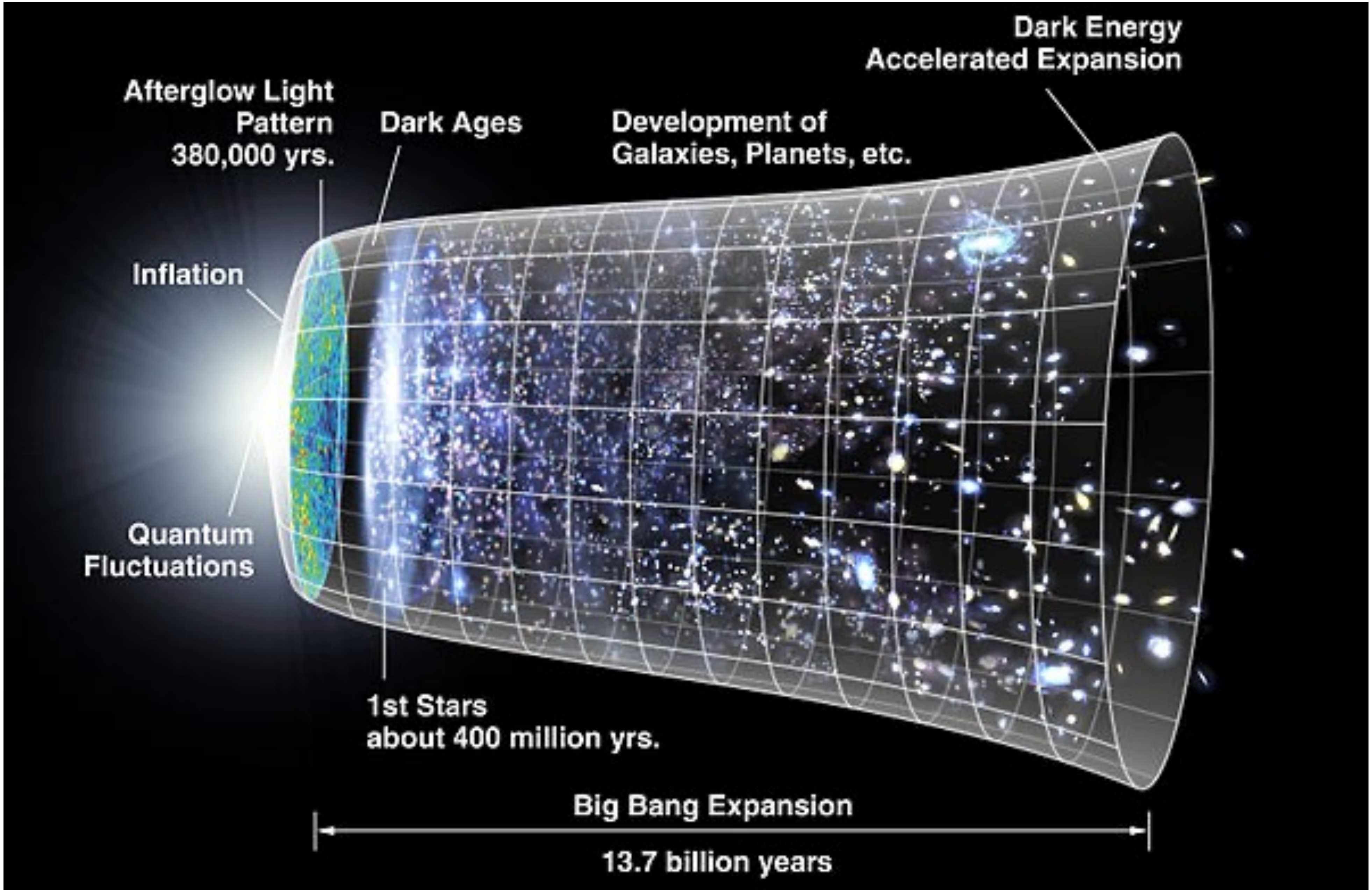
2025 수치상대론 및 중력파 여름학교
(2025.07.28~08.01, 한국천문연구원)

중력파 기초
(Basics of Gravitational Waves)

