Stochastic Gravitational Waves and Pulsar Timing Arrays

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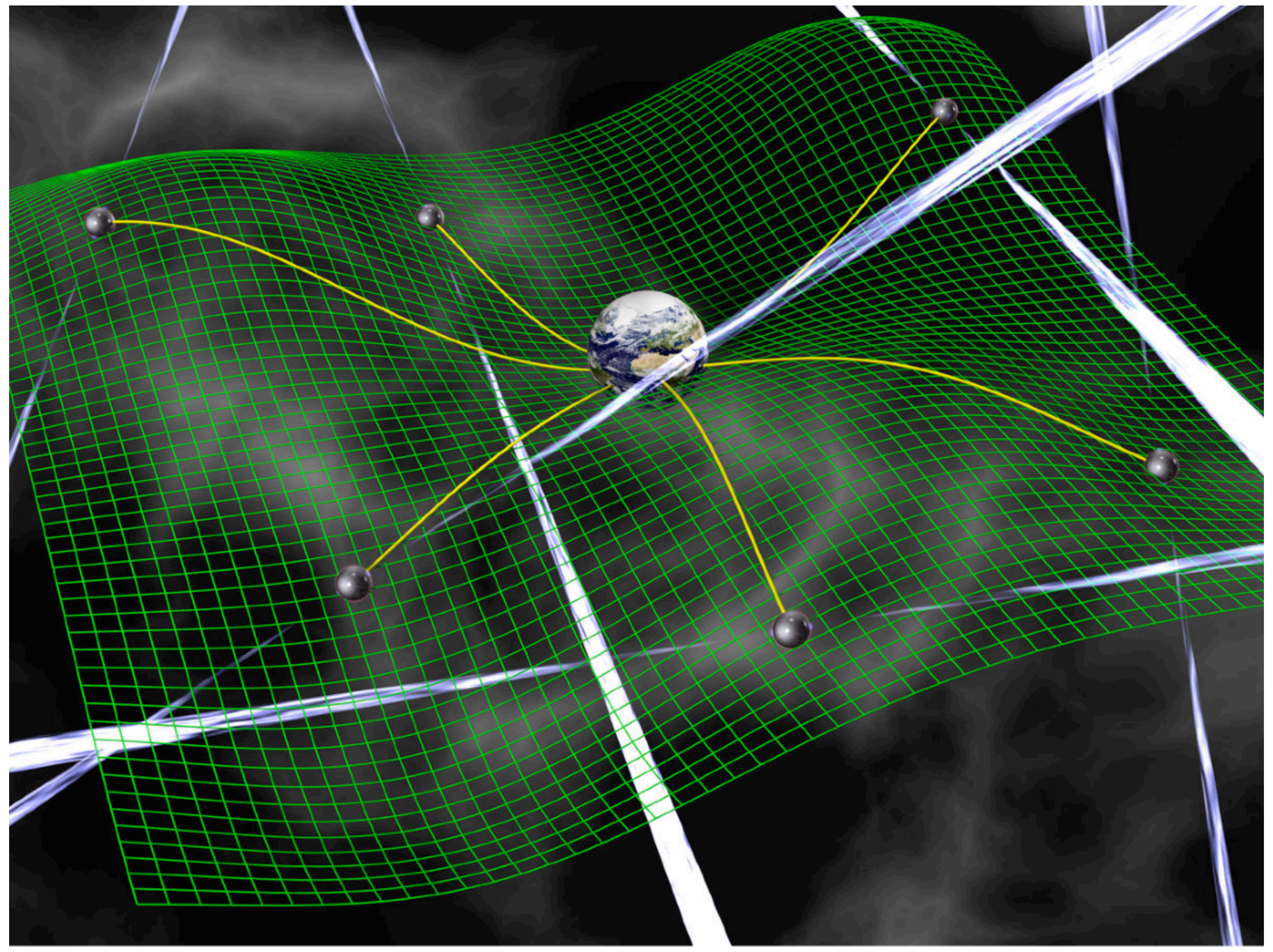
2025 Summer School on Numerical Relativity and Gravitational Waves

Lecture Materials

- Lecture slide
  - https://bit.ly/3U668II
- Lecture note
  - https://bit.ly/459yV42

Motivations

- Pulsar Timing Arrays around the world are on the verge of detecting the gravitational wave background (GWB).
- What is the GWB?
- From what kind of astrophysical phenomena is it generated?
- Why do we say the detection is imminent?
- What is a stochastic gravitational wave (SGW)?
- In this lecture, we explore to answer these questions.

