048}{105}(n_{12}v_2)(v_1v_2) + \frac{1432}{35}(n_{12}v_1)v_2^2 - \frac{5752}{105}(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^2m_1m_2}{r_{12}^3}\Big(- 56(n_{12}v_{12})^5 + 60(n_{12}v_1)^3v_{12}^2 - 180(n_{12}v_1)^2(n_{12}v_2)v_{12}^2 \\
&\qquad + 174(n_{12}v_1)(n_{12}v_2)^2v_{12}^2 - 54(n_{12}v_2)^3v_{12}^2 - \frac{246}{35}(n_{12}v_{12})v_1^4 \\
&\qquad + \frac{1068}{35}(n_{12}v_1)v_1^2(v_1v_2) - \frac{984}{35}(n_{12}v_2)v_1^2(v_1v_2) - \frac{1068}{35}(n_{12}v_1)(v_1v_2)^2
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \frac{180}{7}(n_{12}v_2)(v_1v_2)^2 - \frac{534}{35}(n_{12}v_1)v_1^2v_2^2 + \frac{90}{7}(n_{12}v_2)v_1^2v_2^2 \\
&\qquad + \frac{984}{35}(n_{12}v_1)(v_1v_2)v_2^2 - \frac{732}{35}(n_{12}v_2)(v_1v_2)v_2^2 - \frac{204}{35}(n_{12}v_1)v_2^4 \\
&\qquad + \frac{24}{7}(n_{12}v_2)v_2^4\bigg)\bigg]\boldsymbol{n}_{12} \\
&+ \bigg[-\frac{184}{21}\frac{G^4m_1^3m_2}{r_{12}^5} + \frac{6224}{105}\frac{G^4m_1^2m_2^2}{r_{12}^6} + \frac{6388}{105}\frac{G^4m_1m_2^3}{r_{12}^6} \\
&\quad + \frac{G^3m_1^2m_2}{r_{12}^4}\bigg(\frac{52}{15}(n_{12}v_1)^2 - \frac{56}{15}(n_{12}v_1)(n_{12}v_2) - \frac{44}{15}(n_{12}v_2)^2 - \frac{132}{35}v_1^2 + \frac{152}{35}(v_1v_2) \\
&\qquad - \frac{48}{35}v_2^2\bigg) \\
&\quad + \frac{G^3m_1m_2^2}{r_{12}^4}\bigg(\frac{454}{15}(n_{12}v_1)^2 - \frac{372}{5}(n_{12}v_1)(n_{12}v_2) + \frac{854}{15}(n_{12}v_2)^2 - \frac{152}{21}v_1^2 \\
&\qquad + \frac{2864}{105}(v_1v_2) - \frac{1768}{105}v_2^2\bigg) \\
&\quad + \frac{G^2m_1m_2}{r_{12}^3}\bigg(60(n_{12}v_{12})^4 - \frac{348}{5}(n_{12}v_1)^2v_{12}^2 + \frac{684}{5}(n_{12}v_1)(n_{12}v_2)v_{12}^2 \\
&\qquad - 66(n_{12}v_2)^2v_{12}^2 + \frac{334}{35}v_1^4 - \frac{1336}{35}v_1^2(v_1v_2) + \frac{1308}{35}(v_1v_2)^2 + \frac{654}{35}v_1^2v_2^2 \\
&\qquad - \frac{1252}{35}(v_1v_2)v_2^2 + \frac{292}{35}v_2^4\bigg)\bigg]\boldsymbol{v}_{12}\bigg\} + \mathcal{O}\left(\frac{1}{c^8}\right). \qquad (203)
\end{aligned}
$$

When retaining the "even" relativistic corrections at the 1PN, 2PN and 3PN orders, and neglecting the "odd" radiation reaction terms at the 2.5PN and 3.5PN orders, we find that the equations of motion admit a conserved energy (and a Lagrangian, as we shall see); that energy can be straightforwardly obtained by guess-work starting from Eq. (203), with the result