강궁원
문의: gwkang@cau.ac.kr

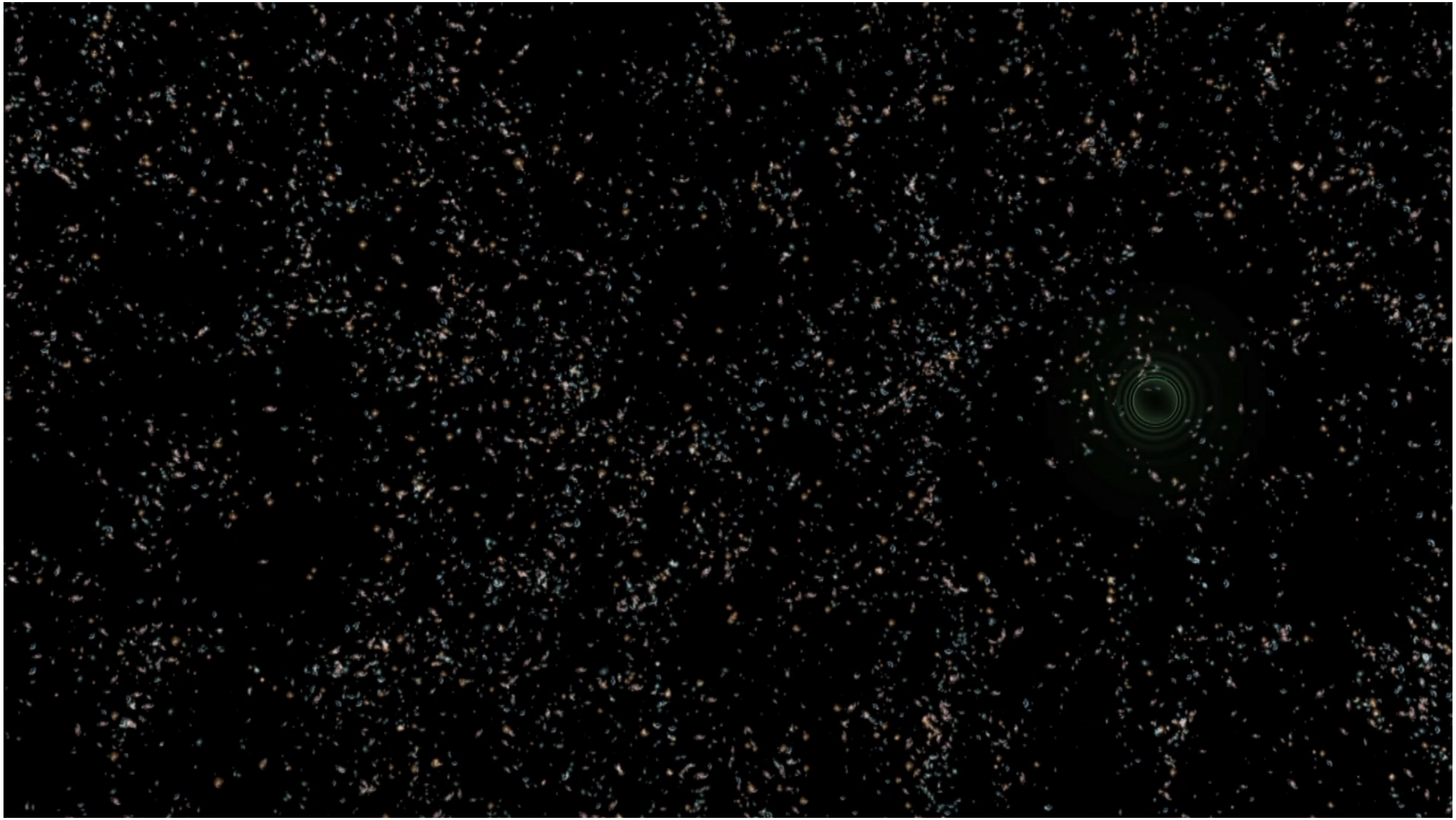
목 차

Mars, left, and the Milky Way are visible in the clear night sky as photographed near Salgotarjan, some 110 kms northeast of Budapest, Hungary, Aug. 03, 2018. (MTVA - Media Service Support and Asset Management Fund)