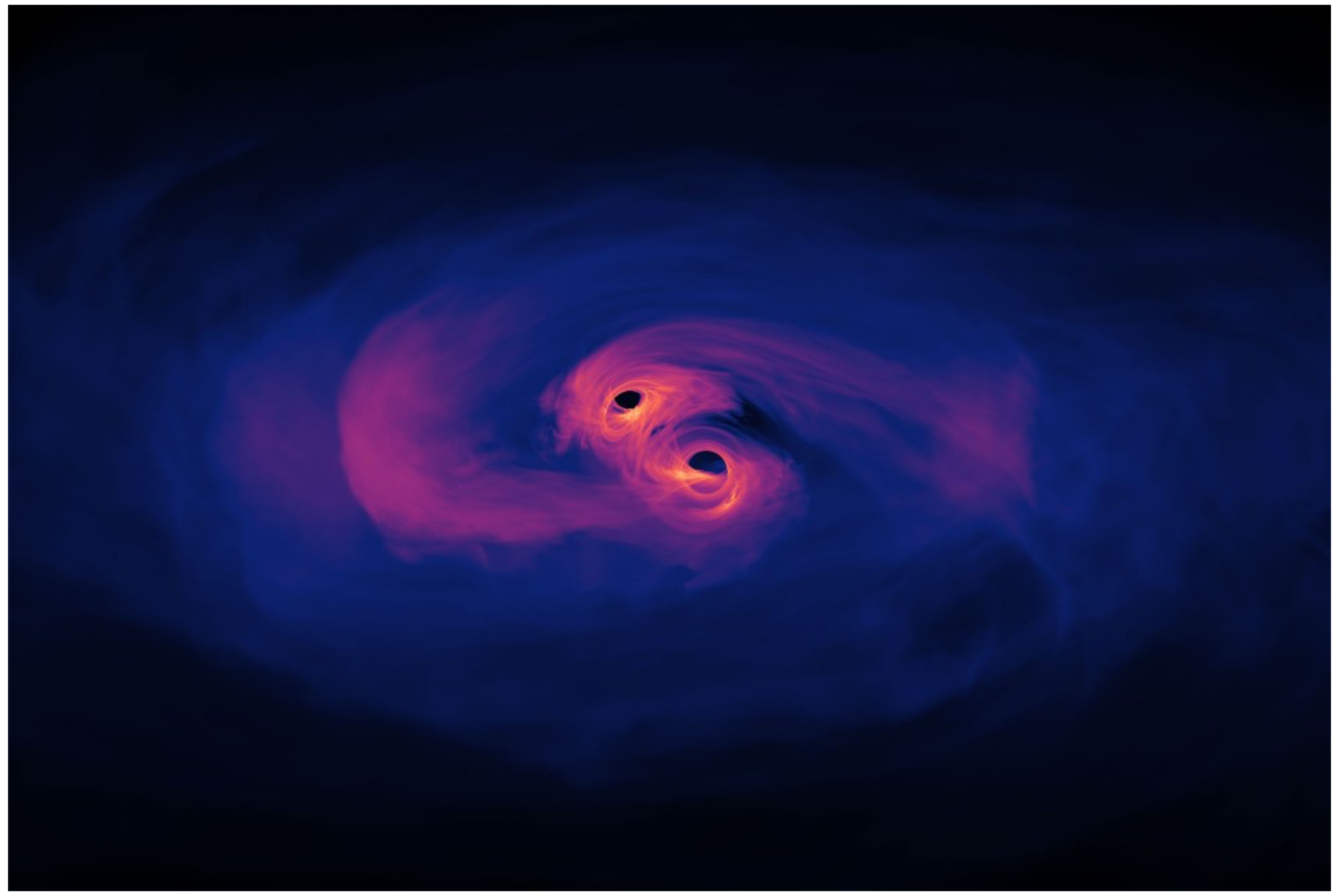


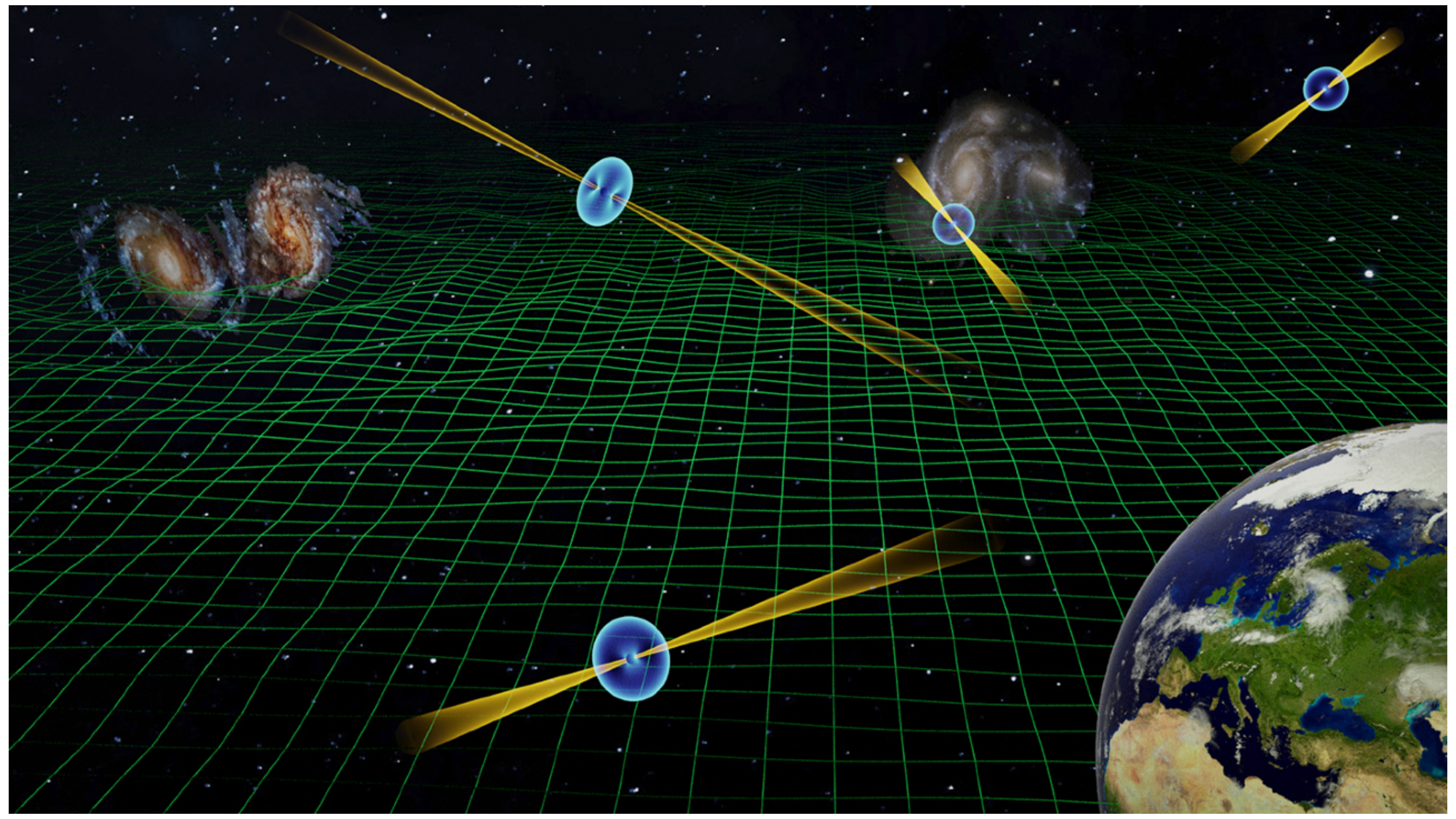
Pulsar Timing Array

Square Kilometre Array

Overview

- Brief Introduction
- Stochastic Gravitational Waves
- Pulsar Timing Arrays

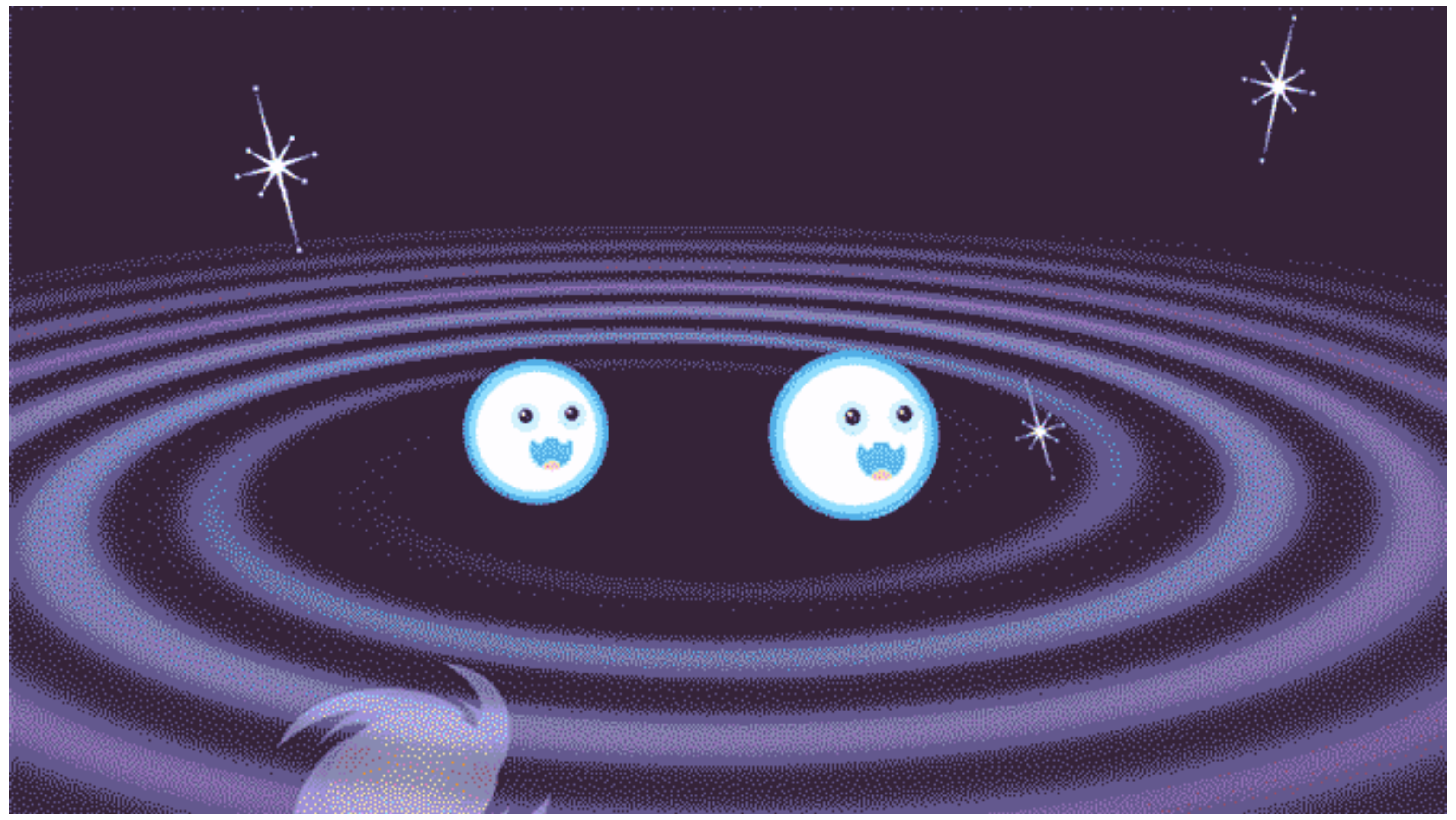




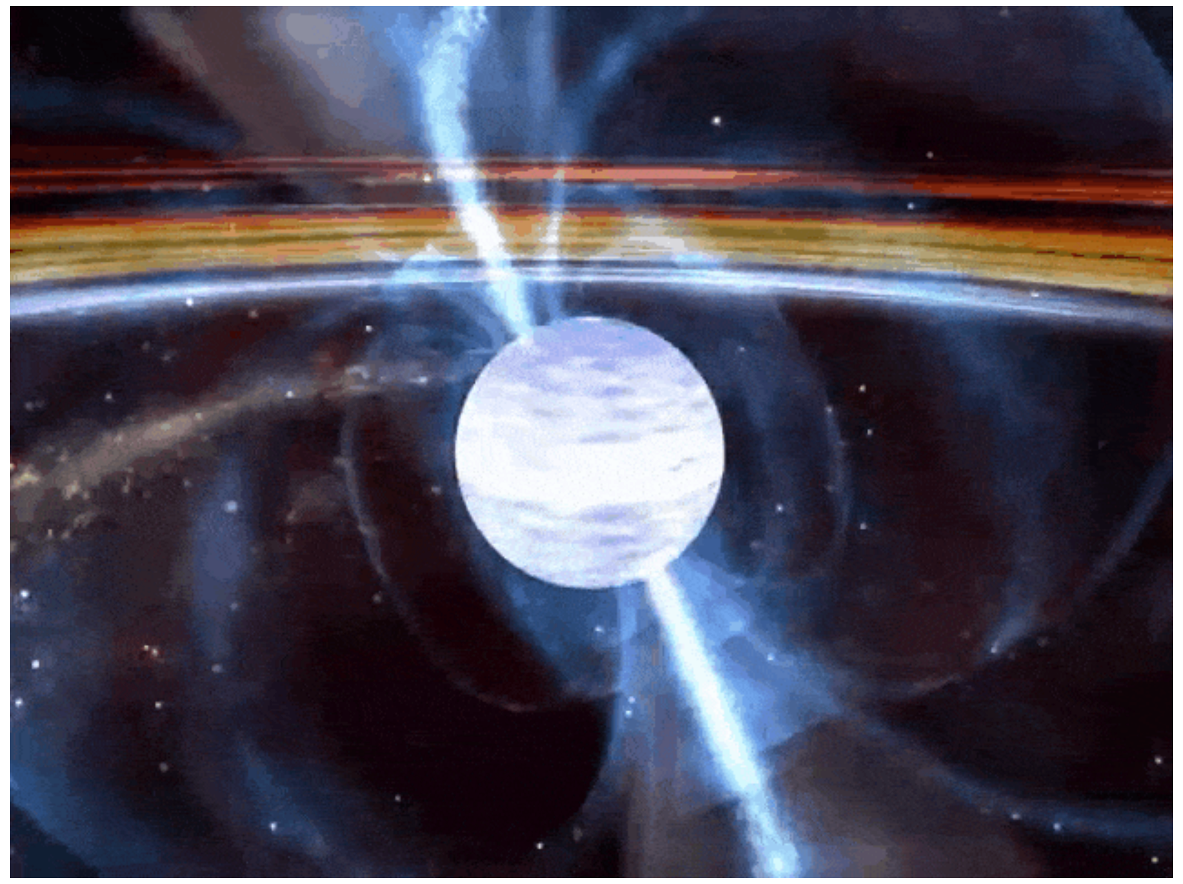
Introduction

Gravitational Wave (GW) Signal Types

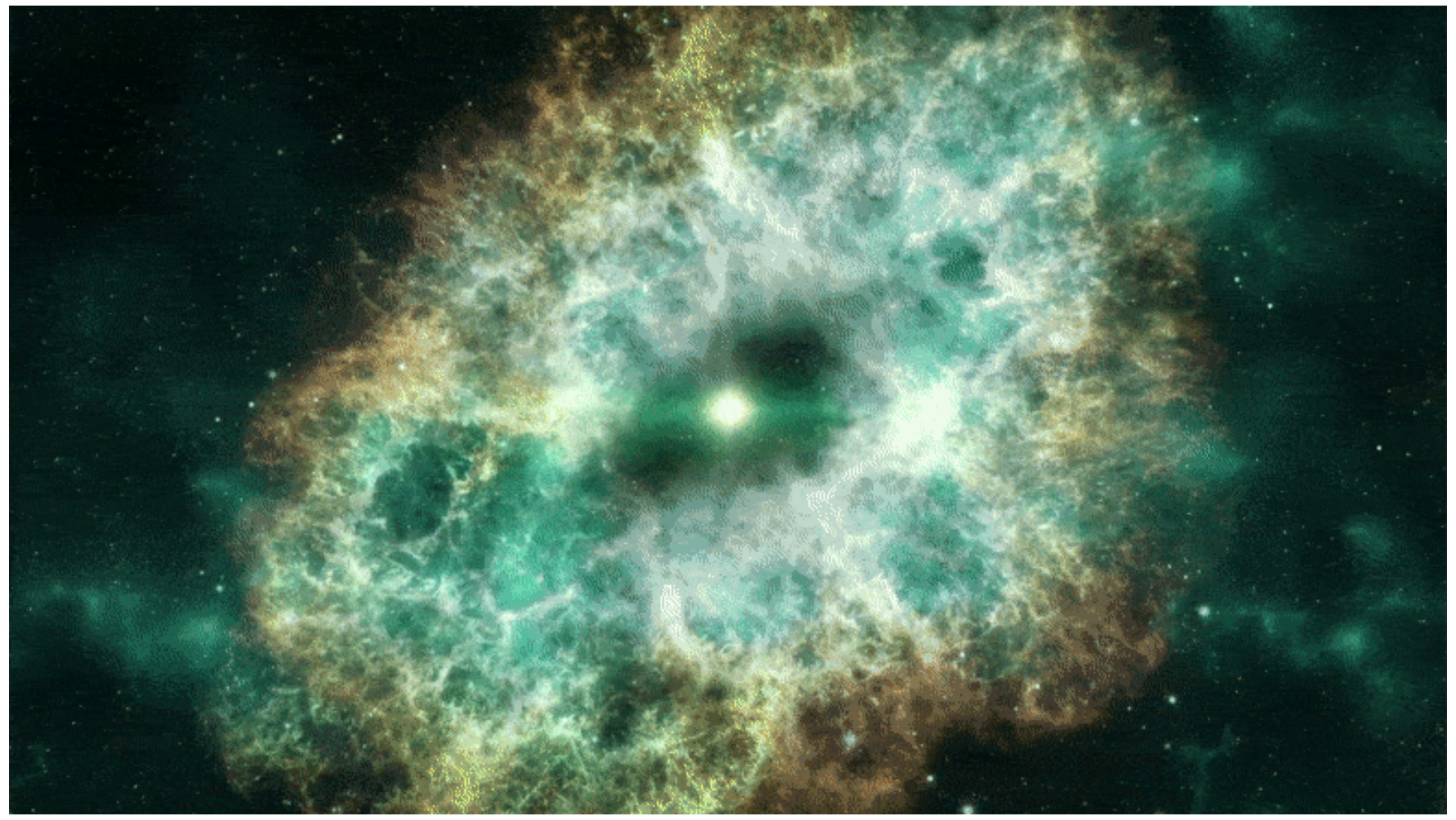
- Compact Binary Coalescence



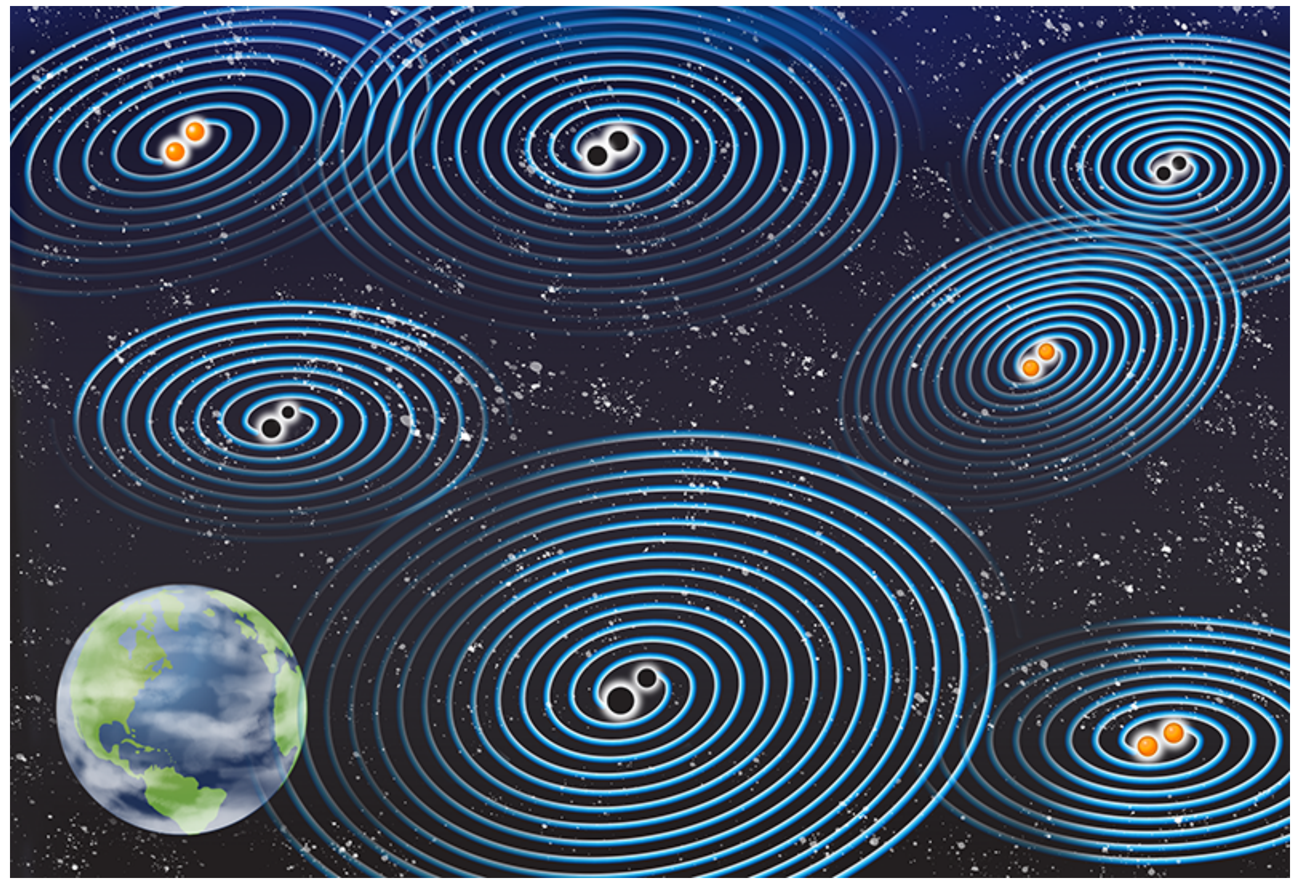
- Continuous



- Burst

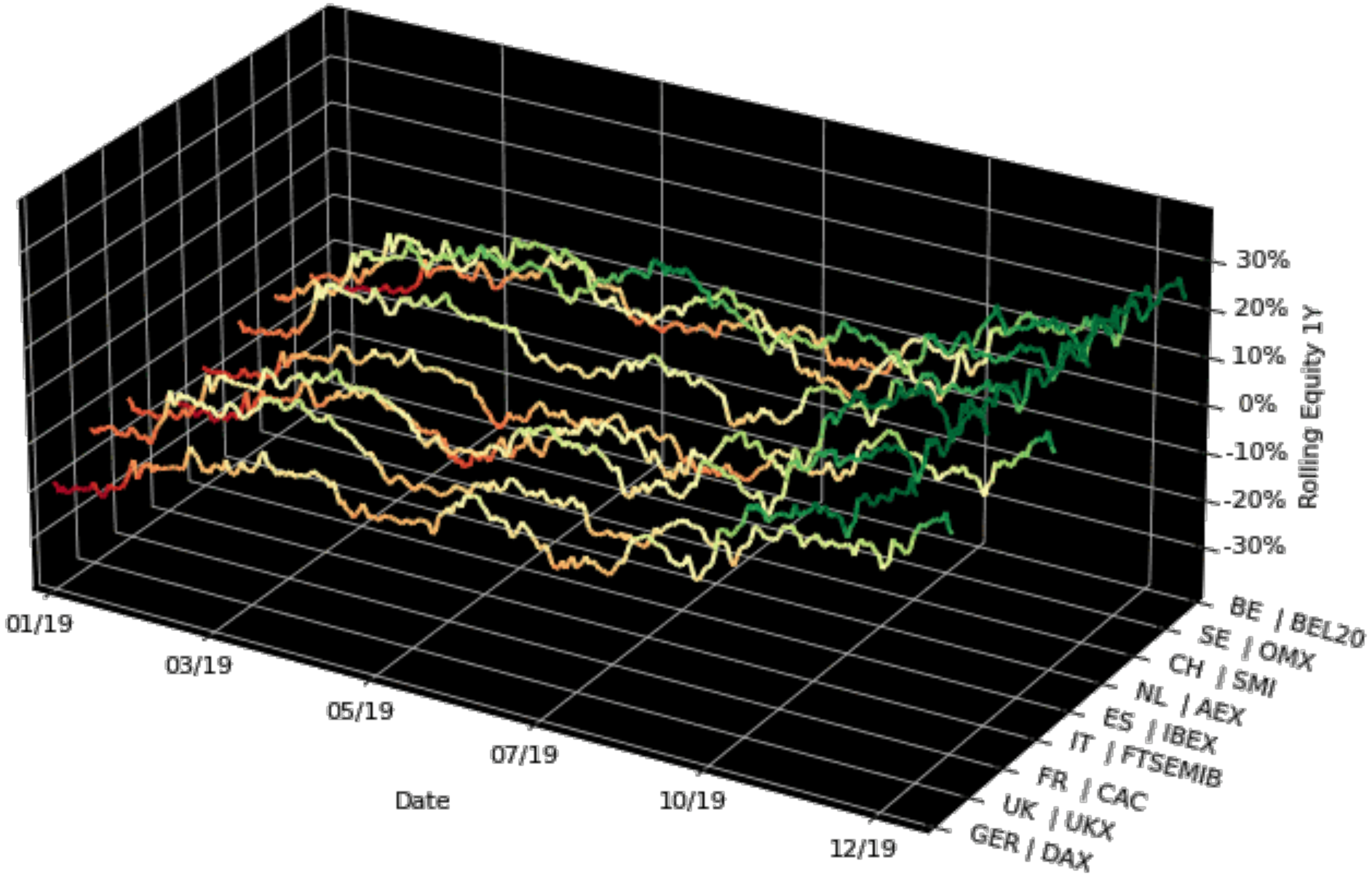


- Stochastic



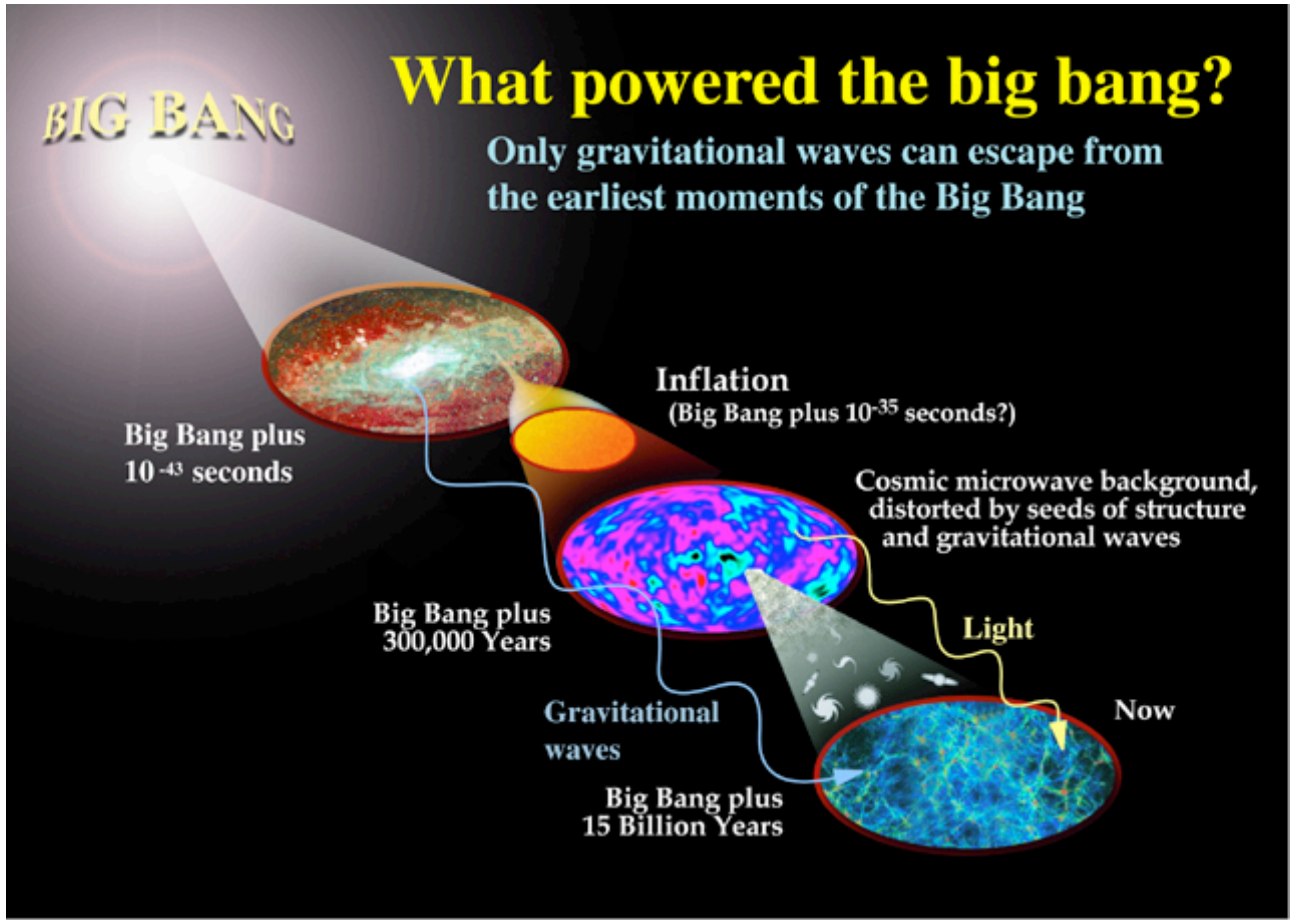
Stochastic GWs (SGWs)

- Stochastic process is described by a random function.
- The metric perturbation $h_{ab}\left(t,\vec{x}\right)$ becomes a statistical random function.



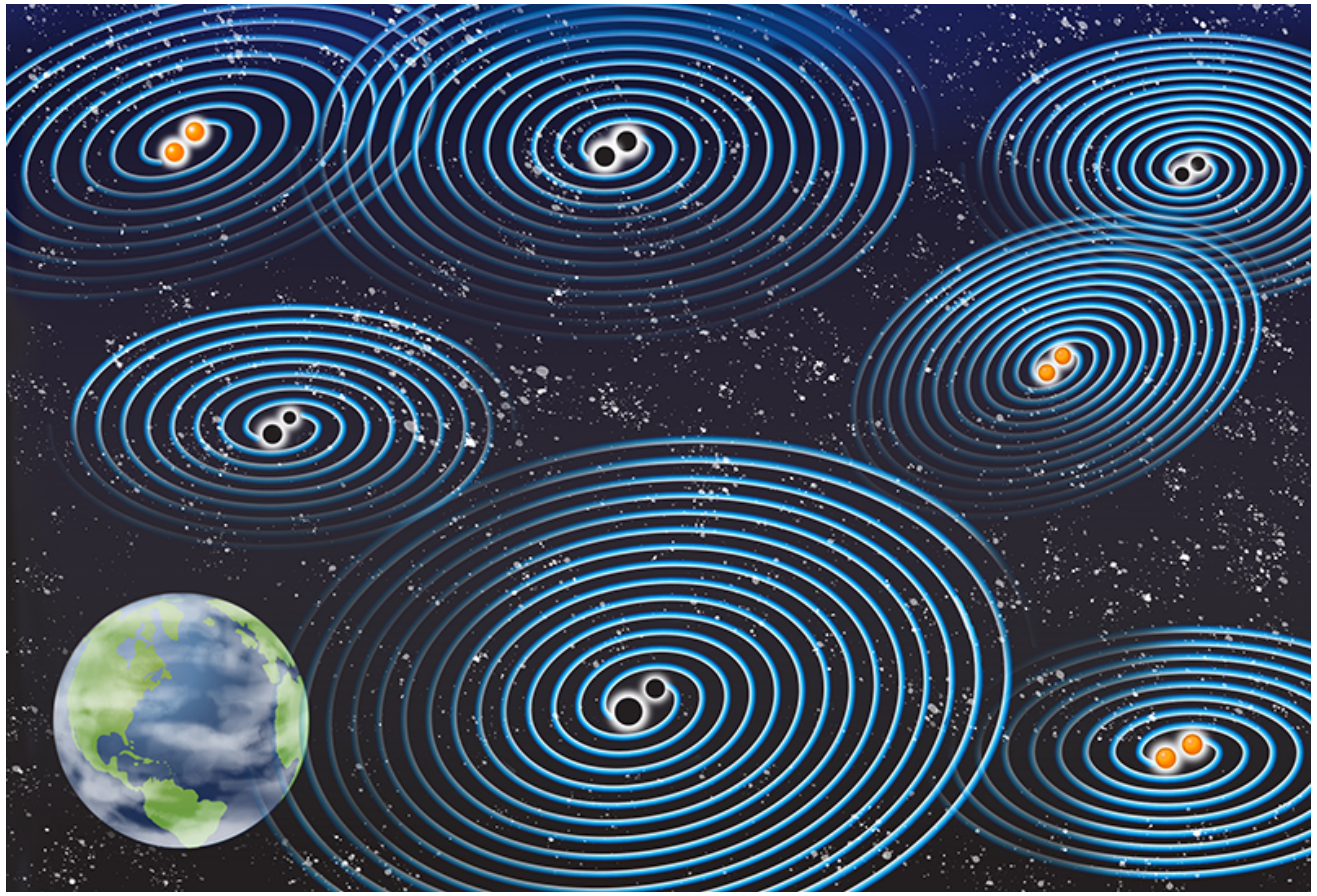
GW Background (GWB)

- Cosmological origin



Quantum state in early universe

- Astrophysical origin

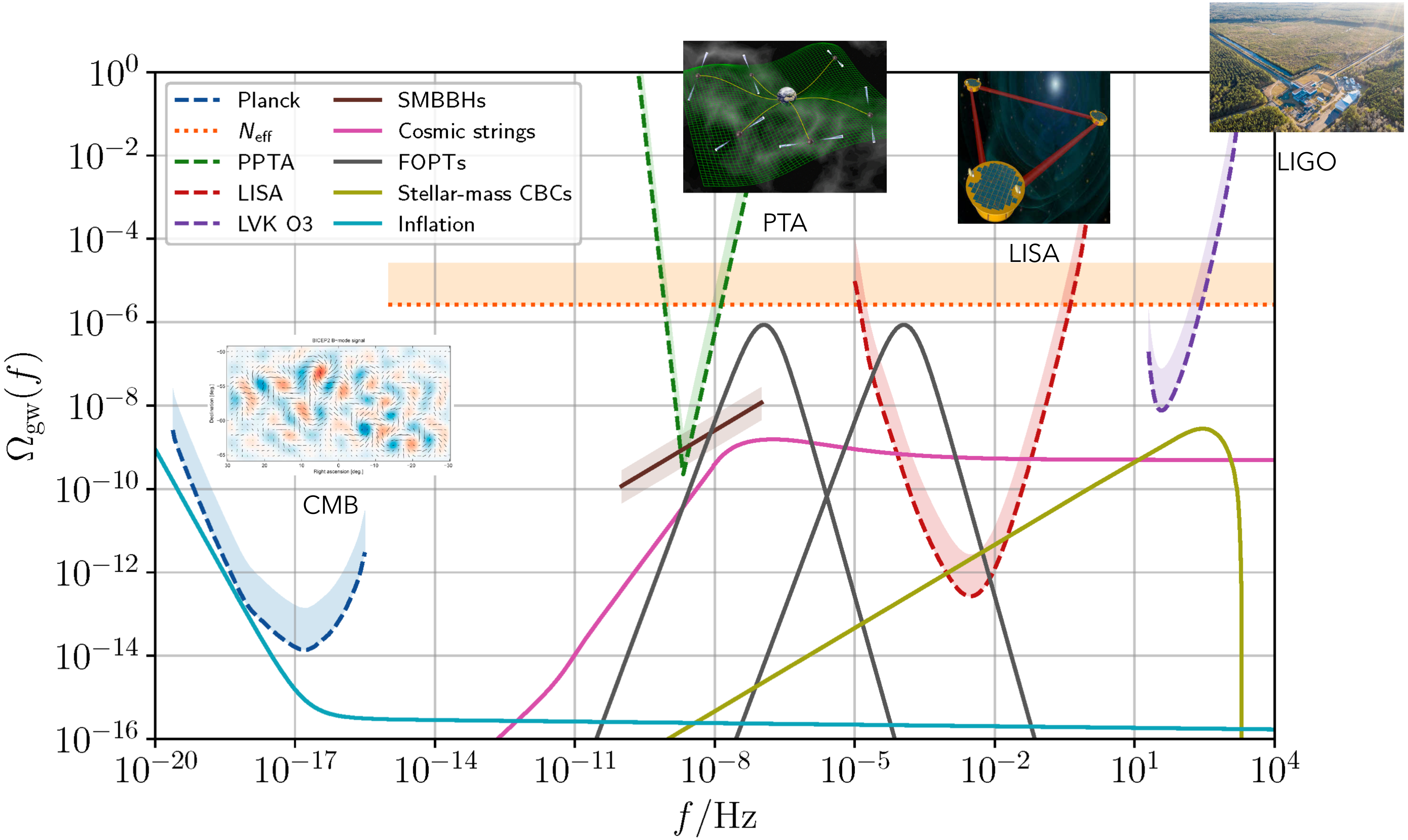


Distribution of compact binaries

Statistical Assumptions on GWB

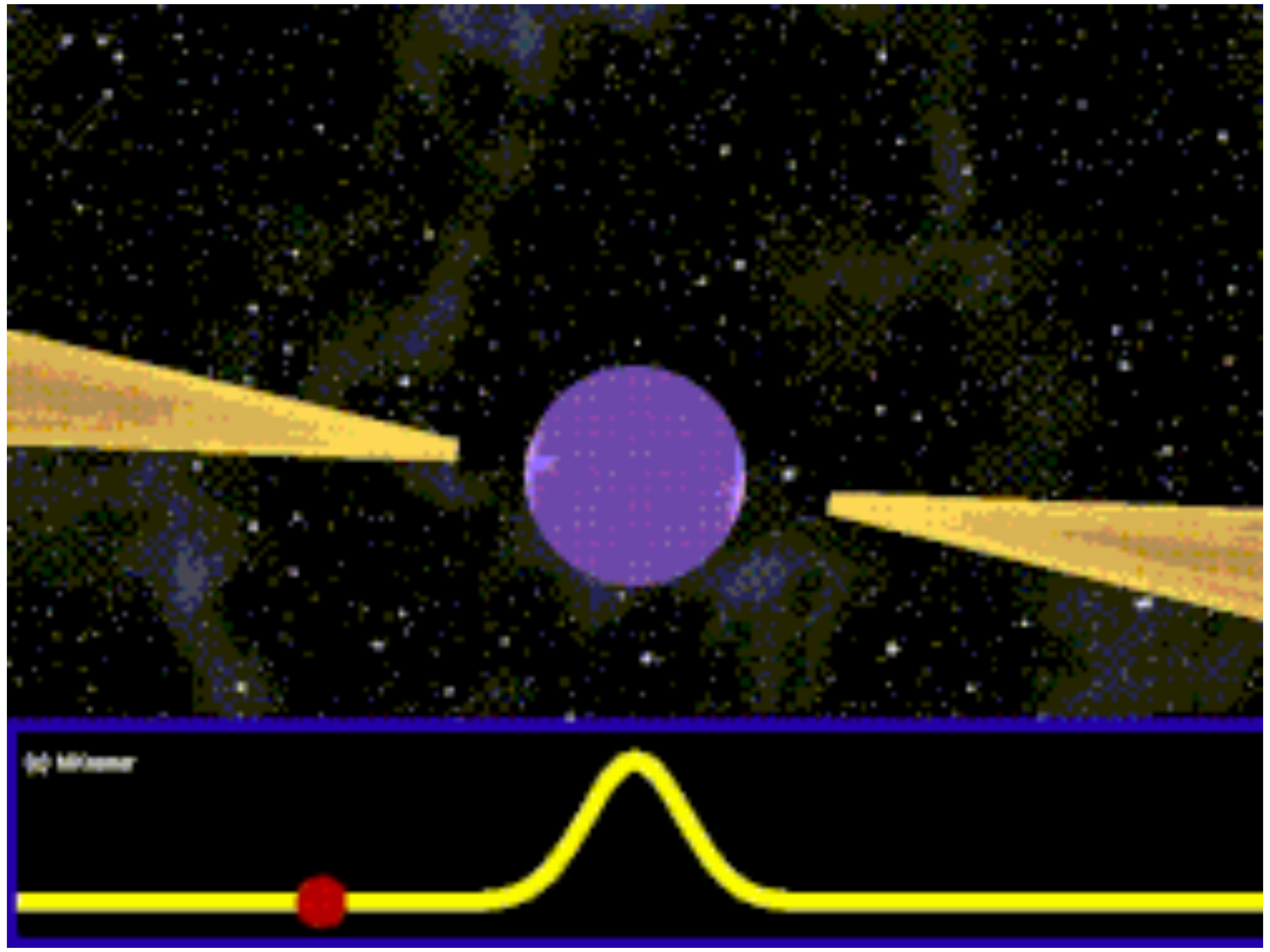
- Gaussian
  - The expectation and correlation (the first and second moments) fully determine all statistical properties.
  - $\left\langle h_{ab}\left(t,\vec{x}\right)\right\rangle = 0$
  - $\left\langle h_{ab}\left(t,\vec{x}\right)h_{cd}\left(t',\vec{x}'\right)\right\rangle = \cdots$
- Stationary and Homogeneous
  - The correlation depends only on the differences in their temporal and spatial separations.
  - $\left\langle h_{ab}\left(t,\vec{x}\right)h_{cd}\left(t',\vec{x}'\right)\right\rangle = R^{h}_{abcd}\left(t'-t,\vec{x}'-\vec{x}\right)$
- Isotropic
  - No directional preference
- Evenly polarized
  - No preferred polarization