$$
\begin{aligned}
E = {} & \frac{m_1 v_1^2}{2} - \frac{G m_1 m_2}{2 r_{12}} \\
& \boxed{+\frac{1}{c^2}} \left\{ \frac{G^2 m_1^2 m_2}{2 r_{12}^2} + \frac{3 m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}} \left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1 v_2) \right) \right\} \\
& \boxed{+\frac{1}{c^4}} \left\{ -\frac{G^3 m_1^3 m_2}{2 r_{12}^3} - \frac{19 G^3 m_1^2 m_2^2}{8 r_{12}^3} + \frac{5 m_1 v_1^6}{16} \right. \\
& \quad + \frac{G m_1 m_2}{r_{12}} \left(\frac{3}{8}(n_{12}v_1)^3 (n_{12}v_2) + \frac{3}{16}(n_{12}v_1)^2 (n_{12}v_2)^2 - \frac{9}{8}(n_{12}v_1)(n_{12}v_2) v_1^2 \right. \\
& \qquad - \frac{13}{8}(n_{12}v_2)^2 v_1^2 + \frac{21}{8} v_1^4 + \frac{13}{8}(n_{12}v_1)^2 (v_1 v_2) + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1 v_2) \\
& \qquad \left. - \frac{55}{8} v_1^2 (v_1 v_2) + \frac{17}{8}(v_1 v_2)^2 + \frac{31}{16} v_1^2 v_2^2 \right) \\
& \quad \left. + \frac{G^2 m_1^2 m_2}{r_{12}^2} \left(\frac{29}{4}(n_{12}v_1)^2 - \frac{13}{4}(n_{12}v_1)(n_{12}v_2) + \frac{1}{2}(n_{12}v_2)^2 - \frac{3}{2}v_1^2 + \frac{7}{4}v_2^2 \right) \right\} \\
& \boxed{+\frac{1}{c^6}} \left\{ \frac{35 m_1 v_1^8}{128} \right. \\
& \quad + \frac{G m_1 m_2}{r_{12}} \left(-\frac{5}{16}(n_{12}v_1)^5 (n_{12}v_2) - \frac{5}{16}(n_{12}v_1)^4 (n_{12}v_2)^2 - \frac{5}{32}(n_{12}v_1)^3 (n_{12}v_2)^3 \right. \\
& \qquad + \frac{19}{16}(n_{12}v_1)^3 (n_{12}v_2) v_1^2 + \frac{15}{16}(n_{12}v_1)^2 (n_{12}v_2)^2 v_1^2 + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)^3 v_1^2 \\
& \qquad + \frac{19}{16}(n_{12}v_2)^4 v_1^2 - \frac{21}{16}(n_{12}v_1)(n_{12}v_2) v_1^4 - 2(n_{12}v_2)^2 v_1^4 \\
& \qquad + \frac{55}{16} v_1^6 - \frac{19}{16}(n_{12}v_1)^4 (v_1 v_2) - (n_{12}v_1)^3 (n_{12}v_2)(v_1 v_2) \\
& \qquad - \frac{15}{32}(n_{12}v_1)^2 (n_{12}v_2)^2 (v_1 v_2) + \frac{45}{16}(n_{12}v_1)^2 v_1^2 (v_1 v_2) \\
& \qquad + \frac{5}{4}(n_{12}v_1)(n_{12}v_2) v_1^2 (v_1 v_2) + \frac{11}{4}(n_{12}v_2)^2 v_1^2 (v_1 v_2) - \frac{139}{16} v_1^4 (v_1 v_2) \\
& \qquad - \frac{3}{4}(n_{12}v_1)^2 (v_1 v_2)^2 + \frac{5}{16}(n_{12}v_1)(n_{12}v_2)(v_1 v_2)^2 + \frac{41}{8} v_1^2 (v_1 v_2)^2 + \frac{1}{16}(v_1 v_2)^3 \\
& \qquad \left. - \frac{45}{16}(n_{12}v_1)^2 v_1^2 v_2^2 - \frac{23}{32}(n_{12}v_1)(n_{12}v_2) v_1^2 v_2^2 + \frac{79}{16} v_1^4 v_2^2 - \frac{161}{32} v_1^2 (v_1 v_2) v_2^2 \right) \\
& \quad + \frac{G^2 m_1^2 m_2}{r_{12}^2} \left(-\frac{49}{8}(n_{12}v_1)^4 + \frac{75}{8}(n_{12}v_1)^3 (n_{12}v_2) - \frac{187}{8}(n_{12}v_1)^2 (n_{12}v_2)^2 \right. \\
& \qquad + \frac{247}{24}(n_{12}v_1)(n_{12}v_2)^3 + \frac{49}{8}(n_{12}v_1)^2 v_1^2 + \frac{81}{8}(n_{12}v_1)(n_{12}v_2) v_1^2 \\
& \qquad - \frac{21}{4}(n_{12}v_2)^2 v_1^2 + \frac{11}{2} v_1^4 - \frac{15}{2}(n_{12}v_1)^2 (v_1 v_2) - \frac{3}{2}(n_{12}v_1)(n_{12}v_2)(v_1 v_2)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \frac{21}{4}(n_{12}v_2)^2(v_1v_2) - 27v_1^2(v_1v_2) + \frac{55}{2}(v_1v_2)^2 + \frac{49}{4}(n_{12}v_1)^2v_2^2 \\
&\qquad\qquad - \frac{27}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{3}{4}(n_{12}v_2)^2v_2^2 + \frac{55}{4}v_1^2v_2^2 - 28(v_1v_2)v_2^2 + \frac{135}{16}v_2^4\bigg) \\
&\quad + \frac{3G^4m_1^4m_2}{8r_{12}^4} + \frac{G^4m_1^3m_2^2}{r_{12}^4}\left(\frac{9707}{420} - \frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right) \\
&\quad + \frac{G^3m_1^2m_2^2}{r_{12}^3}\bigg(\frac{547}{12}(n_{12}v_1)^2 - \frac{3115}{48}(n_{12}v_1)(n_{12}v_2) - \frac{123}{64}(n_{12}v_1)(n_{12}v_{12})\pi^2 - \frac{575}{18}v_1^2 \\
&\qquad\qquad + \frac{41}{64}\pi^2(v_1v_{12}) + \frac{4429}{144}(v_1v_2)\bigg) \\
&\quad + \frac{G^3m_1^3m_2}{r_{12}^3}\bigg(-\frac{44627}{840}(n_{12}v_1)^2 + \frac{32027}{840}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}(n_{12}v_2)^2 + \frac{24187}{2520}v_1^2 \\
&\qquad\qquad - \frac{27967}{2520}(v_1v_2) + \frac{5}{4}v_2^2 + 22(n_{12}v_1)(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right) - \frac{22}{3}(v_1v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\bigg)\bigg\} \\
&+ 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right). \qquad\qquad (205)
\end{aligned}
$$

Post Newtonian Energy & Flux

$$E = -\frac{\mu c^2 x}{2}\left\{1 + \left(-\frac{3}{4} - \frac{1}{12}\nu\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right)x^2 + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)x^3\right\} + \mathcal{O}\left(\frac{1}{c^8}\right).$$

nonspinnig & circular orbit binary

$$\mathcal{L} = \frac{32c^5}{5G}\nu^2 x^5\left\{1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{856}{105}\ln(16x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}.$$

$$\boxed{x \equiv \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3} \qquad m = m_1 + m_2 \qquad \mu = m_1 m_2/m \qquad \nu \equiv \frac{\mu}{m} \equiv \frac{m_1 m_2}{(m_1+m_2)^2}.}$$

Phase evolution

Energy balance equation : orbital binding energy loss = GW emission energy

$$\boxed{\frac{dE}{dt} = -L} \qquad \frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} \qquad \frac{dx}{dt} = -\frac{L}{(dE/dx)}$$

expand with x , $\quad x \equiv \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3}$

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\eta(M\omega)^{5/3}\left(1 - \frac{743+924\eta}{336}(M\omega)^{2/3} + \left(\frac{34\,103}{18\,144} + \frac{13\,661}{2016}\eta + \frac{59}{18}\eta^2\right)(M\omega)^{4/3} - \frac{1}{672}(4159+14\,532\eta)\pi(M\omega)^{5/3}\eta\right.$$

