일반상대론 기초

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✓ 여름학교 주제

- **수치상대론**
  - 3+1 형식화
  - 수치상대론 시뮬레이션
  - 펄서와 중성자별의 기초
  - 중성자별 구조와 상태방정식
  - 블랙홀 준정규 모드

- **중력파**
  - 중력파 이론의 기초
  - 중력파 데이터 분석
  - 중력파 검출기
  - 한국의 중력파 검출기 개발 연구
  - CBC 중력파형과 파형 모델
  - 중력파 사선 후속 전자기파 관측을 통한 다중신호 천문학 연구
  - 중력파 렌징
  - 중력파 우주론

일반상대론, 천체물리/천문학, 광학, ……

목 차

✓ 동영상 자료

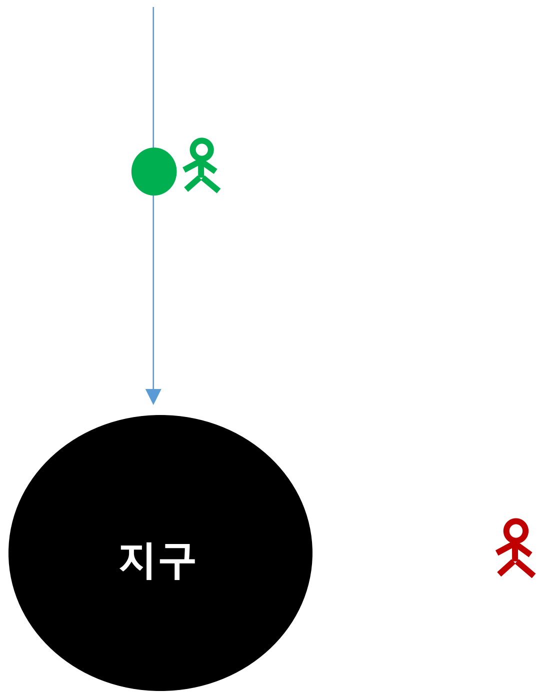
- 카오스 강연 "상대성 이론" 안내
  - 재단 홈페이지: https://ikaos.org/kaos/apply/view.php?kc_idx=152
  - 강연 전체 목록: https://www.youtube.com/playlist?list=PL6JaoI65FbZYrePQLflU446W2pJQ-Jy2n

- 특수상대론 강연 목록
  - 상대성 이론 개론: https://www.youtube.com/watch?v=UAmM1vbeREc
  - 특수 상대성 이론(I): https://www.youtube.com/watch?v=hujwRCgJx1A
  - 특수 상대성 이론(II): https://www.youtube.com/watch?v=02ER8JzIyCg

✓ References

- James Hartle; *Gravity: An Introduction to Einstein's General Relativity*, (Addison Wesley 2021).
- Berbard F. Schutz; *A first course in general relativity*, (Cambridge University Press 2009).
- K.S. Thorne & R.D. Blandford; *Modern Classical Physics* (Princeton University Press, 2017)
- Michele Maggiore; Gravitational Waves, Vo.1 (Oxford University Press, 2017).
- ……

✓ 핵심 질문:

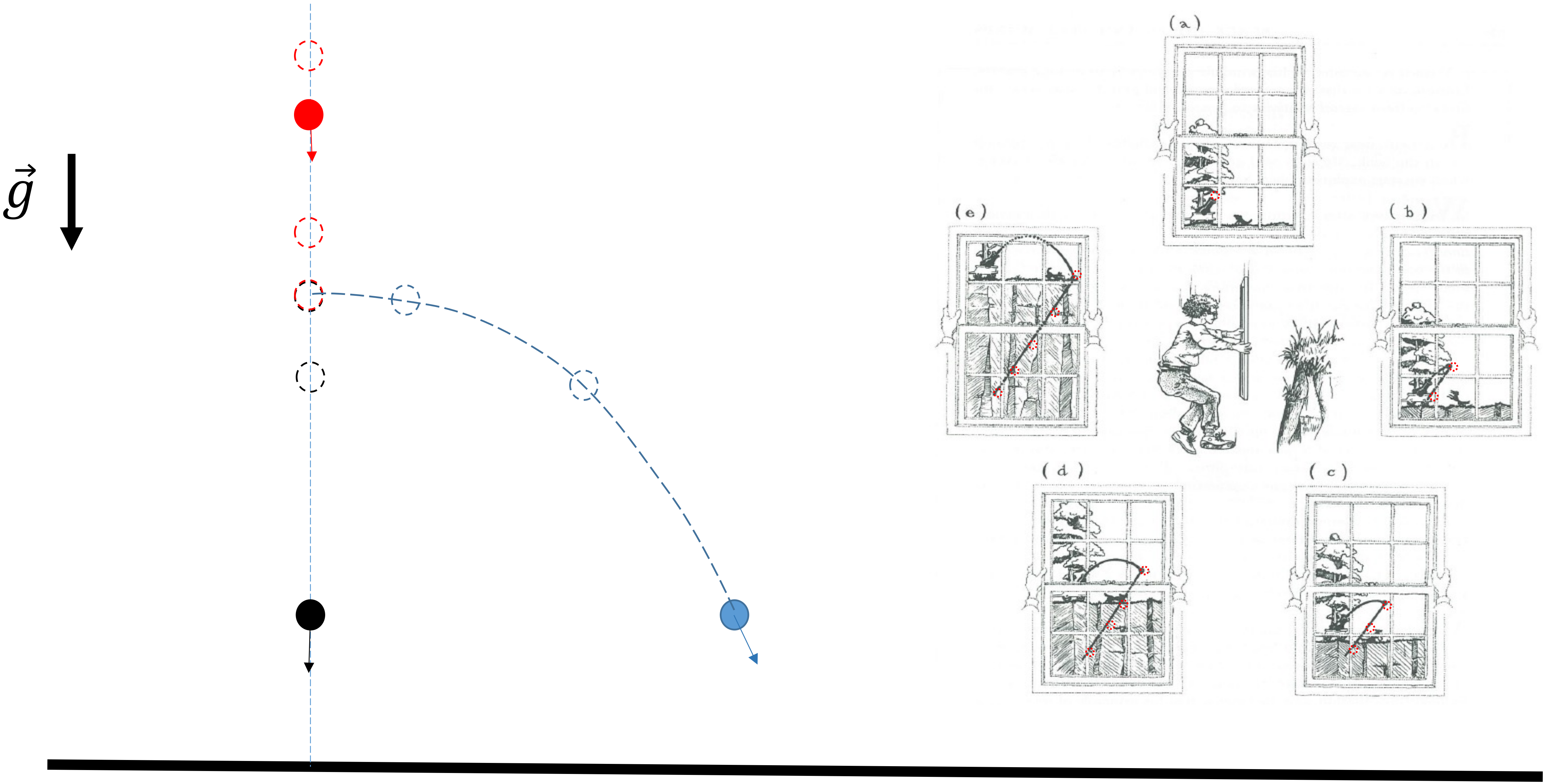


자유낙하가

- 가속 운동...?

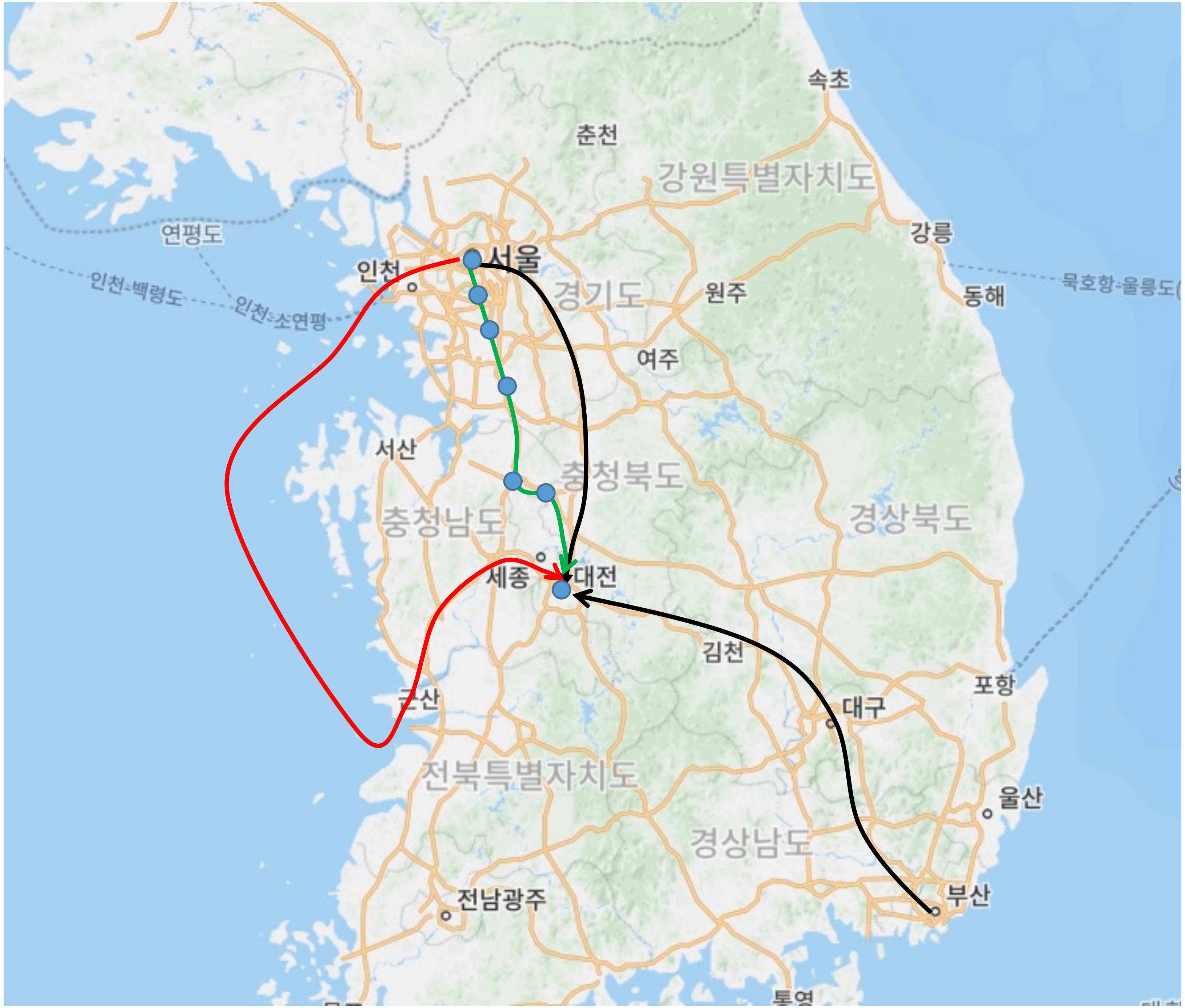
vs

- 관성 운동...?



$\vec{g}$
(a)
(e)
(b)
(d)
(c)

I. 특수상대론



✓ 어디에?
- 서울>수원>천안>...>대전역>천문연
- “KASI”, “학교장님 직장“, sk연구소 옆, ...
- $(36°\,23\,'\ \ 58.5, 127°\,22\,'\ \ 30.8,)$
- 해발 $(70\sim80)m$

➔ 공간좌표: (x, y, z)

✓ 언제?
- 10:04 > 10:30 > ... > 11:07 > 11:40

➔ 시간좌표: (t)

✓ 사건: $(t, x, y, z) = x^{\mu}$
✓ 이동경로: $x^{\mu}(\tau)$

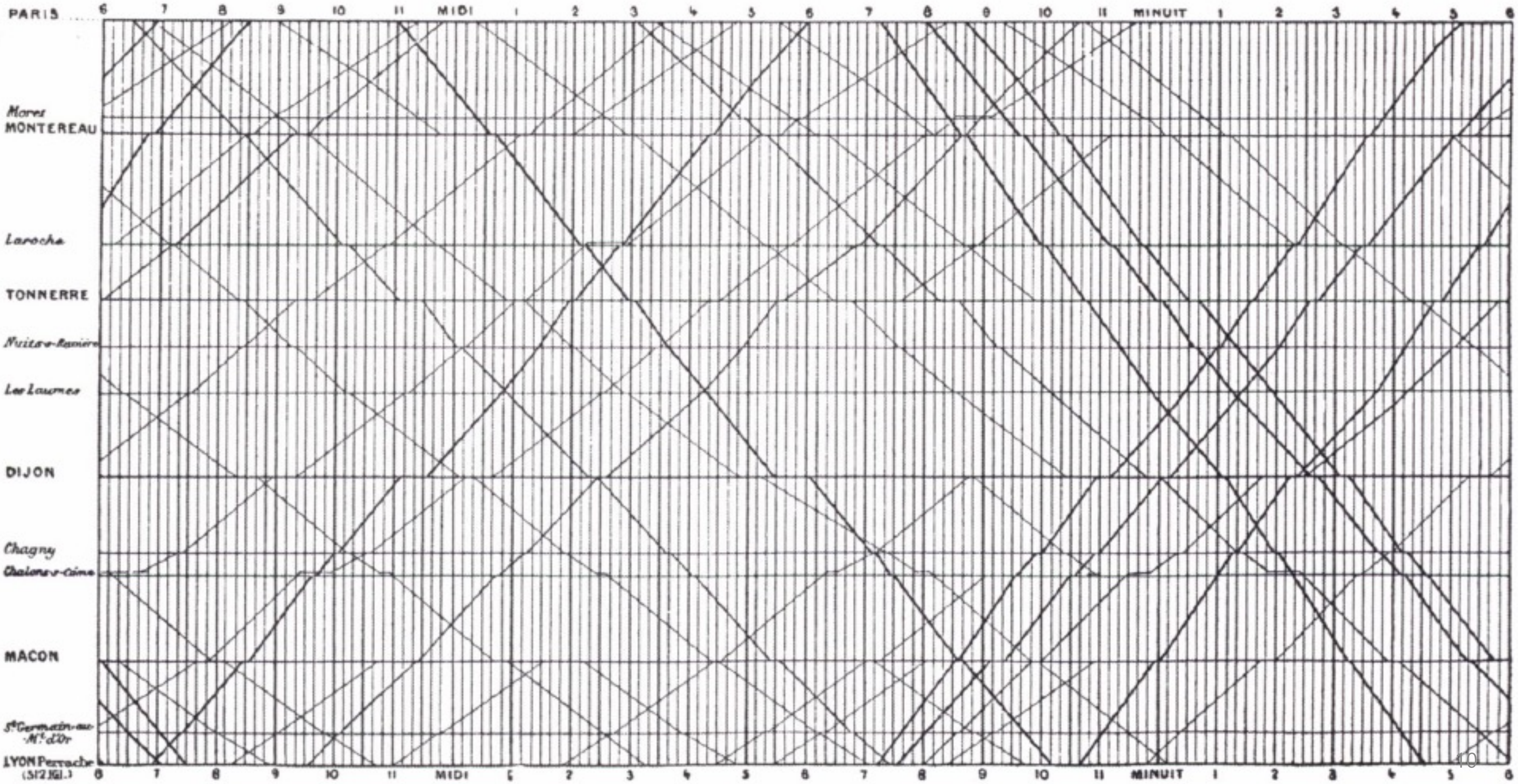
✓ 시간과 공간 or 시공간

BOX 4.2 Railway Trains in Spacetime : 파리 - 리옹(1885(?))