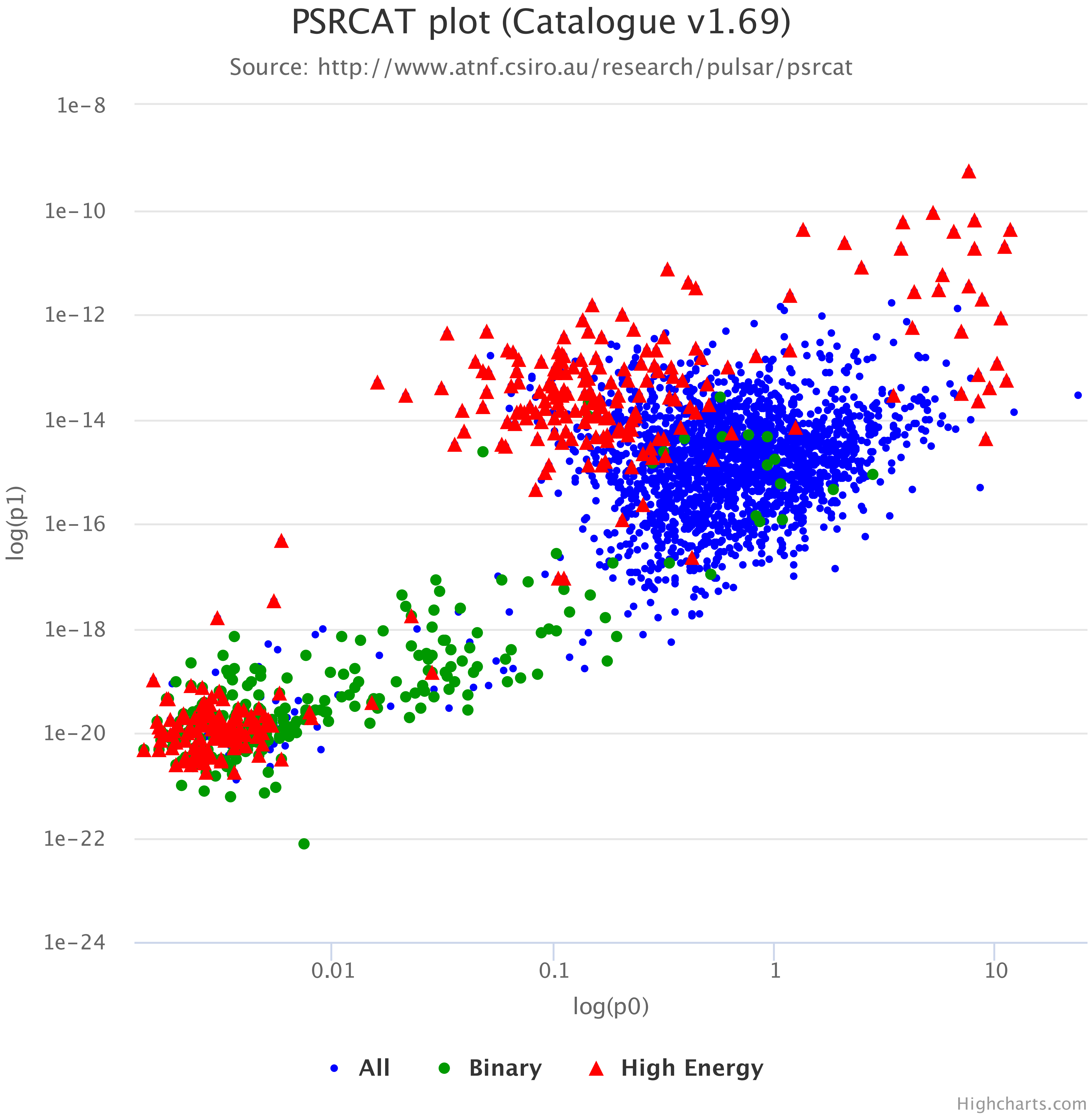
GWB Spectrum



Millisecond Pulsars (MSPs)

- Rotational period < 10 ms
- Extremely stable rotation
- Usually recycled by a companion star

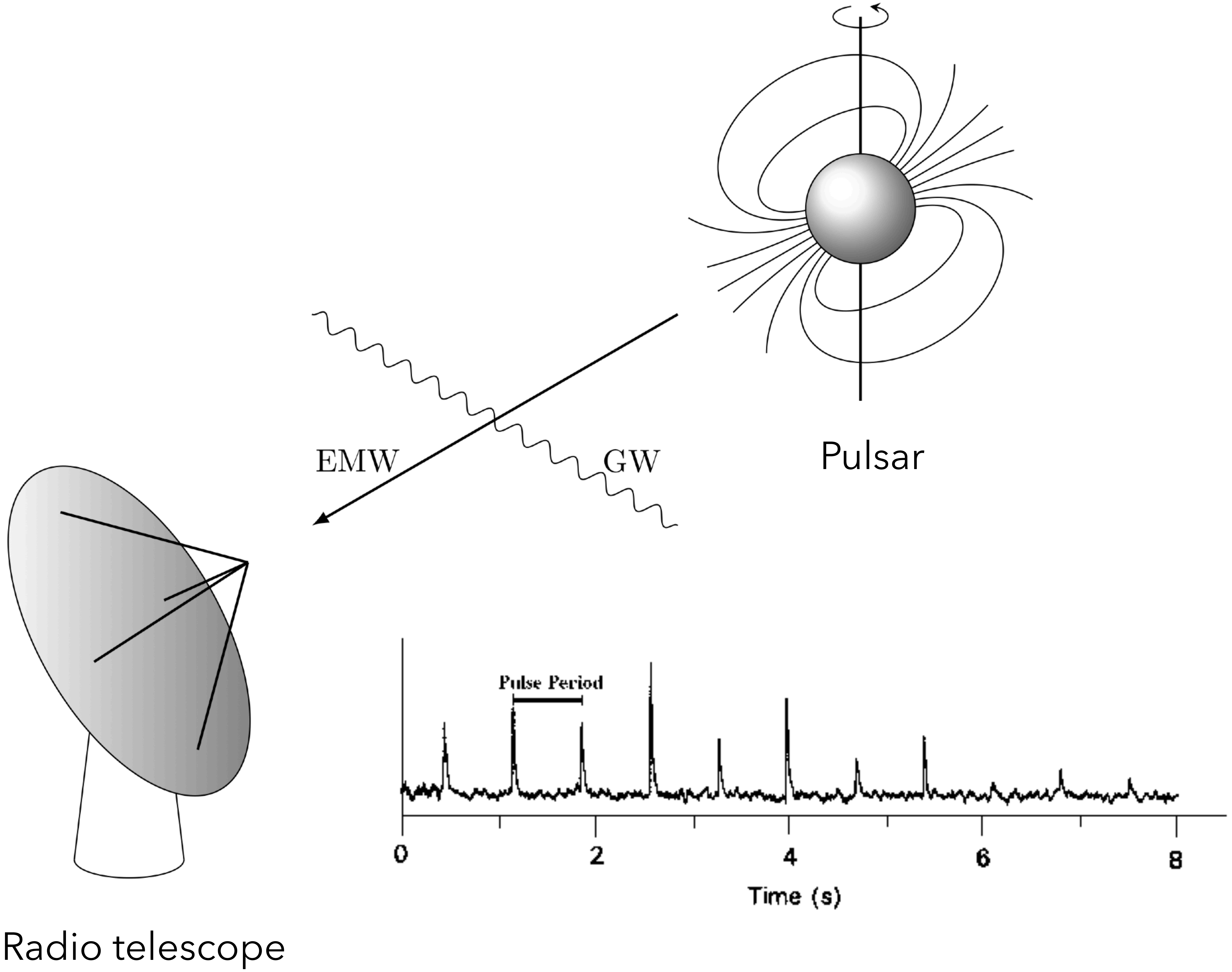




ATNF Pulsar Catalogue

Pulsar Timing

- Electromagnetic pulses are perturbed by GWs.
- Fractional change of the period
  - $\frac{\delta T}{T} \propto h_{ab}$



Properties of Noise

- Detector output
  - $s(t) = h(t) + n(n)$
  - where
  - $h(t) = \frac{\delta T}{T}$: GW signal
- Gaussian and Stationary Noise
  - $\langle n(t) \rangle = 0$
  - $\langle n(t)\, n(t+\tau) \rangle = R_n(\tau)$
- How to distinguish the GW signal and noise?

Random + Random = Random?

Correlation from Two Detector Outputs

- Detectors are far enough apart that their noises are uncorrelated.

$$
\begin{aligned}
\langle s_1 s_2 \rangle &= \langle h_1 h_2 \rangle + \langle h_1 n_2 \rangle + \langle n_1 h_2 \rangle + \langle n_1 n_2 \rangle \\
&= \langle h_1 h_2 \rangle + \langle h_1 \rangle \langle n_2 \rangle + \langle n_1 \rangle \langle h_2 \rangle + \langle n_1 \rangle \langle n_2 \rangle \\
&= \langle h_1 h_2 \rangle
\end{aligned}
$$

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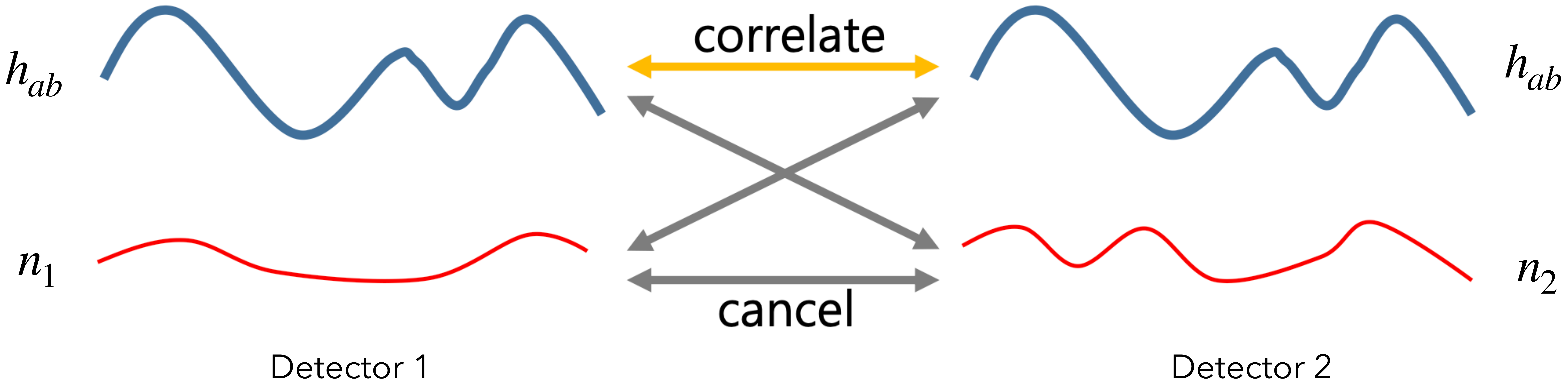
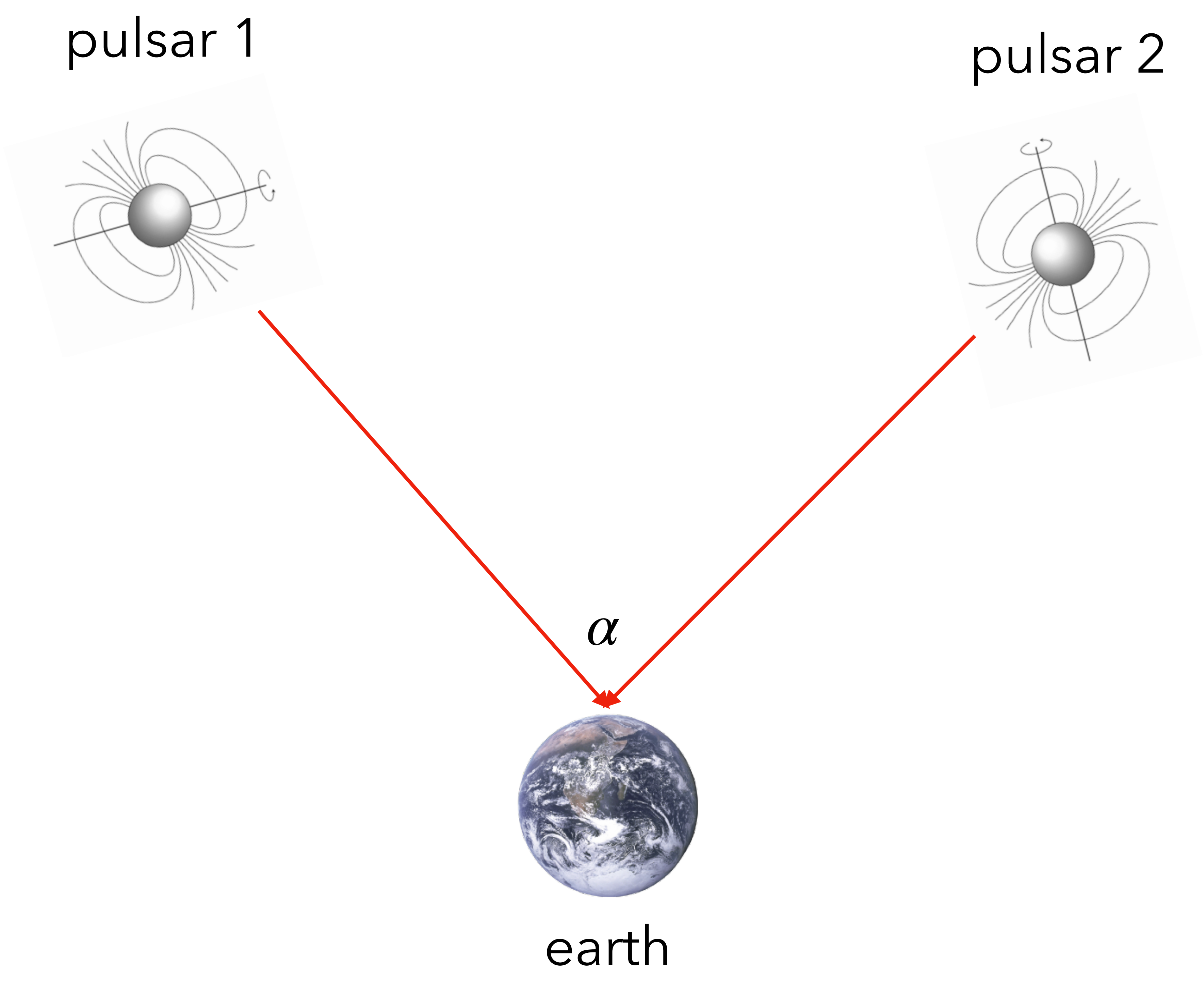
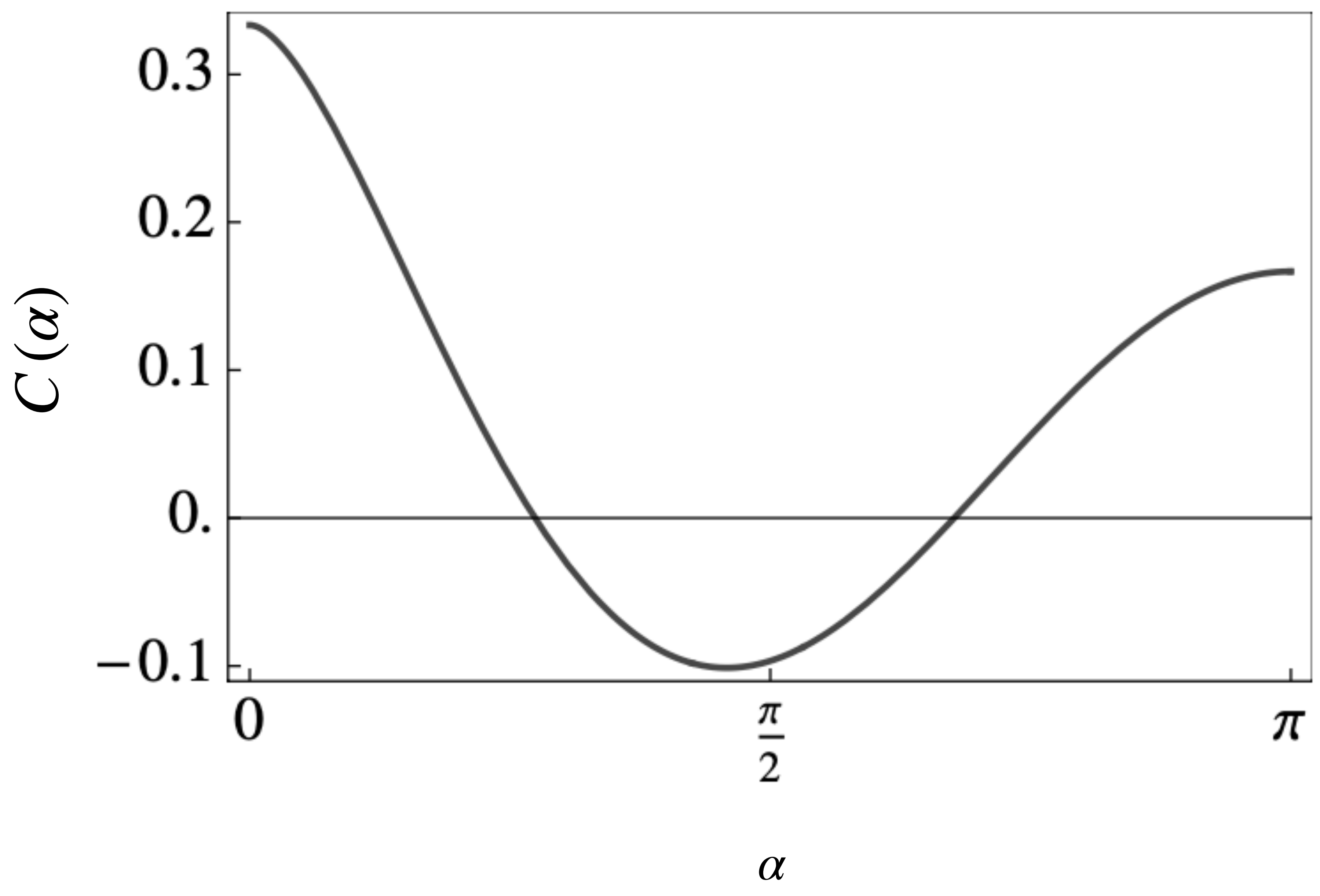


Pulsar Timing Array (PTA)

- Correlation of GW signal from two pulsars
  - $\langle h_1 h_2 \rangle = C(\alpha) \langle h_{ab} h^{ab} \rangle$



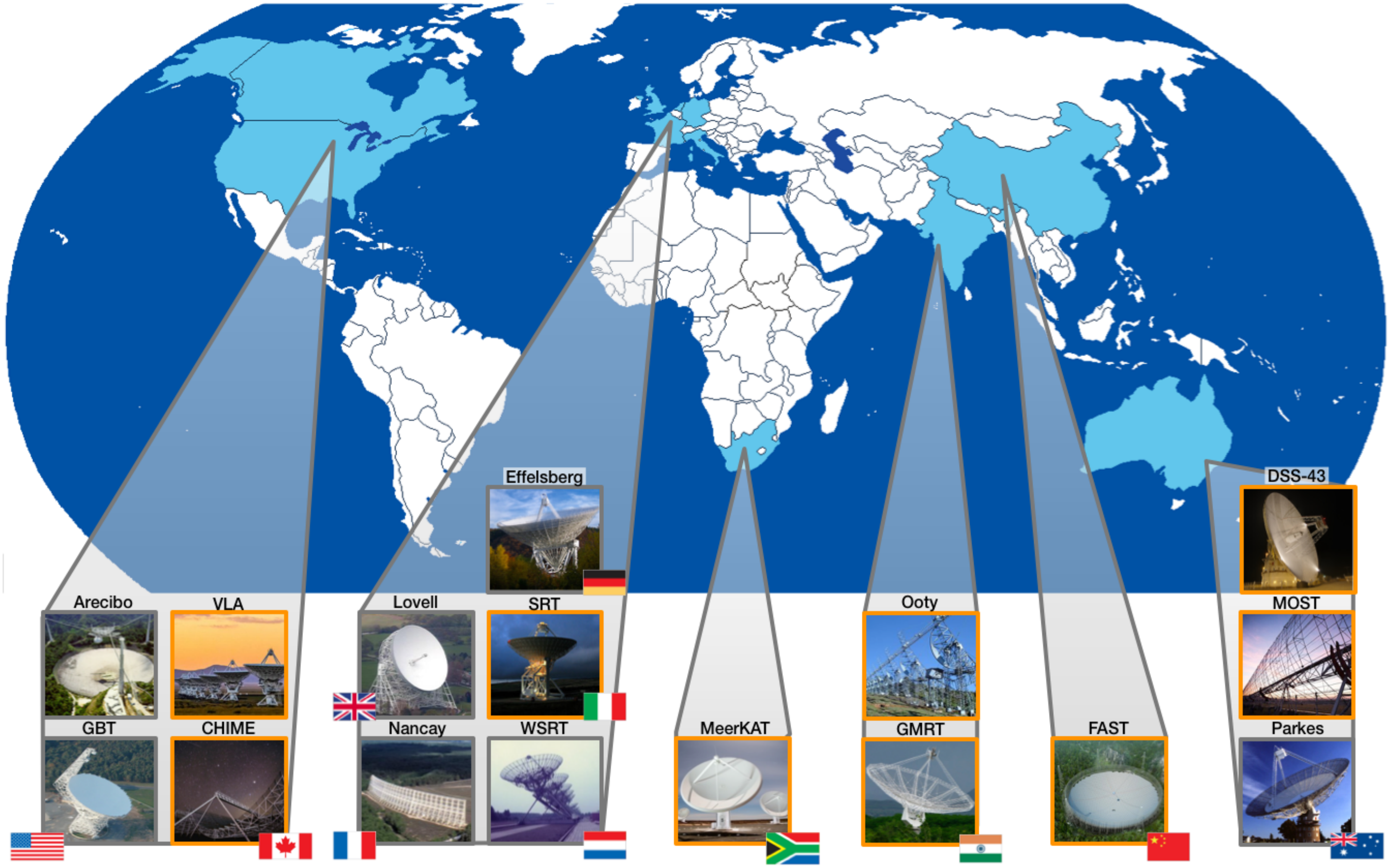
- Signal to Noise Ratio (SNR)
  - $S/N \propto \sqrt{\frac{1}{2} N(N+1) T}$