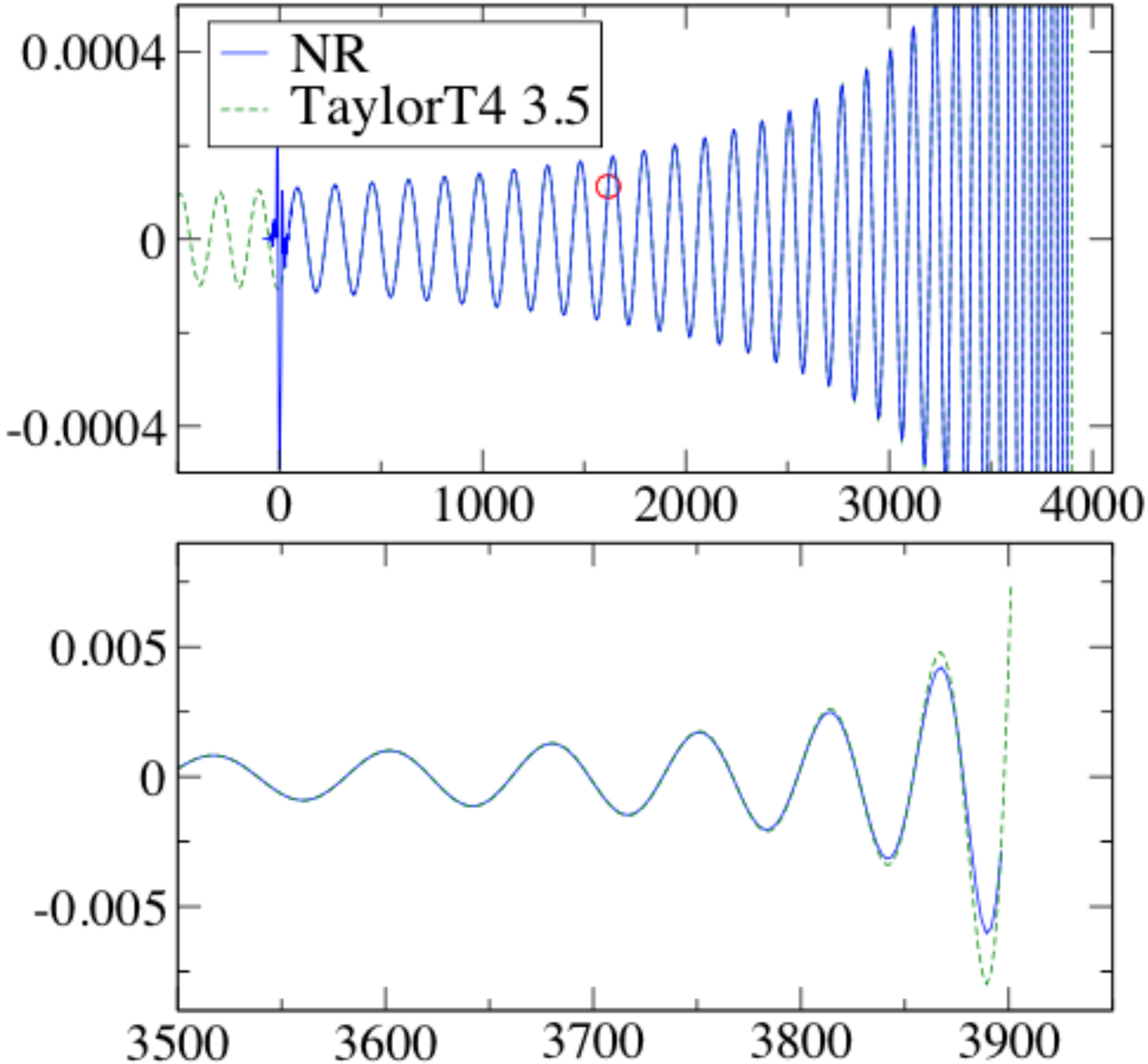
$$+\left[\left(\frac{16\,447\,322\,263}{139\,708\,800} - \frac{1712}{105}\gamma_E + \frac{16}{3}\pi^2\right) + \left(-\frac{273\,811\,877}{1\,088\,640} + \frac{451}{48}\pi^2 - \frac{88}{3}\hat{\theta}\right)\eta + \frac{541}{896}\eta^2 - \frac{5605}{2592}\eta^3\right.$$

$$\left.\left. - \frac{856}{105}\log[16(M\omega)^{2/3}]\right](M\omega)^2 + \left(-\frac{4\,415}{4\,032} + \frac{661\,775}{12\,096}\eta + \frac{149\,789}{3\,024}\eta^2\right)\pi(M\omega)^{7/3}\right),$$

$$w(t) = \int_0^t \dot{w}(w)dt, \qquad \psi(t) = \int_0^t w(t)dt$$ ---> Numerical integration

TaylorT4, T1,2,3,.. : time domain models

Comparing Waveforms



TaylorF2 : frequency-domain model

$$h(f) = \frac{M_c^{5/6}}{\pi^{2/3} D_{\rm eff}} \sqrt{\frac{5}{24}} \underline{f^{-7/6}} e^{i\Psi(f)},$$

stationary phase approx.

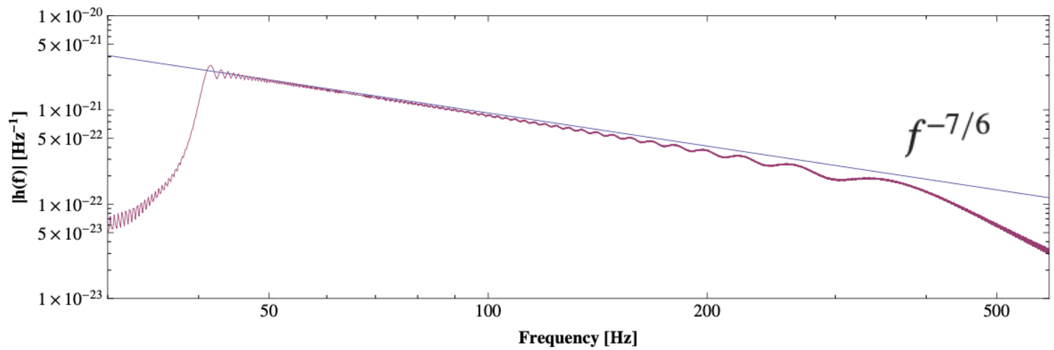


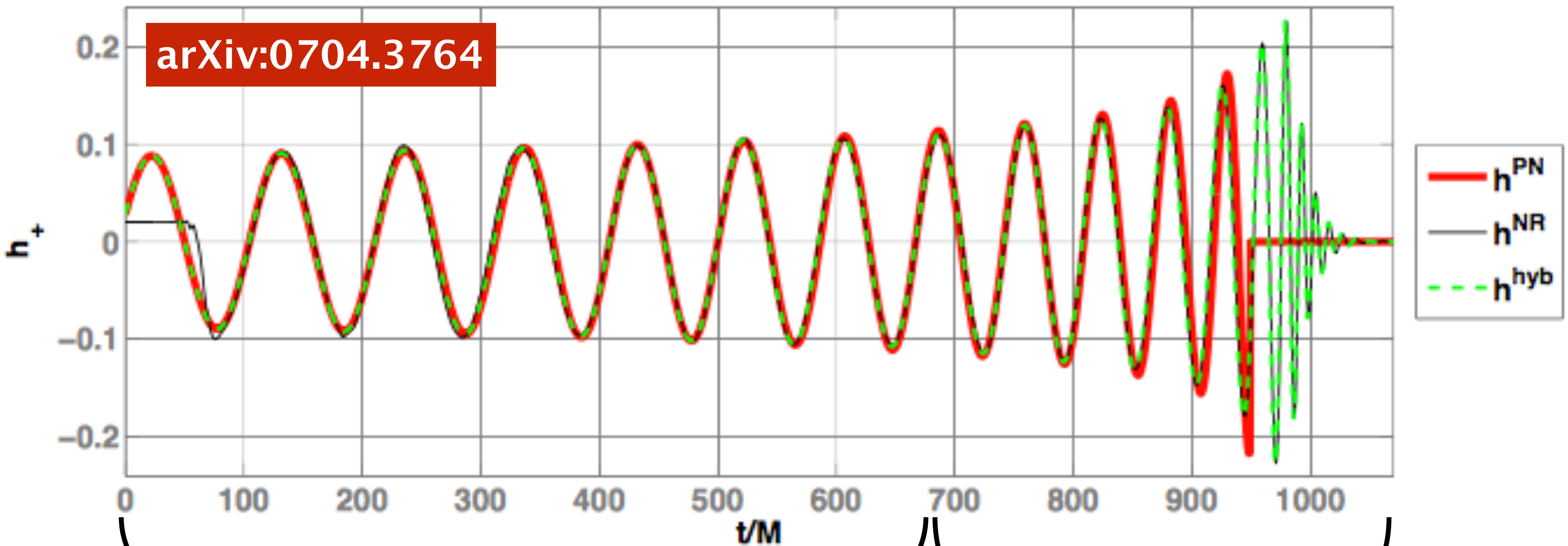
Figure 3.2: **Fourier domain TaylorT4 and SPA waveforms from a non-spinning binary.** We assume the same binary model as in figure 3.1. TaylorT4 (red) and SPA (blue) waveforms coincide at 40 Hz.