시간

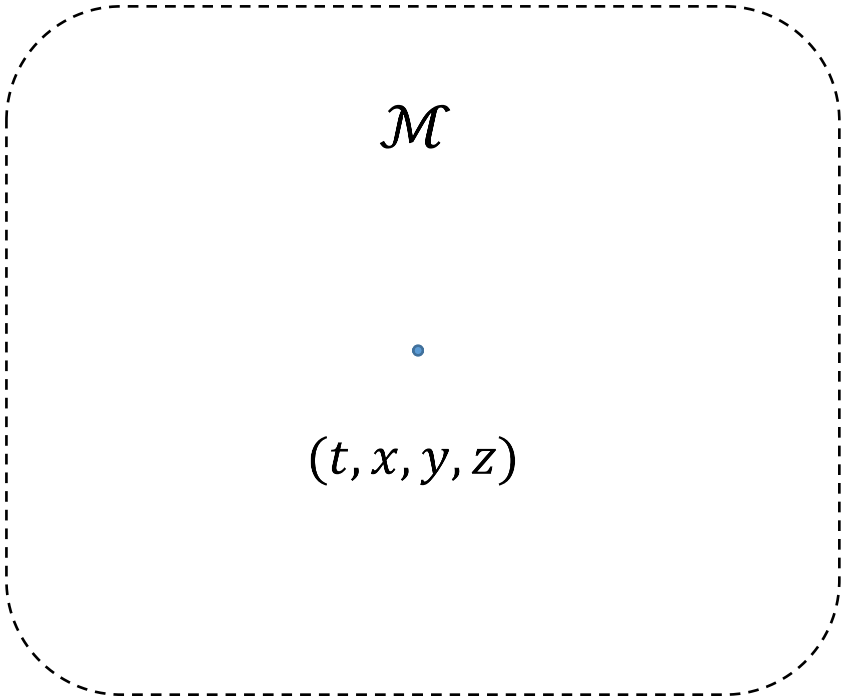
(그림 출처: Hartle)

공간

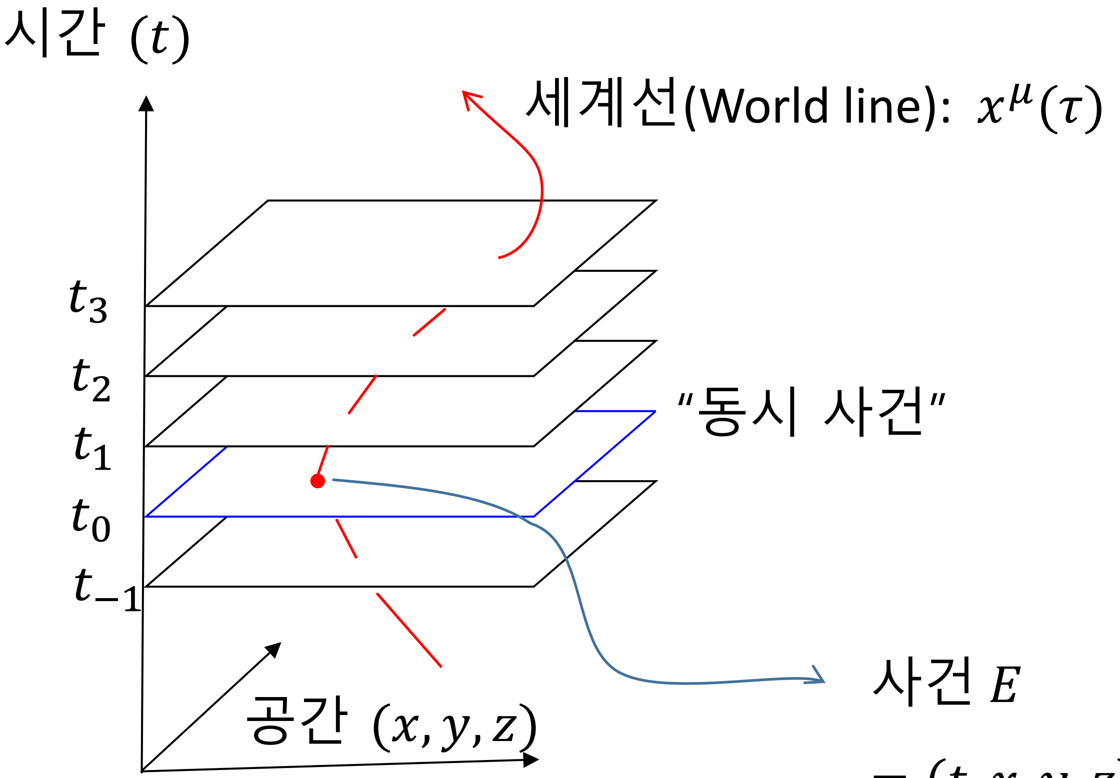


시공간($\boldsymbol{\mathcal{M}}$) = “사건의 집합”

= "사건의 **4**차원 연속체"



- **시공간 도표(Spacetime Diagram)**



사건 E

$= (t, x, y, z)$

$= (x^0, x^1, x^2, x^3)$

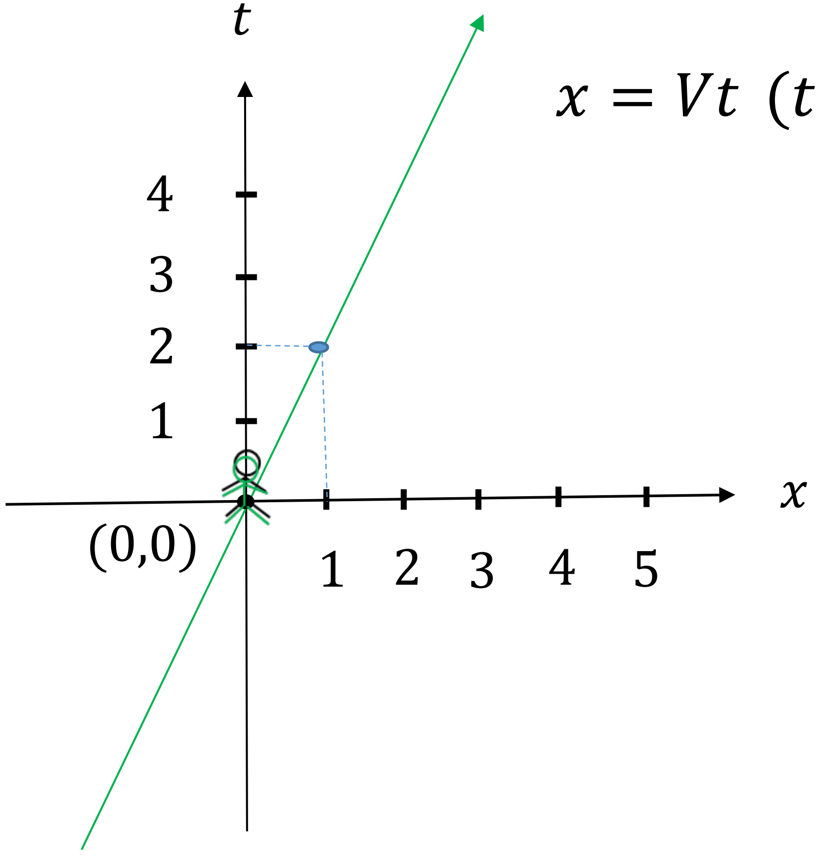
$\equiv x^\mu \quad (\mu = 0,1,2,3)$

- **세계선 연습**(1): 등속 운동

$\Rightarrow$ V = 일정

x = 속도×시간 = Vt

(t,x)	(t,x)
(0,0)	(0,0)
(1,0)	(1,0.5)
(2,0)	(2,1.0)
(3,0)	(3,1.5)
⋮	⋮



$x = Vt \ \left(t = \frac{1}{V}x\right)$

- **세계선 연습**(2): 등가속 운동

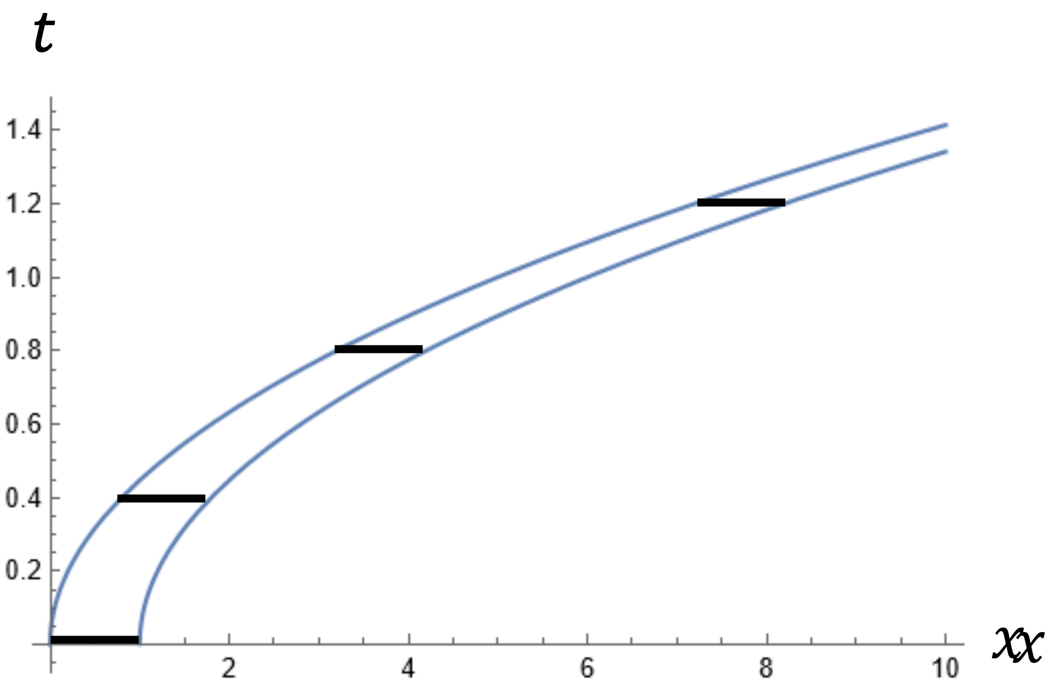
v

$v(t) =$ 일정하게 증가

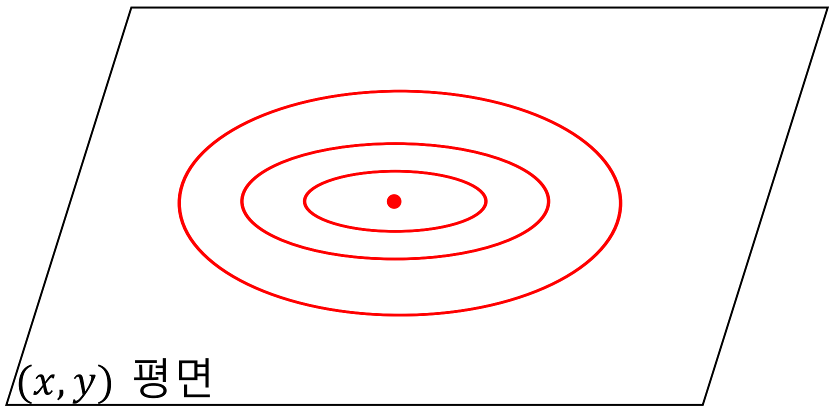
➔ $x = x_0 + \frac{1}{2}at^2$

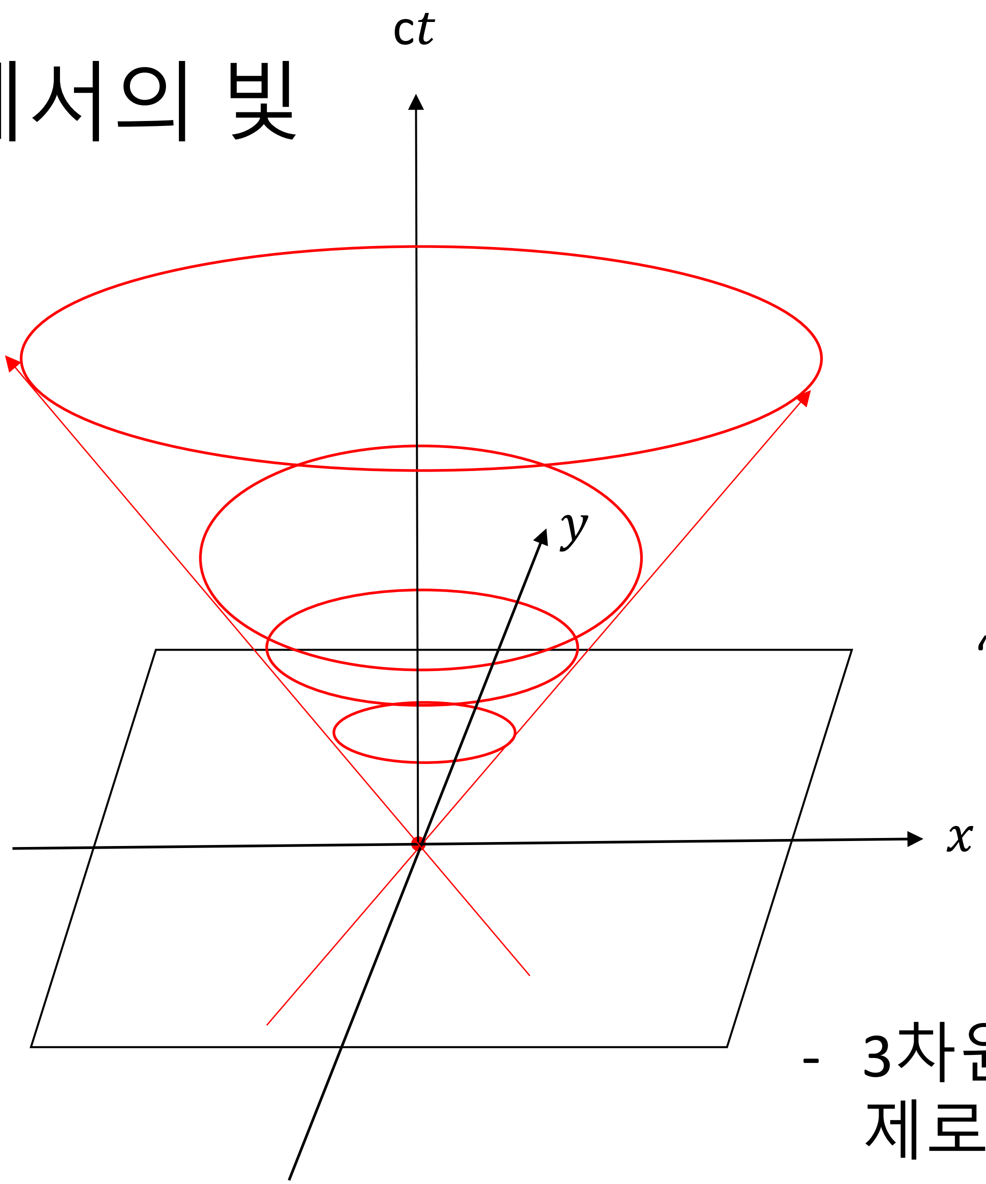
v

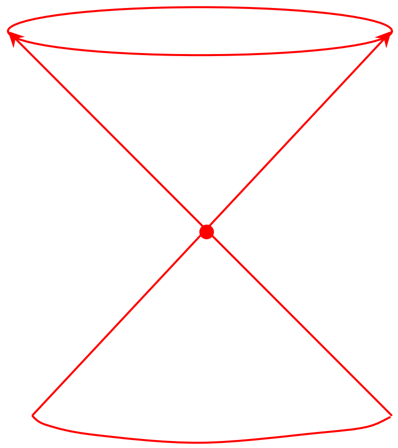
막대: L



- **세계선 연습**(4): 평면에서의 빛



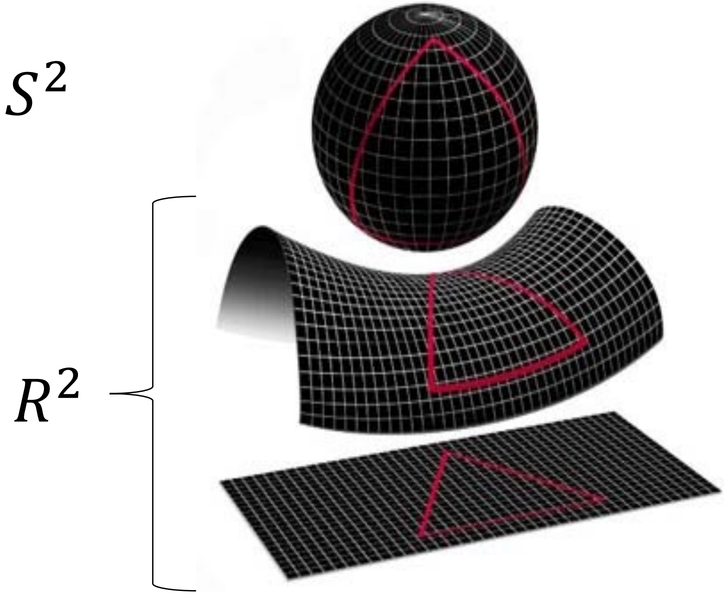




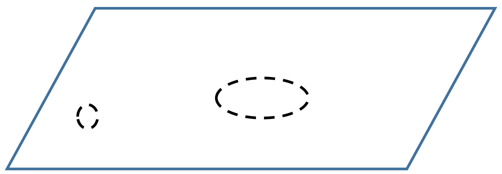
“광뿔(Light-cone)”