Hellings and Downs / ApJ (1983)

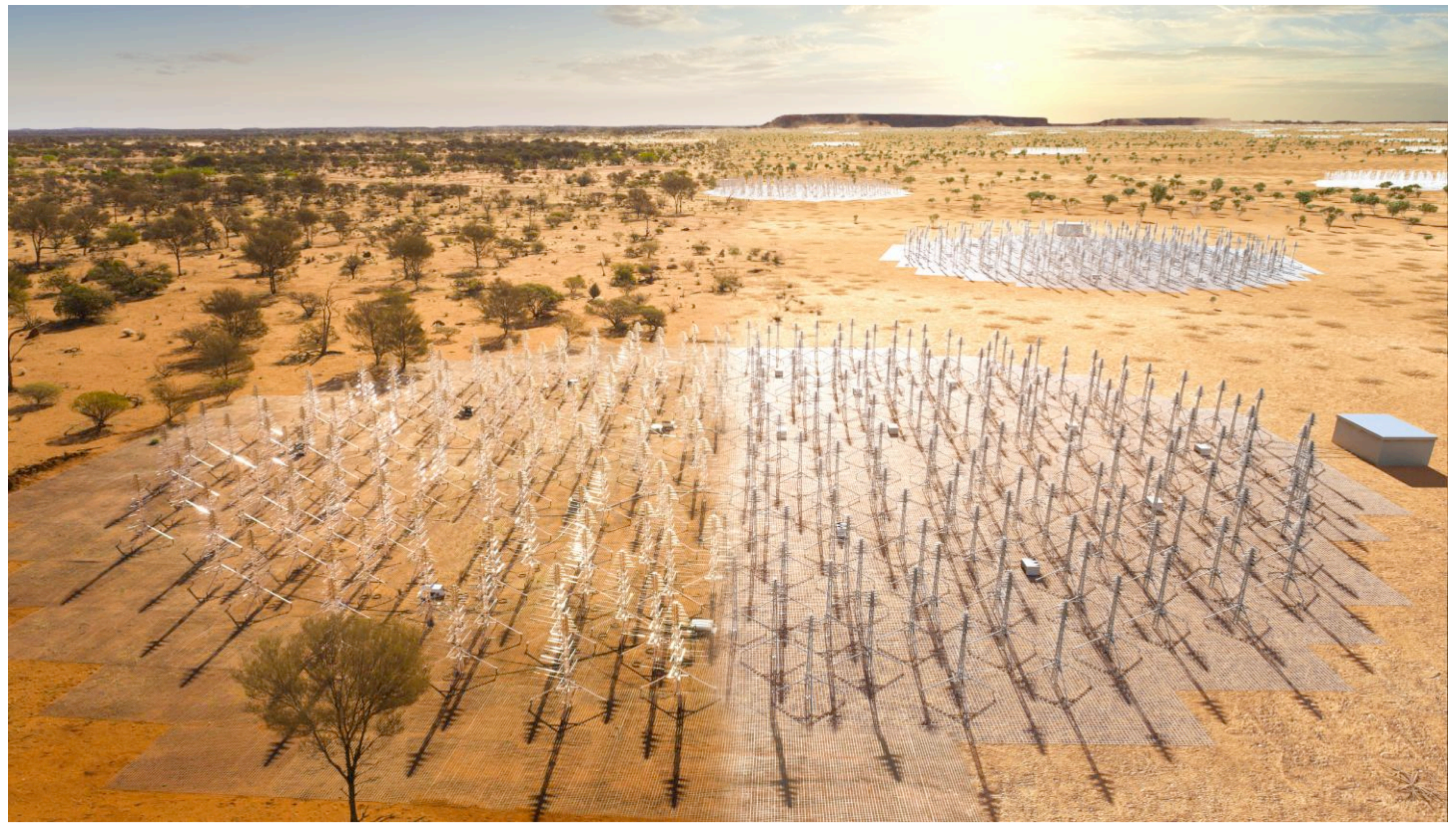
PTAs

- International PTA (IPTA)
  - EPTA+InPTA+NanoGrav+PPTA
- Target GW
  - Supermassive black hole binary (SMBHB)
  - Black hole mass ~ $10^{10}\,M_\odot$
  - frequency ~ $10\,\mathrm{nHz}$
  - GWB from SMBHB



Square Kilometer Array Observatory (SKAO)

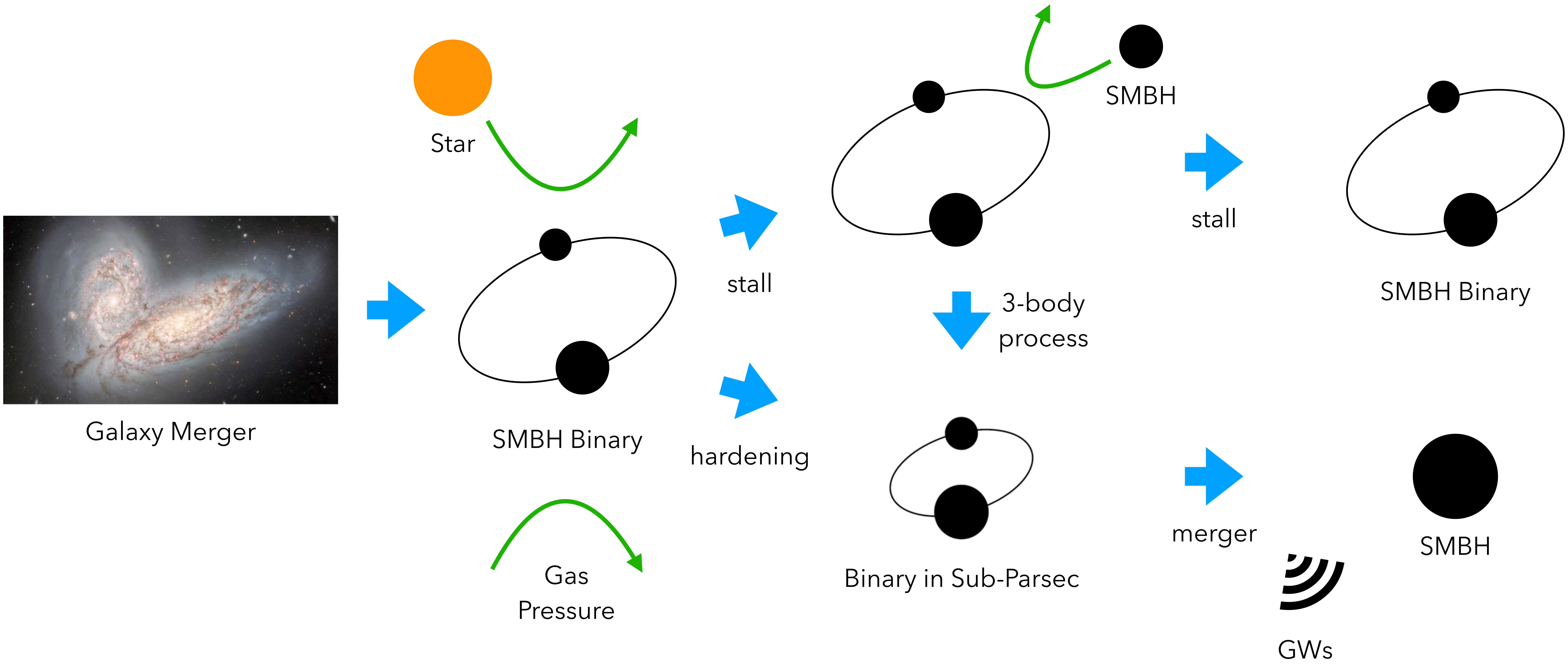
- SKA Low: 50 - 350 MHz / 131,072 antennas
- SKA Mid: 350 MHz - 15.4 GHz / 197 dishes
- Start from 2035



SKA Low / Australia

SKA Mid / South Africa

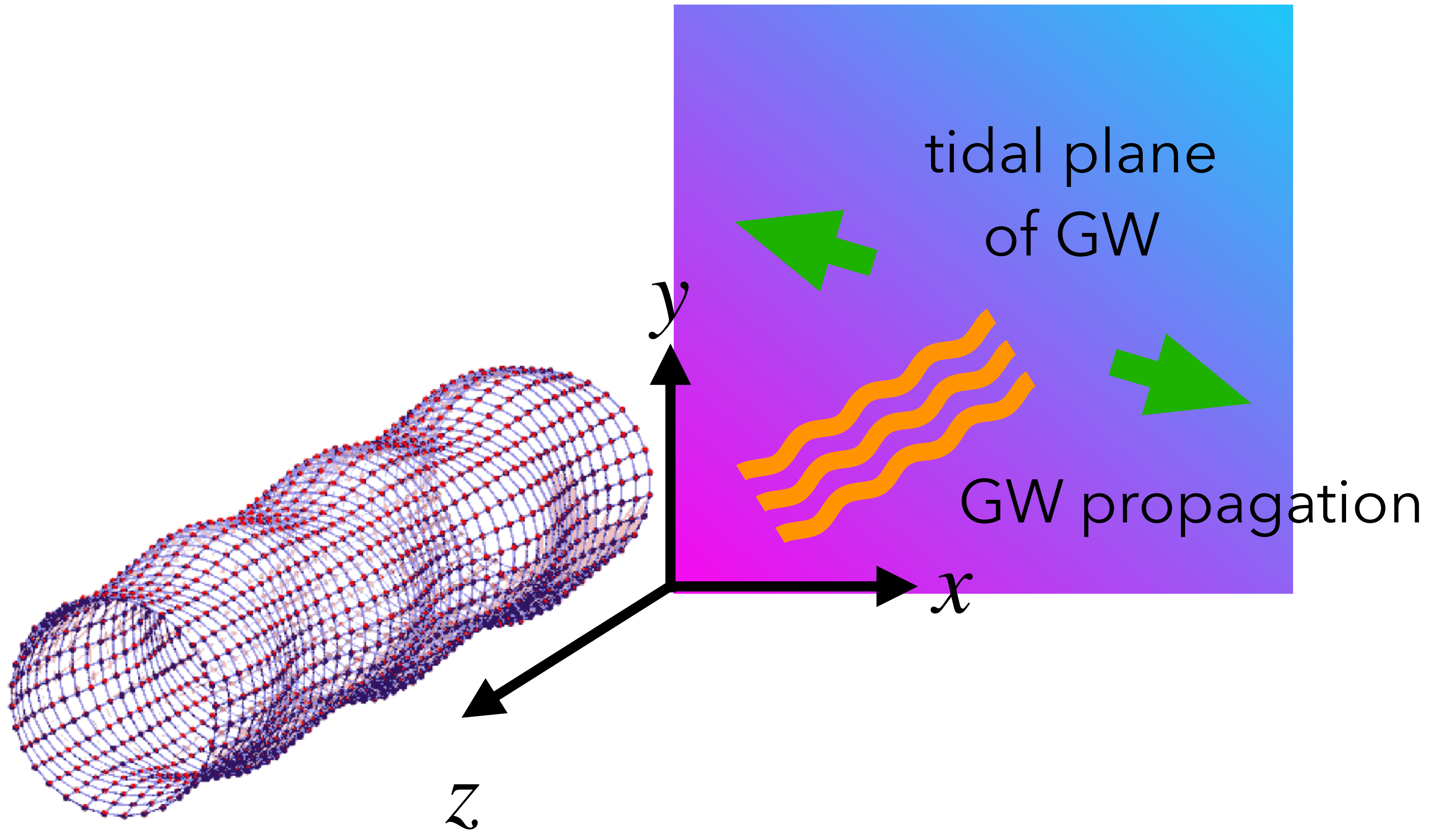
Final Parsec Problem in SMBHB Merger

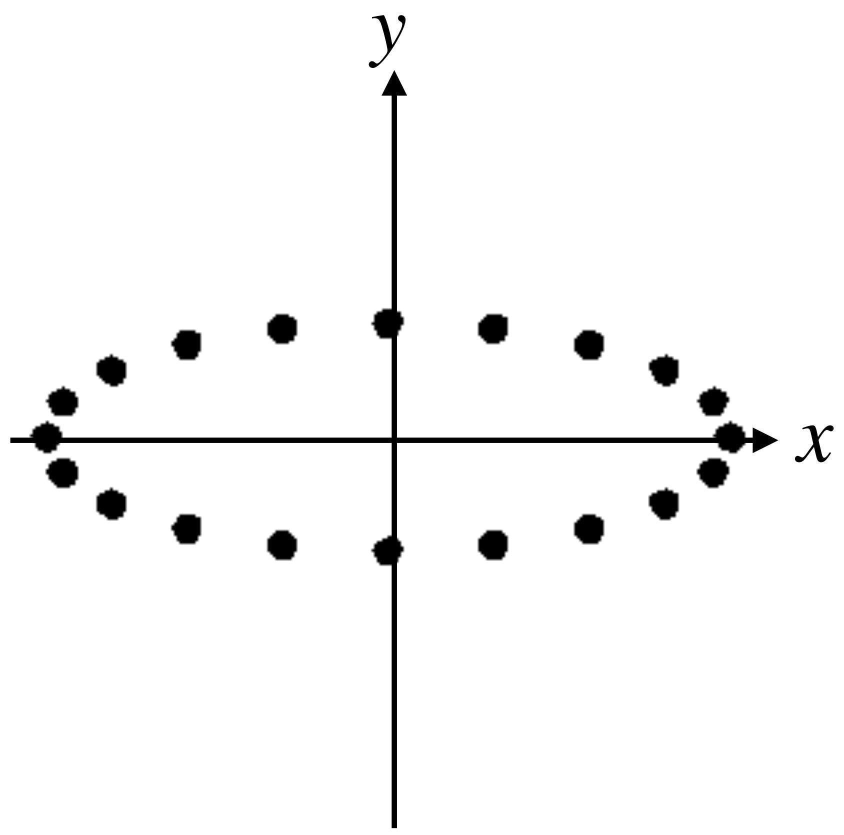


Stochastic Gravitational Waves

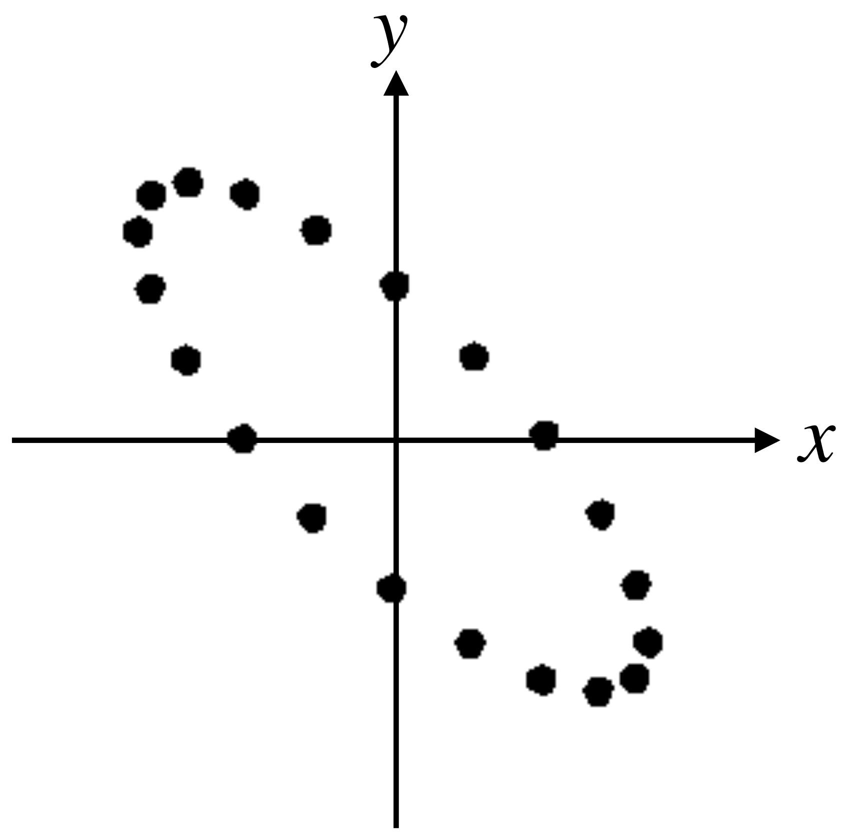
Properties of Gravitational Waves

- Propagation speed: speed of light
- Transverse wave: propagation direction ⊥ tidal direction
- No expansion of tidal plane: GWs do not change the area, but the shape.
- Two polarization modes: plus polarization and cross polarization





