Stochastic Gravitational Waves and Pulsar Timing Arrays

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Abstract

This lecture introduces a basic theory for stochastic gravitational waves (GWs). Pulsar timing arrays

1 Gravitational Waves

Let us consider gravitational waves (GWs) in the transverse-traceless (TT) gauge with a globally inertial coordinate system $\{t, \vec{x}\}$ as

$$h_{ab}\left(t, \vec{x}\right) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d^2\kappa \, \tilde{h}_{ab}\left(\omega, \kappa\right) e^{iP(t,\vec{x};\omega,\kappa)}, \tag{1}$$

where ω is the frequency, κ is the unit spatial vector for the propagation, and P is the phase, a function of spacetime with parameters (ω, κ), given by

$$P\left(t, \vec{x}; \omega, \kappa\right) \equiv \omega\left(-t + \kappa \cdot \vec{x}\right). \tag{2}$$

The wave vector for GWs is defined by

$$k^a \equiv \nabla^a P = \omega\left(n^a + \kappa^a\right), \tag{3}$$

where $n^a \equiv -g^{ab}\left(dt\right)_b = \left(\partial/\partial t\right)^a$. The TT gauge condition gives

$$\tilde{h}_{ab}\left(\omega, \kappa\right) n^b = 0, \tag{4}$$

$$\tilde{h}_{ab}\left(\omega, \kappa\right) \kappa^b = 0, \tag{5}$$

$$\tilde{h}^a{}_a\left(\omega, \kappa\right) = 0. \tag{6}$$

Because h is real, we have

$$\tilde{h}_{ab}\left(-\omega, \kappa\right) = \tilde{h}^*_{ab}\left(\omega, \kappa\right). \tag{7}$$

When we introduce two parameters (θ, ϕ) for κ as

$$\kappa^a\left(\theta, \phi\right) = \sin\theta\cos\phi\hat{x}^a + \sin\theta\sin\phi\hat{y}^a + \cos\theta\hat{z}^a, \tag{8}$$

where $\{\hat{x}, \hat{y}, \hat{z}\}$ is the orthonormal basis in a Cartesian coordinate system, the integradtion for $d^2\kappa$ becomes

$$\int d^2\kappa = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta = \int_0^{2\pi} d\phi \int_{-1}^{1} d\left(\cos\theta\right) \tag{9}$$

2 Polarization

Let us consider a vector space $\mathcal{S}$ for symmetric traclss rank $(0, 2)$ tensors orthogonal to κ. The GW amplitude $\tilde{h}_{ab}\left(\omega, \kappa\right)$ belongs to $\mathcal{S}$. We choose a basis $\{e_A : A = +, \times\}$ for $\mathcal{S}$ as

$$e^+_{ab} = \frac{1}{\sqrt{2}}\left(u_a u_b - v_a v_b\right), \tag{10}$$

$$e^\times_{ab} = \frac{1}{\sqrt{2}}\left(u_a v_b + v_a u_b\right). \tag{11}$$

where $\{u, v, \kappa\}$ is a right-handed orthonormal basis that is the rotation of the orthonormal basis $\{\hat{x}, \hat{y}, \hat{z}\}$ in the Cartesian coordinate system by the Euler angle (θ, ϕ, ψ) as

$$\begin{bmatrix} u^a \\ v^a \\ \kappa^a \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}^a \\ \hat{y}^a \\ \hat{z}^a \end{bmatrix}. \tag{12}$$

Note that the $1/\sqrt{2}$ factors in eqs. (10) and (11) ensure the normalization as

$$e^A_{ab} e^B_{cd} g^{ac} g^{bd} = \delta^{AB}. \tag{13}$$

Then, $\tilde{h}_{ab}$ is decomposed into

$$\begin{aligned} \tilde{h}_{ab} &= \tilde{h}_+ e^+_{ab} + \tilde{h}_\times e^\times_{ab} \\ &= \tilde{h}_A e^A_{ab}, \end{aligned} \tag{14}$$

where $\tilde{h}_A$ are components in the basis and we intoduced the Einstein summation convetion for the index A.

Let us consider the projection operator from general vectors to vectors orthogonal to n and κ defined by

$$P^a{}_b \equiv \delta^a{}_b + n^a n_b - \kappa^a \kappa_b. \tag{15}$$

Then, the projection operator from general rank $(0,2)$ tensors to $\mathcal{S}$ is given by

$$\Lambda^{ab}{}_{cd} \equiv P^a{}_{(c} P^b{}_{d)} - \frac{1}{2} P^{ab} P_{cd}. \tag{16}$$

Note that

$$\begin{aligned} \Lambda^{ab}{}_{ab} &= \frac{1}{2} P^a{}_a P^b{}_b + \frac{1}{2} P^a{}_b P^b{}_a - \frac{1}{2} P^{ab} P_{ab} \\ &= 2 + 1 - 1 \\ &= 2. \end{aligned} \tag{17}$$

The dual basis $\left\{ e^A : A = +, \times \right\}$ uniquely exists such that

$$e^A_{ab} e^{ab}_B = \delta^A{}_B. \tag{18}$$

Since the basis is complete, we obtain

$$e^{ab}_A e^A_{cd} = \Lambda^{ab}{}_{cd}. \tag{19}$$

3 Stochastic Gravitational Waves

We assume that h_{ab} of stochastic gravitational waves (SGWs) is Gaussian such that its statistical status is fully described by the expectation value and the correlations. The expectaion value is given as

$$\langle h_{ab}(t, \vec{x}) \rangle = 0, \tag{20}$$

which implies

$$\left\langle \tilde{h}_{ab}(\omega, \kappa) \right\rangle = 0. \tag{21}$$

We also assume that h of SGWs is temporally stationary and spatiallly homogeneous. Then, its correlations R^h_{abcd} are given by

$$R^h_{abcd}(\tau, \vec{y}) = \langle h_{ab}(t, \vec{x}) h_{cd}(t + \tau, \vec{x} + \vec{y}) \rangle. \tag{22}$$