$$\begin{aligned}
\Phi_{\rm SPA}(f) &= 2\pi f t_c - \Phi_c - \frac{\pi}{4} + \frac{3}{128\nu v^5}\left\{1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11}{4}\nu\right)v^2 - 16\pi v^3\right. \\
&+ 10\left(\frac{3058673}{1016064} + \frac{5429}{1008}\nu + \frac{617}{144}\nu^2\right)v^4 \qquad (3.20) \\
&+ \left(\frac{38645}{756} - \frac{65}{9}\nu\right)\left[1 + 3\log\left(\frac{v}{v_{\rm iso}}\right)\right]\pi v^5 \left[\left(\frac{11583231236531}{4694215680}\right.\right. \\
&- \left.\frac{640}{3}\pi^2 - \frac{6848}{21}\gamma_E - \frac{6848}{21}\log(4v)\right) + \left(\frac{2255}{12}\pi^2 - \frac{15737765635}{3048192}\right)\nu \\
&+ \left.\left.\frac{76055}{1728}\nu^2 - \frac{127825}{1296}\nu^3\right]v^6 + \left(\frac{77096675}{254016} + \frac{378515}{1512}\nu - \frac{74045}{756}\nu^2\right)\pi v^7\right\}
\end{aligned}$$

$$\boxed{v \equiv [\pi f(m_1 + m_2)]^{1/3}}$$

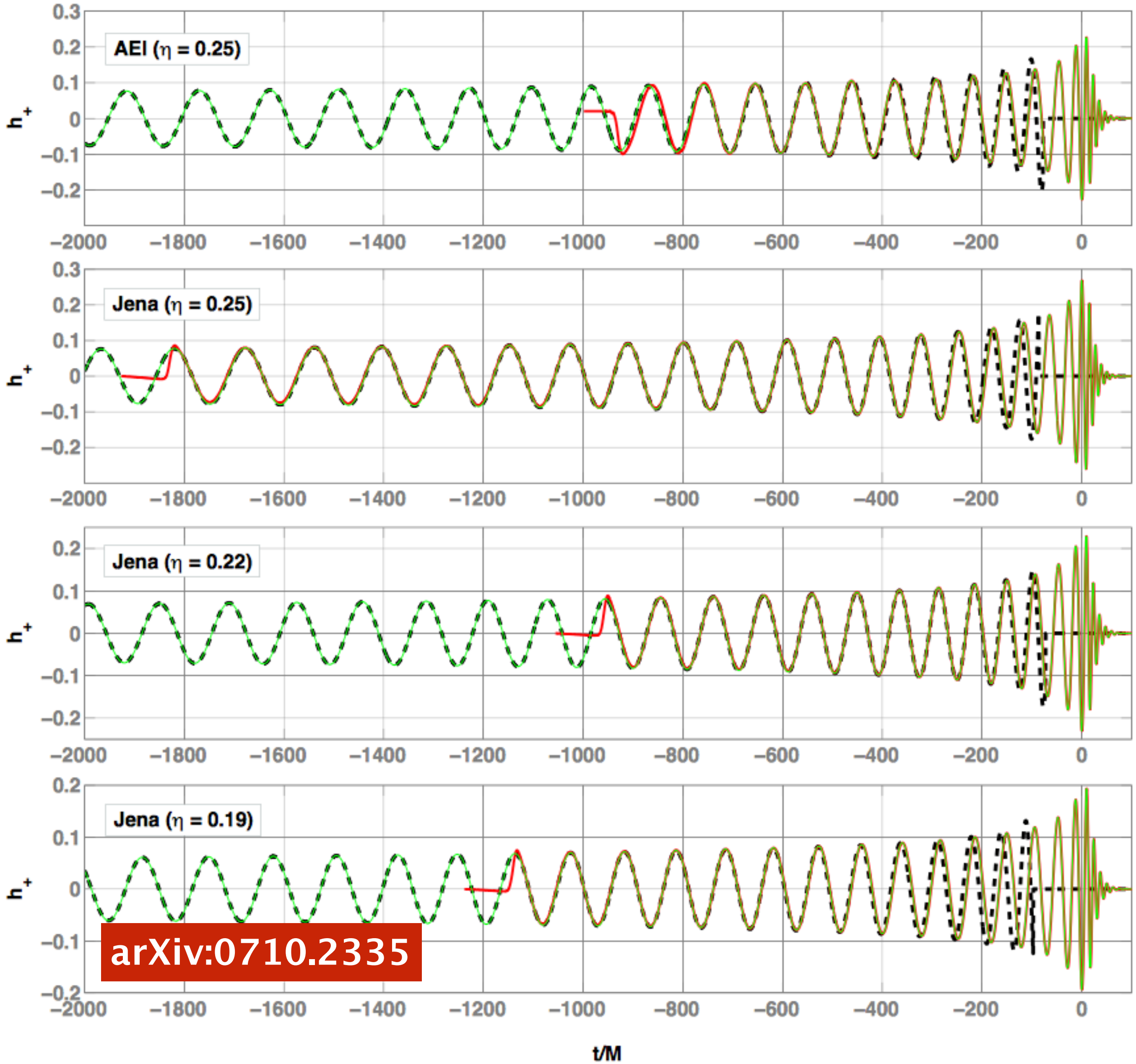
Phenomenological model : IMRPhenom

frequency-domain model !!