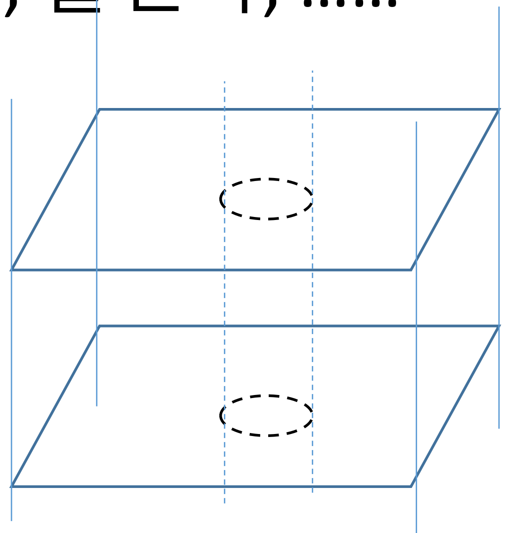
- 3차원의 경우 원은 실제로는 구면을 나타냄

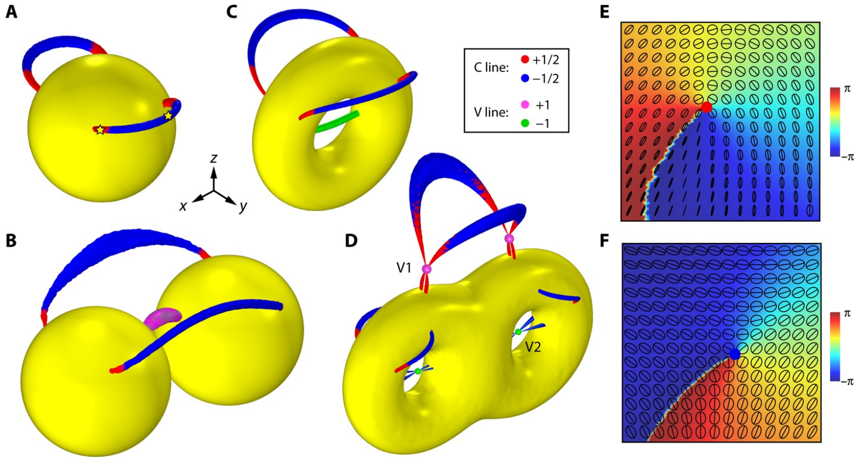
- 위상 구조(Topological structure)
- 미분 구조(Differentiability)
- 차원(Dimension)
- 메트릭 구조(Metric structure)
- 유한, 무한, 연속, 불연속, ……



(출처: NASA)

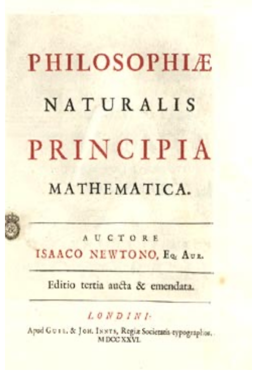






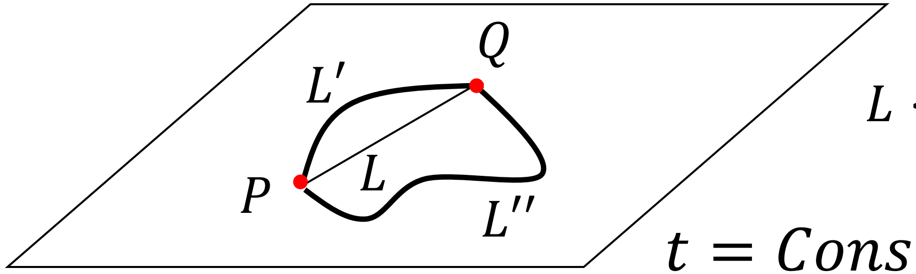
Peng *et al*, Science Advances (2022)

✓ 거리(메트릭) 구조: "두 사건 간의 거리"


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.
AUCTORE
ISAACO NEWTONO, Eq. Aur.
Editio tertia aucta & emendata.
LONDINI:
MDCCXXVI.

$$P = (t, x, y, z) \sim Q = (t', x', y', z')$$

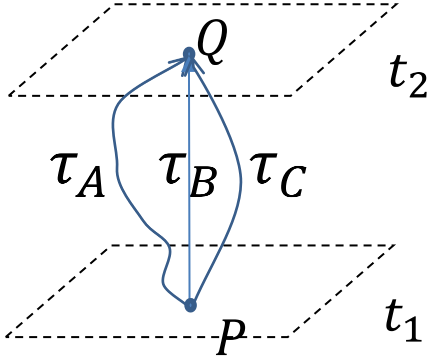
- **다른 위치 동시 사건($t = t'$):** "공간적" 거리



$L < L' < L''$

$t = Const.$

- **다른 시각($t_1 \neq t_2$):** "시간적" 거리



$$\tau_A = t_2 - t_1 = \tau_B = \tau_C$$

- **뉴튼의 절대공간:**
  - 두 사건을 잇는 경로에 따라 다름
  - 관측자의 운동 상태와는 무관

➔ **편평한 유클리드 공간**

$$\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 = \Delta l'^2$$

- **뉴튼의 절대시간:**
  - 두 사건을 잇는 경로에 무관
  - 관측자의 운동 상태와도 무관

➔ $\Delta t = \Delta t'$

- **시간과 공간 모두 물질의 출현에 영향 받지 않음**

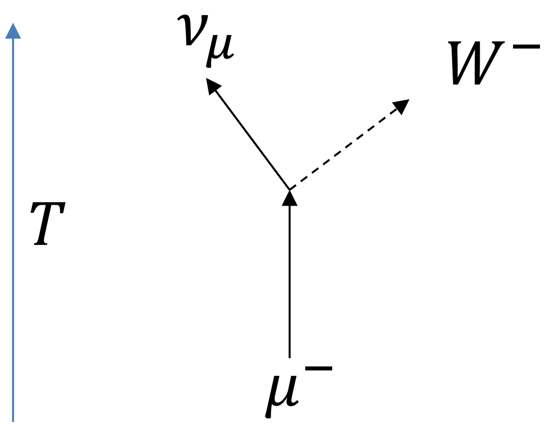
✓ 정말로 그러 한가...?

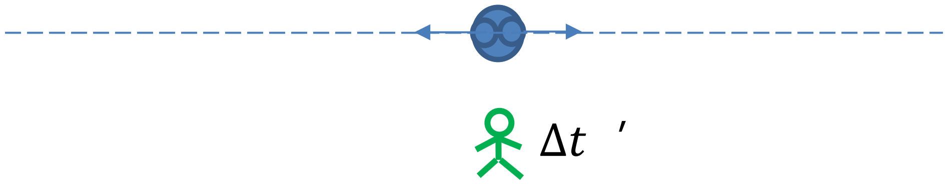
뉴튼적 거리구조

vs

특수상대론적 거리 구조

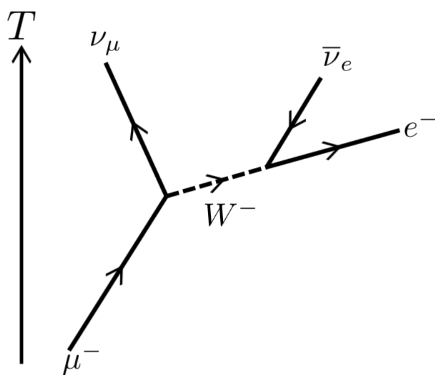
✓Ex) 뮤온의 수명

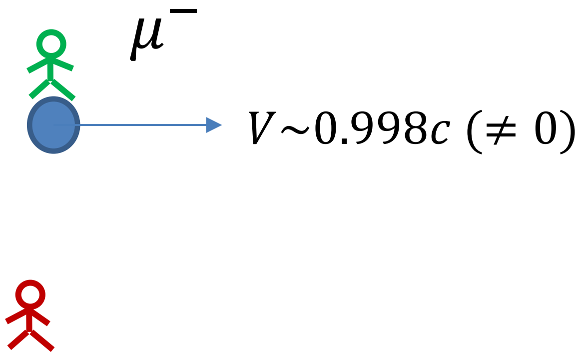


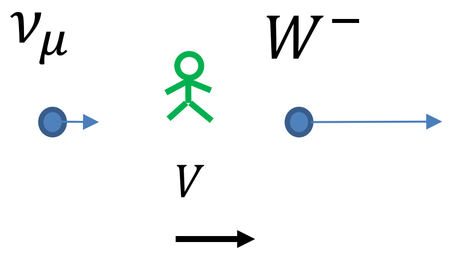


➔ 수명: $\Delta t\ ' = 2.1969811 \times 10^{-6}\ s \sim 2.2\ \mu s$

➔ 수명: $\Delta t = 2.2\ \mu s$?

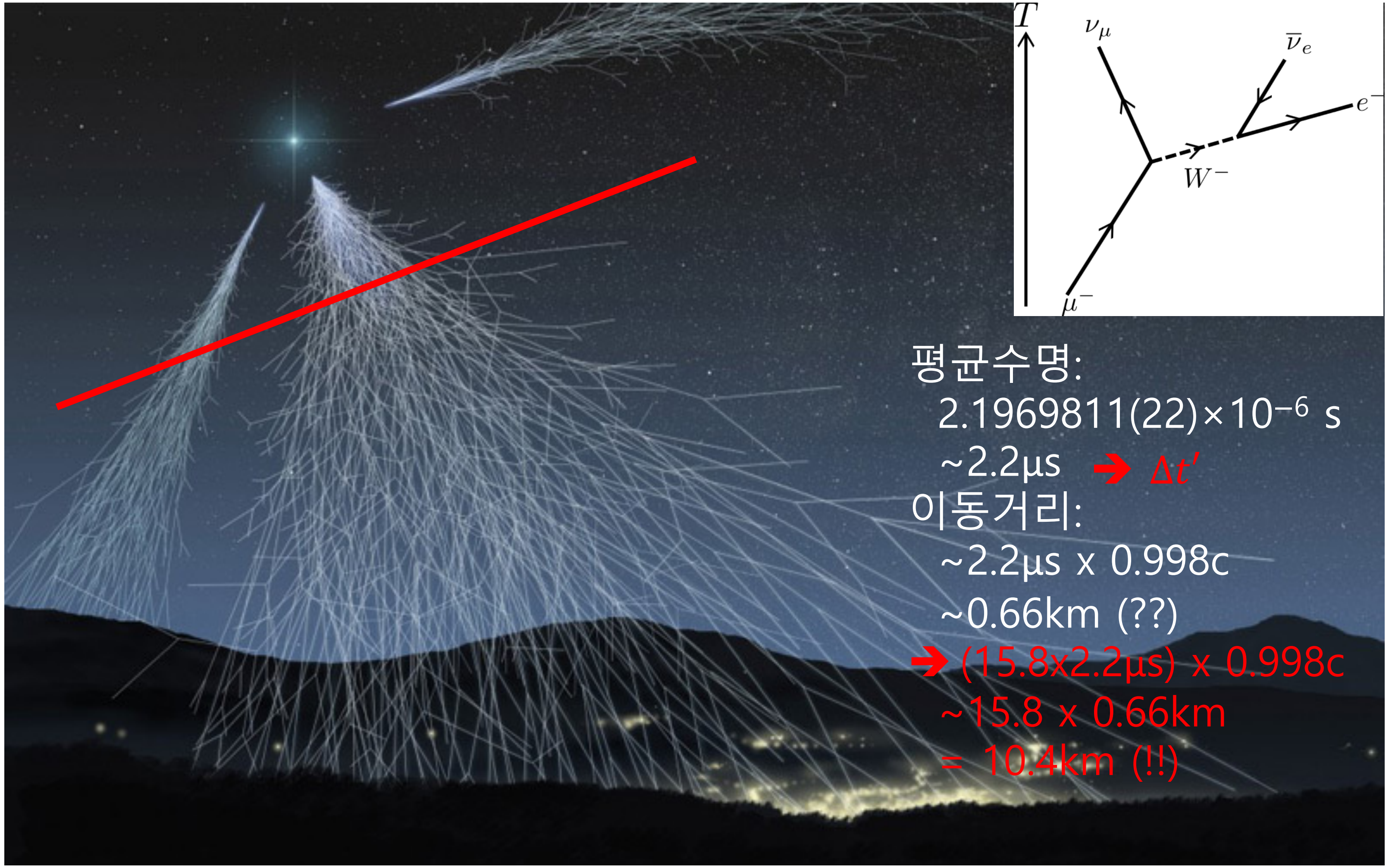






➔ 수명: $\Delta t = 2.2\ \mu s$?

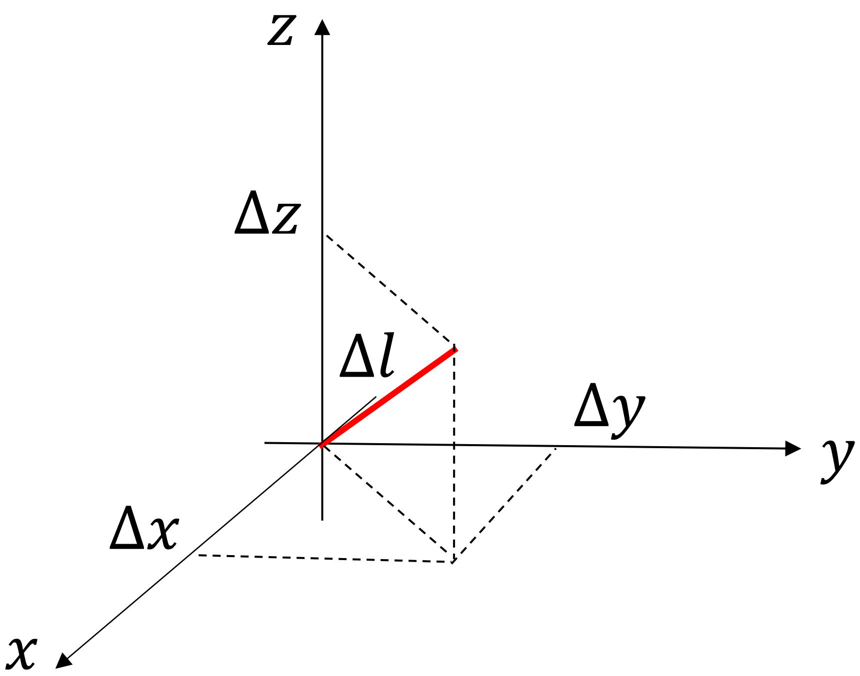
❖뮤온 검출의 미스터리:



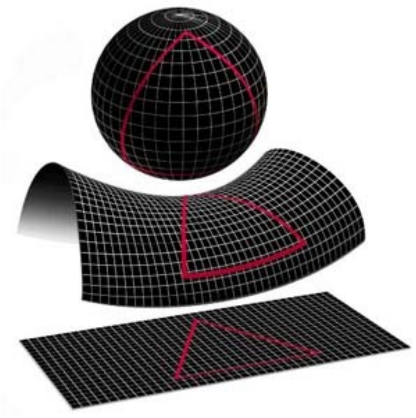
$V_{\text{뮤온}} \sim 0.998\ c$

$$\Delta t = \Gamma_{\text{뮤온}} \Delta t' \sim 15.8\ \Delta t'$$

• 편평한 유클리드 3차원 공간:



$$\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$
$$= \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$



(출처: NASA)

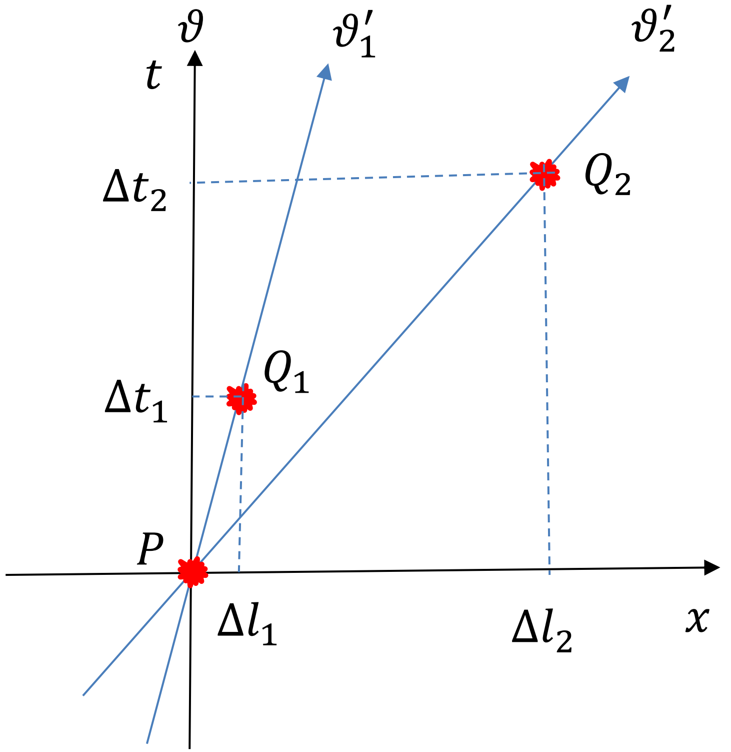
$dl^2 = d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$: 3차원 구