Note that it only depends on the time difference τ and the position difference $\vec{y}$. The correlation has properties as

$$R^h_{abcd} = R^h_{cdab}, \tag{23}$$

$$\begin{aligned} R^h_{abcd}(-\tau, -\vec{y}) &= \langle h_{ab}(t, \vec{x}) h_{cd}(t - \tau, \vec{x} - \vec{y}) \rangle \\ &= \langle h_{ab}(t' + \tau, \vec{x}' + \vec{y}) h_{cd}(t', \vec{x}') \rangle \\ &= R^h_{cdab}(\tau, \vec{y}) \\ &= R^h_{abcd}(\tau, \vec{y}). \end{aligned} \tag{24}$$

The wave equation for R^h is given by

$$\begin{aligned} \Box_{(\tau,\vec{y})} R^h_{abcd} &= \left\langle h_{ab}(t, \vec{x}) \Box_{(\tau,\vec{y})} h_{cd}(t + \tau, \vec{x} + \vec{y}) \right\rangle \\ &= 0, \end{aligned} \tag{25}$$

where $\square_{(\tau,\vec{y})}$ is the D'Alembertian operator with partial derivatives with respect to $(\tau,\vec{y})$. Then, R^h has wave solution as

$$R^h_{abcd}(\tau,\vec{y}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\, S^h_{abcd}(\omega,\kappa)\, e^{i\omega(-\tau+\kappa\cdot\vec{y})}, \tag{26}$$

where $S^h_{abcd}(\omega,\kappa)$ is the power spectral density having properties

$$S^h_{abcd} = S^h_{cdab}, \tag{27}$$
$$S^h_{abcd}(-\omega,\kappa) = S^h_{abcd}(\omega,\kappa), \tag{28}$$
$$S^h_{abcd}(-\omega,\kappa) = S^{h*}_{abcd}(\omega,\kappa). \tag{29}$$

The first equation is derived by eq. (23), the second is deduced from eq. (24), and the third comes from that R^h is real. Therefore, we conclude that S^h is also real. From eq. (22), we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\, S^h_{abcd}(\omega,\kappa)\, e^{i\omega(-\tau+\kappa\cdot\vec{y})}\\
&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int d^2\kappa'\left\langle \tilde{h}_{ab}(\omega',\kappa')\,\tilde{h}_{cd}(\omega,\kappa)\right\rangle e^{i\omega'\left(-t+\kappa'\cdot\vec{x}\right)}e^{i\omega(-t-\tau+\kappa\cdot(\vec{x}+\vec{y}))}\\
&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\left[\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int d^2\kappa'\left\langle \tilde{h}^*_{ab}(\omega',\kappa')\,\tilde{h}_{cd}(\omega,\kappa)\right\rangle e^{i\left(\omega'-\omega\right)t}e^{i\left(\omega\kappa-\omega'\kappa'\right)\cdot\vec{x}}\right]e^{i\omega(-\tau+\kappa\cdot\vec{y})}.
\end{aligned} \tag{30}$$

Note that the expression inside the square bracket has nothign to do with t and $\vec{x}$. Therefore, we obtain

$$\left\langle \tilde{h}^*_{ab}(\omega,\kappa)\,\tilde{h}_{cd}(\omega',\kappa')\right\rangle = 2\pi\delta(\omega'-\omega)\,\delta^2(\kappa'-\kappa)\,S^h_{abcd}(\omega,\kappa) \tag{31}$$

For $\kappa(\theta,\phi)$, the factor $\delta^2(\kappa'-\kappa)$ is explictly written as

$$\delta^2(\kappa'-\kappa) = \delta(\cos\theta'-\cos\theta)\,\delta(\phi'-\phi). \tag{32}$$

Assuming no preference to SGW polarization, we get

$$S^h_{abcd}(\omega,\kappa) = \frac{1}{2}S_h(\omega,\kappa)\,\Lambda_{abcd}(\kappa), \tag{33}$$

where S_h is a scalar function of (ω,κ). In addition, we assume the isotropy of SGW. Then,

$$S_h(\omega,\kappa) = \frac{1}{4\pi}S_h(\omega), \tag{34}$$

where $S_h(\omega)$ is the scalar function of ω. The factors $1/2$ in eq. (33) and $1/4\pi$ in eq. (34) are chosen such that

$$\begin{aligned}
\left\langle h_{ab}(t,\vec{x})\,h^{ab}(t,\vec{x})\right\rangle &= R^h_{abcd}(0,0)\,g^{ac}g^{bd}\\
&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\, S^h_{abcd}(\omega,\kappa)\,g^{ac}g^{bd}\\
&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\,\frac{1}{2}S_h(\omega,\kappa)\,\Lambda_{abcd}(\kappa)\,g^{ac}g^{bd}\\
&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\, S_h(\omega,\kappa)\\
&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\,\frac{1}{4\pi}S_h(\omega)\\
&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,S_h(\omega).
\end{aligned} \tag{35}$$

Therefore, $S_h(\omega)$ becomes the power spectral density for $\left\langle h_{ab}h^{ab}\right\rangle$. In these assumptions, we have

$$\left\langle \tilde{h}^*_{ab}(\omega,\kappa)\,\tilde{h}_{cd}(\omega',\kappa')\right\rangle = 2\pi\delta(\omega'-\omega)\,\frac{1}{4\pi}\delta^2(\kappa'-\kappa)\,\frac{1}{2}\Lambda_{abcd}(\kappa)\,S_h(\omega). \tag{36}$$