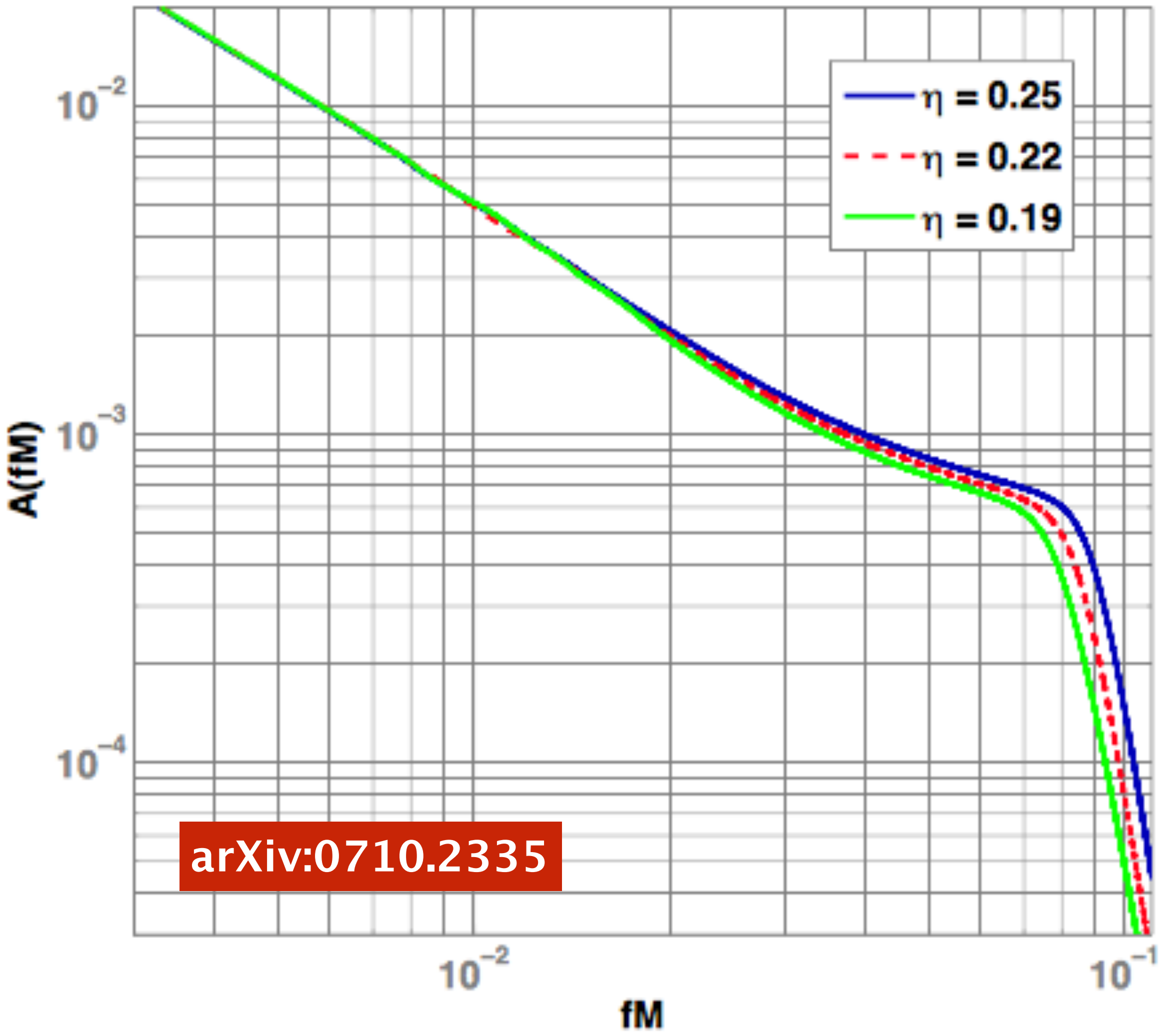


Hybrid = early inspiral:PN + late inspiral-M-R:NR

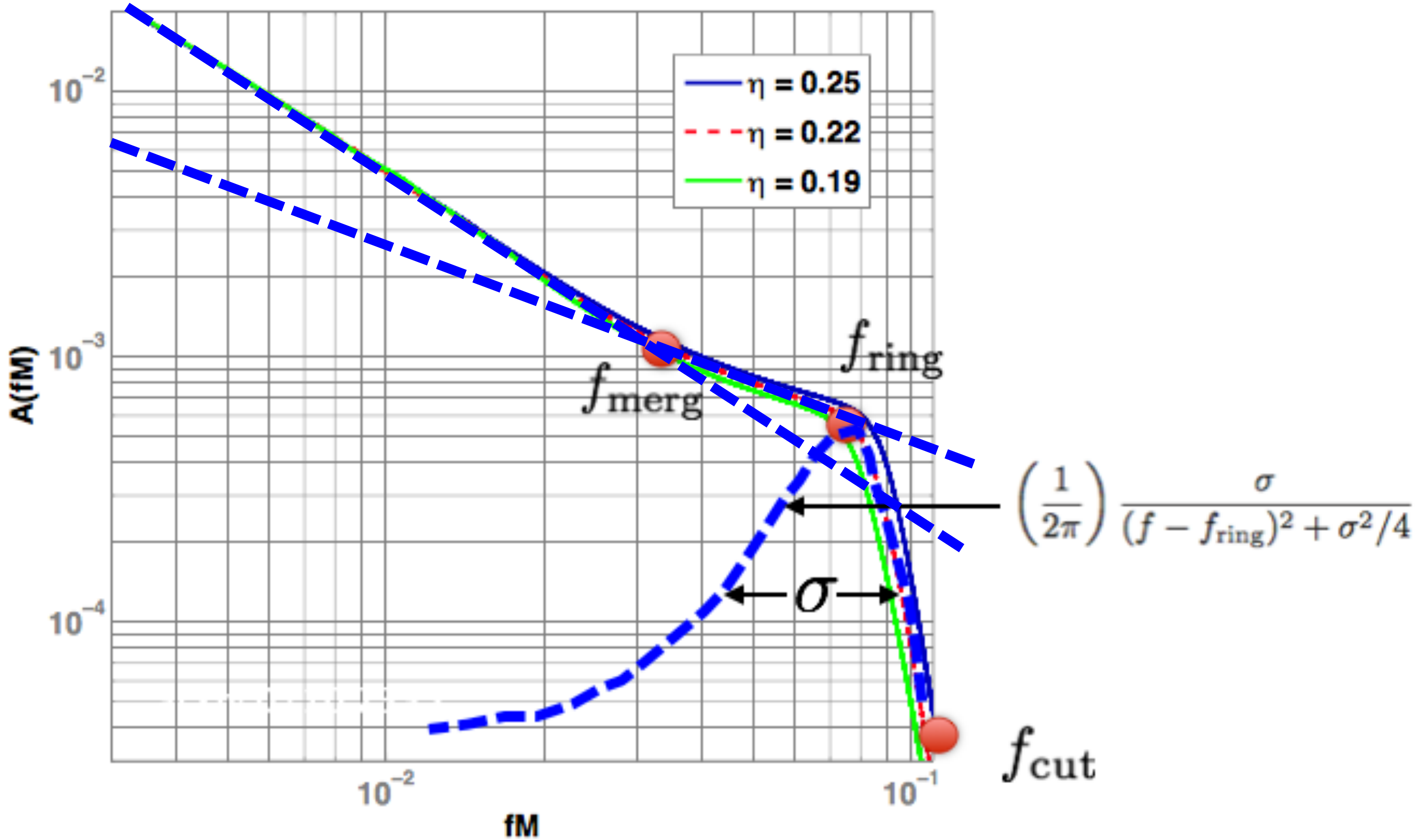
Constructing Hybrid waveforms



Fourier amplitudes of hybrid waveforms



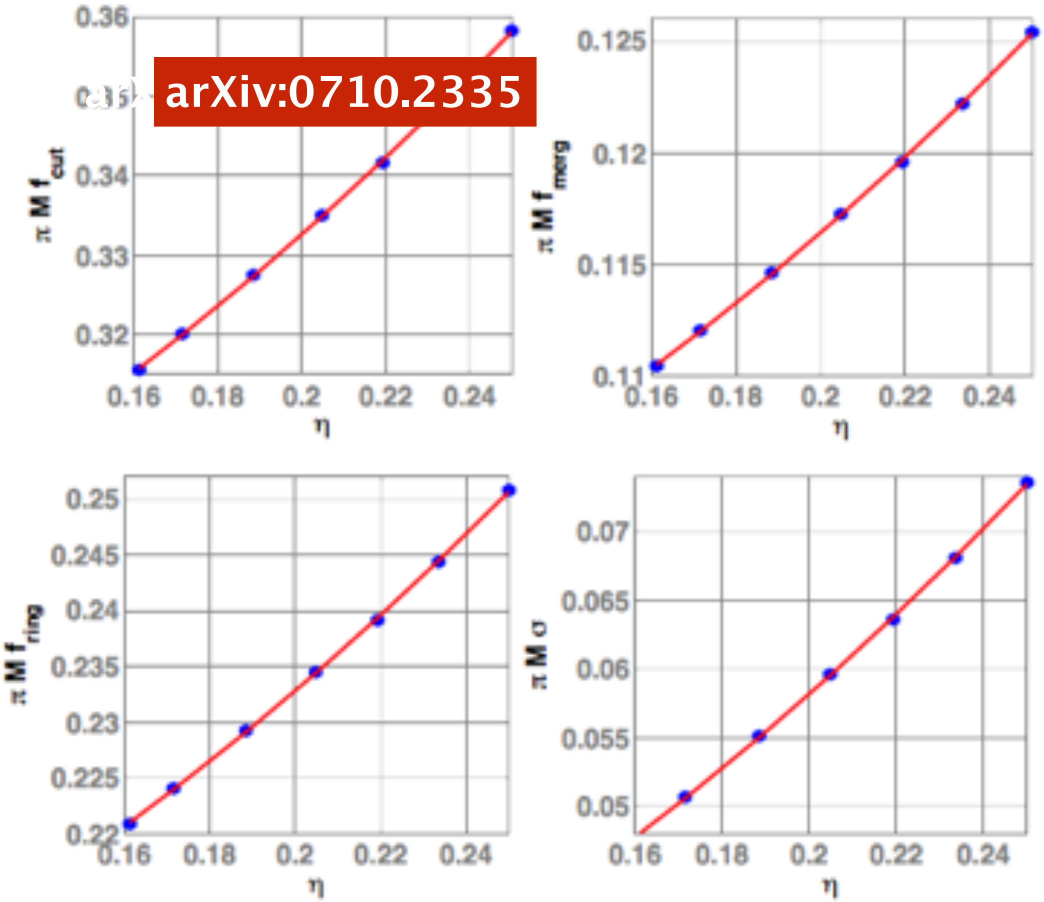
Fourier amplitudes of hybrid waveforms



$$A_{\rm eff}(f) \equiv C \begin{cases} (f/f_{\rm merg})^{-7/6} & \text{if } f < f_{\rm merg} \\ (f/f_{\rm merg})^{-2/3} & \text{if } f_{\rm merg} \leq f < f_{\rm ring} \\ w\,\mathcal{L}(f, f_{\rm