$dl^2 = d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$: 3차원 말안장

$dl^2 = d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta\, d\phi^2) = dx^2 + dy^2 + dz^2$

특수상대론적 시공간의 거리 구조

- **가상 실험**



예1) $V_1 = 1\ m/s$ 의 속도로 움직이는 ϑ_1' :

$(\Delta t_1', \Delta l_1') = (1\ sec, 0\ m) \leftrightarrow (\Delta t_1, \Delta l_1)_{뉴튼} = (1\ s, 1\ m)$ $\leftrightarrow (\Delta t_1, \Delta l_1) = (1.000 \cdots 6\ s, 1.000 \cdots 6\ m)$

예2) $V_2 = 0.9 \times (3 \times 10^8\ m/s)$ 의 속도로 움직이는 ϑ_2' :

$(\Delta t_2', \Delta l_2') = (1\ sec, 0\ m) \leftrightarrow (\Delta t_2, \Delta l_2)_{뉴튼} = (1\ s, 2.7 \times 10^8\ m)$ $\leftrightarrow (\Delta t_2, \Delta l_2) = (2.29\ s, 6.19 \times 10^8\ m)$

$$\Delta s_1'^2 = -(c\Delta t_1')^2 + (\Delta l_1')^2 = -\left(3\times 10^8 \frac{m}{s}\times 1\ s\right)^2 + 0 = -9.00\cdots 0\times 10^{16}\ m^2$$

$$\Delta s_1^2 = -(c\Delta t_1)^2 + (\Delta l_1)^2 = -(3\times 10^8\times 1.00\cdots 6)^2 + (1.00\cdots 6\ m)^2$$

$$= (-9\times 10^{16}\ \times 1.00\cdots 4 + 1.00\cdots 4)\ m^2$$

$$= (-9\times 1.00\cdots 4 + 1.00\cdots 4\times 10^{-16})\times 10^{16} m^2 \cong -9.00\cdots 4\times 10^{16}\ m^2$$

$$\boxed{\rightarrow \quad \Delta s_1^2 \cong \Delta s_1'^2}$$

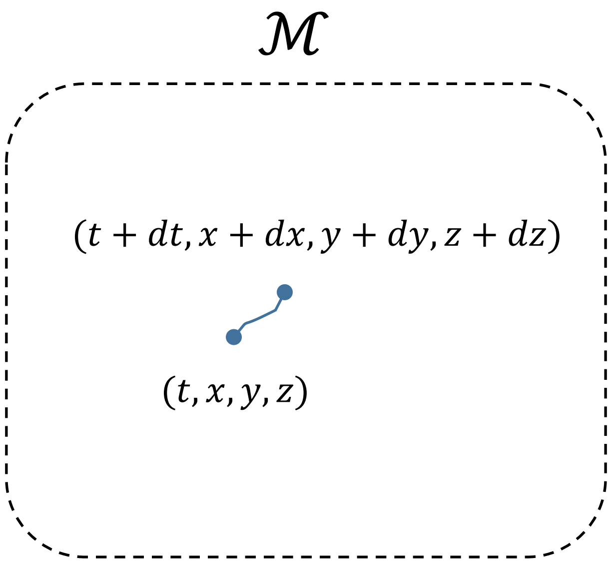
$$\Delta s_2'^2 = -(c\Delta t_2')^2 + \Delta l_2'^2 = -\left(3\times 10^8 \frac{m}{s}\times 1\ s\right)^2 + 0 = -9.00\times 10^{16}\ m^2$$

$$\Delta s_2^2 = -(c\Delta t_2)^2 + \Delta l_1^2 = -(3\times 10^8\ m/s\times 2.29\ s)^2 + (6.19\times 10^8\ m)^2$$

$$= (-47.2 + 38.3)\times 10^{16}\ m^2 = -8.90\times 10^{16}\ m^2$$

$$\boxed{\rightarrow \quad \Delta s_2^2 \cong \Delta s_2'^2}$$

✓ 특수상대론적 시공간(Special relativistic ST)



- **거리(메트릭) 구조(Line element):**

$$\boxed{ds^2 = -d(ct)^2 + dx^2 + dy^2 + dz^2}$$

- 밍코프스키(Minkowski) 시공간
- 편평한 시공간: "곡률 (Curvature)" = 0
- 로렌츠 공간 vs 유클리드 공간
- **"시간"과 "공간"은 분리할 수 없고 절대적이지도 않다!**

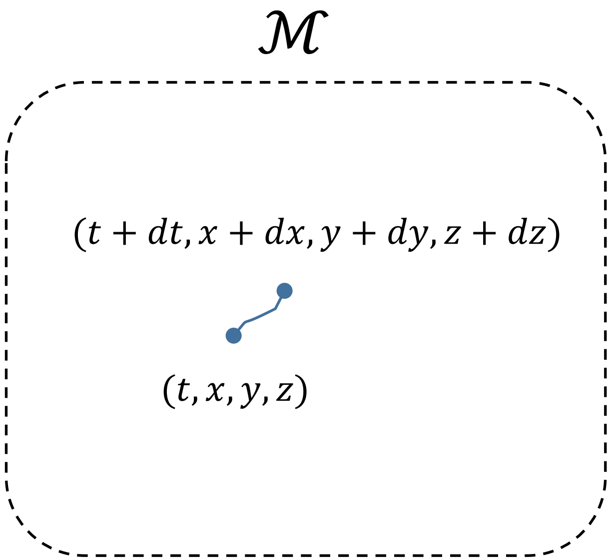
➔ 시공간 복합체!

$\Delta l = \Delta l', \ \Delta t = \Delta t'$ 이 아니고,

$$\Delta s^2 = -(c\Delta t)^2 + \Delta l^2$$
$$= -(c\Delta t')^2 + \Delta l'^2 = \Delta s'^2$$

관성계 구분 불가능

➔ $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt'^2 + dx'^2 + dy'^2 + dz'^2$



$$ds^2 = -d(ct)^2 + dx^2 + dy^2 + dz^2$$
$$= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\mathrm{ds}^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$
$$= \eta_{00}(dx^0)^2 + \eta_{01} dx^0 dx^1 + \cdots$$

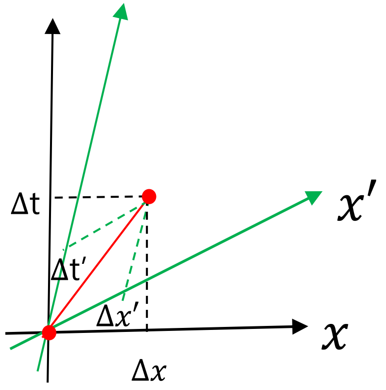
Minkowski metric:

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

직교 좌표계 극좌표계

- **모든 특수상대론적 성질들은 이러한 메트릭 구조의 산물:**

Ex1) 광속 불변:



$(c = 1)$

특별한 경로: $\Delta s^2 = 0$

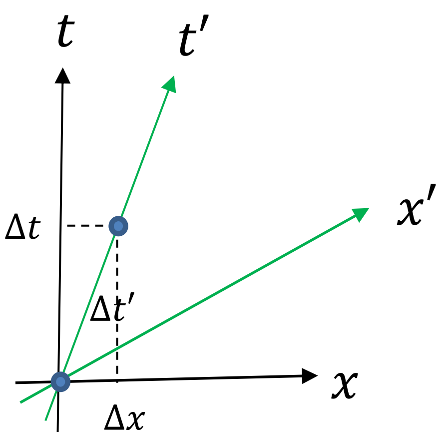
$$0 = \Delta s^2 = -(c\Delta t)^2 + \Delta x^2 = -(c\Delta t')^2 + \Delta x'^2$$

$$\Rightarrow \quad c_\vartheta = \frac{\Delta x}{\Delta t} = \frac{c\Delta t}{\Delta t} = c$$

$$c_{\vartheta'} = \frac{\Delta x'}{\Delta t'} = \frac{c\Delta t'}{\Delta t'} = c = c_\vartheta$$: 광속 동일(불변)!

Ex2) 시간 지연 효과:

- 관측자 ϑ': "동일 위치" $\Delta x' = 0 = \Delta y' = \Delta z'$
- 관측자 ϑ: "다른 위치" $\Delta x \neq 0 = \Delta y = \Delta z$



$\left(\frac{\Delta x}{\Delta t} = V \right)$

$$\Delta s^2 = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

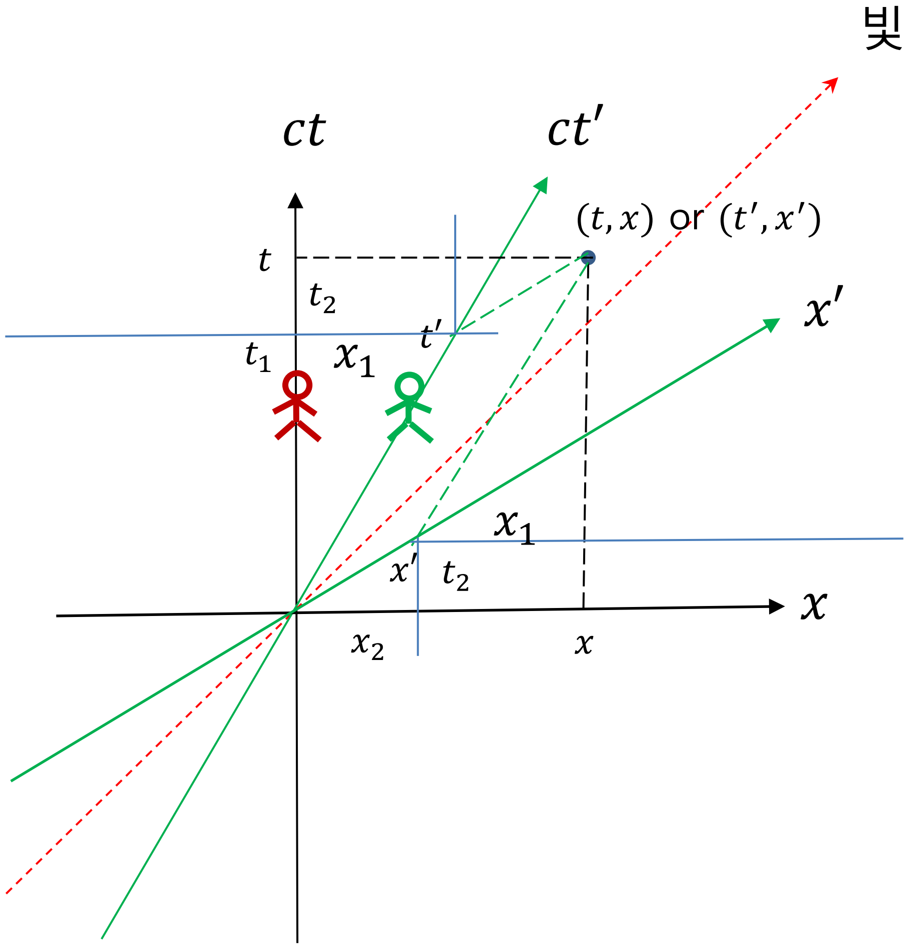
$$\Delta s'^2 = -(c\Delta t')^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

$$\Delta s^2 = \Delta s'^2 \Rightarrow -(c\Delta t)^2 + \Delta x^2 = -(c\Delta t')^2 + 0^2$$

$$(c\Delta t)^2[1 - (\frac{\Delta x}{c\Delta t})^2] = (c\Delta t)^2[1 - (\frac{V}{c})^2] = (c\Delta t')^2$$

$$\Rightarrow \Delta t = \frac{1}{\sqrt{1 - (V/c)^2}} \Delta t'$$

Ex3) 로렌츠 변환:



$x_1 = vt_1, -t'^2 + 0^2 = -t_1^2 + x_1^2 = -t_1^2(1 - v^2)$

➔ $t_1 = 1/\sqrt{1 - v^2}\, t'$

$t_2 = vx_2, \;\; -0^2 + x'^2 = -t_2^2 + x_2^2 = -t_2^2\left(1 - \frac{1}{v^2}\right)$

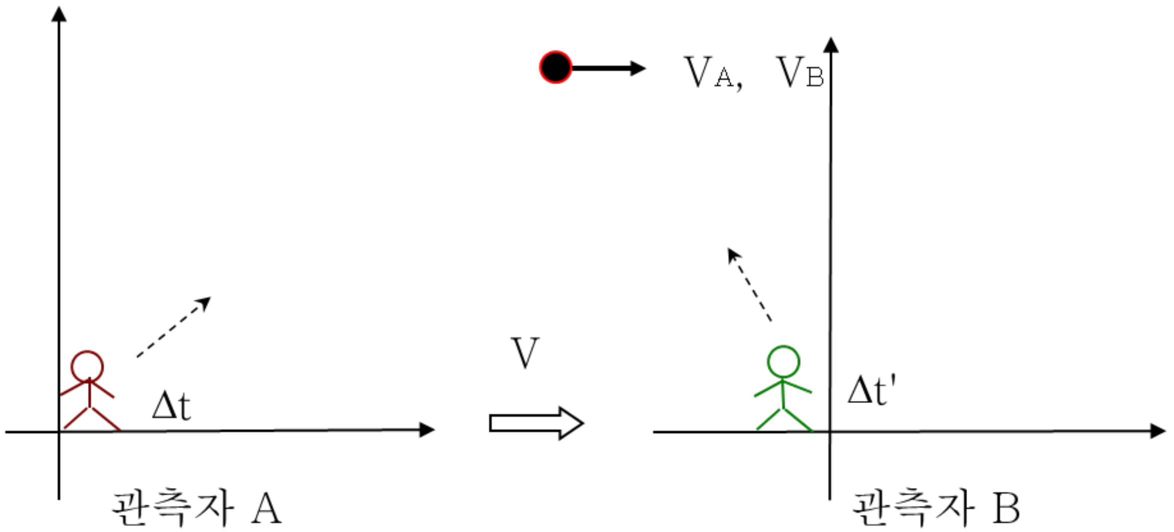
➔ $t_2 = \frac{v}{\sqrt{1-v^2}}\, x'$

$\therefore\; t = t_1 + t_2 = \frac{1}{\sqrt{1-v^2}} t' + \frac{v}{\sqrt{1-v^2}}\, x' = \frac{1}{\sqrt{1-v^2}} (t' + vx')$

마찬가지로,

$x = x_1 + x_2 = \frac{v}{\sqrt{1-v^2}}\, t' + \frac{1}{\sqrt{1-v^2}}\, x' = \frac{1}{\sqrt{1-v^2}} (x' + vt')$

Ex4) 속도 합성:



$$\Delta t = \frac{1}{\sqrt{1-(v/c)^2}}\left(\Delta t' + \frac{v}{c}\frac{\Delta x'}{c}\right)$$

$$\Delta x = \frac{1}{\sqrt{1-(v/c)^2}}\left(\Delta x' + v\,\Delta t'\right)$$

$$\Delta y = \Delta y' = 0$$

$$\Delta z = \Delta z' = 0$$

$$v_A = \frac{\Delta x}{\Delta t} = \frac{\frac{1}{\sqrt{1-(-v/c)^2}}\left(\Delta x' + v\,\Delta t'\right)}{\frac{1}{\sqrt{1-(-v/c)^2}}\left(\Delta t' + \frac{v}{c}\frac{\Delta x'}{c}\right)} = \frac{\Delta x' + v\,\Delta t'}{\Delta t' + \frac{v}{c}\frac{\Delta x'}{c}} = \frac{\frac{\Delta x'}{\Delta t'} + v}{1 + \frac{v}{c}\frac{\Delta x'/\Delta t'}{c}} = \frac{v_B + v}{1 + \frac{v}{c}\frac{v_B}{c}}$$

$$\boxed{\boldsymbol{V_A} = \frac{\boldsymbol{V_B + V}}{1 + V\,V_B/c^2}}$$

Note:

i) $V_A < c$ for $V_B, V < c$

ii) $V_A = c$ for $V_B = c$ (or $V = c$)

It never exceed the speed of light!