4 Energy density of GW background

The energy density of GWB is given by

$$\begin{aligned}
\rho_{\rm gw} &= \frac{1}{32\pi}\left\langle \partial_t h_{ab}\partial_t h^{ab}\right\rangle \\
&= \frac{1}{32\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int d^2\kappa'\,\left(-i\omega\right)\left(-i\omega'\right)\left\langle \tilde{h}_{ab}\left(\omega,\kappa\right)\tilde{h}^{ab}\left(\omega',\kappa'\right)\right\rangle \\
&= \frac{1}{32\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int d^2\kappa'\,\left(i\omega\right)\left(-i\omega'\right)\left\langle \tilde{h}^*_{ab}\left(\omega,\kappa\right)\tilde{h}^{ab}\left(\omega',\kappa'\right)\right\rangle \\
&= \frac{1}{32\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int d^2\kappa'\,\omega\omega' 2\pi\delta\left(\omega'-\omega\right)\frac{1}{4\pi}\delta^2\left(\kappa'-\kappa\right)S_h\left(\omega\right) \\
&= \frac{1}{32\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\int d^2\kappa'\,\omega^2\frac{1}{4\pi}\delta^2\left(\kappa'-\kappa\right)S_h\left(\omega\right) \\
&= \frac{1}{32\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega^2 S_h\left(\omega\right) \\
&= \frac{1}{32\pi}\int_{0}^{\infty}df\,\left(2\pi f\right)^2 S_h^{\rm one-sided}\left(f\right) \\
&= \frac{1}{32\pi}\int_{0}^{\infty}df\,\left(2\pi f\right)^2 S_h^{\rm one-sided}\left(f\right). \qquad (37)
\end{aligned}$$

Let us define $\Omega_{\rm gw}$ as the log frequency density for the cosmological parameter of GWB:

$$\begin{aligned}
\frac{\rho_{\rm gw}}{\rho_{\rm c}} &= \int_0^{\infty} d\ln f\,\Omega_{\rm gw}\left(f\right) \\
&= \frac{1}{12H_0^2}\int_0^{\infty}df\,\left(2\pi f\right)^2 S_h^{\rm one-sided}\left(f\right) \\
&= \frac{\pi^2}{3H_0^2}\int_0^{\infty}d\ln f\,f^3 S_h^{\rm one-sided}\left(f\right) \qquad (38)
\end{aligned}$$

where $\rho_{\rm c} = 3H_0^2/8\pi$ is the critical density. Therfore, $\Omega_{\rm gw}$ is related to $S_h^{\rm one-sided}$ as

$$\Omega_{\rm gw}\left(f\right) = \frac{\pi^2}{3H_0^2}f^3 S_h^{\rm one-sided}\left(f\right). \qquad (39)$$

5 Detector Output Correlation

Detector output is given by

$$s\left(t\right) = h\left(t\right) + n\left(t\right). \qquad (40)$$

GW signal h is given by

$$h\left(t\right) = D^{ab}h_{ab}\left(t,\vec{x}_0\right), \qquad (41)$$

where D^{ab} is the detector tensor and $\vec{x}_0$ is the position of detector. We assume that the noise n is Gaussian and stationary as

$$\left\langle n\left(t\right)\right\rangle = 0 \qquad (42)$$

$$\left\langle n\left(t\right)n\left(t+\tau\right)\right\rangle = R_n\left(\tau\right) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,S_n\left(\omega\right)e^{-i\omega\tau}, \qquad (43)$$

where $R_n\left(\tau\right)$ is the auto-correlation depends only on the time difference τ and $S_n\left(\omega\right)$ is the noise spectral density.

Let us define a correlation measurement from two detector outputs as

$$Y \equiv \int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt'\,s_1\left(t\right)s_2\left(t'\right)Q\left(t'-t\right), \qquad (44)$$

where Q is the real filter function. We assume the noises of each detector are not correlated as

$$\begin{aligned}
\left\langle n_1\left(t\right)n_2\left(t'\right)\right\rangle &= \left\langle n_1\left(t\right)\right\rangle\left\langle n_2\left(t'\right)\right\rangle \\
&= 0. \qquad (45)
\end{aligned}$$

Then, we obtain

$$\begin{aligned}\langle s_1(t)\, s_2(t')\rangle &= \langle h_1(t)\, h_2(t')\rangle + \langle h_1(t)\, n_2(t')\rangle + \langle n_1(t)\, h_2(t')\rangle + \langle n_1(t)\, n_2(t')\rangle \\ &= \langle h_1(t)\, h_2(t')\rangle + \langle h_1(t)\rangle \langle n_2(t')\rangle + \langle n_1(t)\rangle \langle h_2(t')\rangle + \langle n_1(t)\rangle \langle n_2(t')\rangle \\ &= \langle h_1(t)\, h_2(t')\rangle\,. \end{aligned} \tag{46}$$