ring}, \sigma) & \text{if } f_{\rm ring} \leq f < f_{\rm cut} \end{cases}$$

4 parameters

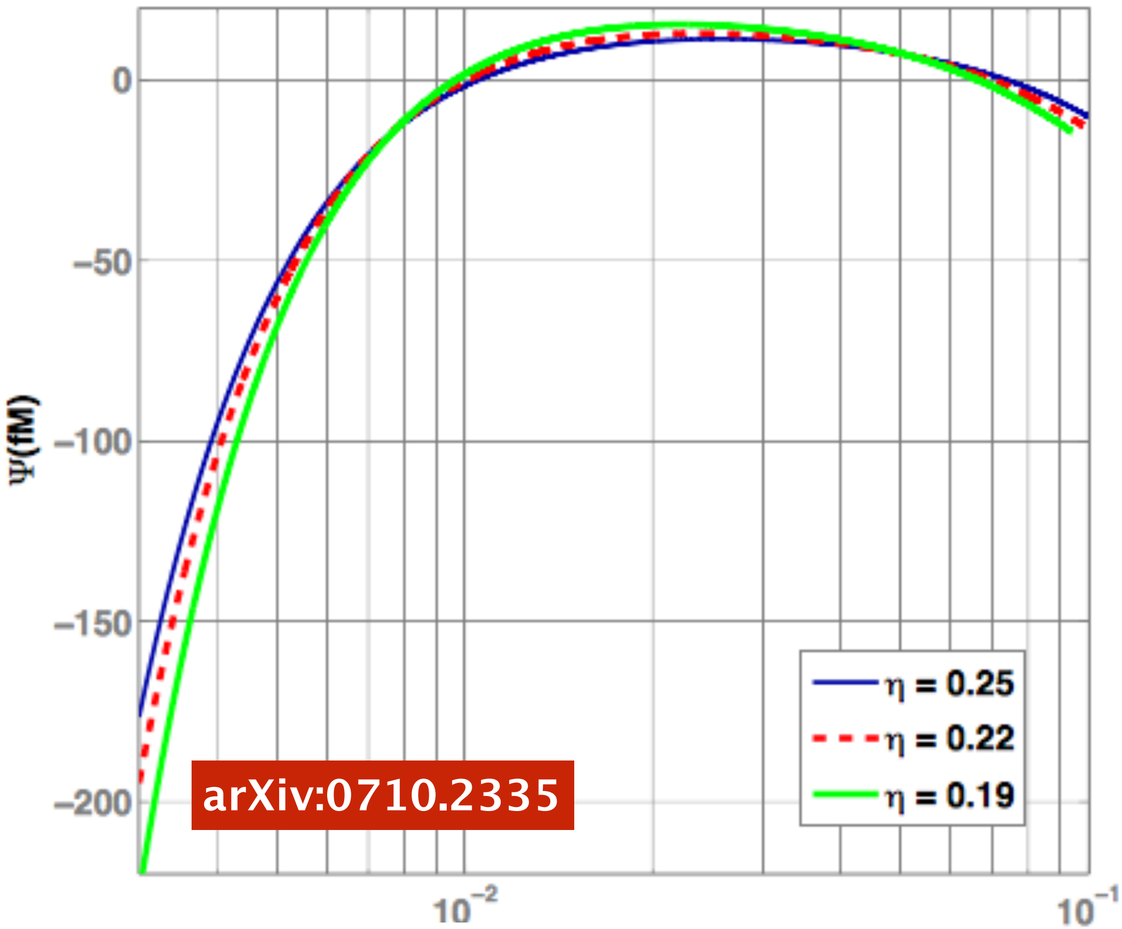
Best-match amplitude parameters



$$\alpha_{j\ \mathrm{int}} = \frac{a_j\,\eta^2 + b_j\,\eta + c_j}{\pi M}$$

Parameter	a_k	b_k	c_k
f_{merg}	2.9740×10^{-1}	4.4810×10^{-2}	9.5560×10^{-2}
f_{ring}	5.9411×10^{-1}	8.9794×10^{-2}	1.9111×10^{-1}
σ	5.0801×10^{-1}	7.7515×10^{-2}	2.2369×10^{-2}
f_{cut}	8.4845×10^{-1}	1.2848×10^{-1}	2.7299×10^{-1}