Ex5) 동시 사건:

- 두 사건: $\mathrm{P}_1 = (t_1, x_1)$ & $P_2 = (t_2, x_2)$

  $dt = t_2 - t_1 = 0$ $\rightarrow$ “동시에 발생한 사건“

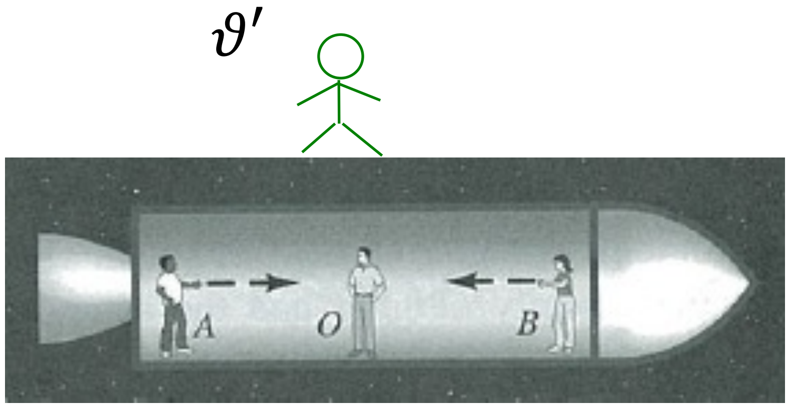
- $\mathrm{P}_1 = (t'_1, x'_1)$ & $P_2 = (t'_2, x'_2)$
  - $dt' = t'_2 - t'_1 = 0$?

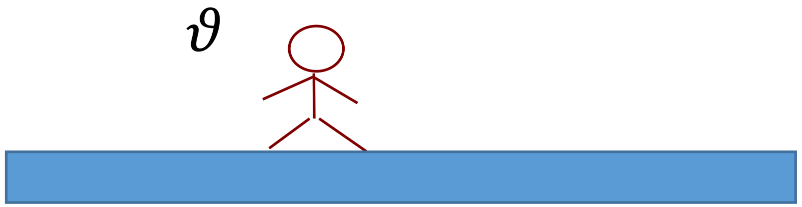
- Note: $ds^2 = -dt^2 + dx^2 = dx^2 = -dt'^2 + dx'^2$

  ➔ Not necessarily $dt' = 0$ $(t'_1 = t'_2)$

  ➔ **$t'_1 \neq t'_2$ for other observers!**

✓ 동시에 발생한 사건



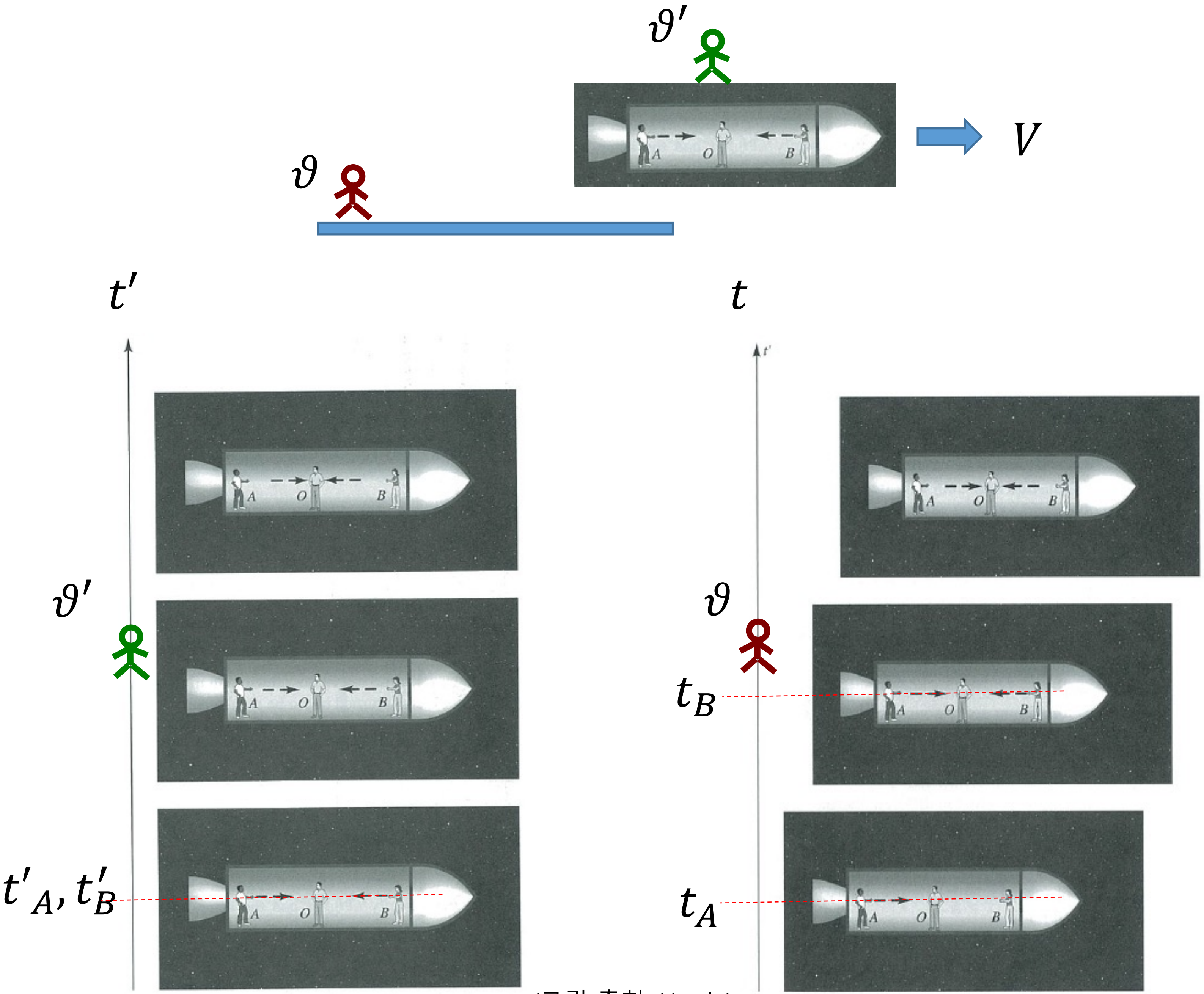


- **ϑ' 관성계:**
  - **"A와 B에서 쏜 빛이 가운데 O에서 만남"**

  ➔ A와 B에선 빛을 쏜 사건 $P(t_A'), Q(t_B')$은 동시에 발생한 사건: $t_A' = t_B'$

- **관측자 $\boldsymbol{\vartheta}$** 에게도 이 두 사건 $P(t_A), Q(t_B)$은 동시에 발생했을까?: $t_A = t_B$??

(그림 출처: Hartle)



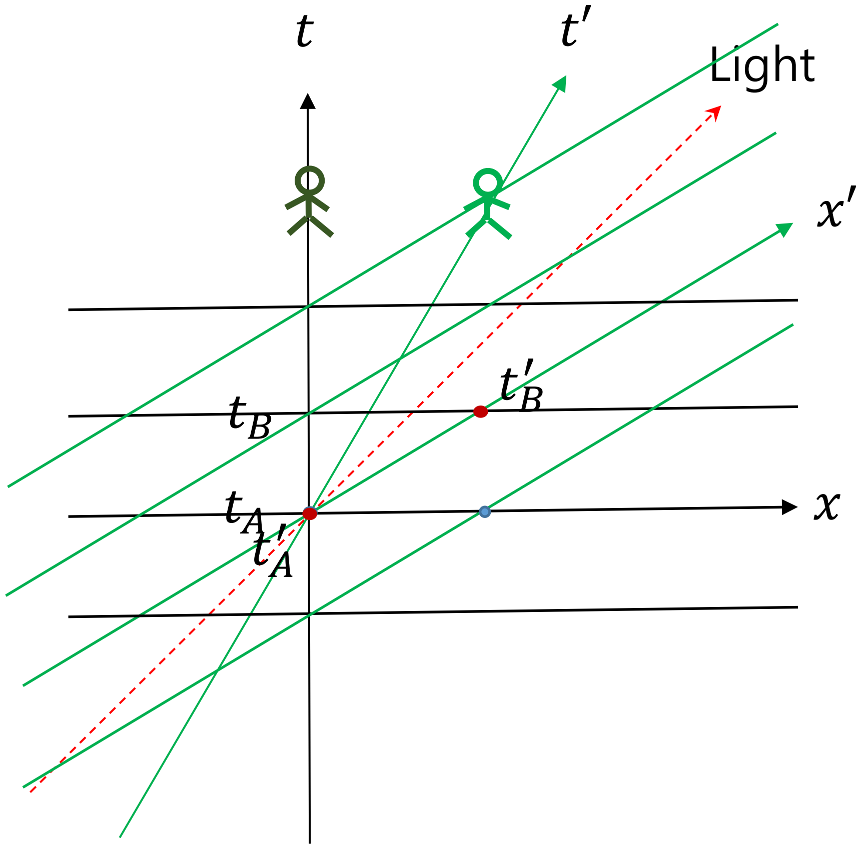
(그림 출처: Hartle)

- **관성계 $\boldsymbol{\vartheta}$:** $P(t_A)$ 와 $Q(t_B)$은 다른 시각에 발생한 사건. 즉,

$$t_A \neq t_B \; (t_A < t_B)$$

- 만약 동시에 쏘았다면 빛은 O에서 만나지 못하고 O의 왼편에서 만날 것이다.(광속 동일, O는 오른편으로 움직이고 있음)

→ 동시성의 상대성

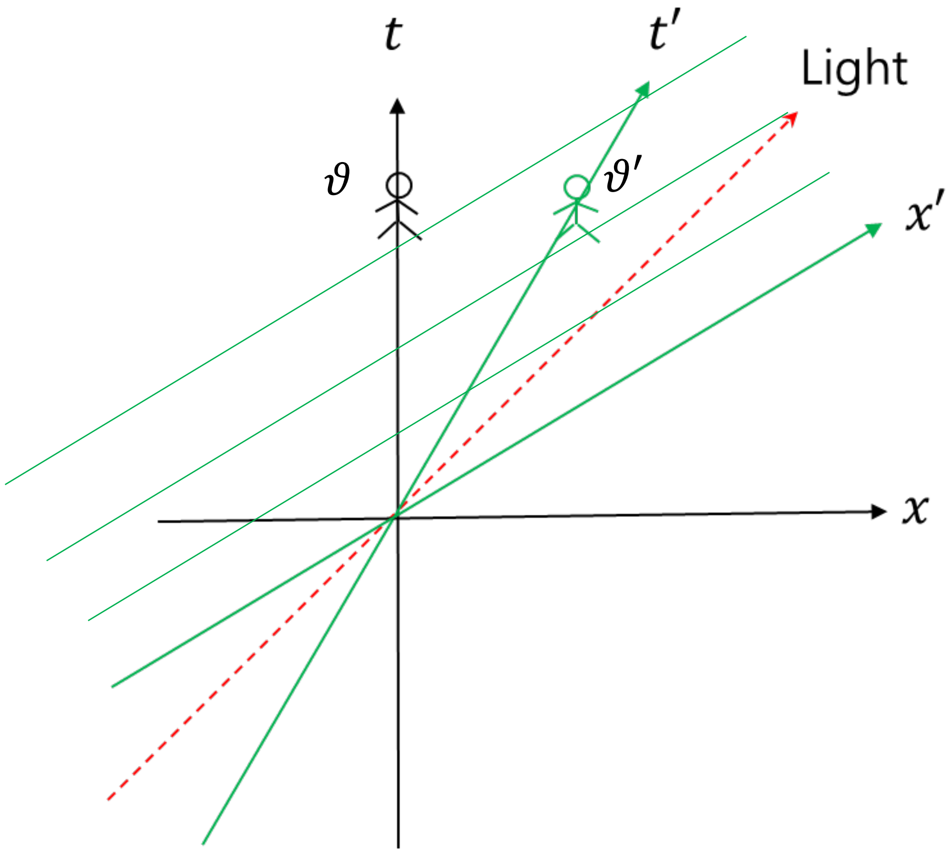


관측자 ϑ 의 관점

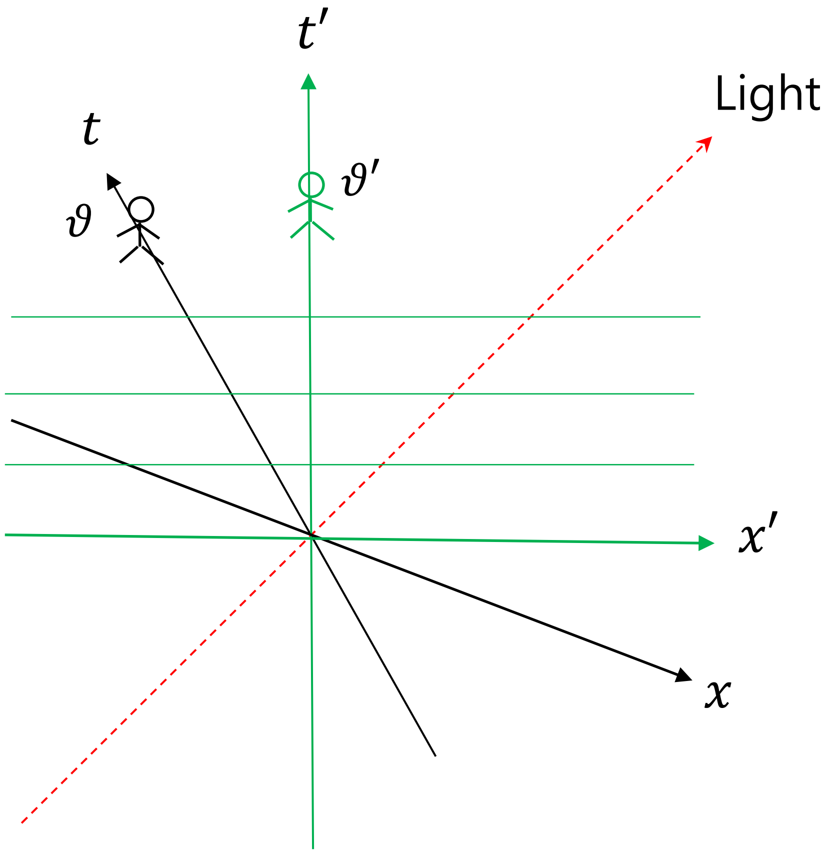
$t'_B = t'_A$

$t_B > t_A$

- 관측자 ϑ 가 보는 동시 사건들
- 관측자 ϑ 에서의 동시 사건들의 시간 진행
- 관측자 ϑ' 이 보는 동시 사건들
- 관측자 ϑ' 에서의 동시 사건들의 시간 진행

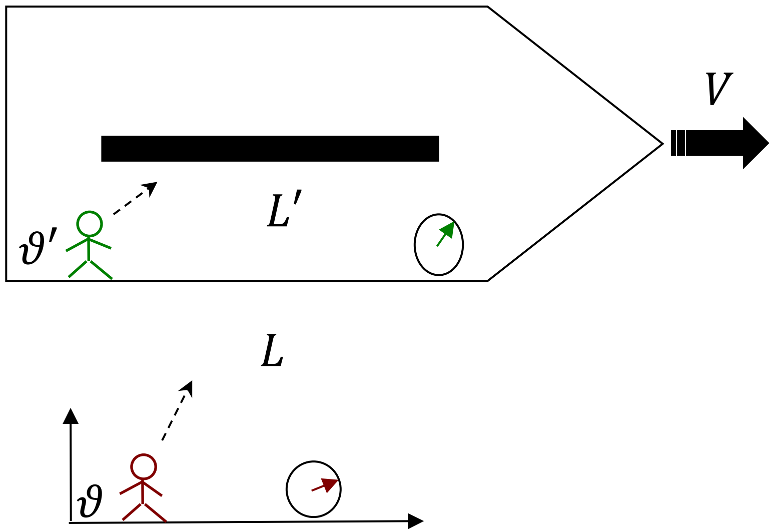


관측자 ϑ 의 관점

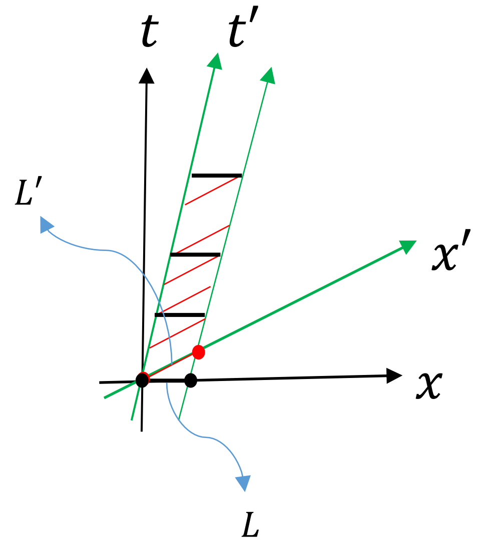


관측자 ϑ' 의 관점

Ex6) 길이 수축:



- **<u>길이란?</u>:** 동시에 발생한 두 사건의 거리 ➔
  $L' = \Delta x' = x'_2 - x'_1$ w/ $\Delta t' = 0$ (막대 양 끝)
- 마찬가지로 '정지' 관측자(ϑ)도 움직이는 막대의 양 끝을 동시에 측정하여 길이 구함: L
- Note: ϑ' 의 두 사건은 ϑ 에게는 동시에 발생한 두 사건이 아님

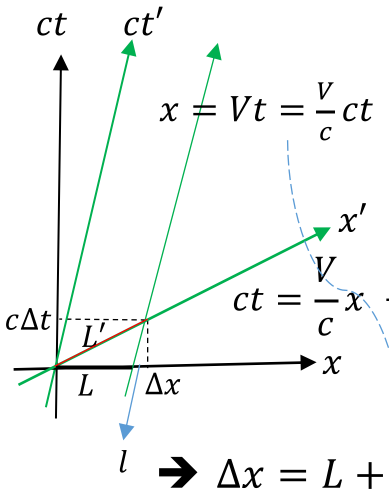


$$\Delta s^2 = -(c\Delta t)^2 + \Delta x^2 = -0^2 + L'^2$$

$$-\left(\frac{V}{c}\Delta x\right)^2 + \Delta x^2 = \Delta x^2\left[1 - \left(\frac{V}{c}\right)^2\right] = L'^2$$

$$\rightarrow \frac{L^2}{[1-(v/c)^2]^2}[1 - (v/c)^2] = L'^2$$

$$\boxed{L = \sqrt{1 - \left(\frac{V}{c}\right)^2}\, L'}$$



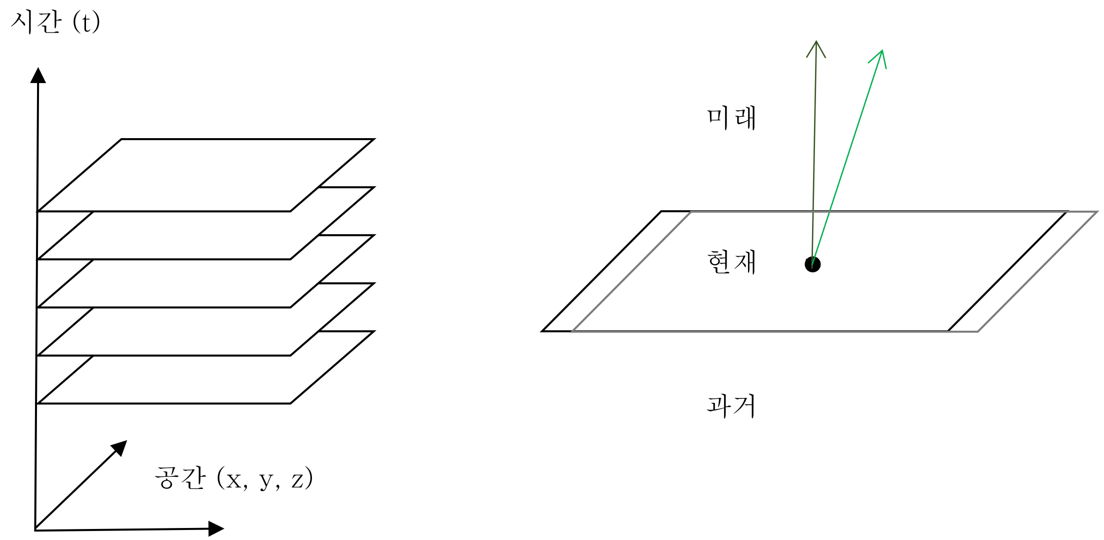
$x = Vt = \frac{V}{c}ct$

$ct = \frac{V}{c}x \rightarrow c\Delta t = \frac{V}{c}\Delta x$

➔ $\Delta x = L + l = L + \frac{V}{c}c\Delta t = L + (\frac{V}{c})^2\Delta x$ ➔ $\Delta x = L/[1 - \left(\frac{V}{c}\right)^2]$

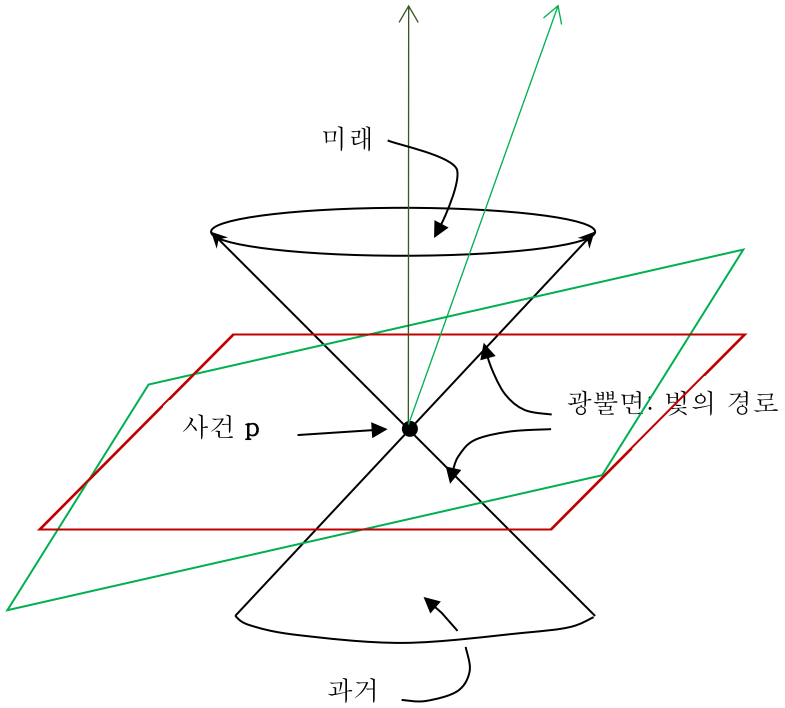
- **뉴튼적 시공간의 인과구조**

  (Newtonian ST):

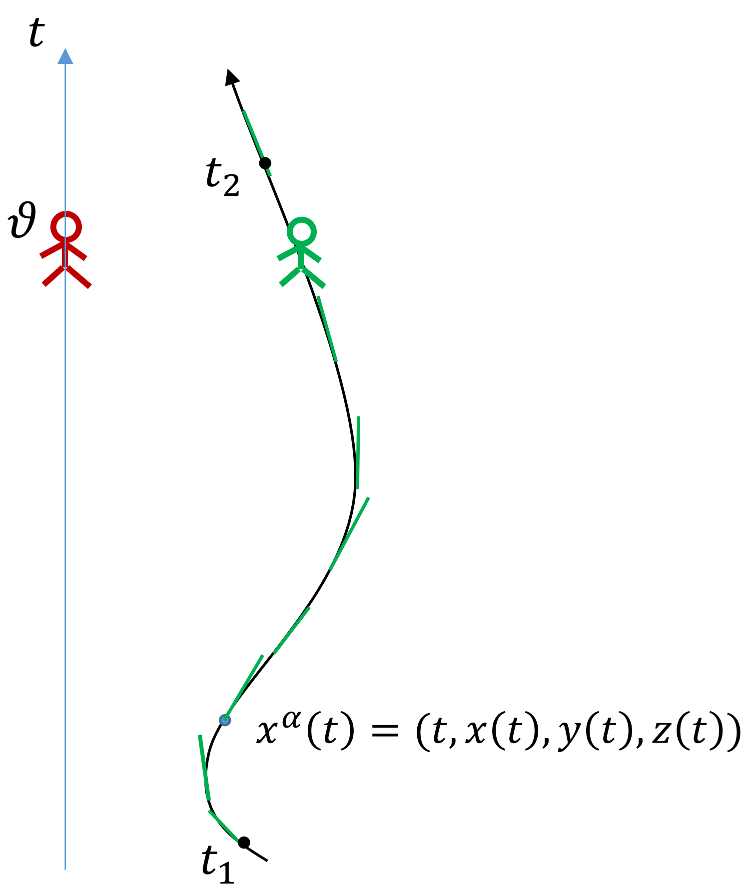


- **특수상대론적 시공간의 인과구조**

  (Special relativity):



- **질문:** 두 사람의 나이는?

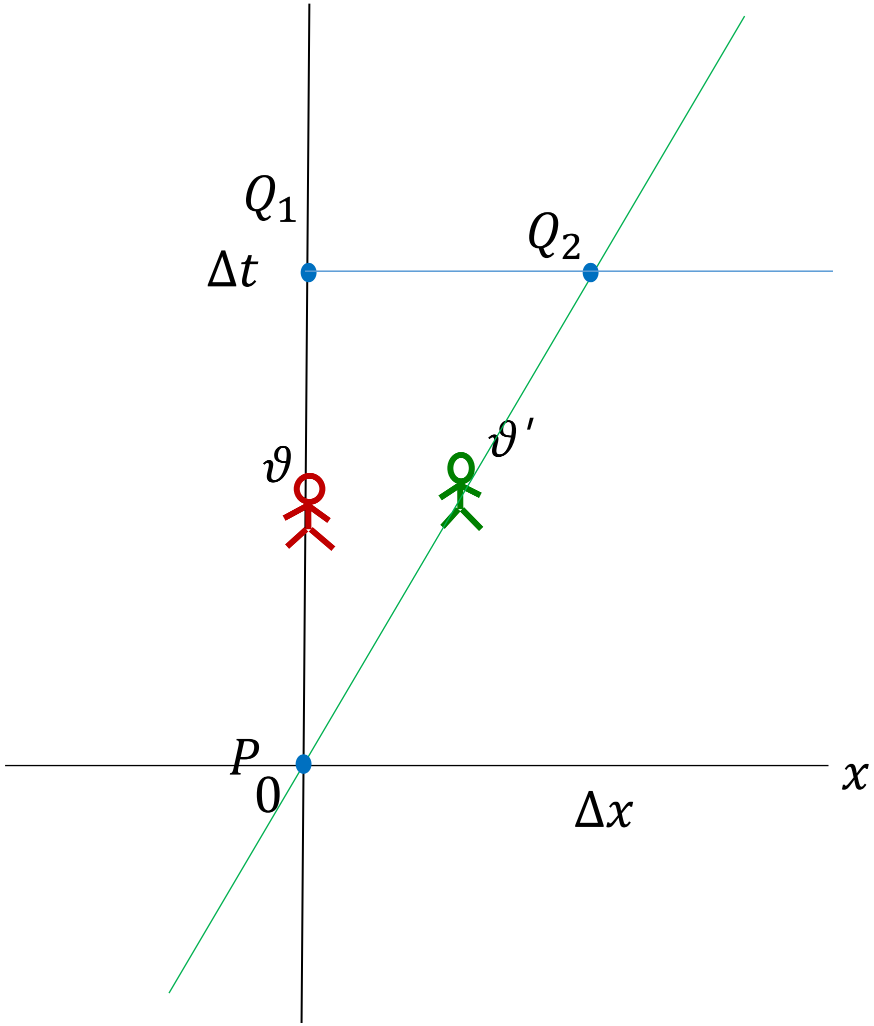


$\tau_\vartheta, \tau_{\vartheta'} = ?$

➔ "고유 시간 "

- **고유 시간(Proper time)의 계산:**

(시간 지연: $\Delta t = \frac{1}{\sqrt{1-(v/c)^2}} \Delta t'$)



- ϑ 의 경로에 대한 고유 시간: $\tau_{\vartheta} = \Delta t$
- ϑ' 의 경로에 대한 고유 시간:

$$\tau_{\vartheta'} = \Delta t' = \sqrt{1-(v/c)^2}\,\Delta t \quad (< \tau_{\vartheta})$$

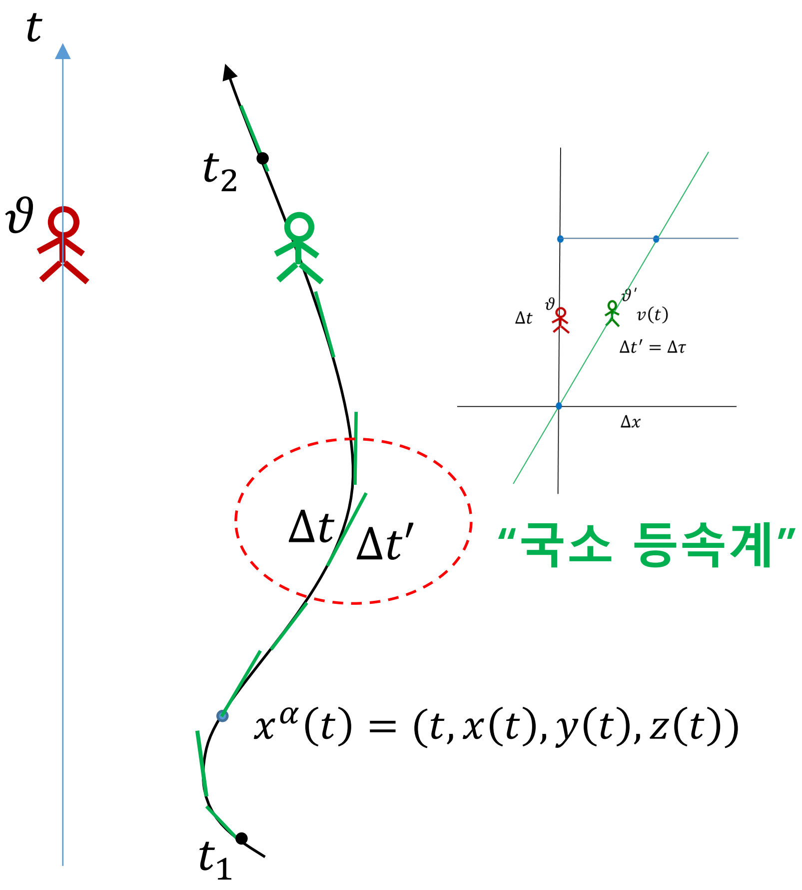
- Note: $\tau_{\vartheta'} = \Delta\tau = \sqrt{1-(v/c)^2}\,\Delta t = \sqrt{c^2 - v^2}\,\frac{\Delta t}{c}$

$$= \sqrt{c^2 - (\Delta x/\Delta t)^2}\,\frac{\Delta t}{c} = \sqrt{(c\Delta t)^2 - \Delta x^2]}\,\frac{1}{c}$$

$$= \sqrt{-\Delta s^2}\,\frac{1}{c}$$

$$\rightarrow \boxed{\Delta\tau = \sqrt{-\Delta s^2}\,\frac{1}{c} \xRightarrow{\Delta t \to 0} d\tau = \sqrt{-ds^2}/c}$$

- **일반적인 시간적 경로:** 경로 상 어느 지점에서도 $ds^2 < 0$



$$\Delta\tau_i = \Delta t_i' = \frac{1}{c}\sqrt{-\Delta s_i^2}$$

$$\Longrightarrow \quad \lim_{\Delta\tau\to 0}\left(\textstyle\sum_i \Delta\tau_i\right) = \lim_{\Delta\tau\to 0}\left(\frac{1}{c}\sum_i \sqrt{-\Delta s_i^2}\right)$$

$$\rightarrow \quad \tau = \int d\tau = \frac{1}{c}\int\sqrt{-ds^2}$$

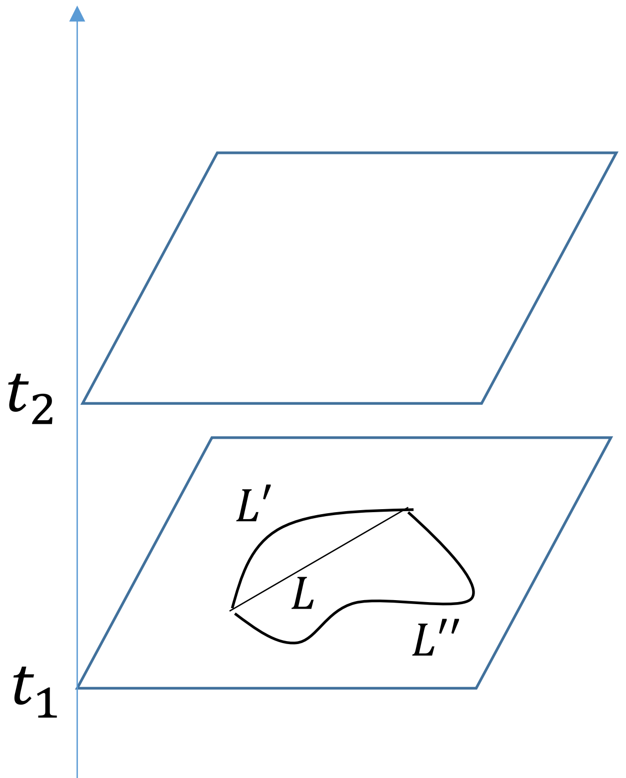
$$= \frac{1}{c}\int\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$$

$$= \frac{1}{c}\int_{t_1}^{t_2}\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\,dt$$

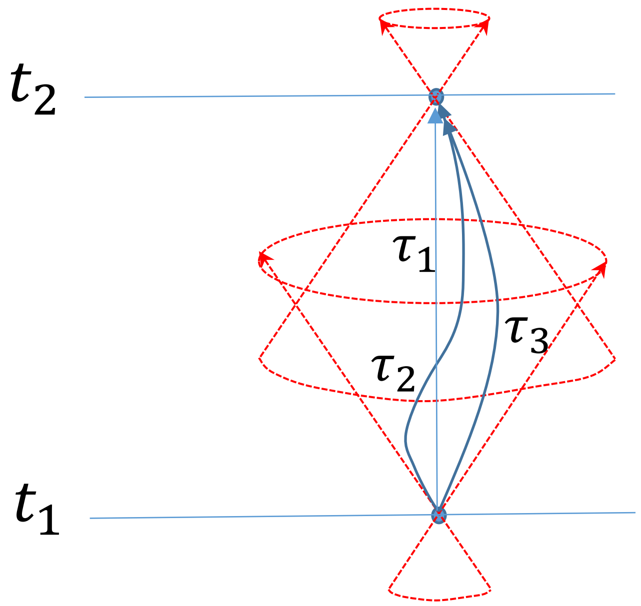
$$= \frac{1}{c}\int_{t_1}^{t_2}\sqrt{c^2 - v(t)^2}\,dt$$

여기서 $\dot{x}^\alpha(t) = \frac{dx^\alpha}{dt}$.