The signal S for the measurement is defined by

$$\begin{aligned} S &\equiv \langle Y\rangle \\ &= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'\ \langle s_1(t)\, s_2(t')\rangle\, Q(t'-t) \\ &= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'\, D_1^{ab} D_2^{cd} \langle h_{ab}(t, \vec{x}_1)\, h_{cd}(t', \vec{x}_2)\rangle\, Q(t'-t) \\ &= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'\, D_1^{ab} D_2^{cd} R^h_{abcd}(t'-t, \vec{x}_2 - \vec{x}_1)\, Q(t'-t) \\ &= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'\, D_1^{ab} D_2^{cd} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d^2\kappa\, S^h_{abcd}(\omega, \kappa)\, e^{i\omega\left(-\left(t'-t\right)+\kappa\cdot(\vec{x}_2-\vec{x}_1)\right)} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \tilde{Q}(\omega')\, e^{-i\omega'\left(t'-t\right)} \\ &= D_1^{ab} D_2^{cd} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d^2\kappa\, \frac{1}{4\pi} S_h(\omega)\, \frac{1}{2} \Lambda_{abcd}(\kappa)\, e^{i\omega(\kappa\cdot(\vec{x}_2-\vec{x}_1))} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \tilde{Q}^*(\omega')\, \delta_T(\omega'-\omega)\, \delta_T(\omega'-\omega) \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S_h(\omega)\, \tilde{\Gamma}(\omega) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \tilde{Q}^*(\omega')\, \delta_T(\omega'-\omega)\, \delta_T(\omega'-\omega) \\ &\simeq T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S_h(\omega)\, \tilde{\Gamma}(\omega)\, \tilde{Q}^*(\omega) \end{aligned} \tag{47}$$

where δ_T is defined by

$$\begin{aligned} \delta_T(\omega) &\equiv \int_{-T/2}^{T/2} dt\, e^{i\omega t} \\ &= T\, \mathrm{sinc}(\omega T/2)\,, \end{aligned} \tag{48}$$

and $\tilde{\Gamma}(\omega)$ is the overlap reduction function given by

$$\tilde{\Gamma}(\omega) \equiv \frac{1}{4\pi} \int d^2\kappa\, \frac{1}{2} \Lambda_{abcd}(\kappa)\, D_1^{ab} D_2^{cd} e^{i\omega(\kappa\cdot(\vec{x}_2-\vec{x}_1))}. \tag{49}$$

The noise N for the measurement is defined by

$$\begin{aligned}
N^2 &\equiv \mathrm{Var}\left([Y]_{h=0}\right) \\
&= \left\langle \left[Y^2\right]_{h=0} \right\rangle - \left\langle [Y]_{h=0} \right\rangle^2 \\
&= \left\langle \left[Y^2\right]_{h=0} \right\rangle - \left(\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \, \left\langle n_1(t)\, n_2(t') \right\rangle Q(t'-t) \right)^2 \\
&= \left\langle \left[Y^2\right]_{h=0} \right\rangle - \left(\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \, \left\langle n_1(t) \right\rangle \left\langle n_2(t') \right\rangle Q(t'-t) \right)^2 \\
&= \left\langle \left[Y^2\right]_{h=0} \right\rangle \\
&= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \int_{-T/2}^{T/2} dt'' \int_{-T/2}^{T/2} dt''' \, \left\langle n_1(t)\, n_2(t')\, n_1(t'')\, n_2(t''') \right\rangle Q(t'-t)\, Q(t'''-t'') \\
&= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \int_{-T/2}^{T/2} dt'' \int_{-T/2}^{T/2} dt''' \, \left\langle n_1(t)\, n_1(t'') \right\rangle \left\langle n_2(t')\, n_2(t''') \right\rangle Q(t'-t)\, Q(t'''-t'') \\
&= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \int_{-T/2}^{T/2} dt'' \int_{-T/2}^{T/2} dt''' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'''}{2\pi} \\
&\quad \times S_1^n(\omega)\, e^{-i\omega\left(t''-t\right)} S_2^n(\omega')\, e^{-i\omega'\left(t'''-t'\right)} \tilde{Q}(\omega'')\, e^{-i\omega''\left(t'-t\right)} \tilde{Q}(\omega''')\, e^{-i\omega'''\left(t'''-t''\right)} \\
&= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \int_{-T/2}^{T/2} dt'' \int_{-T/2}^{T/2} dt''' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'''}{2\pi} \\
&\quad \times S_1^n(\omega)\, S_2^n(\omega')\, \tilde{Q}(\omega'')\, \tilde{Q}^*(\omega''')\, e^{-i\omega\left(t-t''\right)} e^{-i\omega'\left(t'''-t'\right)} e^{-i\omega''\left(t'-t\right)} e^{-i\omega'''\left(t''-t'''\right)} \\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'''}{2\pi} \\
&\quad \times S_1^n(\omega)\, S_2^n(\omega')\, \tilde{Q}(\omega'')\, \tilde{Q}^*(\omega''')\, \delta_T(\omega''-\omega)\, \delta_T(\omega''-\omega')\, \delta_T(\omega'''-\omega)\, \delta_T(\omega'''-\omega') \\
&\simeq T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_1^n(\omega)\, S_2^n(\omega) \left| \tilde{Q}(\omega) \right|^2
\end{aligned} \tag{50}$$

Then, the signal to noise ratio (SNR) is given by

$$\frac{S}{N} \simeq \sqrt{T} \frac{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_h(\omega)\, \tilde{\Gamma}(\omega)\, \tilde{Q}^*(\omega)}{\sqrt{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_1^n(\omega)\, S_2^n(\omega) \left|\tilde{Q}(\omega)\right|^2}} \tag{51}$$

$$= \sqrt{T} \frac{\left\langle S_h \tilde{\Gamma} / S_1^n S_2^n, \tilde{Q} \right\rangle}{\sqrt{\left\langle \tilde{Q}, \tilde{Q} \right\rangle}} \tag{52}$$

where the inner product is defined by

$$\langle A, B \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_1^n(\omega)\, S_2^n(\omega)\, A(\omega)\, B^*(\omega)\,. \tag{53}$$

The SNR is maximized when we choose the filter Q as

$$\tilde{Q} \propto \frac{S_h \tilde{\Gamma}}{S_1^n S_2^n}. \tag{54}$$

In this case,

$$\frac{S}{N} = \sqrt{T} \sqrt{\left\langle \frac{S_h \tilde{\Gamma}}{S_1^n S_2^n}, \frac{S_h \tilde{\Gamma}}{S_1^n S_2^n} \right\rangle} \tag{55}$$

$$= \sqrt{T} \sqrt{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{S_h^2(\omega)}{S_1^n(\omega)\, S_2^n(\omega)} \left| \tilde{\Gamma}(\omega) \right|^2} \tag{56}$$