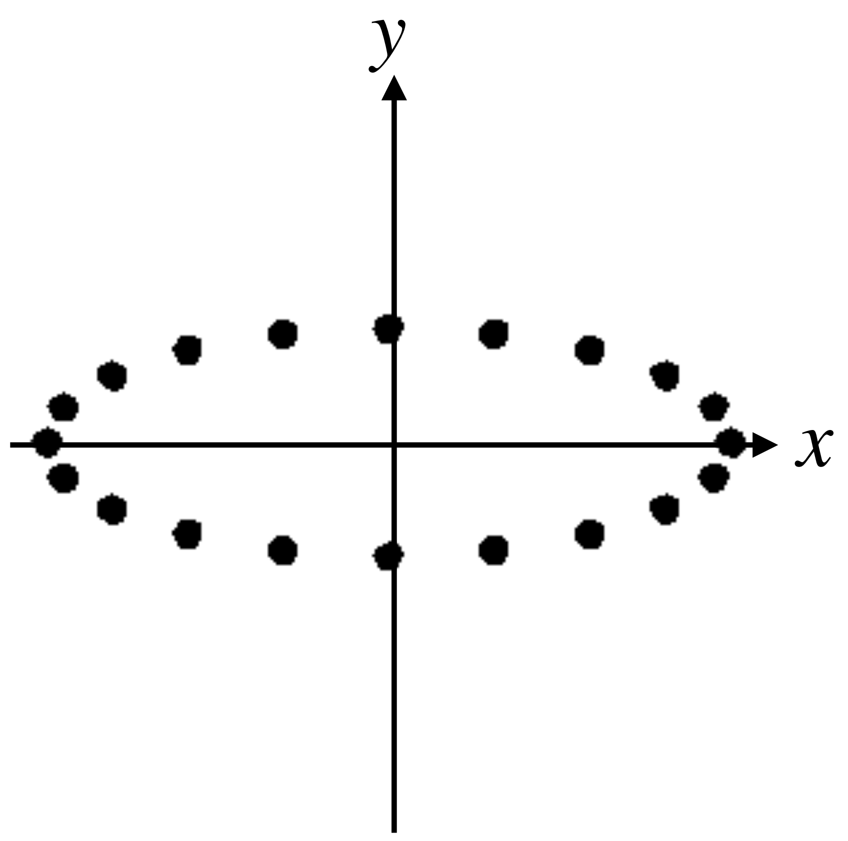
+ polarization



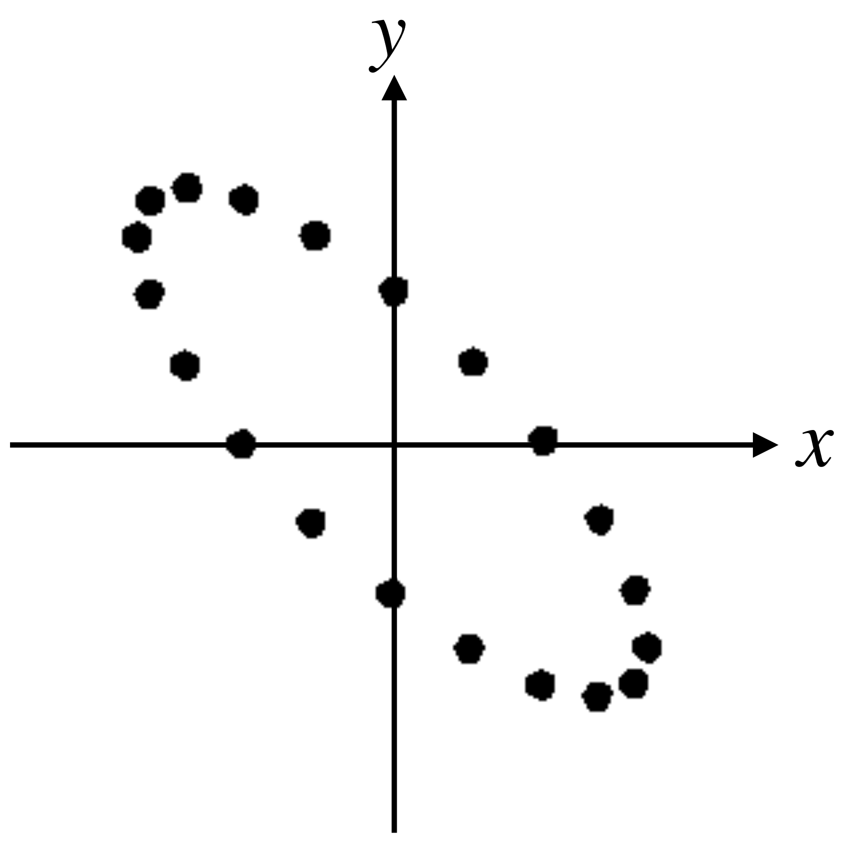
× polarization

Gravitational Waves

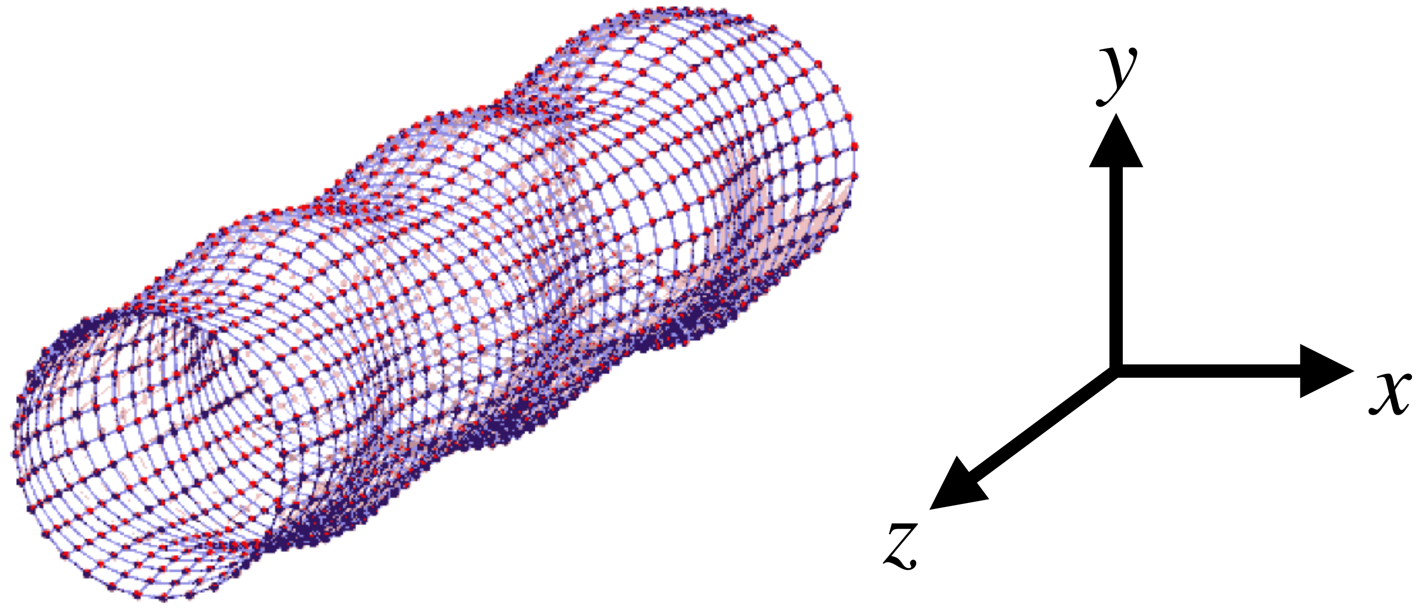
- Let us consider a metric perturbation $h_{\mu\nu} \ll 1$ in a globally inertial coordinate system $\{t, x, y, z\}$
- Monochromatic Plane GWs propagating to $+z$ axis in Transverse-Traceless gauge
- $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cos\{\omega(-t+z)+\phi\}$$



+ polarization



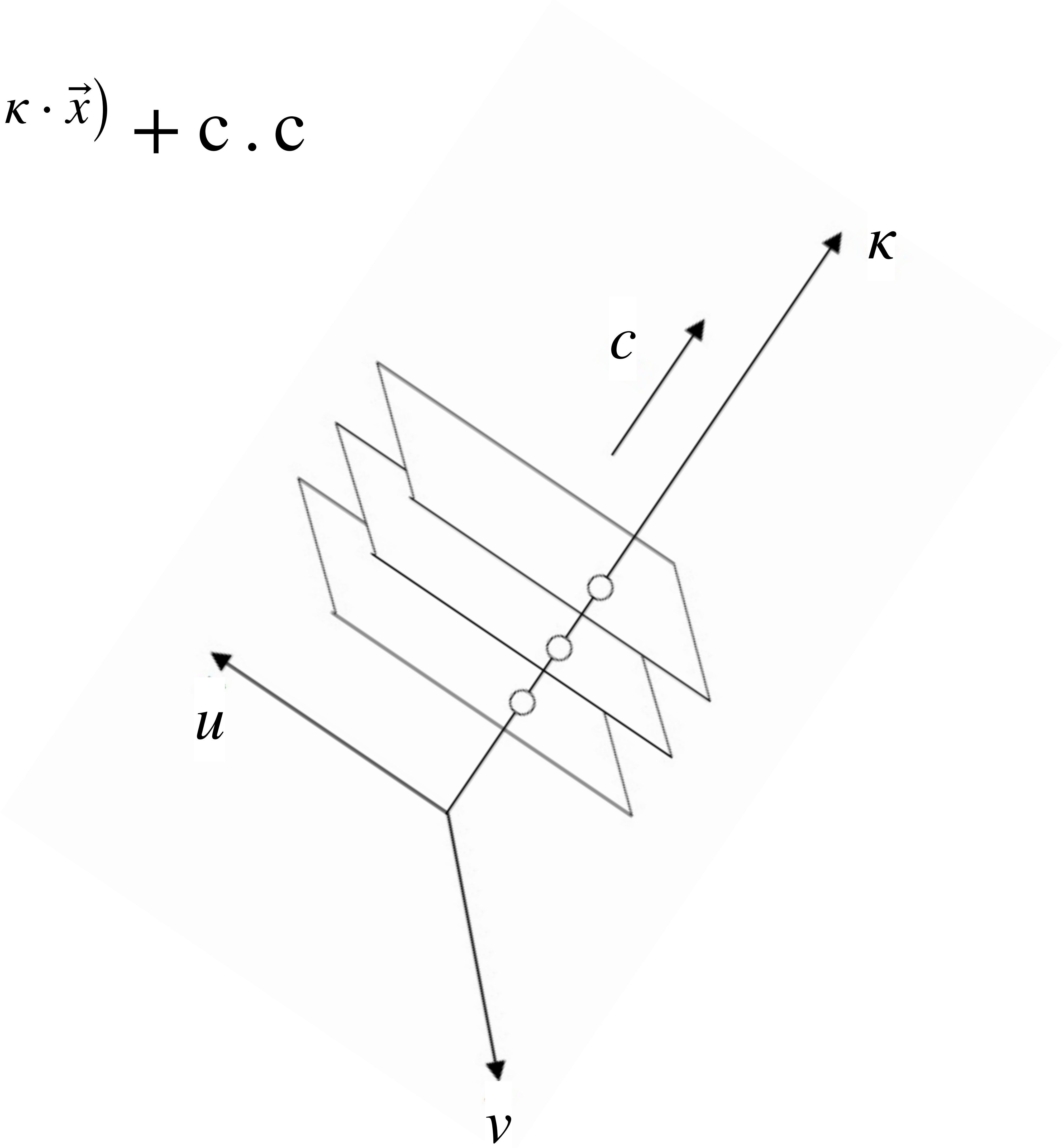
× polarization



propagation of GWs

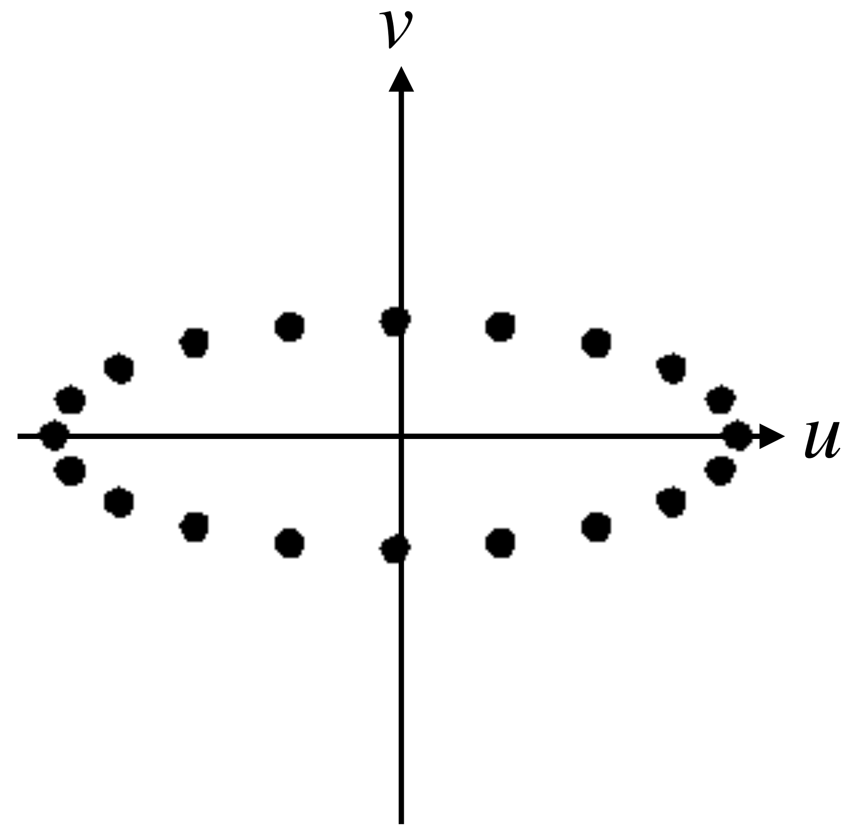
Monochromatic Plane GWs

- $h_{ab}\left(t,\vec{x}\right) = 2\Re\left[\tilde{h}_{ab}e^{i\omega\left(-t+\kappa\cdot\vec{x}\right)}\right] = \tilde{h}_{ab}e^{i\omega\left(-t+\kappa\cdot\vec{x}\right)} + \mathrm{c.c}$
- Complex amplitude: $\tilde{h}_{ab}$
- Angular Frequency: ω
- Unit vector of propagation direction: κ
- Phase: $P\left(t,\vec{x}\right) \equiv \omega\left(-t+\kappa\cdot\vec{x}\right)$
- Transverse-Traceless Gauge condition
  - $\tilde{h}^a{}_a = 0$ $\quad\quad$ $\tilde{h}_{ab}n^b = 0$ $\quad\quad$ $\tilde{h}_{ab}\kappa^b = 0$
  - where $n^a \equiv -g^{ab}\left(dt\right)_b = \left(\partial/\partial t\right)^a$

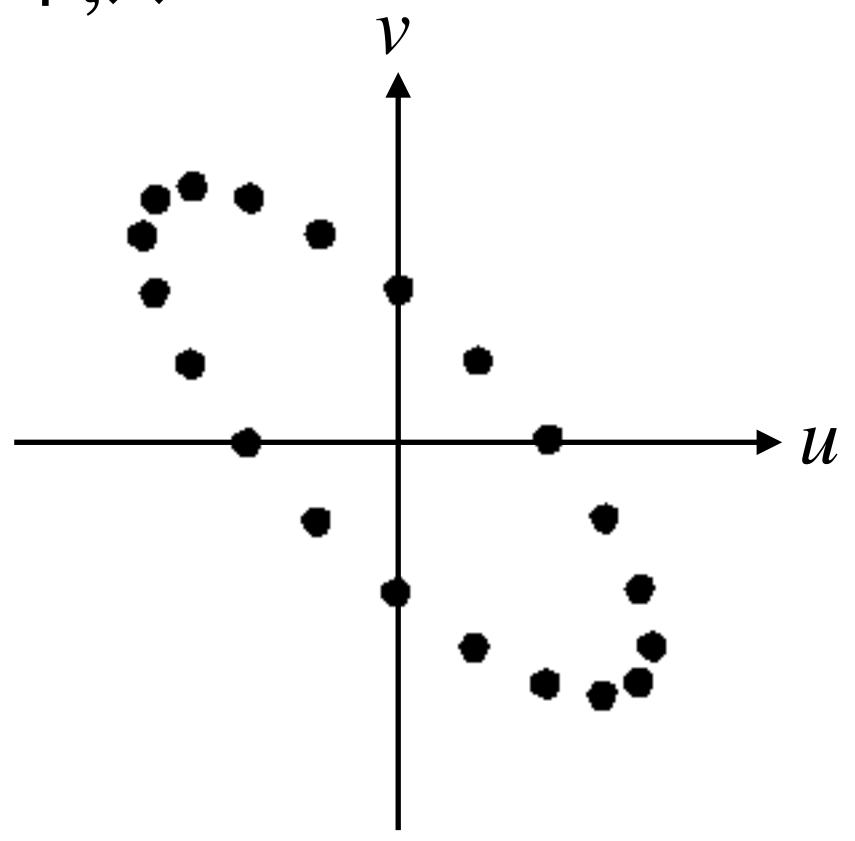


Introducing Orthonormal Basis

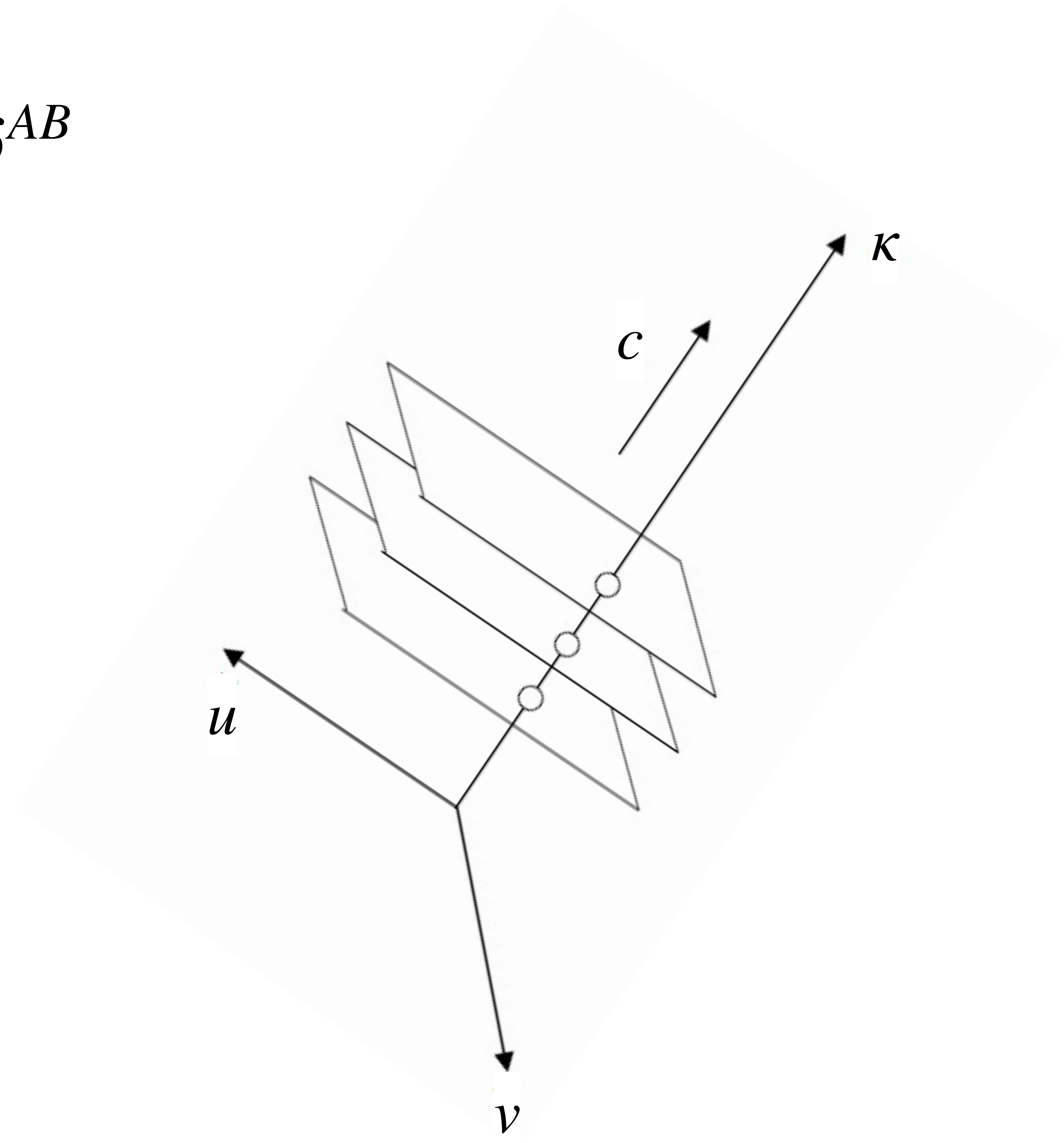
- $e^{+}_{ab} = \frac{1}{\sqrt{2}}\left(u_a u_b - v_a v_b\right)$ s.t. $e^{A}_{ab} e^{B}_{cd} g^{ac} g^{bd} = \delta^{AB}$
- $e^{\times}_{ab} = \frac{1}{\sqrt{2}}\left(u_a v_b + v_a u_b\right)$
- $\tilde{h}_{ab} = \tilde{h}_{+} e^{+}_{ab} + \tilde{h}_{\times} e^{\times}_{ab} = \sum_{A=+,\times} \tilde{h}_A e^{A}_{ab} = \tilde{h}_A e^{A}_{ab}$



+ polarization

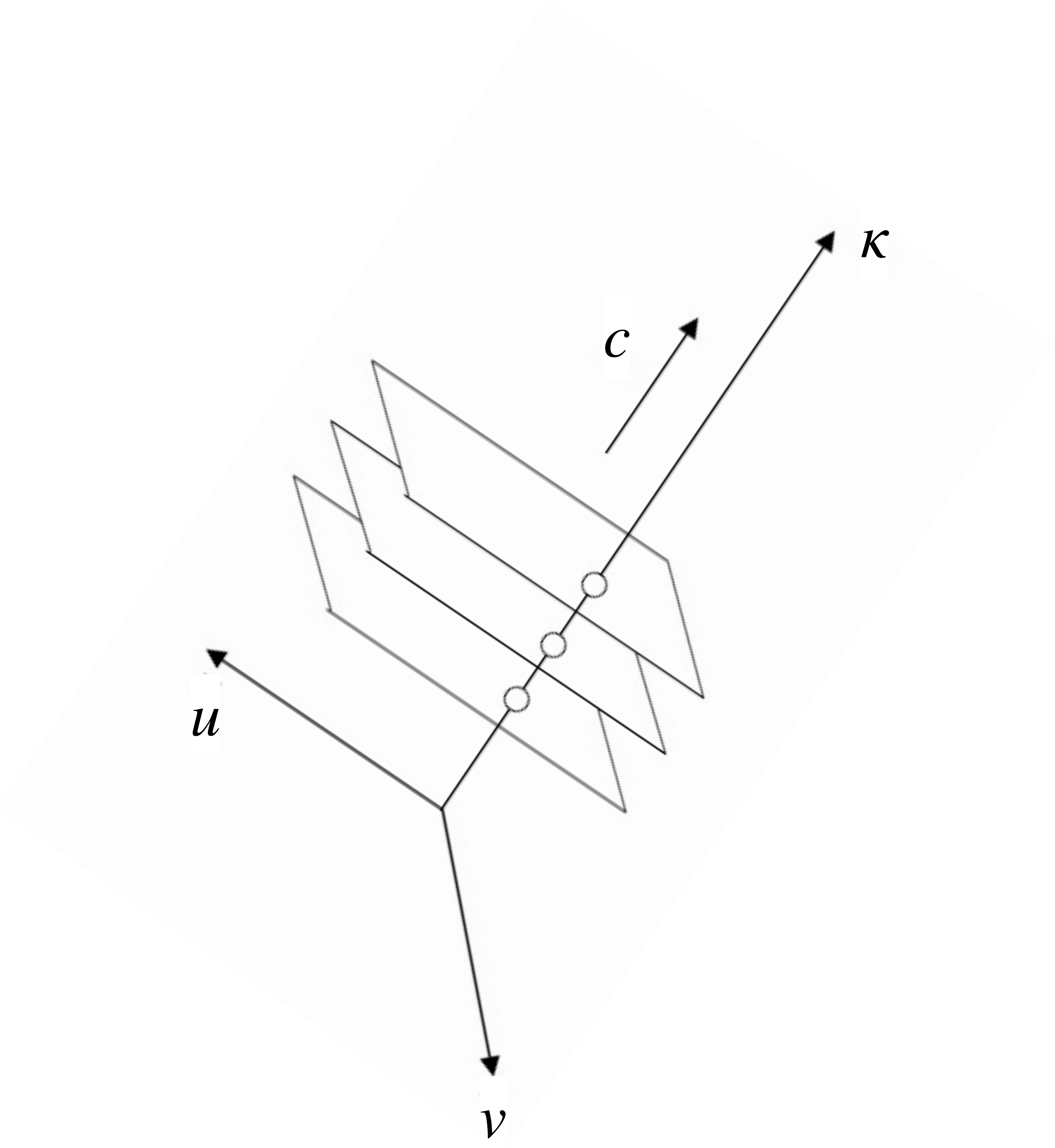


× polarization



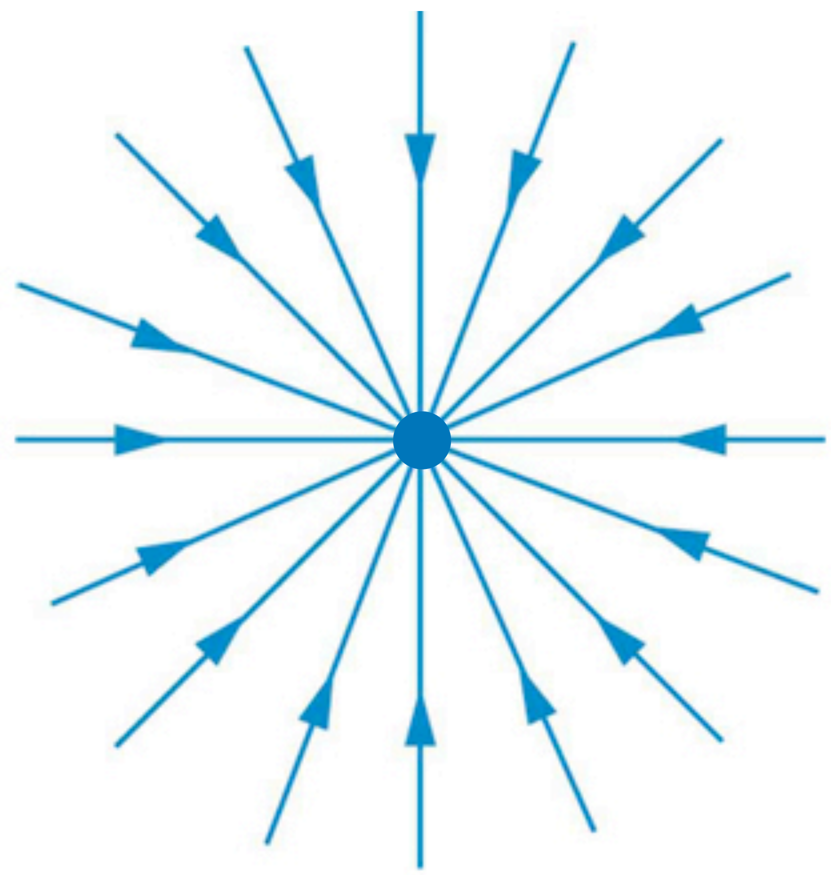
Plane GWs

- $h_{ab}\left(t,\vec{x}\right) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{h}_{ab}(\omega)\, e^{i\omega\left(-t + \kappa \cdot \vec{x}\right)}$
- Complex amplitude: $\tilde{h}_{ab}(\omega) = \tilde{h}_A(\omega)\, e^A_{ab}$
- Note that $\tilde{h}_{ab}(-\omega) = \tilde{h}^*_{ab}(\omega)$



General Gravitational Waves

- $h_{ab}\left(t, \vec{x}\right) = \int d^2\kappa \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{h}_{ab}(\omega, \kappa)\, e^{i\omega\left(-t + \kappa \cdot \vec{x}\right)}$
- Integration over all directions: $\int d^2\kappa = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta$
  - where $\kappa = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$
- Complex amplitude: $\tilde{h}_{ab}(\omega, \kappa) = \tilde{h}_A(\omega, \kappa)\, e^A_{ab}(\kappa)$
- Note that $\tilde{h}_{ab}(-\omega, \kappa) = \tilde{h}^*_{ab}(\omega, \kappa)$
- All information of GWs are encoded in $\tilde{h}_{ab}(\omega, \kappa)$.