- ds^2 은 관측자에 대해 불변(스칼라량)이어서 어느 관측자가 측정해도 동일



고유 길이: $L < L' < L''$



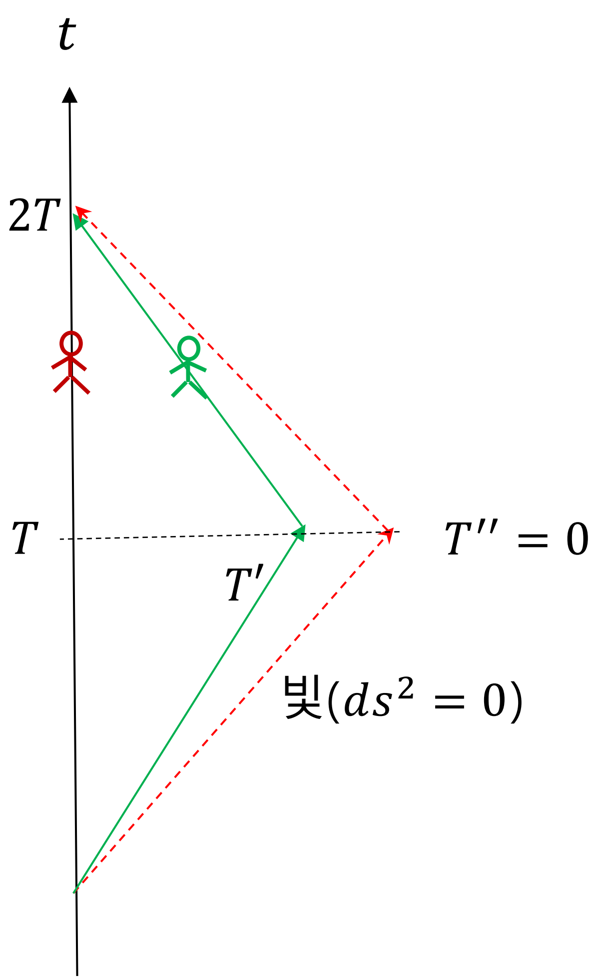
- 두 사건을 잇는 경로에 따라 시간 흐름 달라짐:

$$\tau_1 = t_2 - t_1 \neq \tau_2 \neq \tau_3$$

- 뉴튼의 절대시공간에서는 경로에 무관:

$$\tau_1 = t_2 - t_1 = \tau_2 = \tau_3$$

- "쌍둥이 역설":



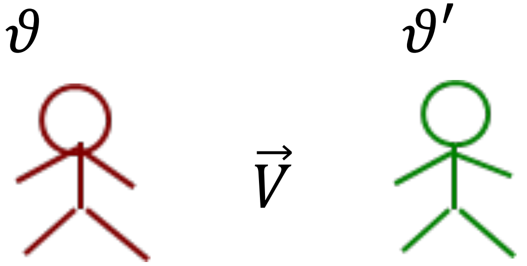
$$d\tau = \sqrt{-ds^2} = \sqrt{(c\,dt)^2 - dx^2}$$

$\rightarrow$ 0 (빛의 경로)

$$2T > 2T' > 2T'' = 0$$

상대성 원리와 특수상대론적 운동 역학

✓ 상대성 원리



“물리법칙은 모든 관성계에서 동일”

- <u>절대 시공간</u>: 갈릴레오 변환에 대해 물리법칙이 같은 형태

$$t' = t$$

$$x' = x - Vt$$

➔

- <u>특수상대론적 시공간</u>: 로렌츠 변환에 대해 동일한 형태

$$t' = \frac{1}{\sqrt{1-V^2/c^2}}\left(t - \frac{V}{c}\frac{x}{c}\right)$$

$$x' = \frac{1}{\sqrt{1-V^2/c^2}}\left(x - \frac{V}{c}ct\right)$$

- **뉴튼 역학:** $m\vec{a} = m\frac{d^2\vec{x}}{dt^2} = \vec{F}$

  - 갈릴레오 변환에 대해 동형:

    $t' = t$

    $x' = x - Vt$

    ➔ $a_x = \frac{d^2x}{dt^2} = \frac{d}{dt'}\frac{d(x'+Vt')}{dt'} = \frac{d^2x'}{dt'^2} + 0 = a'_x$

    ➔ $m\vec{a}' = \vec{F}'$

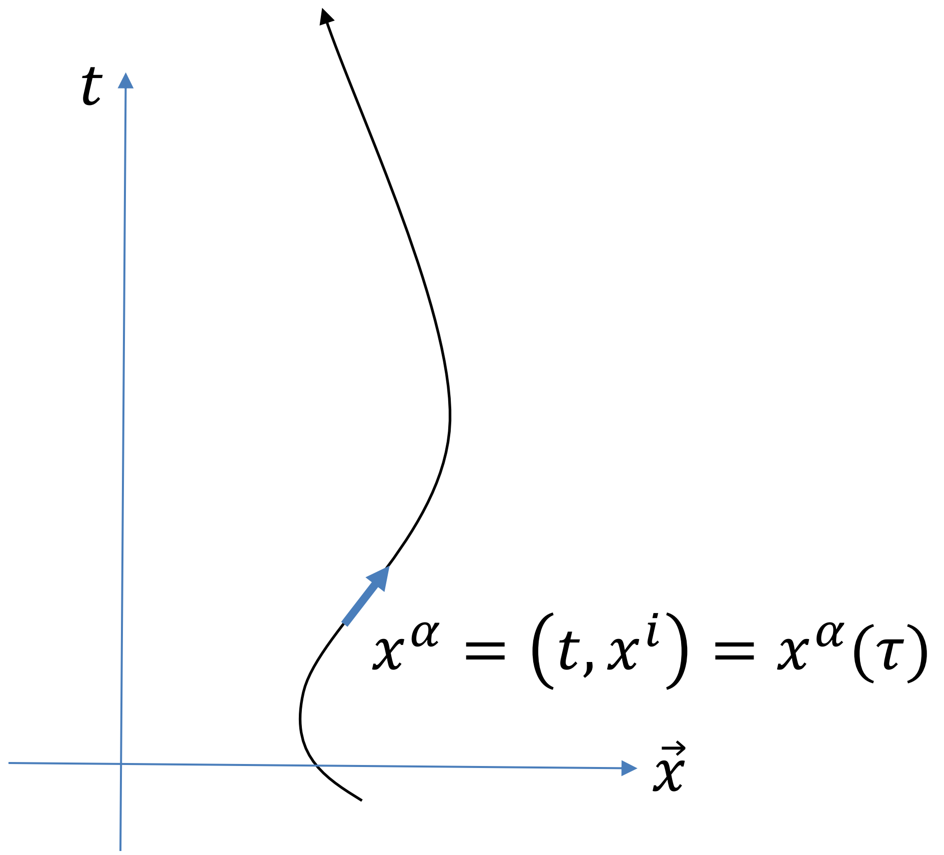
  - 로렌츠 변환에 대해서는 같은 형태가 아님

    - $ma^i = F^i, i = 1,2,3$ ➔ "시간" 성분이 없음.

    - $t' = \gamma(t - Vx) \rightarrow t = \gamma(t' + Vx')$
      $x' = \gamma\,(x - Vt) \rightarrow x = \gamma\,(x' + Vt')$

    ➔ $a_x = \frac{d}{d(\gamma(t'+Vx'))}\frac{d(\gamma\left(x'+Vt'\right))}{d(\gamma(t'+Vx'))} = \frac{1}{\gamma}\frac{d}{d(t'+Vx')}\frac{d\left(x'+Vt'\right)}{d(t'+Vx')} = \frac{1}{\gamma}\left(\frac{d^2x'}{dt'^2} + \cdots\right) = \frac{1}{\gamma}(a'_x + \cdots)$

✓ 특수상대론적 운동 역학:



- For a time-like trajectory, the <u>four-velocity</u> is defined as follows;

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right) : \text{ a 4-vector}$$

$d\tau \equiv \sqrt{-ds^2} = \sqrt{-\eta_{\alpha\beta} dx^\alpha dx^\beta}$

Or, $(d\tau)^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta$

- Three-velocity: $\vec{V} \equiv \frac{d\vec{x}}{dt}$

- Note: $u^t = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-V^2}} \equiv \gamma,\ u^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \gamma V^i$

  ➔ $u^\alpha = (\gamma, \gamma\vec{V})$: 모두 $\vec{V}$ 에 의해 결정됨

- Four-acceleration: $a^\alpha \equiv \frac{du^\alpha}{d\tau}$

- Four-momentum: $p^\alpha \equiv m u^\alpha = \left(\frac{m}{\sqrt{1-V^2}}, \frac{m\vec{V}}{\sqrt{1-V^2}}\right) = (E, \vec{p})$

- **Note:** (스칼라곱) $u \cdot v \equiv \eta_{\alpha\beta} u^\alpha v^\beta = -u^0 v^0 + \delta_{ij} u^i v^j$
  - $u \cdot u \equiv \eta_{\alpha\beta} u^\alpha u^\beta = \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{\eta_{\alpha\beta} dx^\alpha dx^\beta}{(d\tau)^2} = \frac{-(d\tau)^2}{(d\tau)^2} = -1 \rightarrow u \cdot u = -1,$
  - $0 = \frac{d(u \cdot u)}{d\tau} = 2u \cdot \frac{du}{d\tau} \rightarrow a \cdot u = 0$ (Orthogonal), $p \cdot p = -E^2 + \vec{p}^2 = -m^2 \rightarrow E^2 = (mc^2)^2 + \vec{p}^2$

- **특수상대론적 운동 방정식:**

$$\boxed{ma^\mu = f^\mu \quad (\mu = 0, 1, 2, 3)}$$

여기서 $f^\mu = (f^0, \vec{f})$: Four-force

Note:

- 임의의 관성계에서 같은 형태: $\Lambda^{\alpha'}_{\mu}(ma^\mu = f^\mu) \rightarrow ma'^\alpha = f'^\alpha$

Since $a \cdot u = 0$, $f \cdot u = 0$
$\rightarrow -f^0u^0 + \vec{f} \cdot \vec{u} = -f^0\gamma + (\gamma\vec{F}) \cdot (\gamma\vec{V})$
$\rightarrow f^0 = \gamma\vec{F} \cdot \vec{V}$ ➔ $f^\mu = (\gamma\vec{F} \cdot \vec{V}, \gamma\vec{F})$

- $ma^\mu = m\dfrac{du^\mu}{d\tau} = \dfrac{dp^\mu}{d\tau} = f^\mu$, or $\dfrac{dp^0}{d\tau} = \dfrac{dE}{d\tau} = f^0$ & $\dfrac{d\vec{p}}{d\tau} = \vec{f}$

  - $\dfrac{d\vec{p}}{d\tau} = \dfrac{dt}{d\tau}\dfrac{d\vec{p}}{dt} = \gamma\dfrac{d\vec{p}}{dt} = \gamma\vec{F}$ with "three-force" defined as $\vec{F} \equiv \dfrac{d\vec{p}}{dt}$. Then, $\vec{f} = \gamma\vec{F}$ and

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\frac{d\vec{x}}{d\tau}\right) = \boxed{m\frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2}}\frac{d\vec{x}}{dt}\right) = \vec{F}: \text{ 상대론적 운동방정식}}$$

  - $\dfrac{dp^0}{d\tau} = \dfrac{dt}{d\tau}\dfrac{dE}{dt} = \gamma\dfrac{dE}{dt} = f^0 = \gamma\vec{F} \cdot \vec{V}$ ➔ $\boxed{\dfrac{dE}{dt} = \vec{F} \cdot \vec{V}: \text{ "단위 시간당 해준 일" or 에너지 보존법칙}}$

    ➔ $p^0 = E$ 를 에너지로 해석하는 이유

  - If $f^\mu = 0$, $p^\mu = \text{Constant.}$: Conservation of the four-momentum

• 특수상대론적 운동에너지 (Relativistic energy):

m

$V = 0$ $\qquad \vec{V}\ (\neq 0)$

운동에너지: KE = 0 ➔ $\frac{1}{2}mV^2$ (뉴튼 역학)

$$p^\alpha = \left(\frac{m}{\sqrt{1-V^2}}, \frac{m\vec{V}}{\sqrt{1-V^2}}\right) = (E, \vec{p})$$

$$\frac{dE}{dt} = \vec{F} \cdot \vec{V}$$

- 테일러 전개:
  $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$

$$E \equiv \frac{mc^2}{\sqrt{1-(V/c)^2}} = mc^2[1-(V/c)^2]^{-\frac{1}{2}} = mc^2\left[1 + \left(-\frac{1}{2}\right)(-1)\left(\frac{V}{c}\right)^2 + \frac{3}{8}\left(\frac{V}{c}\right)^4 + \cdots\right] \qquad (V/c \ll 1)$$

$$\rightarrow \quad E = \underbrace{mc^2}_{\text{"정지 질량 에너지"}} + \underbrace{\underbrace{\frac{1}{2}mV^2}_{\text{고전적 운동에너지}} + \frac{3}{8}mc^2\left(\frac{V}{c}\right)^4 + \cdots}_{\text{상대론적 운동에너지}}$$

➔ $\left(\frac{mc^2}{\sqrt{1-(V/c)^2}} - mc^2\right)$: **Relativistic Kinetic Energy**

$$\boxed{E = \frac{mc^2}{\sqrt{1-(V/c)^2}} \equiv \tilde{m}c^2}$$

- 빛의 속도로 증가시키려면$(V \rightarrow c)$ 무한의 에너지 필요!!
- 따라서 물체의 속도는 빛의 속도를 넘지 못한다.

Ex1) Uniform motion in special relativity:

a) $\frac{d\vec{p}}{dt} = m\frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2}}\frac{d\vec{x}}{dt}\right) = \vec{F} = 0 \ \rightarrow \vec{p} = \frac{\vec{v}}{\sqrt{1-v^2}} = \text{Constant.} \rightarrow \vec{v} = \text{Constant.}$

Ex2) Uniformly accelerating motion in special relativity:

a) What would be wrong if it satisfies $\frac{dv}{dt} = a = \frac{F}{m} = Constant.?$

Sol) $v(t) = v_0 + at \ \rightarrow \infty$. Its velocity exceeds the speed of light as the time increases.