6 Detection of Gravitational Waves by Light

6.1 Maxwell's Equations

Maxwell equation for 4-potential A is given by

$$2\nabla^b \nabla_{[a} A_{b]} = 4\pi J_a, \tag{57}$$

where J is the electromagnetic 4-current and we introduce Gaussian unit, $\epsilon_0 = 1/4\pi$ and $\mu_0 = 4\pi$. Using Lorenz gauge,

$$\nabla^a A_a = 0, \tag{58}$$

we get

$$\nabla^b \nabla_b A_a = R^b{}_a A_b - 4\pi J_a. \tag{59}$$

Exercise: As you know, Maxwell's equations consist of 4 equations. However, with 4-potential A, we only need Gauss's law and Ampère's law as in eq. (57). Why?

Exercise: Prove that the conservation of electric charge, $\nabla^a J_a = 0$.

6.2 Geometrical Optics

Let us consider a local Lorentz frame $\{t, \vec{x}\}$. In the frame, $\mathcal{L}$ is defined as a typical length over which the waves vary and $\mathcal{R}$ is defined as a typical components of the Riemann curvature tensor. When the frequency of electromagnetic waves ω is much larger than $1/L$ where $L \equiv \min(\mathcal{L}, \mathcal{R})$, geometrical optics is valid. Let us consider electromagnetic waves given by

$$A_a(t, \vec{x}) = 2\Re\left[\left\{\tilde{A}_a + \omega^{-1}\tilde{B}_a + O\left(\omega^{-2}\right)\right\} e^{i\omega q(t,\vec{x})}\right], \tag{60}$$

such that $\hat{l}^a \equiv \nabla^a q$ is future-directed. In vacuum, eq. (59) becomes the wave equation for A as

$$\nabla^b \nabla_b A_a = 0 \tag{61}$$

Through,

$$\begin{aligned}
\nabla_b A_a &= 2\Re\left[\left\{i\omega\hat{l}_b\left(\tilde{A}_a + \omega^{-1}\tilde{B}_a\right) + \nabla_b\tilde{A}_a + O\left(\omega^{-1}\right)\right\} e^{i\omega q}\right] \\
&= 2\Re\left[\left\{i\omega\hat{l}_b\tilde{A}_a + i\hat{l}_b\tilde{B}_a + \nabla_b\tilde{A}_a + O\left(\omega^{-1}\right)\right\} e^{i\omega q}\right],
\end{aligned} \tag{62}$$

$$\nabla^a A_a = 2\Re\left[\left\{i\omega\left(\hat{l}\cdot\tilde{A}\right) + i\left(\hat{l}\cdot\tilde{B}\right) + \nabla^a\tilde{A}_a + O\left(\omega^{-1}\right)\right\} e^{i\omega q}\right], \tag{63}$$

$$\begin{aligned}
\nabla_c \nabla_b A_a &= 2\Re\left[\left\{i\omega\hat{l}_c\left(i\omega\hat{l}_b\tilde{A}_a + i\hat{l}_b\tilde{B}_a + \nabla_b\tilde{A}_a\right) + i\omega\nabla_c\left(\hat{l}_b\tilde{A}_a\right) + O(1)\right\} e^{i\omega q}\right] \\
&= 2\Re\left[\left\{-\omega^2\hat{l}_c\hat{l}_b\tilde{A}_a + \omega\left(-\hat{l}_b\hat{l}_c\tilde{B}_a + i\hat{l}_c\nabla_b\tilde{A}_a + i\nabla_c\left(\hat{l}_b\tilde{A}_a\right)\right) + O(1)\right\} e^{i\omega q}\right],
\end{aligned} \tag{64}$$

$$\nabla^b \nabla_b A_a = 2\Re\left[\left\{-\omega^2\left(\hat{l}\cdot\hat{l}\right)\tilde{A}_a + \omega\left(-\left(\hat{l}\cdot\hat{l}\right)\tilde{B}_a + 2i\hat{l}^b\nabla_b\tilde{A}_a + i\tilde{A}_a\nabla_b\hat{l}^b\right) + O(1)\right\} e^{i\omega q}\right], \tag{65}$$

we get

$$0 = \hat{l}\cdot\hat{l}, \tag{66}$$

in the leading-order of ω and

$$0 = \hat{l}\cdot\tilde{A}, \tag{67}$$

$$0 = 2\hat{l}^b\nabla_b\tilde{A}_a + \tilde{A}_a\nabla_b\hat{l}^b \tag{68}$$

in the next-to-leading-order. Rewriting results in $Q \equiv \omega q$ and $l \equiv \omega\hat{l}$, we obtain the evolution equations along l as

$$\begin{aligned}
l^a\nabla_a Q &= l\cdot l \\
&= 0,
\end{aligned} \tag{69}$$

$$\begin{aligned}
l^b\nabla_b l^a &= g^{ac} l^b \nabla_b \nabla_c Q \\
&= g^{ac} l^b \nabla_c \nabla_b Q \\
&= \frac{1}{2} g^{ac} \nabla_c (l\cdot l) \\
&= 0,
\end{aligned} \tag{70}$$

in the leading-order and

$$l^b\nabla_b\tilde{A}_a = -\frac{1}{2}\tilde{A}_a\nabla_b l^b \tag{71}$$

$$l\cdot\tilde{A} = 0 \tag{72}$$

in the next-to-leading-order. Introducing the real amplitude $\mathcal{A} \equiv \sqrt{\tilde{A}\cdot\tilde{A}^*}$ and polarization vector $\tilde{f}_a \equiv \tilde{A}_a/\mathcal{A}$, we get

$$0 = \nabla_b\left(\mathcal{A}^2 l^b\right), \tag{73}$$

$$0 = l^b\nabla_b\tilde{f}_a, \tag{74}$$

$$0 = l\cdot\tilde{f}. \tag{75}$$

The first equation is conservation of the number of light rays, the second equation is the parallel transport of polarization, and the third equation is the transverse condition of polarization.