All directions

Statistical Assumptions on GWB

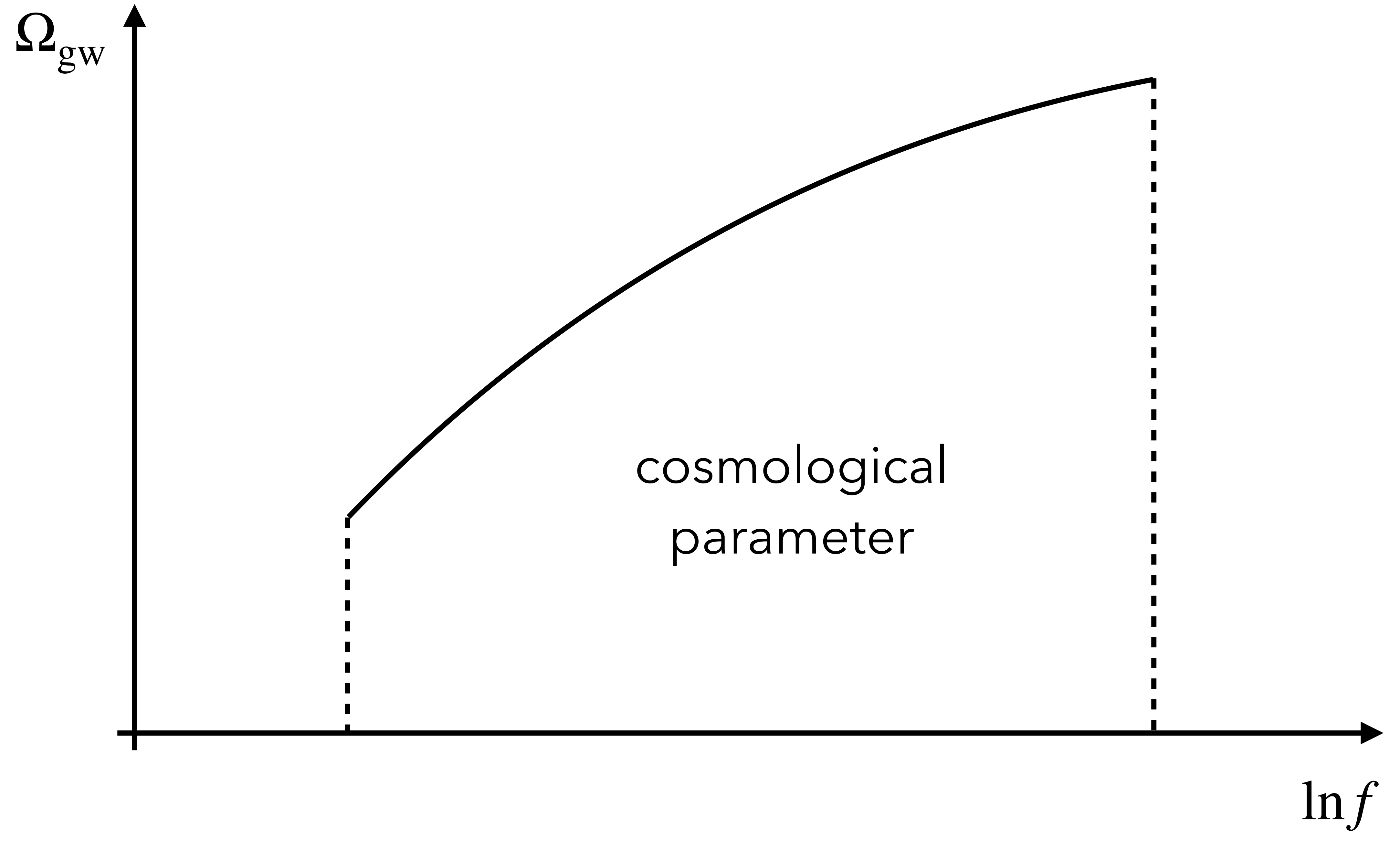
- Gaussian and stationary assumption
  - $\left\langle h_{ab}\left(t,\vec{x}\right)\right\rangle = 0$
  - $\left\langle h_{ab}\left(t,\vec{x}\right)h_{cd}\left(t+\tau,\vec{x}+\vec{y}\right)\right\rangle = R^h_{abcd}\left(\tau,\vec{y}\right)$
  - $R^h_{abcd}\left(\tau,\vec{y}\right) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\kappa\, S^h_{abcd}(\omega,\kappa)\, e^{i\omega\left(-\tau+\kappa\cdot\vec{y}\right)}$
  - R^h: correlation
  - S^h: power spectral density
  - Note that $S^h_{abcd}(\omega,\kappa)$ is real.
- No prefer polarization assumption
  - $S^h_{abcd}(\omega,\kappa) = \frac{1}{2}S_h(\omega,\kappa)\,\Lambda_{abcd}(\kappa)$
  - $\Lambda^{ab}{}_{cd} \equiv P^a{}_{(c}P^b{}_{d)} - \frac{1}{2}P^{ab}P_{cd}$: projection operator for symmetric traceless rank-2 tensors in $u-v$ plane
  - $P^a{}_b = \delta^a{}_b + n^a n_b - \kappa^a\kappa_b$: projection operator for vector to $u-v$ plane
- Isotropic assumption
  - $S_h(\omega,\kappa) = \frac{1}{4\pi}S_h(\omega)$

Statistical Assumptions on GWB

- Final results
  - $\left\langle h_{ab}\left(t,\vec{x}\right)h_{cd}\left(t+\tau,\vec{x}+\vec{y}\right)\right\rangle = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^2\kappa}{4\pi}\frac{1}{2}\Lambda_{abcd}(\kappa)\,S_h(\omega)\,e^{i\omega\left(-\tau+\kappa\cdot\vec{y}\right)}$
  - $\left\langle \tilde{h}_{ab}(\omega,\kappa)\,\tilde{h}^{*}_{cd}(\omega',\kappa')\right\rangle = 2\pi\delta\left(\omega-\omega'\right)\frac{1}{4\pi}\delta^2\left(\kappa-\kappa'\right)\frac{1}{2}\Lambda_{abcd}(\kappa)\,S_h(\omega)$
- All information of GWB are encoded in $S_h(\omega)$
- Amplitude square of GWs
  - $\left\langle h_{ab}h^{ab}\right\rangle = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S_h(\omega)$

"Omega GW" of GWB

- Energy density of GWB
  - $\rho_{gw} = \frac{1}{32\pi}\left\langle \partial_t h_{ab} \partial_t h^{ab} \right\rangle = \frac{1}{32\pi}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\omega^2 S_h(\omega) = \frac{1}{32\pi}\int_0^{\infty} df\,\left(2\pi f\right)^2 S_h^{\text{one-sided}}\left(f\right)$
- Omega GW
  - $\frac{\rho_{\text{gw}}}{\rho_{\text{c}}} = \int_0^{\infty} d\ln f\,\Omega_{\text{gw}}\left(f\right)$
  - $\Omega_{\text{gw}}\left(f\right) = \frac{\pi^2}{3H_0^2} f^3 S_h^{\text{one-sided}}\left(f\right)$
  - where $\rho_c = \frac{3H_0^2}{8\pi}$

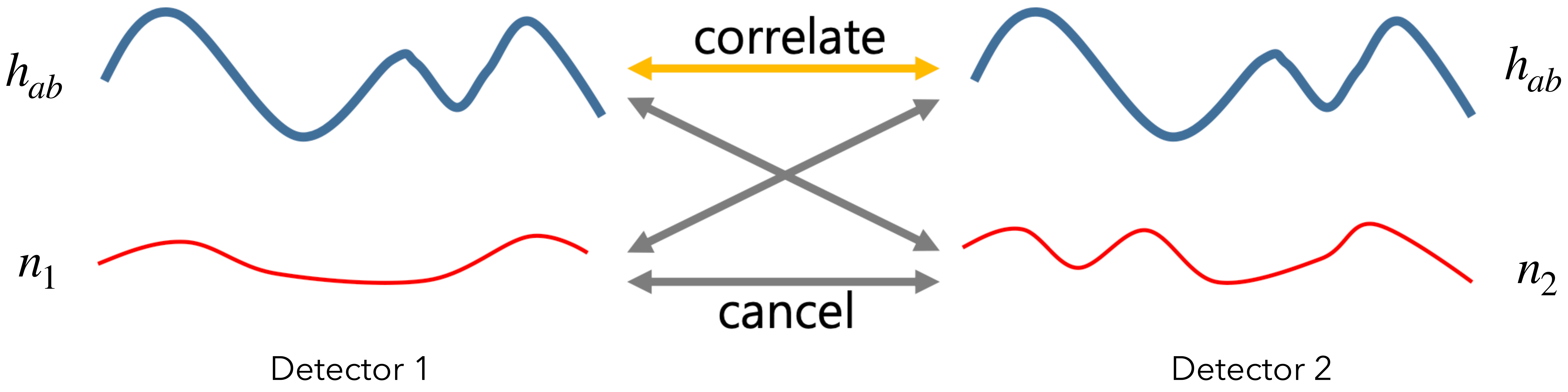


Output, Signal and Noise

- Output
  - $s(t) \equiv h(t) + n(t)$
- GW Signal
  - $h(t) \equiv D^{ab} h_{ab}\left(t, \vec{x}_0\right)$
  - D^{ab}: detector tensor
  - $\vec{x}_0$: detector position
- Gaussian and stationary noise
  - $\langle n(t) \rangle = 0$
  - $\langle n(t)\, n(t+\tau) \rangle = R_n(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_n(\omega)\, e^{-i\omega\tau}$
  - $S_n(\omega)$: noise spectral density
  - Note that $S_n(\omega)$ is real and even function.

Two-detector Correlation Method

- $\langle s_1 s_2 \rangle = \langle h_1 h_2 \rangle + \langle h_1 n_2 \rangle + \langle n_1 h_2 \rangle + \langle n_1 n_2 \rangle = \langle h_1 h_2 \rangle$
- In practice,
  - $\langle h_1 h_2 \rangle = O(T)$
  - other terms $= O\left(\sqrt{T}\right)$



Correlation Measure

- Correlation Measure
  - $Y \equiv \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \, s_1(t)\, s_2(t')\, Q(t'-t)$
  - $Q(t)$: real filter function
- Signal for the Correlation Measure

  $S \equiv \langle Y \rangle$

  $= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \left\langle h_1(t)\, h_2(t') \right\rangle Q(t'-t)$
  - $\simeq T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_h(\omega)\, \tilde{\Gamma}(\omega)\, \tilde{Q}^*(\omega)$
- where
  - $\tilde{\Gamma}(\omega) = \int \frac{d^2\kappa}{4\pi} \frac{1}{2} \Lambda_{abcd}(\kappa)\, D_1^{ab} D_2^{cd} e^{i\omega\kappa\cdot(\vec{x}_1 - \vec{x}_2)}$

  : overlap reduction function

Noise for Correlation Measure

- Noise for the Correlation Measure

$$N^2 = \mathrm{Var}\left([Y]_{h=0}\right) = \left[\left\langle Y^2 \right\rangle - \langle Y \rangle^2\right]_{h=0}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\langle \tilde{n}_1(\omega)\, \tilde{n}_2^*(\omega)\, \tilde{n}_1^*(\omega')\, \tilde{n}_2(\omega') \right\rangle \tilde{Q}(\omega)\, \tilde{Q}^*(\omega') + \left[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\langle \tilde{n}_1(\omega)\, \tilde{n}_2^*(\omega) \right\rangle \tilde{Q}(\omega)\right]^2$$

-

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\langle \tilde{n}_1(\omega)\, \tilde{n}_1^*(\omega') \right\rangle \left\langle \tilde{n}_2(\omega')\, \tilde{n}_2^*(\omega) \right\rangle \tilde{Q}(\omega)\, \tilde{Q}^*(\omega')$$

$$\simeq T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_1^n(\omega)\, S_2^n(\omega) \left|\tilde{Q}(\omega)\right|^2$$

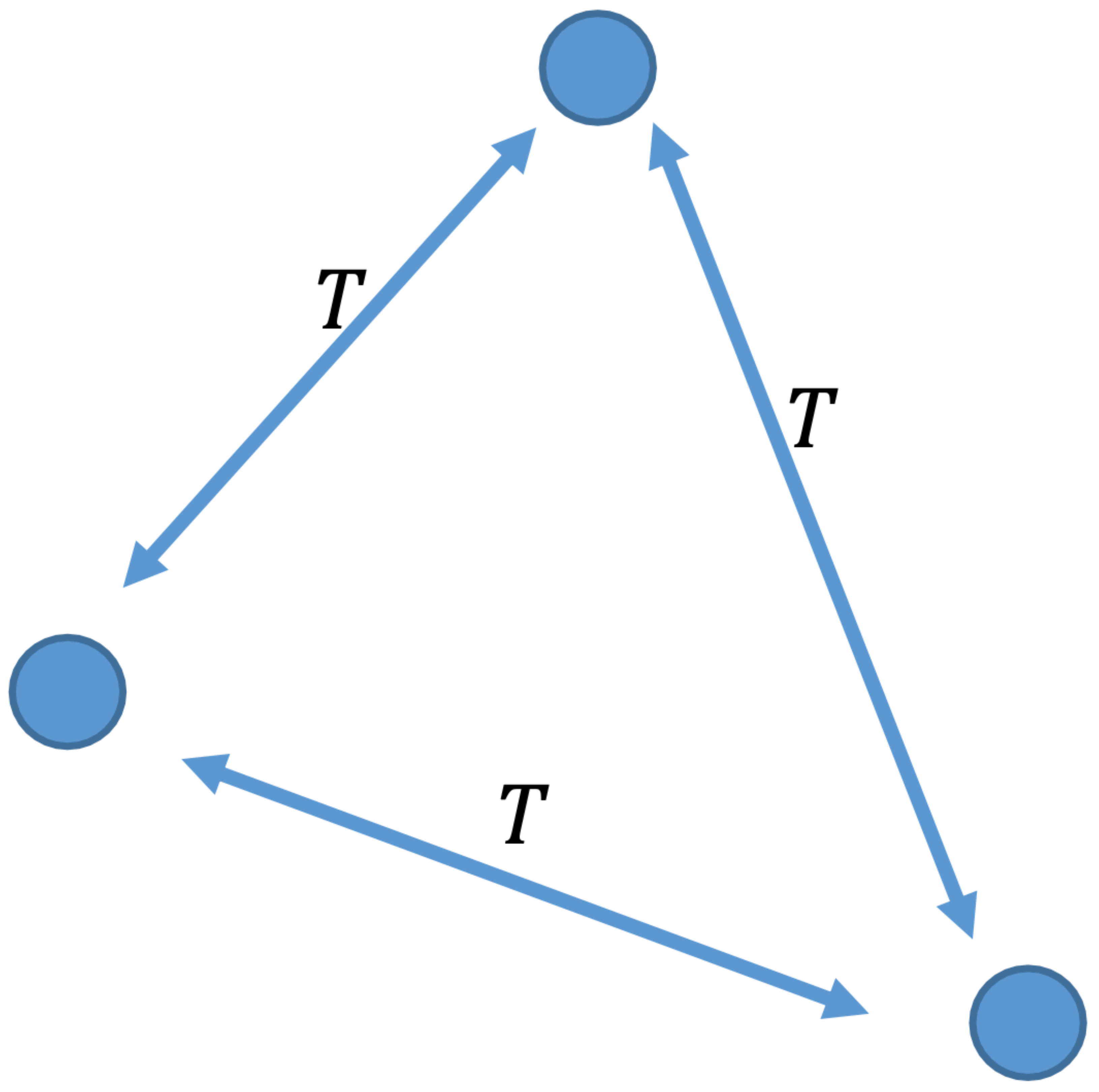
- $S_1^n(\omega)$, $S_2^n(\omega)$: noise spectral densities for detectors

Signal to Noise Ratio (SNR)

- Signal to Noise Ratio for the Correlation Measure
  - $$\frac{S}{N} = \sqrt{T}\frac{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S_h(\omega)\,\tilde{\Gamma}(\omega)\,\tilde{Q}^*(\omega)}{\sqrt{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S_{n,1}(\omega)\,S_{n,2}(\omega)\left|\tilde{Q}(\omega)\right|^2}}$$
- We have to determine $Q(t)$ to maximize SNR.
  - $$\frac{S}{N} = \sqrt{T}\frac{\left\langle \tilde{\Gamma}S_h/S_1^nS_2^n, \tilde{Q}\right\rangle}{\sqrt{\left\langle \tilde{Q}, \tilde{Q}\right\rangle}}$$
  - where
    $$\left\langle \tilde{A}, \tilde{B}\right\rangle = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S_1^n(\omega)\,S_2^n(\omega)\,\tilde{A}(\omega)\,\tilde{B}^*(\omega)$$
- SNR is maximized when $\tilde{Q} \propto \tilde{\Gamma}S_h/S_1^nS_2^n$
- Then, maximal SNR is
  - $$\frac{S}{N} = \sqrt{T}\left[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{S_h^2(\omega)}{S_1^n(\omega)\,S_2^n(\omega)}\left|\tilde{\Gamma}(\omega)\right|^2\right]^{1/2}$$

N-detector network

- Total measurement time: $T_N = \frac{N(N-1)}{2}T$



Pulsar Timing Array

Electromagnetism in Curved Spacetime

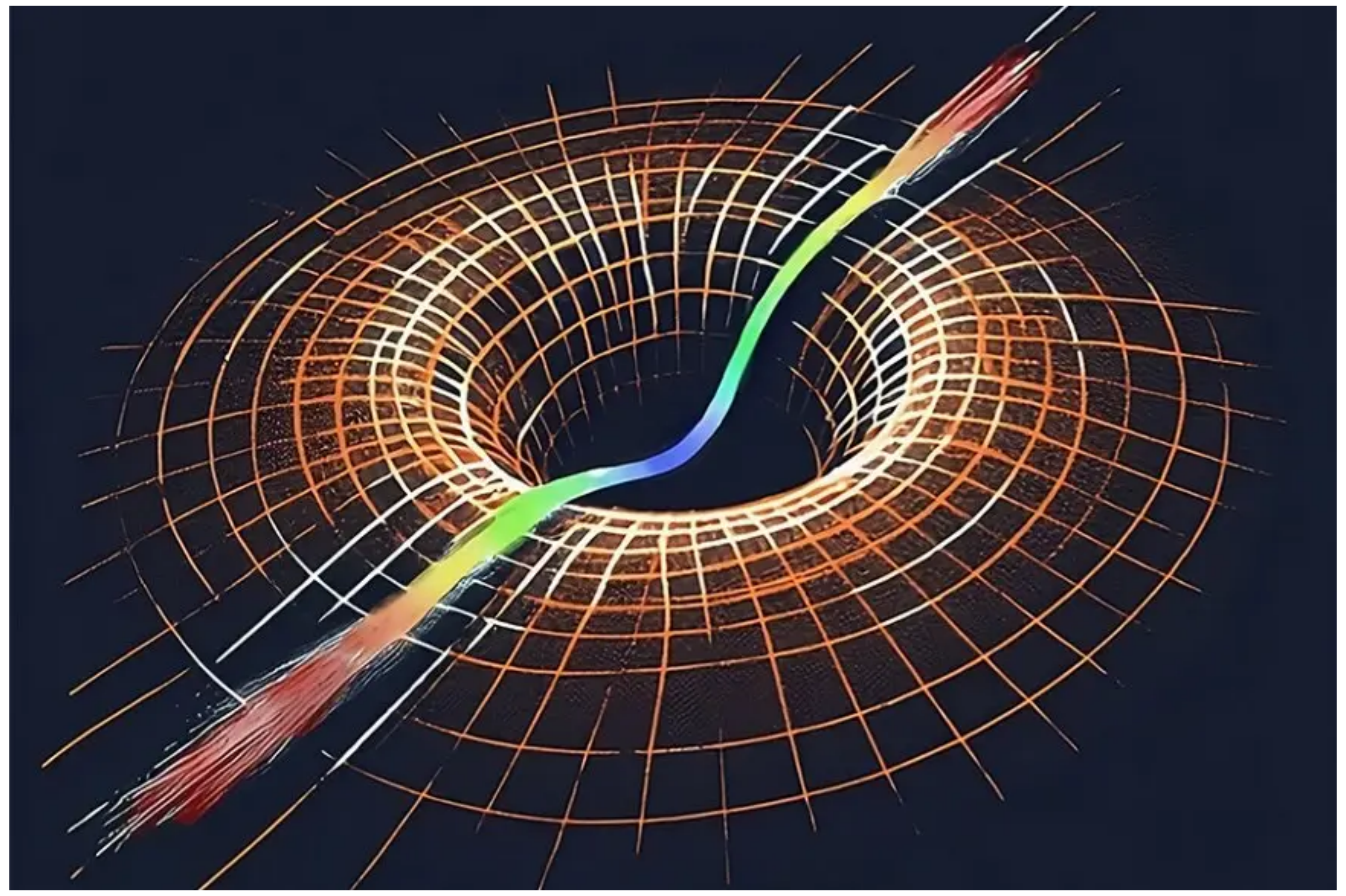
- Maxwell's equations without EM charges
  - $\nabla^b F_{ab} = 0$
  - $dF = 0$
  - F: field strength tensor
  - d: exterior derivative
- EM Potential
  - We can introduce EM Potential A such that $F = dA$ at least locally.