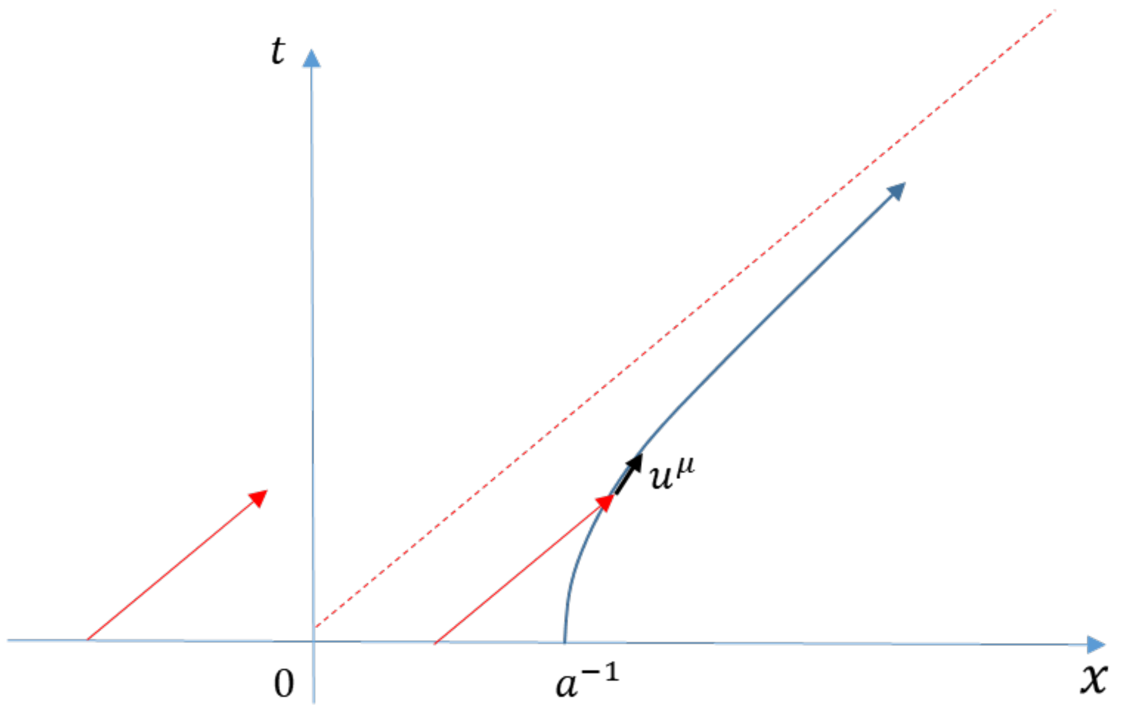
b) Instead, solve the equation of motion, $m\frac{d(\gamma v)}{dt} = F = Constant.$, with $\gamma = (1 - v^2)^{-1/2}$. What are $v(t)$ as $t{\sim}0$ and ∞ ? Roughly draw the world line of a uniformly accelerating motion. Here, this object is at rest initially, and $c = 1$.

Sol) $d\left(\frac{v}{\sqrt{1-v^2}}\right) = \frac{F}{m}dt = adt \ \rightarrow \ \frac{v}{\sqrt{1-v^2}} = at + C = at$ since $v(t = 0) = 0$. Thus,

$$v^2 = (1 - v^2)(at)^2 \ \rightarrow [1 + (at)^2]v^2 = (at)^2 \ \rightarrow v(t) = \frac{at}{\sqrt{1+a^2t^2}}.$$

Note that $v{\sim}at$ for $t{\sim}0$, *e.g.,* slow motion, and $v \rightarrow 1 = c$ as $t \rightarrow \infty$.

$$x^\mu(\tau) = (t, x) = (a^{-1}\sinh(a\tau), a^{-1}\cosh(a\tau))$$

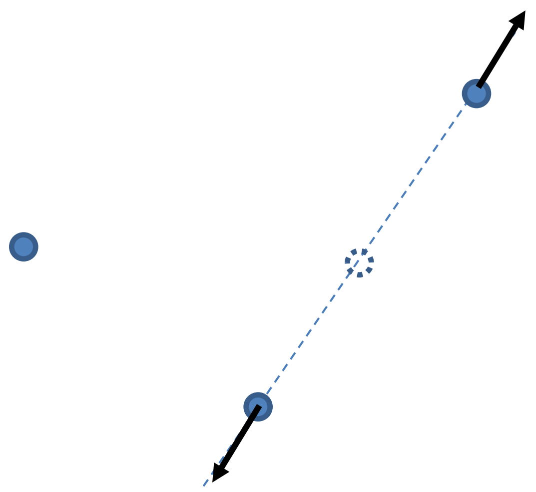


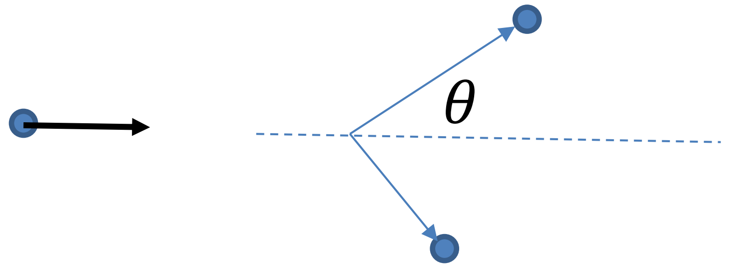
$$a^{-1} = \left[\frac{s}{m/s}\right] = a^{-1}c^2 \sim \frac{(3\times 10^5\ km)^2}{10m/s^2} \sim 9\times 10^{15}\ m = 9\times 10^{12} km$$

Ex3): Consider the reaction $A \rightarrow B + C$ with particle rest masses m_A, m_B and m_C, respectively.

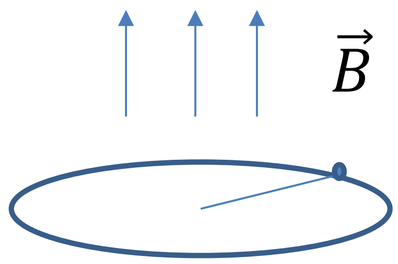
a) If A is at rest in the lab frame, what is the momentum four-vector $P_A{}^{\mu}$ in the lab frame? Show also that, in this lab frame, particle B has an energy $E_B = (m_A{}^2 + m_B{}^2 - m_C{}^2)/2m_A$.

b) If A decays while moving in the lab frame, find the relation between the angle θ at which B comes off, and the energies of A and B.





Ex4): A relativistic charged particle in a magnetic field



$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-v^2}}\right) = \vec{F}, \qquad \vec{F} = q\vec{v}\times\vec{B}$$

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-v^2}}\right) = \frac{m}{\sqrt{1-v^2}}\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B} \quad \rightarrow \quad m\gamma\frac{v^2}{R} = qvB \quad \rightarrow \quad R = \frac{m\gamma v}{qB} = \frac{|\vec{p}|}{qB} = \frac{\sqrt{E^2-m^2}}{qB}$$

$$f^t = 0, f^r = \gamma F^r = \gamma qvB = \frac{qB}{m}\sqrt{E^2-m^2}$$

Ex5) 전자기학의 상대론적 표현

- 고전적 표현:

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

여기서

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix},$$

$$J^\alpha = (c\rho, \vec{J})$$

- 특수상대론적 표현:

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0,$$

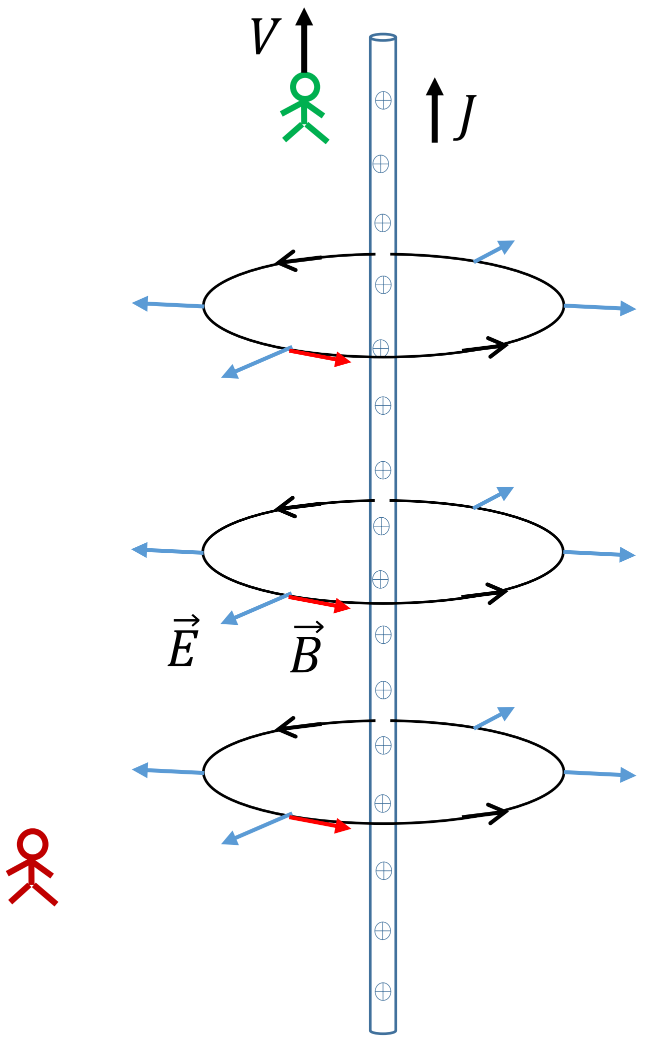
$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

특수상대론적 시공간에서의 텐서 방정식

➔ 상대성 원리를 만족한다는 것을 보장:

$$\boldsymbol{A_{\alpha\beta} = 0 \iff A'_{\alpha\beta} = \Lambda^{\mu}_{\alpha'} \Lambda^{\nu}_{\beta'} A_{\mu\nu} = 0}$$

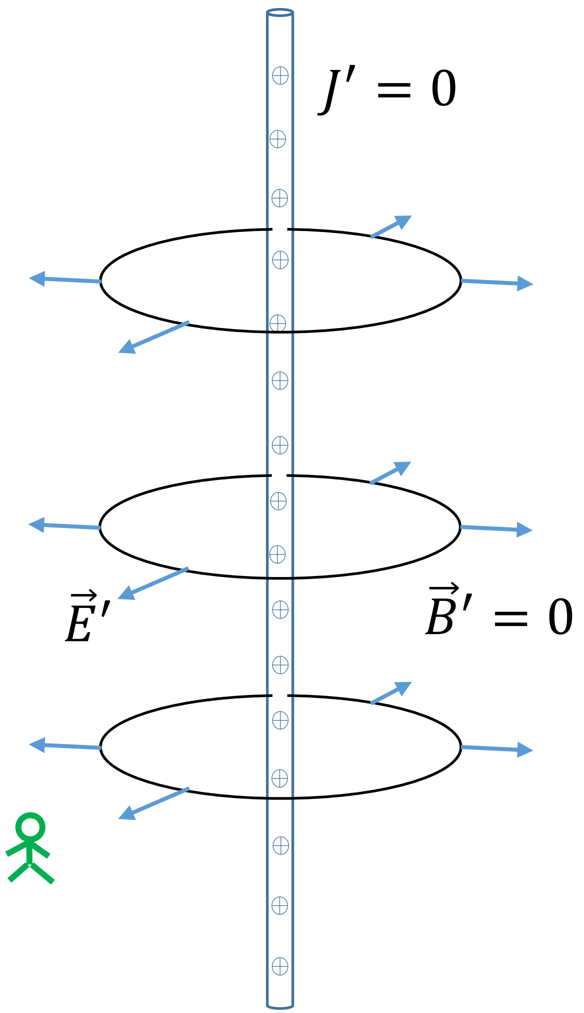
Ex) 전류와 자기장:



$$-B_x = F^{yz} = \Lambda^{y}_{\alpha'}\Lambda^{z}_{\beta'}F^{\alpha'\beta'}$$

$$= \gamma\left(\frac{V}{c}E'_y - \cancelto{0}{B'_x}\right)$$

$$\Longrightarrow \boldsymbol{B_x = -\frac{\gamma V}{c}E'_y}$$



✓ 뉴튼 극한: $c \to \infty$

i) 로렌츠 변환 ➔ 갈릴레오 변환

$$t' = \frac{1}{\sqrt{1-v^2/c^2}}\left(t - \frac{v}{c}\frac{x}{c}\right)$$
$$x' = \frac{1}{\sqrt{1-v^2/c^2}}\left(x - \frac{v}{c}\,ct\right)$$
$$y' = y,\ z' = z$$

⇩

$$t' = t$$
$$x' = x - vt$$
$$y' = y,\ z' = z$$

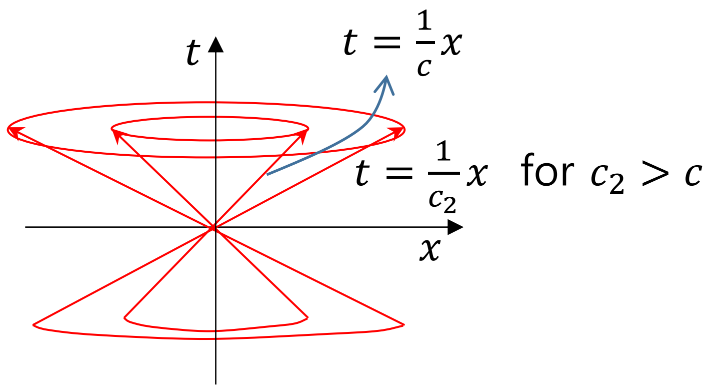
ii) 절대 시공간의 거리 구조

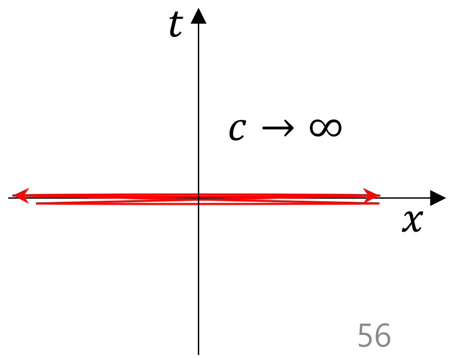
$$\Delta s^2 = -(c\Delta t)^2 + (\Delta l)^2 \cong -(c\Delta t)^2 = \Delta s'^2 \cong -(c\Delta t')^2$$

➔ $\Delta t = \Delta t'$.

Consequently, we also obtain $\Delta l = \Delta l'$

iii) 인과 구조





✓ Question and Answer:

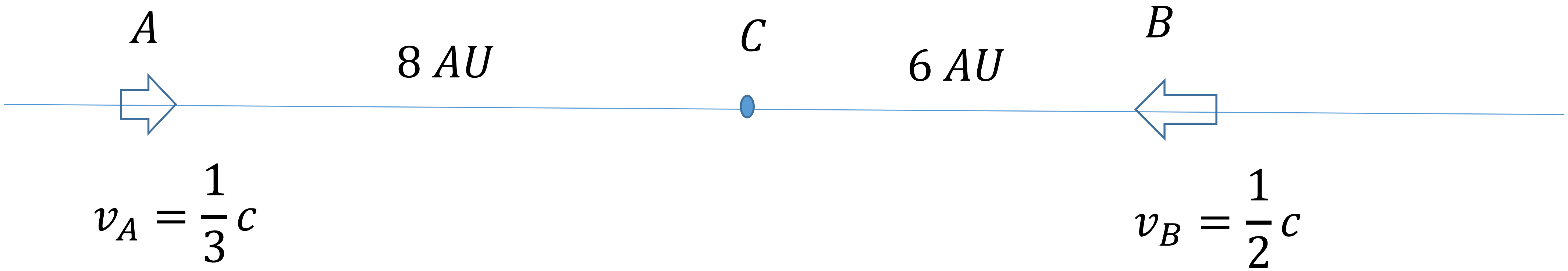
Homework Problems

Prob.1) 본인의 세계선을 간략히 그리시오(탄생부터 현재까지).

Prob.2) Ex3) 로렌츠 변환을 유도하시오.

Prob.3) Ex6) 길이 수축을 유도하시오.

Prob.4) 우주에서 다음과 같이 C 지점에서 1시간(각자의 고유시간) 후 친구를 만나려면 A 출발 후 B는 언제 출발해야 하는가?



Prob.5) Consider the reaction $\mathrm{A} \rightarrow \mathrm{B} + \mathrm{C}$ with particle rest masses $\mathrm{m_A}$, $\mathrm{m_B}$ and $\mathrm{m_C}$, respectively.

a) If A is at rest in the lab frame, what is the momentum four-vector $\mathrm{P_A}^{\mu}$ in the lab frame? Show also that, in this lab frame, particle B has an energy $\mathrm{E_B} = (\mathrm{m_A}^2 + \mathrm{m_B}^2 - \mathrm{m_C}^2)/2\mathrm{m_A}$.

b) If A decays while moving in the lab frame, find the relation between the angle θ at which B comes off, and the energies of A and B.