6.3 Perturbation of Rays

We introduce the Minkowski background spacetime and monochromatic plane electromagnetic wave given by

$$A_a = 2\Re\left[\tilde{A}_a e^{iQ}\right], \tag{76}$$

where $\tilde{A}$ and $l_a \equiv \nabla_a Q$ are constant over spacetime. We impose Lorenz gauge and radiation gauge as

$$0 = \nabla^a A_a, \tag{77}$$
$$0 = u \cdot A, \tag{78}$$

implies

$$0 = \tilde{A} \cdot l, \tag{79}$$
$$0 = \tilde{A} \cdot u. \tag{80}$$

Meanwhile, GWs are in the TT gauge given by

$$0 = k^a \tilde{h}_{ab}(k), \tag{81}$$
$$0 = \tilde{h}^a{}_a(k), \tag{82}$$
$$0 = u^a \tilde{h}_{ab}(k), \tag{83}$$

for all $k \in \mathcal{N}$.

Perturbed phase is given by

$$\tilde{Q}(\epsilon) = Q + \epsilon S + O\left(\epsilon^2\right). \tag{84}$$

The linear perturbation of l becomes

$$\begin{aligned}\mathcal{L}_v l^a &= \mathcal{L}_v\left(g^{ab}\nabla_b Q\right)\\ &= -h^{ab} l_b + \nabla^a S.\end{aligned} \tag{85}$$

Then, eq. (69) provides

$$0 = \left(-h^{ab} l_b + \nabla^a S\right) l_a + l^a \nabla_a S, \tag{86}$$

implies

$$l^a \nabla_a S = \frac{1}{2} h_{ab} l^a l^b. \tag{87}$$

The general solution of S is decomposed into the particular solution and the homogeneous solution as

$$S = S^{\mathrm{p}} + S^{\mathrm{h}}. \tag{88}$$

The particular solution can be solved using

$$S^{\mathrm{p}} = \int_{\mathcal{N}} d^3\mathcal{N}(k)\; \tilde{S}^{\mathrm{p}}(k)\, e^{iP(;k)}. \tag{89}$$

Plugging it into eq. (87), we get

$$\tilde{S}^{\mathrm{p}} = -\frac{1}{2} i \frac{1}{(k \cdot l)} \tilde{h}_{ab}(k)\, l^a l^b. \tag{90}$$

The homogeneous solution have to be satisfied

$$l^a \nabla_a S^{\mathrm{h}} = 0, \tag{91}$$

implies $S^{\mathrm{h}}(t, x, y, z) = X(q = -t + x, y, z)$.

Let us consider the frequency as

$$\omega \equiv -u^a \nabla_a Q. \tag{92}$$

Its perturbed value becomes

$$\tilde{\omega}(\epsilon) = \omega + \epsilon\alpha + O\left(\epsilon^2\right), \tag{93}$$

where

$$\alpha = -v^a l_a - u^a \nabla_a S$$
$$= -u^a \nabla_a S. \tag{94}$$

Note that α is gauge-invariant because ω is constant on $\mathcal{M}_0$. We give boundary condition at the 3-dimensional timelike plane $\mathcal{P}$ that is the congruence of emitters as

$$0 = [\alpha]_{\mathcal{P}} \tag{95}$$
$$= \left[-u^a \nabla_a S^{\rm p} - u^a \nabla_a S^{\rm h}\right]_{\mathcal{P}} \tag{96}$$
$$= \left[-u^a \int_{\mathcal{N}} d^3\mathcal{N}(k)\; ik_a \left(-\frac{1}{2} i \frac{1}{k \cdot l} \tilde{h}_{bc} l^b l^c\right) e^{iP(;k)} + \partial_q X\left(q = -t + x, y, z\right)\right]_{\mathcal{P}}, \tag{97}$$
$$= \left[\frac{1}{2} \int_{\mathcal{N}} d^3\mathcal{N}(k)\; \frac{1}{\hat{k} \cdot l} \tilde{h}_{ab} l^a l^b e^{i\omega_{\rm g}(-t + \kappa \cdot (y\hat{y} + z\hat{z}))} + \partial_q X\left(q = -t, y, z\right)\right]_{\mathcal{P}}, \tag{98}$$

implies

$$\partial_q X(q, y, z) = -\frac{1}{2} \int_{\mathcal{N}} d^3\mathcal{N}(k)\; \frac{1}{\hat{k} \cdot l} \tilde{h}_{ab} l^a l^b e^{i\omega_{\rm g}(q + \kappa \cdot (y\hat{y} + z\hat{z}))}, \tag{99}$$
$$X(q, y, z) = \frac{1}{2} i \int_{\mathcal{N}} d^3\mathcal{N}(k)\; \frac{1}{k \cdot l} \tilde{h}_{ab}(k)\, l^a l^b e^{i\omega_{\rm g}(q + \kappa \cdot (y\hat{y} + z\hat{z}))} + C(y, z) \tag{100}$$
$$S^{\rm h}(t, \vec{x}) = \frac{1}{2} i \int_{\mathcal{N}} d^3\mathcal{N}(k)\; \frac{1}{k \cdot l} \tilde{h}_{ab}(k)\, l^a l^b e^{i\omega_{\rm g}(-t + \lambda \cdot \vec{x} + \kappa \cdot (\vec{x} - (\lambda \cdot \vec{x})\lambda))} + C(y, z). \tag{101}$$