IMR phase

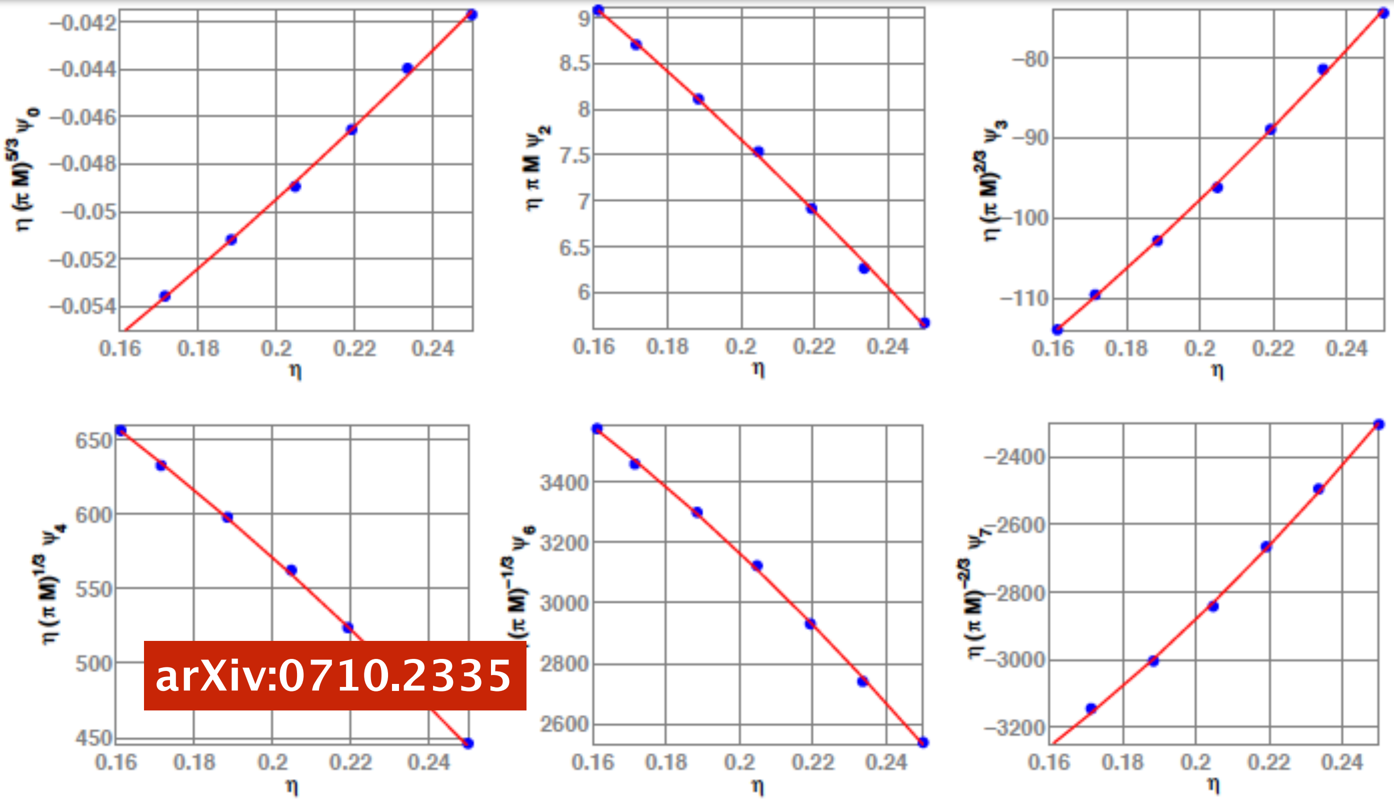


$$\Psi_{\rm eff}(f) = 2\pi f t_0 + \varphi_0 + \sum_{k=0}^{7} \psi_k \, f^{(k-5)/3}\,,$$

$$\boldsymbol{\psi} = \{\psi_0,\, \psi_2,\, \psi_3,\, \psi_4,\, \psi_6,\, \psi_7\}$$

6 parameters

Best-match phase parameters



$$\psi_{k\ \mathrm{int}} = \frac{x_k\,\eta^2 + y_k\,\eta + z_k}{\eta\,(\pi M)^{(5-k)/3}}$$

Parameter	x_k	y_k	z_k
ψ_0	1.7516×10^{-1}	7.9483×10^{-2}	-7.2390×10^{-2}
ψ_2	-5.1571×10^{1}	-1.7595×10^{1}	1.3253×10^{1}
ψ_3	6.5866×10^{2}	1.7803×10^{2}	-1.5972×10^{2}
ψ_4	-3.9031×10^{3}	-7.7493×10^{2}	8.8195×10^{2}
ψ_6	-2.4874×10^{4}	-1.4892×10^{3}	4.4588×10^{3}
ψ_7	2.5196×10^{4}	3.3970×10^{2}	-3.9573×10^{3}

Early inspiral amplitude : TaylorF2

TaylorF2

$$\boxed{h(f) = \frac{\mathcal{M}_c^{5/6}}{\pi^{2/3} D_{\rm eff}} \sqrt{\frac{5}{24}} \underline{f^{-7/6}} e^{i\Psi(f)},}$$

IMRPhenom

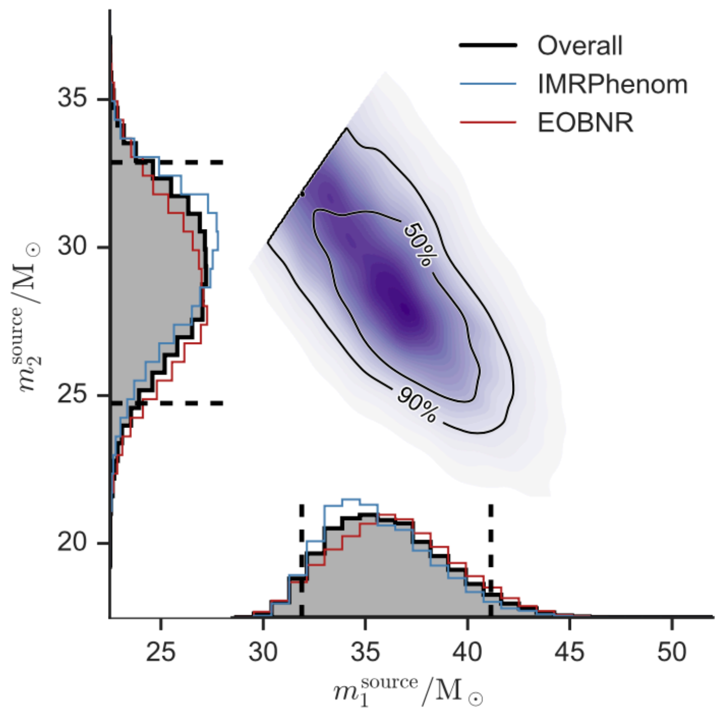
$$u(f) \equiv \mathcal{A}_{\rm eff}(f)\, e^{i\Psi_{\rm eff}(f)}, \qquad \mathcal{A}_{\rm eff}(f) \equiv \begin{cases} \underline{(f/f_{\rm merg})^{-7/6}} & \text{if} \quad f < f_{\rm merg} \\ (f/f_{\rm merg})^{-2/3} & \text{if} \quad f_{\rm merg} \leqslant f < f_{\rm ring} \\ w\mathcal{L}(f, f_{\rm ring}, \sigma) & \text{if} \quad f_{\rm ring} \leqslant f < f_{\rm cut}. \end{cases}$$

$$\Psi_{\rm eff}(f) = 2\pi f t_0 + \phi_0 + \psi_0 f^{-5/3} + \psi_2 f^{-1} + \psi_3 f^{-2/3} + \psi_4 f^{-1/3} + \psi_6 f^{1/3}$$