- $\Box A_a = R^b{}_a A_b$
- with Lorenz gauge $\nabla^a A_a = 0$
- R: Ricci curvature
- $\Box \equiv \nabla^a \nabla_a$: D'Alembertian

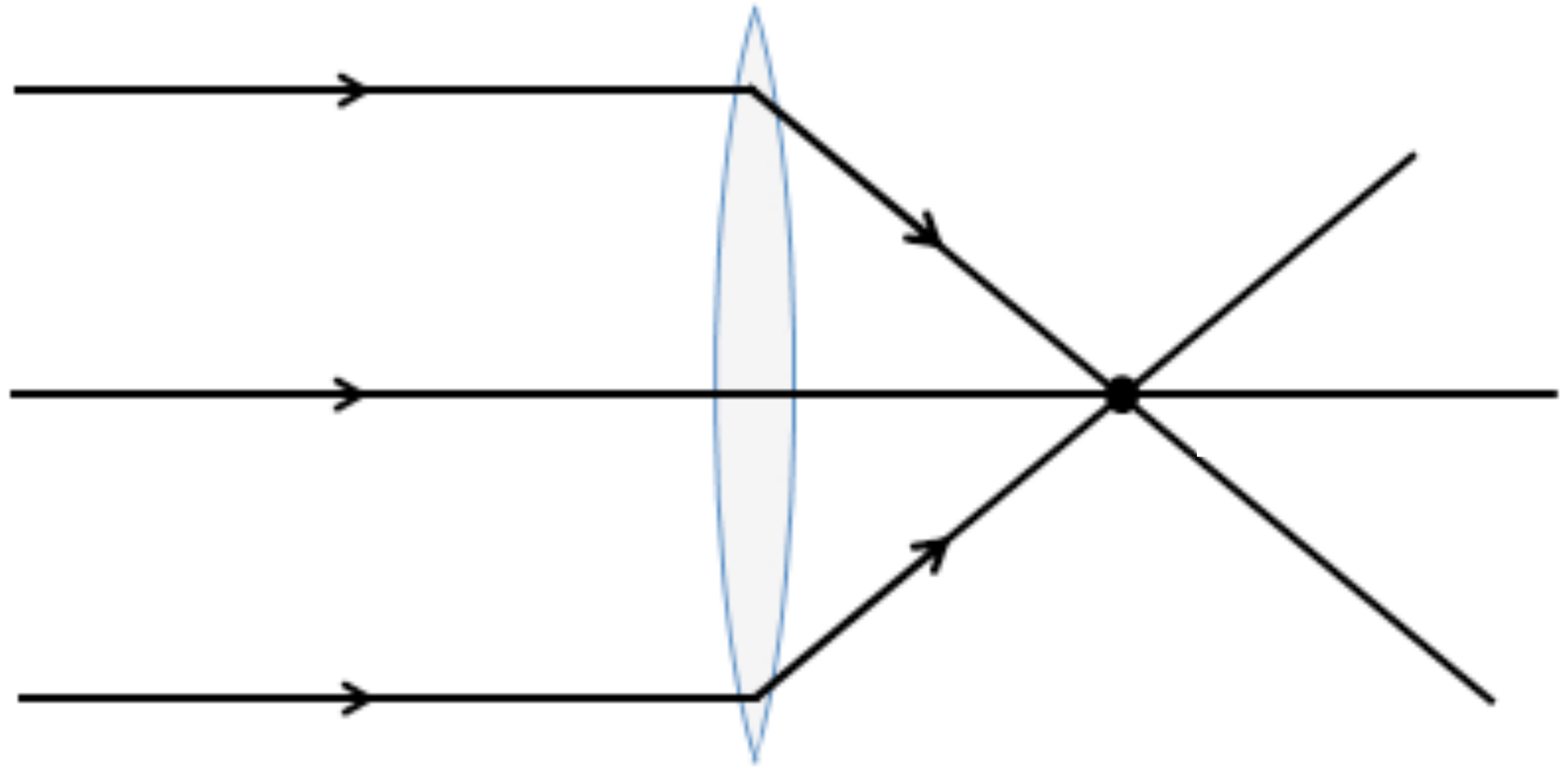
James C. Maxwell

Geometrical Optics in Curved Spacetime

- In a local Lorentz frame $\{t, \vec{x}\}$, if an EM field A can be described as
  - $A_a\left(t, \vec{x}\right) = 2\Re\left[\tilde{A}_a\left(t, \vec{x}\right) e^{iQ\left(t, \vec{x}\right)}\right]$
  - $\tilde{A}$: amplitude Q: phase $l_\alpha \equiv \partial_\alpha Q$: wave vector
- with conditions
  - $\left|l_\alpha\right| \gg \max\left\{1/\mathscr{L}, 1/\mathscr{R}\right\}$
  - $\mathscr{R}$: curvature radius
  - $\mathscr{L}$: typical length scale of medium variation
- It is EMW in the geometrical optics regime.
- Rays can be introduced in geometrical optics.



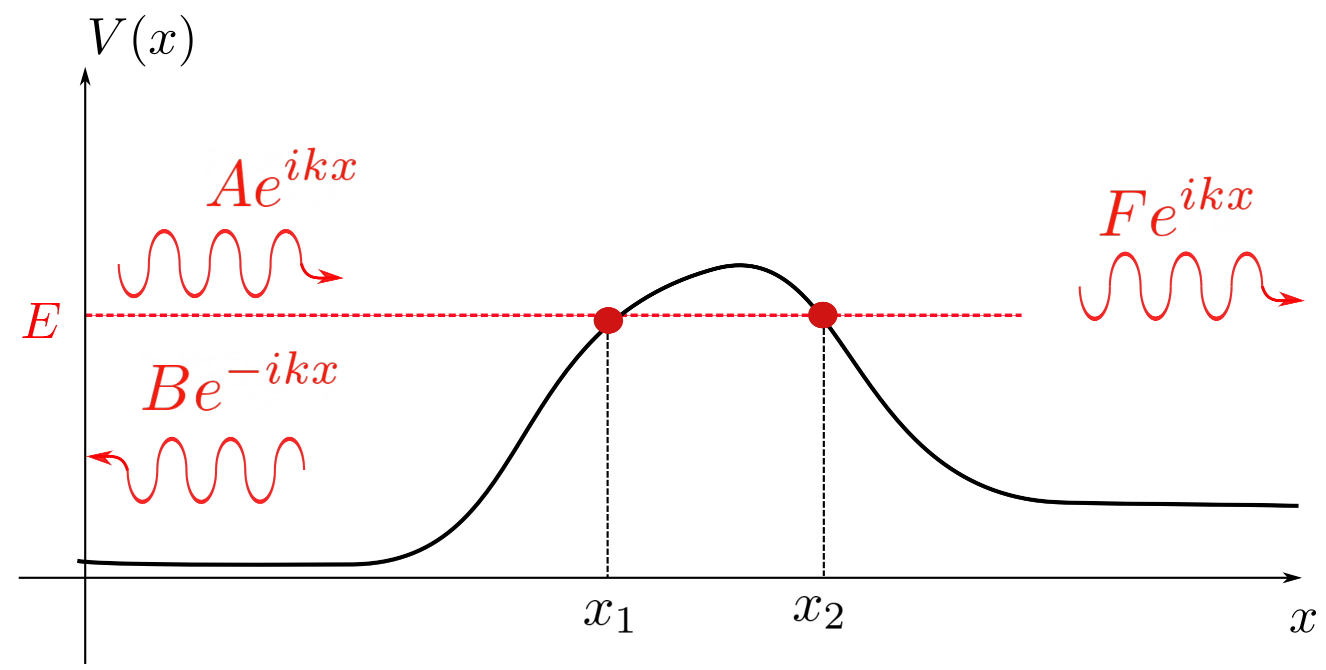
Credit: Matias Koivurova



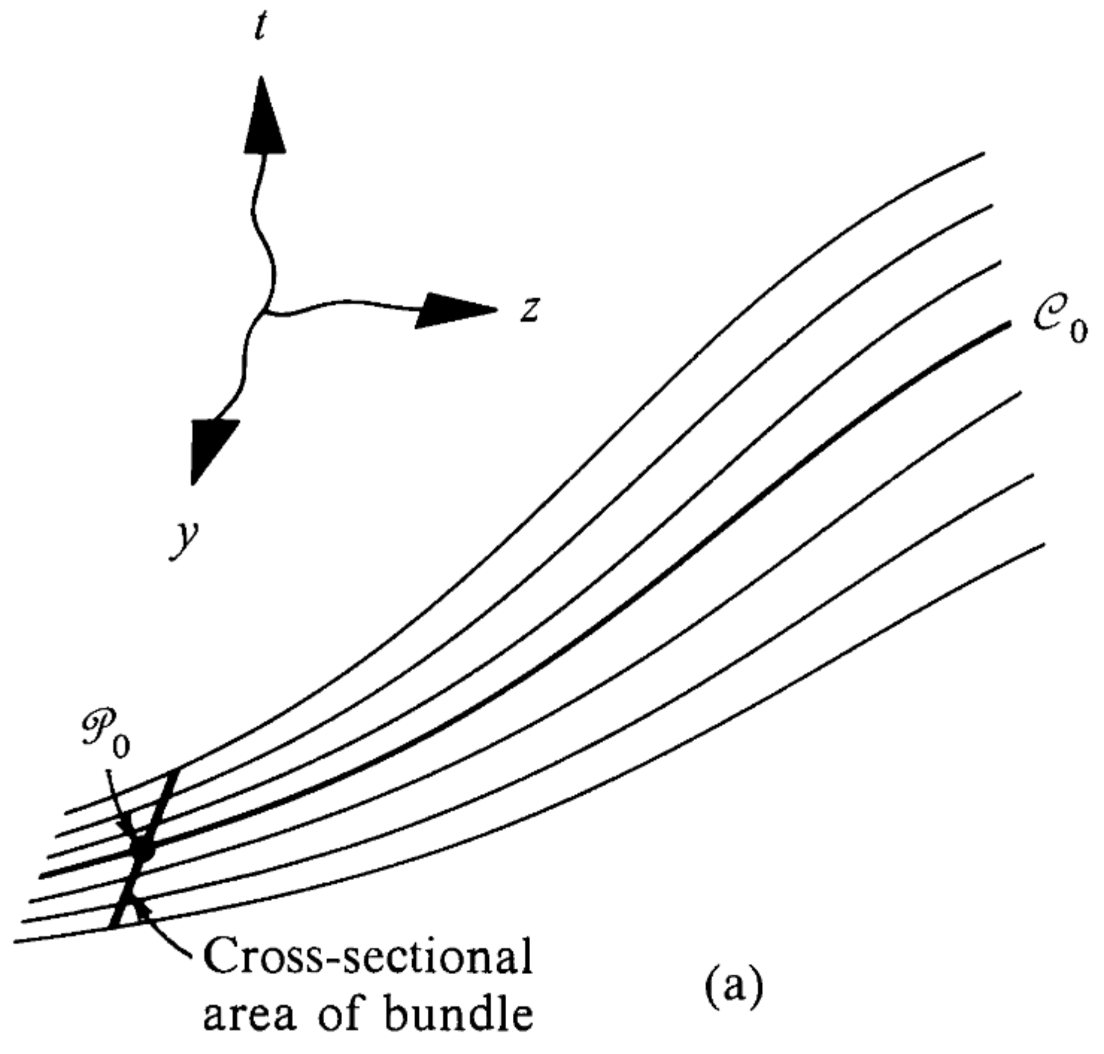
Geometrical Optics with Lens

Geometrical Optics in Curved Spacetime

- By introducing an order analysis parameter $\alpha \to 0$ such that
  - $l_\alpha = O\left(\alpha^{-1}\right)$ $\qquad$ $\tilde{A}_\alpha = O\left(\alpha^{0}\right)$
  - (phase changes much faster than the amplitude)
- Maxwell's equations in the leading-order give
  - $l \cdot l = 0$: wave is propagating to null direction
  - $l^a \nabla_a Q = 0$: phase is constant along wave propagation
  - $l^b \nabla_b l^a = 0$: waves are propagating along null geodesic
- From the next-to-leading-order
  - $0 = \nabla_a \left\{ \left(\tilde{A} \cdot \tilde{A}^* \right) l^a \right\}$: number of rays are conserved → photon number conservation
  - $0 = l \cdot \tilde{A}$: transverse wave



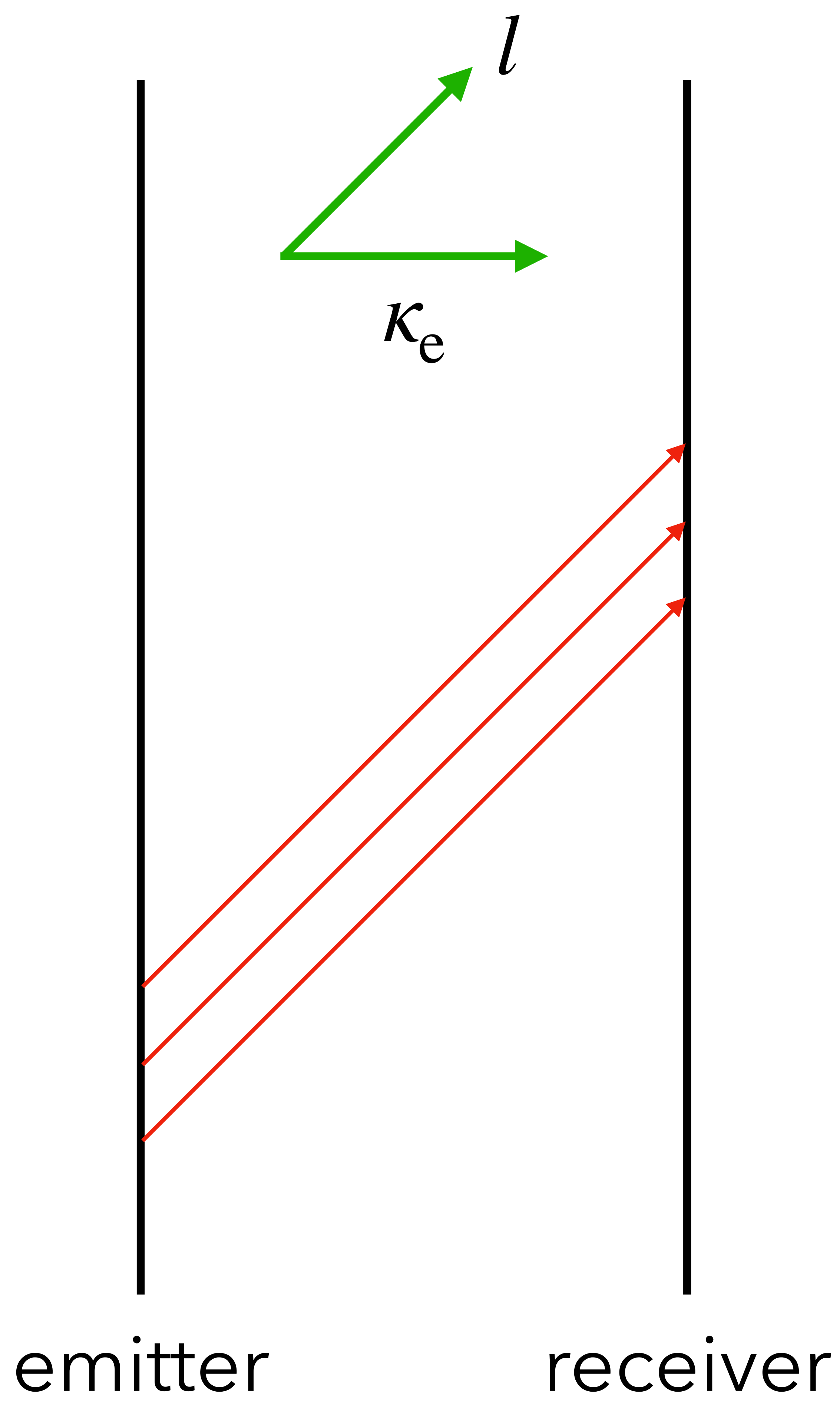
WKB Approximation in Quantum Mechanics / Credit: TU Delft



MTW Figure 22.1

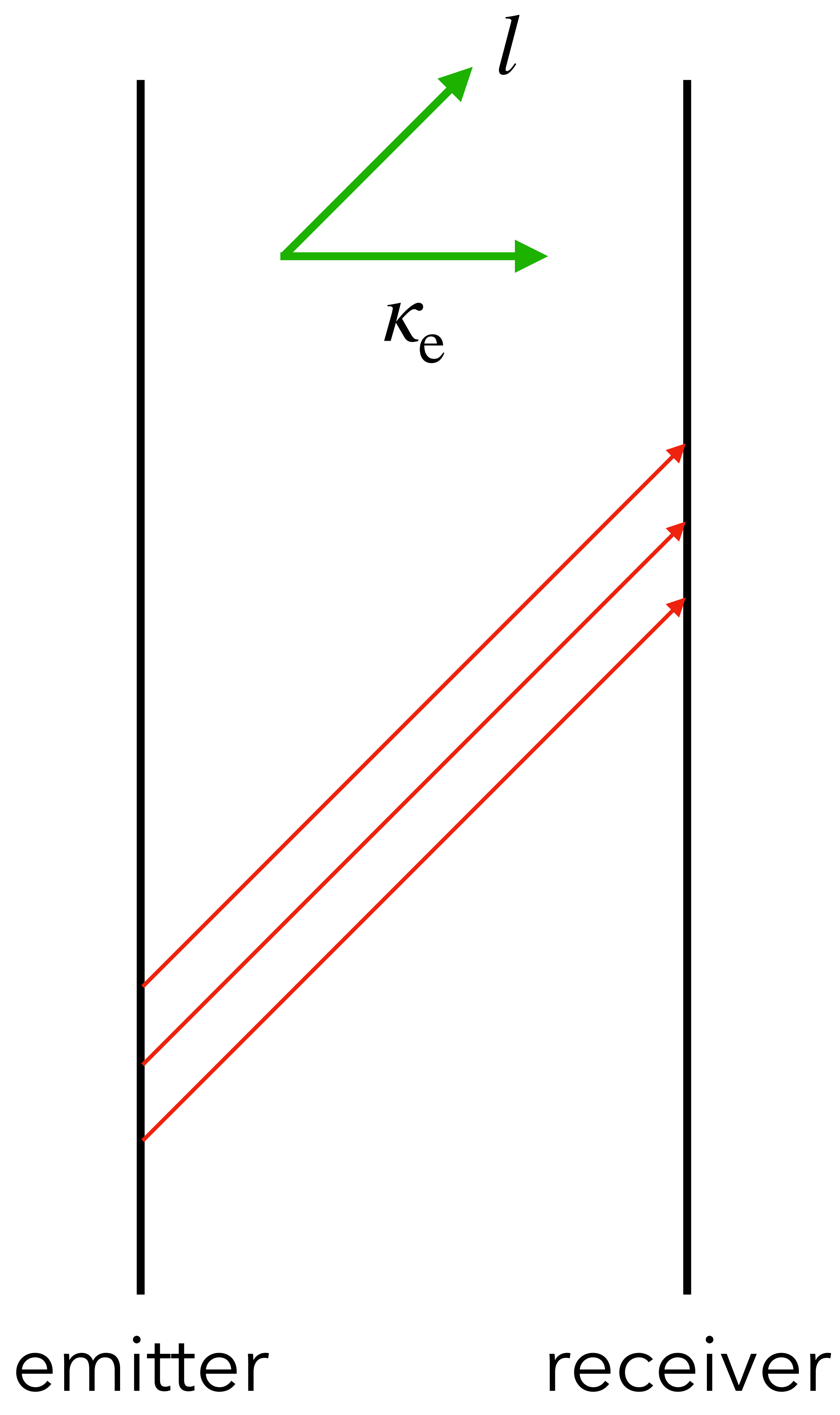
Monochromatic EMW in Flat Spacetime

- Monochromatic Wave Form
  - $A_a(t,\vec{x}) = 2\Re\left[\tilde{A}_a e^{iQ(t,\vec{x})}\right]$
  - $\tilde{A}_a$: constant amplitude
  - $Q(t,\vec{x}) = \omega_{\rm e}\left(-t + \kappa_{\rm e}\cdot\vec{x}\right)$: real phase
  - $\omega_{\rm e} = -u^a\nabla_a Q$: frequency
  - $\kappa_{\rm e}$: unit spatial propagation direction
  - $l^a \equiv \nabla^a Q = \omega_{\rm e}\left(u^a + \kappa_{\rm e}^a\right)$: wave vector
- From the Maxwell's equations
  - $0 = g_{ab}l^a l^b$ (null condition)
  - $0 = l^b\nabla_b l^a$ (null geodesic)