We conclude that

$$S = -\frac{1}{2} i \int_{\mathcal{N}} d^3\mathcal{N}\, \frac{1}{k \cdot l} \tilde{h}_{ab}(k)\, l^a l^b \left(1 - e^{iD(;k,l)}\right) e^{iP(;k)} + C(y, z), \tag{102}$$

where

$$D(t, \vec{x}; k, l) = \omega_{\rm g}(1 - \kappa \cdot \lambda)(\lambda \cdot \vec{x}). \tag{103}$$

Before arrival of GWs, we assume no perturbation as $S(t < t_0, 0 < x < L, y, z) = C(y, z) = 0$. Therefore,

$$S = -\frac{1}{2} i \int_{\mathcal{N}} d^3\mathcal{N}\, \frac{1}{k \cdot l} \tilde{h}_{ab} l^a l^b \left(e^{iP(;k)} - e^{i(P+D)(;k,l)}\right), \tag{104}$$

where

$$(P + D)(t, \vec{x}; k, l) = P\left(t - \lambda \cdot \vec{x}, \vec{x} - \lambda(\lambda \cdot \vec{x}); k\right). \tag{105}$$

Perturbed amplitude is given by

$$\tilde{\tilde{A}}_a(\epsilon) = \tilde{A}_a + \epsilon \tilde{B}_a + O\left(\epsilon^2\right). \tag{106}$$

The evolution of the amplitude, eq. (71), provides

$$l^b \nabla_b \tilde{B}_a - \dot{C}^c{}_{ab} \tilde{A}_c l^b = -\frac{1}{2} \tilde{A}_a \left\{\nabla_b\left(-h^{bc} l_c + \nabla^b S\right) + l^c \dot{C}^b{}_{cb}\right\}, \tag{107}$$
$$l^b \nabla_b \tilde{B}_a = -\frac{1}{2} \tilde{A}_a \nabla_b \nabla^b S + \frac{1}{2}\left(\nabla_a h^c{}_b + \nabla_b h^c{}_a - \nabla^c h_{ab}\right) \tilde{A}_c l^b. \tag{108}$$

It gives

$$\tilde{\tilde{B}}_a = \frac{1}{2(k \cdot l)} \left[\frac{1}{2} \frac{1}{k \cdot l} \tilde{A}_a \tilde{h}_{bc} l^b l^c \left\{k - \left(k \cdot \hat{l}\right)\lambda\right\} \cdot \left\{k - \left(k \cdot \hat{l}\right)\lambda\right\} e^{iD(;k,l)} + \left(k_a \tilde{h}^c{}_b + k_b \tilde{h}^c{}_a - k^c \tilde{h}_{ab}\right) \tilde{A}_c l^b\right] e^{iP(;k)} \tag{109}$$
$$= \frac{1}{2(k \cdot l)} \left[\frac{1}{2} \tilde{A}_a \tilde{h}_{bc} l^b \hat{l}^c k \cdot (u - \lambda)\, e^{iD(;k,l)} + \left(k_a \tilde{h}^c{}_b + k_b \tilde{h}^c{}_a - k^c \tilde{h}_{ab}\right) \tilde{A}_c l^b\right] e^{iP(;k)} \tag{110}$$

$$\tilde{B}_a = \int_{\mathcal{N}} d^3\mathcal{N}\, \frac{1}{2(k \cdot l)} \left[\frac{1}{2} \tilde{A}_a \tilde{h}_{bc} l^b \hat{l}^c k \cdot (u - \lambda)\, e^{iD(;k,l)} + \left(k_a \tilde{h}^c{}_b + k_b \tilde{h}^c{}_a - k^c \tilde{h}_{ab}\right) \tilde{A}_c l^b\right] e^{iP(;k)} + \tilde{C}_a, \tag{111}$$

where $\tilde{C}_a(t,x,y,z) = \tilde{Y}_a(q=-t+x,y,z)$ is quantity satisfying $l^b\nabla_b\tilde{C}_a = 0$. The Lorenz gauge, eq. (72), provides

$$0 = \left(-h^{ab}l_b + \nabla^a S\right)\tilde{A}_a + l\cdot\tilde{B} \tag{112}$$

$$= \left[-h^{ab}l_b + \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{1}{k\cdot l}\tilde{h}_{bc}(k)\,l^b l^c\left\{k^a e^{iP(;k)} - \left(k^a - \left(k\cdot\hat{l}\right)\lambda^a\right)e^{i(P+D)(;k,l)}\right\}\right]\tilde{A}_a$$

$$+\int_{\mathcal{N}} d^3\mathcal{N}\,\tilde{h}_{ab}\left(\tilde{A}^a - \frac{k\cdot\tilde{A}}{2\,(k\cdot l)}l^a\right)l^b e^{iP(;k)} + l\cdot\tilde{C} \tag{113}$$

$$= -\frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot\tilde{A}}{k\cdot l}\tilde{h}_{ab}(k)\,l^a l^b e^{i(P+D)(;k,l)} + l\cdot\tilde{C} \tag{114}$$

$$\tilde{C}_a = \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot\tilde{A}}{k\cdot l}\tilde{h}_{ab}l^b e^{i(P+D)(;k,l)} + C\,(q=-t+x,y,z)\,\hat{l}_a \tag{115}$$