Model	PhenomA	PhenomB	PhenomC
Mass range $[M_\odot]$	$50 \leq M \leq 200$	$M \leq 400$	$M \leq 350$
Mass ratio range	$q \leq 4$	$q \leq 10$	$q \leq 4$
Detector	initial LIGO, Virgo, advanced LIGO	initial LIGO	advanced LIGO

BBH Models

Family	Short name	Full name	Precession	Multipoles $(\ell, \|m\|)$	Reference
EOBNR	EOBNR	SEOBNRv4_ROM	✗	(2, 2)	[57]
	EOBNR HM	SEOBNRv4HM_ROM	✗	(2, 2), (2,1), (3, 3), (4, 4), (5, 5)	[26,32]
	EOBNR P	SEOBNRv4P	✓	(2, 2), (2, 1)	[33,118,119]
	EOBNR PHM	SEOBNRv4PHM	✓	(2, 2), (2, 1), (3, 3), (4, 4), (5, 5)	[33,118,119]
Phenom	Phenom	IMRPhenomD	✗	(2, 2)	[120,121]
	Phenom HM	IMRPhenomHM	✗	(2, 2), (2, 1), (3, 3), (3, 2), (4, 4), (4, 3)	[22]
	Phenom P	IMRPhenomPv2/v3[a]	✓	(2, 2)	[23,122]
	Phenom PHM	IMRPhenomPv3HM	✓	(2, 2), (2, 1), (3, 3), (3, 2), (4, 4), (4, 3)	[24]



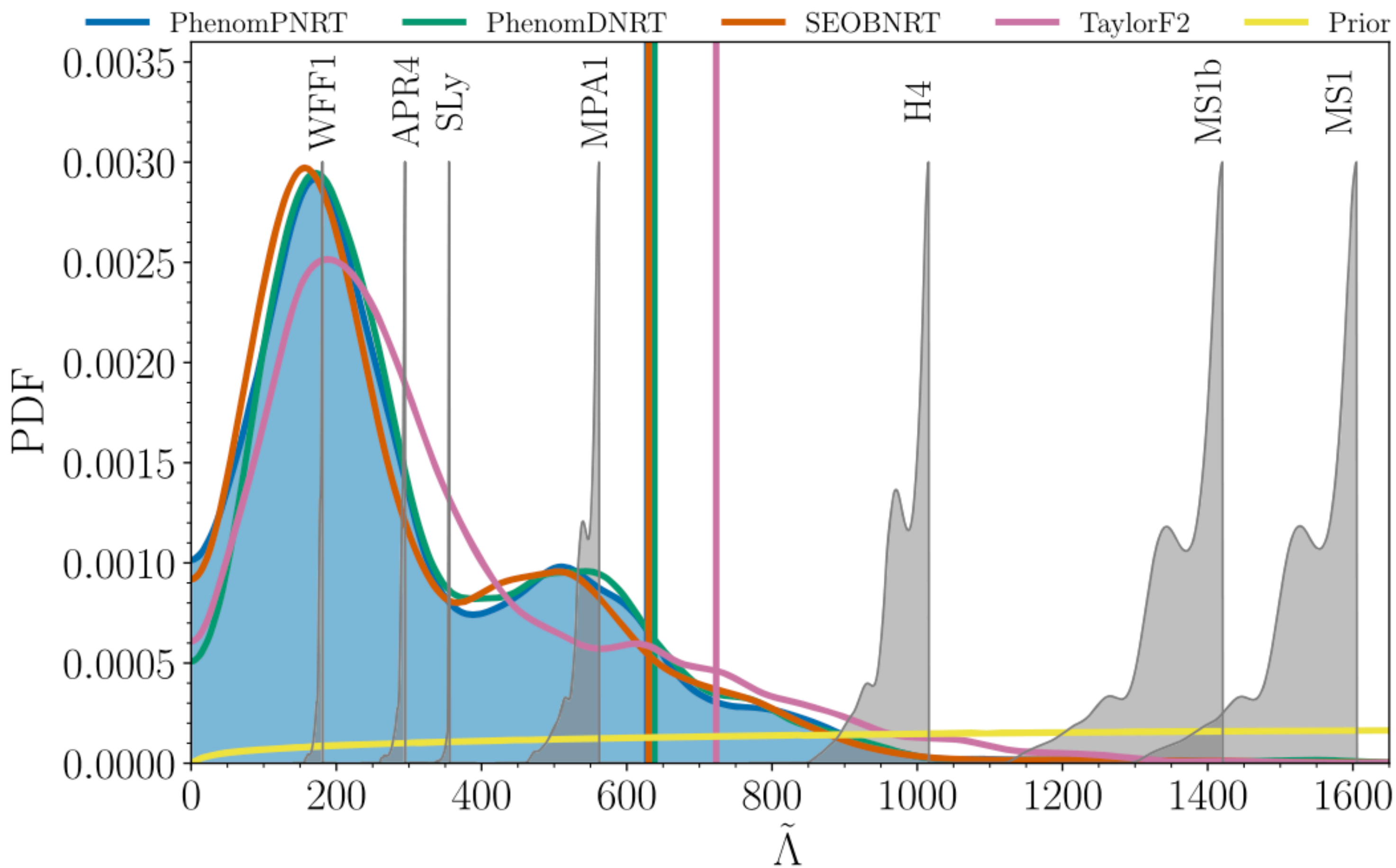
GW150914 PE

BH-NS Models

Full name (implemented in LAL) Short label (used in this work)	References Base model	 Corrections
SEOBNRv4_ROM_NRTidalv2 SEOBNR_T	[14–16]	[17, 18]
SEOBNRv4_ROM_NRTidalv2_NSBH [19] SEOBNR_NSBH	[14–16]	[17, 18, 20]
IMRPhenomPv2_NRTidalv2 IMRPhenomP_T	[21–23]	[17, 18]
IMRPhenomNSBH [24] IMRPhenom_NSBH	[25]	[18, 20]

TABLE I: Waveform models used in our analysis for NSBH systems.

NS-NS Models



GW170817 PE

Thanks !