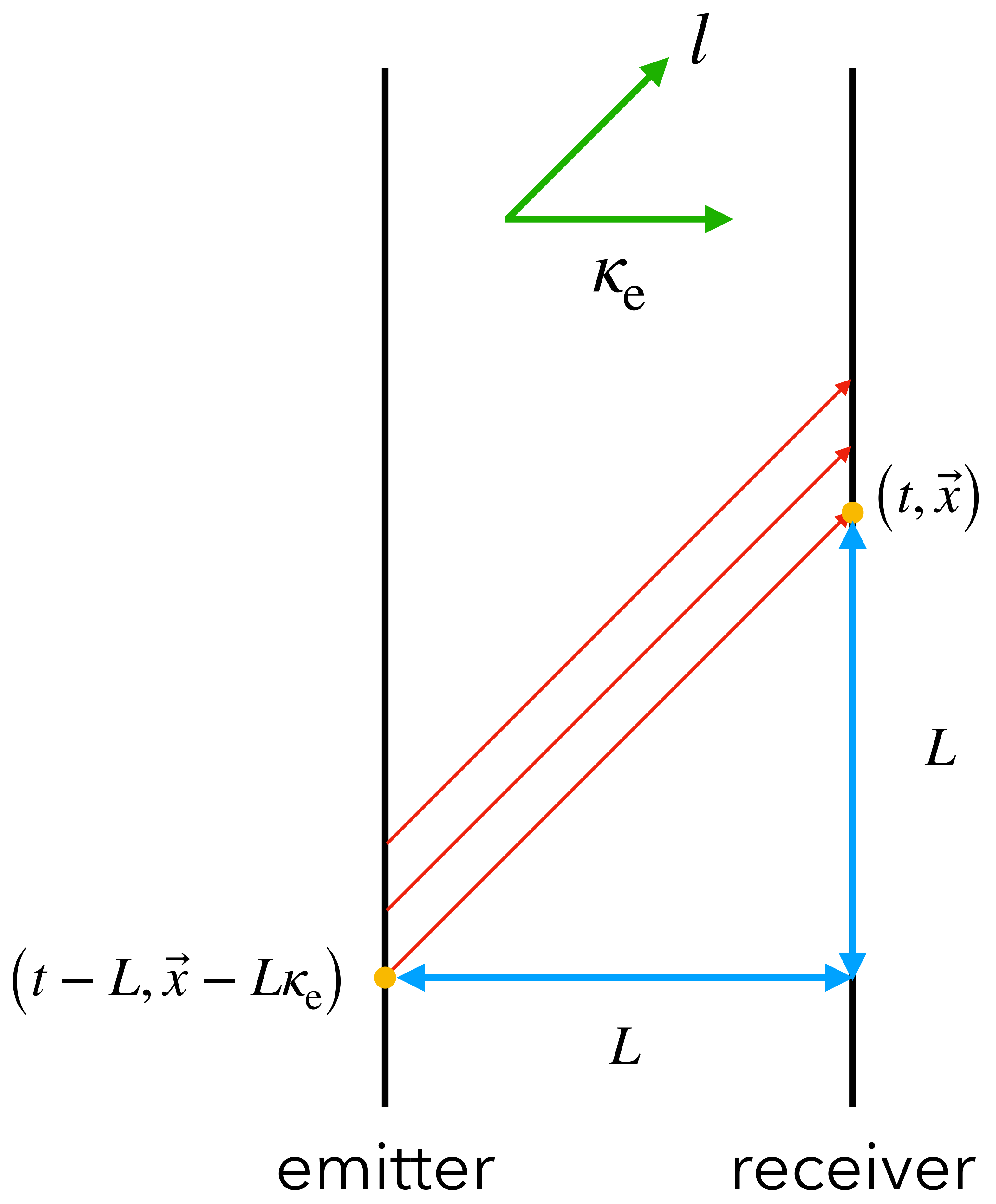
Linear Perturbation of Geometrical Optics

- Perturbed EMWs
  - $\hat{Q}(\epsilon) = Q + \epsilon\delta Q + O\left(\epsilon^2\right)$
  - $\hat{\omega}_\mathrm{e}(\epsilon) = \omega_\mathrm{e} + \epsilon\delta\omega_\mathrm{e} + O\left(\epsilon^2\right)$
  - Note that $\delta\omega_\mathrm{e}$ is gauge invariant because ω_e is constant scalar.
- Perturbation of Geometrical-Optical Equations
  - $l^a \nabla_a \delta Q = \dfrac{1}{2} h_{ab} l^a l^b$
  - $\delta\omega_\mathrm{e} = -u^a \nabla_a \delta Q$ in TT gauge.



Linear Perturbation of Geometrical Optics

- Boundary Condition
  - $\delta\omega_{\rm e} = -\partial_t \delta Q = 0$ at the emitter.
  - For convenience, we just set $\delta Q = 0$ at the emitter.
- Solution
  - $\delta Q = \int d^2\kappa_{\rm g} \int_{-\infty}^{\infty} \frac{d\omega_{\rm g}}{2\pi} \frac{1}{2i\left(k\cdot l\right)} \tilde{h}_{ab}\left(\omega_{\rm g}, \kappa_{\rm g}\right) l^a l^b \left(e^{iP} - e^{iP'}\right)$
- where
  - $P\left(t, \vec{x}\right) = \omega_{\rm g}\left(-t + \kappa_{\rm g}\cdot\vec{x}\right)$: GW phase
  - $P'\left(t, \vec{x}\right) = P\left(t - L, \vec{x} - L\kappa_{\rm e}\right)$: retarded GW phase



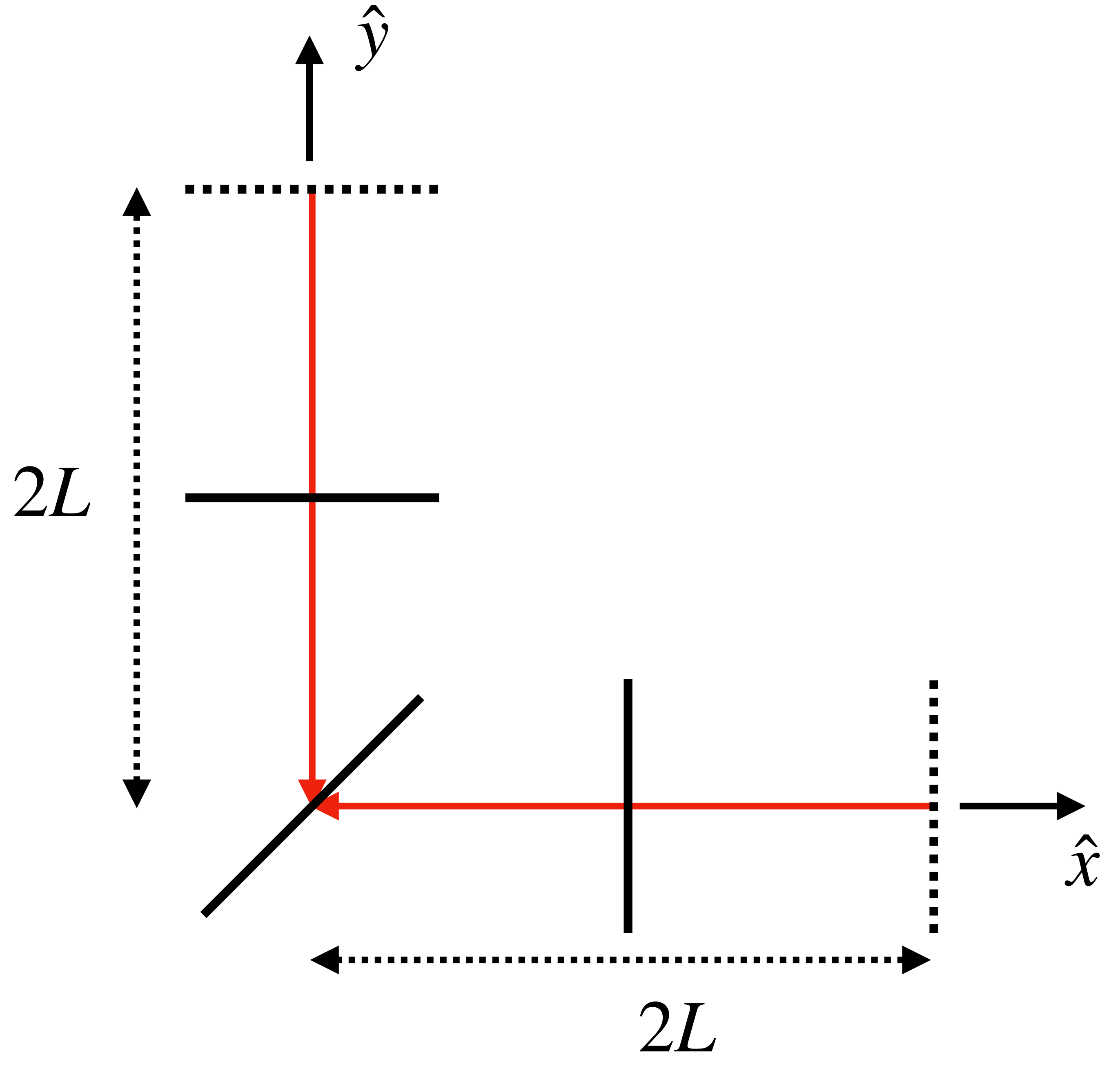
Interferometric GW Detector

- Phase Difference in Dark Port at $\vec{x} = 0$

$$\delta Q^{(x)} - \delta Q^{(y)} = -\frac{1}{2}\omega_{\rm e}\int d^2\kappa_{\rm g}\int_{-\infty}^{\infty}\frac{d\omega_{\rm g}}{2\pi}\frac{1}{i\omega_{\rm g}}\left[\frac{\tilde{h}_{xx}}{1+\kappa_x}\left(e^{iP\left(t,-2L\hat{x}\right)} - e^{iP(t-2L,0)}\right) - \frac{\tilde{h}_{yy}}{1+\kappa_y}\left(e^{iP\left(t,-2L\hat{y}\right)} - e^{iP(t-2L,0)}\right)\right]$$

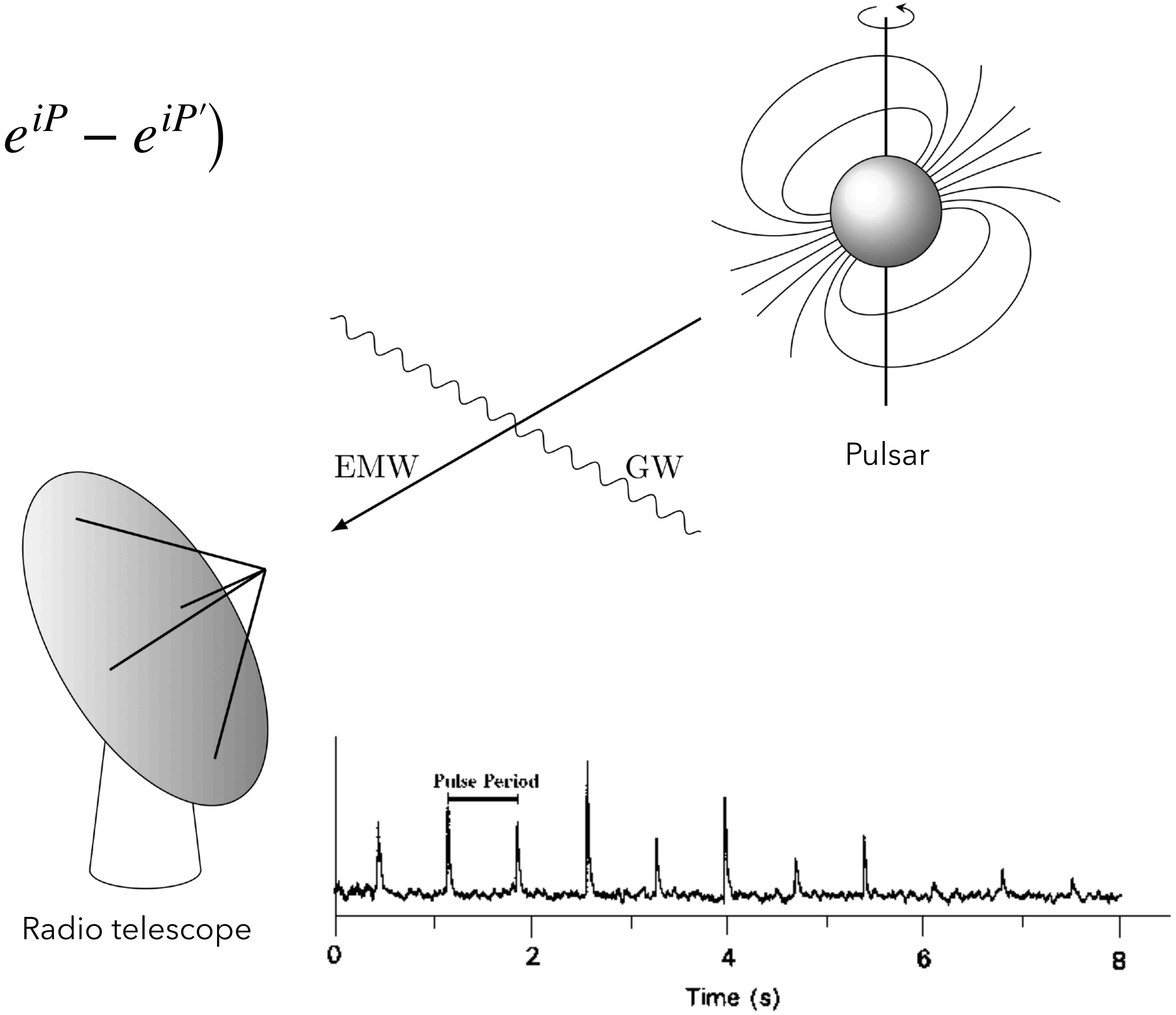
- $$\simeq L\omega_{\rm e}\left(\hat{x}^a\hat{x}^b - \hat{y}^a\hat{y}^b\right)h_{ab}(t,0) \qquad \text{for} \qquad \left(\omega_{\rm g}L \ll 1\right)$$

- For LIGO,
  - $\omega_g \sim 10^2\,\mathrm{Hz}$
  - $L \sim 10^3\,\mathrm{m}$
  - $\omega_g L/c \sim 10^{-3} \ll 1$



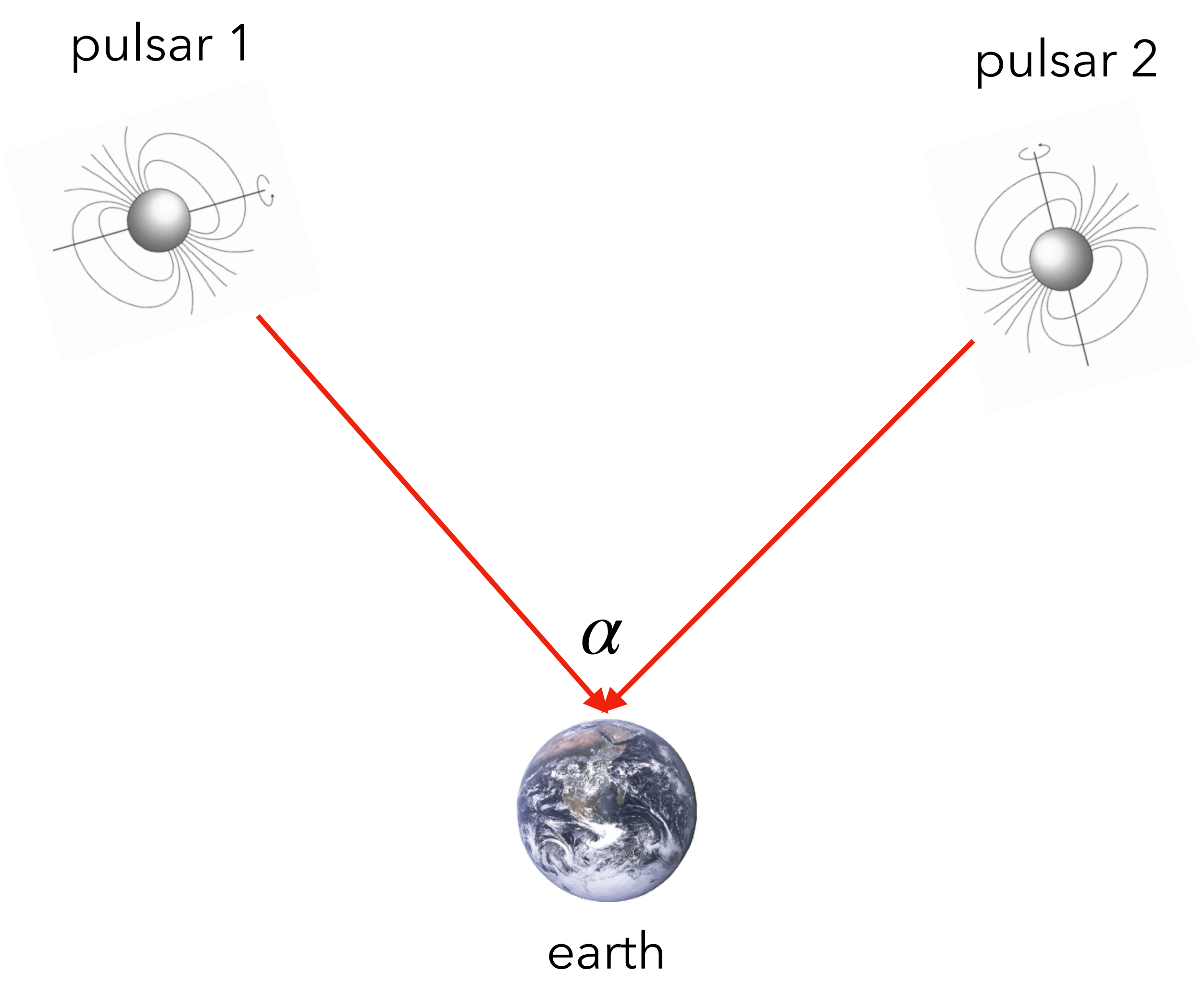
Pulsar Timing

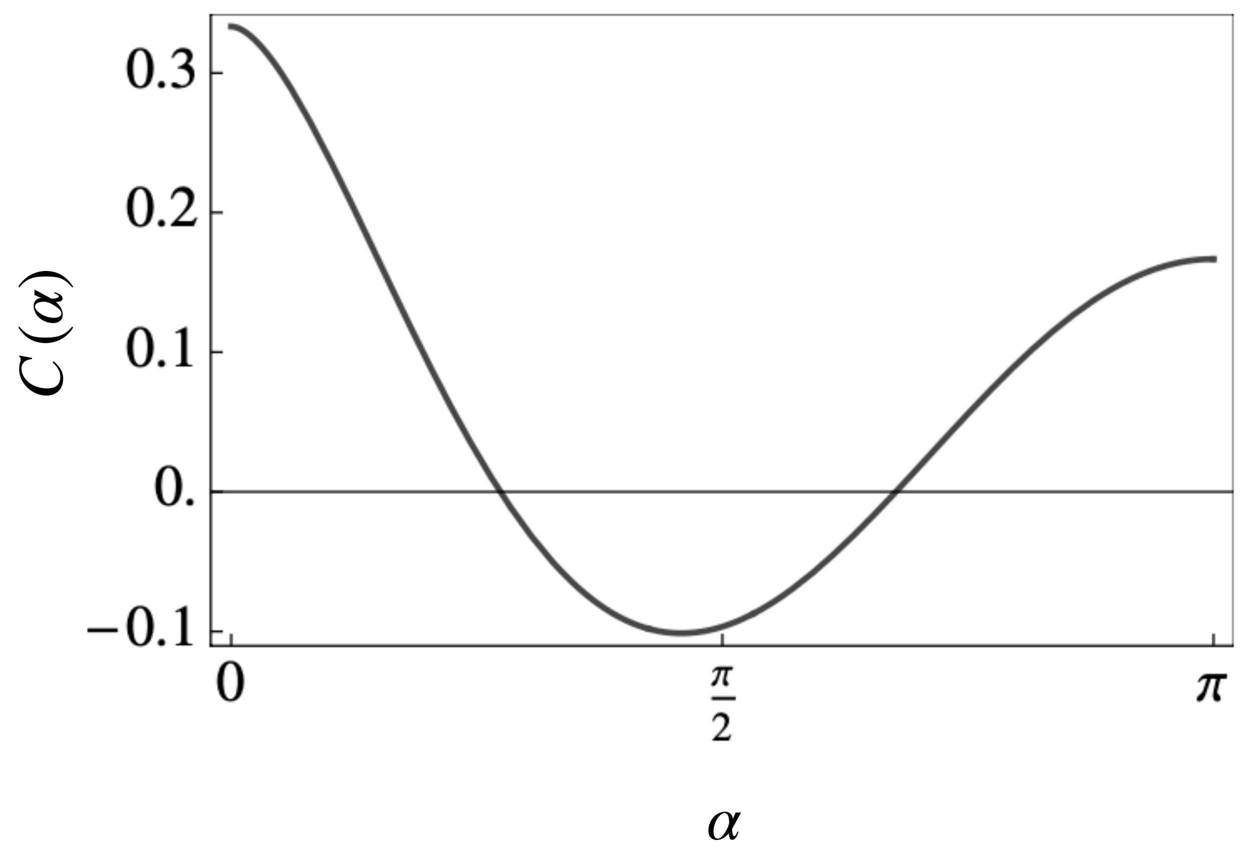
- Change of Pulse Period
  - $\frac{\delta T}{T} = -\frac{\delta\omega_{\rm e}}{\omega_e} = \int d^2\kappa_{\rm g} \int_{-\infty}^{\infty} \frac{d\omega_{\rm g}}{2\pi} \frac{1}{2} \frac{\tilde{h}_{ab}\kappa_{\rm e}^a \kappa_{\rm e}^b}{1 - \kappa_{\rm g}\cdot\kappa_{\rm e}} \left(e^{iP} - e^{iP'}\right)$
- For PTAs,
  - $\omega_{\rm g} \sim 10^{-9}\,{\rm Hz}$
  - $L \sim 10^3\,{\rm pc} \sim 10^{19}\,{\rm m}$
  - $\omega_g L/c \sim 10^2 \gg 1$



Pulsar Timing Array

- Correlation
  - $\langle h_1 h_2 \rangle = \left\langle \frac{\delta T_1}{T_1} \frac{\delta T_2}{T_2} \right\rangle = C(\alpha) \langle h_{ab} h^{ab} \rangle$
  - where
  - $C(\alpha) = \frac{1}{3} - \frac{1}{6}x + x \ln x$
  - $x(\alpha) = \frac{1 - \cos\alpha}{2} = \sin^2(\alpha/2)$





Hellings and Downs / ApJ (1983)

Summary

- The GWB, which is a superposition of GWs originating from various sources in the universe, is stochastic in nature.
- The characteristics of the GWB are that it is Gaussian, stationary, isotropic, and does not prefer any specific polarization.
- PTAs are measuring the GWB in the nHz frequency band, generated by SMBHB.
- To distinguish between noise and GW signals, a correlation method is used.
- Thank you for your listening! 😄

Thank you for dis!
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