The radiation gauge provides

$$0 = \left(\tilde{B} + iS\tilde{A}\right)\cdot u \tag{116}$$

$$= \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot u}{k\cdot l}\tilde{h}_{ab}\tilde{A}^a l^b e^{iP(;k)} - C\,(q=-t+x,y,z) \tag{117}$$

$$C = \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot u}{k\cdot l}\tilde{h}_{ab}\tilde{A}^a l^b e^{iP(;k)} \tag{118}$$

As a result,

$$\tilde{B}_a = \int_{\mathcal{N}} d^3\mathcal{N}\,\frac{1}{2\,(k\cdot l)}\left[\frac{1}{2}\tilde{A}_a\tilde{h}_{bc}l^b\hat{l}^c k\cdot(u-\lambda)\,e^{i(P+D)(;k,l)} + \left(k_a\tilde{h}^c{}_b + k_b\tilde{h}^c{}_a - k^c\tilde{h}_{ab}\right)\tilde{A}_c l^b e^{iP(;k)}\right]$$

$$+\frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot\tilde{A}}{k\cdot l}\tilde{h}_{ab}(k)\,l^b e^{i(P+D)(;k,l)}$$

$$+\frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\,\frac{k\cdot u}{k\cdot l}\tilde{h}_{bc}\tilde{A}^b l^c e^{iP(;k)}\hat{l}_a$$

$$= \int_{\mathcal{N}} d^3\mathcal{N}\,\frac{1}{2\,(k\cdot l)}\left[\left\{\frac{1}{2}\tilde{A}_a\tilde{h}_{bc}l^b\hat{l}^c k\cdot(u-\lambda) + \left(k\cdot\tilde{A}\right)\tilde{h}_{ab}l^b\right\}e^{i(P+D)(;k,l)}\right.$$

$$\left.+\left\{\left(k_a\tilde{h}^c{}_b + k_b\tilde{h}^c{}_a - k^c\tilde{h}_{ab}\right)\tilde{A}_c l^b + \hat{l}_a\,(k\cdot u)\,\tilde{h}_{bc}\tilde{A}^b l^c\right\}e^{iP(;k)}\right] \tag{119}$$

7 Pulsar Timing Array

Pulsar Timing:

$$h(t) \equiv \frac{\delta T}{T} \tag{120}$$

$$= \frac{\delta\omega_{\rm e}}{\omega_{\rm e}} \tag{121}$$

$$= \int d^2\kappa_{\rm g}\int_{-\infty}^{\infty}\frac{d\omega_{\rm g}}{2\pi}\,\frac{1}{2}\frac{\tilde{h}_{ab}(\omega_{\rm g},\kappa_{\rm g})\,\kappa_{\rm e}^a\kappa_{\rm e}^b}{1-\kappa_{\rm g}\cdot\kappa_{\rm e}}\left(e^{iP(t,\vec{x}_0;\omega_{\rm g},\kappa_{\rm g})} - e^{iP'(t,\vec{x}_0;\omega_{\rm g},\kappa_{\rm g})}\right) \tag{122}$$

$$= \int d^2\kappa_{\rm g}\int_{-\infty}^{\infty}\frac{d\omega_{\rm g}}{2\pi}\,\frac{1}{2}\frac{\tilde{h}_{ab}(\omega_{\rm g},\kappa_{\rm g})\,\kappa_{\rm e}^a\kappa_{\rm e}^b}{1-\kappa_{\rm g}\cdot\kappa_{\rm e}}e^{iP(t,\vec{x}_0;\omega_{\rm g},\kappa_{\rm g})}\left(1 - e^{i\omega_{\rm g}L(1-\kappa_{\rm g}\cdot\kappa_{\rm e})}\right) \tag{123}$$

where

$$P(t,\vec{x};\omega_{\rm g},\kappa_{\rm g}) = \omega_{\rm g}(-t+\kappa_{\rm g}\cdot\vec{x}) \tag{124}$$

$$P'(t,\vec{x};\omega_{\rm g},\kappa_{\rm g}) = P(t-L,\vec{x}-L\kappa_{\rm e};\omega_{\rm g},\kappa_{\rm g}) \tag{125}$$

$$= \omega_{\rm g}(-t+\kappa_{\rm g}\cdot\vec{x}) + \omega_{\rm g}L(1-\kappa_{\rm g}\cdot\kappa_{\rm e}) \tag{126}$$

Correlation:

$$\langle h_1(t)\,h_2(t+\tau)\rangle = \int d^2\kappa_{\rm g}\int_{-\infty}^{\infty}\frac{d\omega_{\rm g}}{2\pi}\,\frac{1}{2}\frac{\kappa_{\rm e,1}^a\kappa_{\rm e,1}^b}{1-\kappa_{\rm g}\cdot\kappa_{\rm e,1}}e^{iP(t,\vec{x}_0;\omega_{\rm g},\kappa_{\rm g})}\left(1-e^{i\omega_{\rm g}L_1(1-\kappa_{\rm g}\cdot\kappa_{\rm e,1})}\right)$$

$$\times\int d^2\kappa'_{\rm g}\int_{-\infty}^{\infty}\frac{d\omega'_{\rm g}}{2\pi}\,\frac{1}{2}\frac{\kappa_{\rm e,2}^c\kappa_{\rm e,2}^d}{1-\kappa'_{\rm g}\cdot\kappa_{\rm e,2}}e^{iP\left(t+\tau,\vec{x}_0;\omega'_{\rm g},\kappa'_{\rm g}\right)}\left(1-e^{i\omega'_{\rm g}L_2\left(1-\kappa'_{\rm g}\cdot\kappa_{\rm e,2}\right)}\right)$$

$$\times\left\langle\tilde{h}_{ab}(\omega_{\rm g},\kappa_{\rm g})\,\tilde{h}_{cd}\left(\omega'_{\rm g},\kappa'_{\rm g}\right)\right\rangle \tag{127}$$