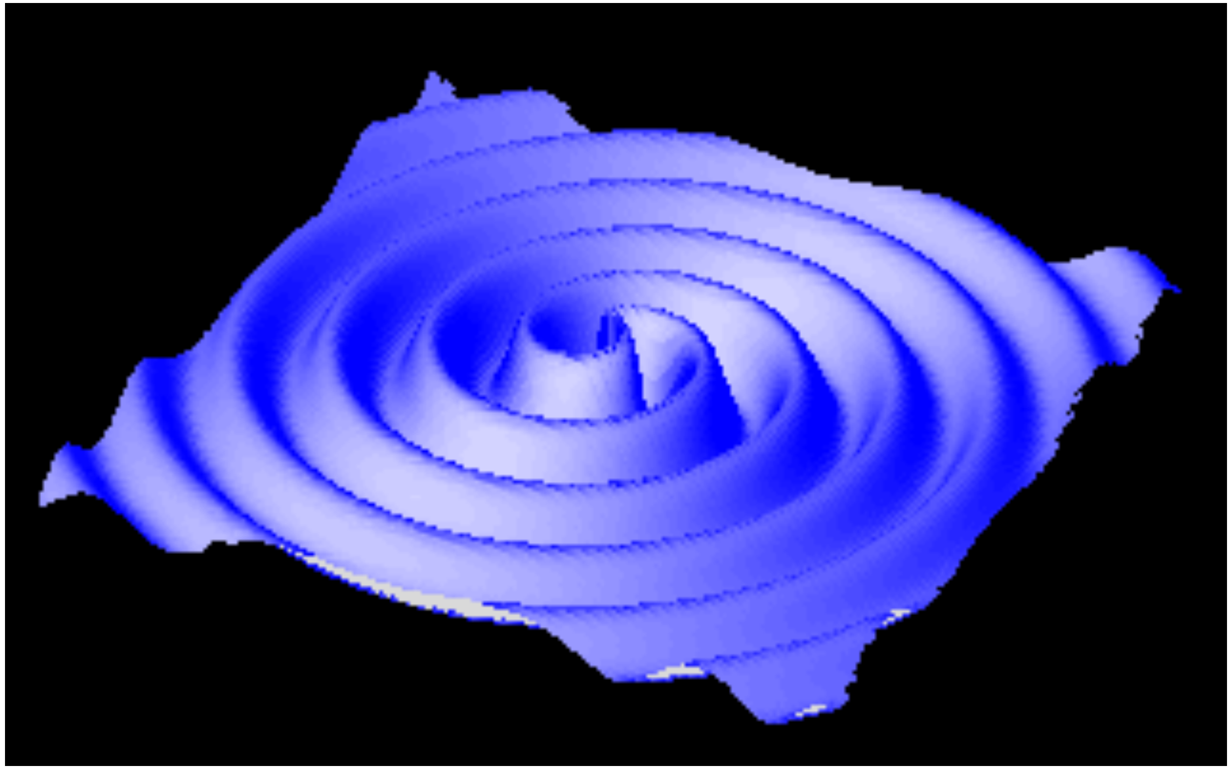
Credit courtesy: NASA/WMAP Science Team

시공간 곡률의 파동



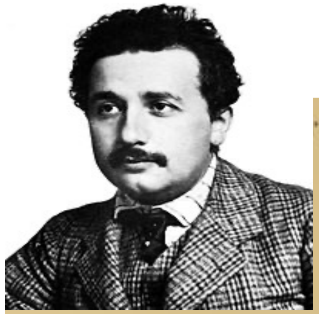
(Credit: LIGO/SXS/R.Hurt and T. Pyle)

I. 중력파



- 가속하는 전하가 전자기파를 발생시키듯이 질량의 요동이 중력파 발생
- 시공간 자체의 파동
- 물질의 요동이 주위의 시공간을 변형시키고 이 시공간의 주름이 빛의 속도로 퍼져 나감

✓ 1916년 아인슈타인이 예측



physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. EINSTEIN.

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \qquad (1)$$

oder

$$\sum_\alpha \frac{\partial^2 \gamma'_{\mu\nu}}{\partial x_\alpha^2} = 2\varkappa T_{\mu\nu}. \qquad (6)$$

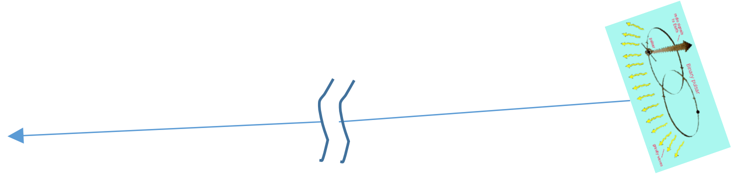
$$\gamma'_{22} = -\frac{\varkappa}{4\pi R}\frac{\partial^2}{\partial t^2}\left(\int \rho y^2 dV\right). \qquad (23)$$

Auf analoge Weise berechnet man

$$\gamma'_{33} = -\frac{\varkappa}{4\pi R}\frac{\partial^2}{\partial t^2}\left(\int \rho z^2 dV\right) \qquad (23a)$$

$$\gamma'_{23} = -\frac{\varkappa}{4\pi R}\frac{\partial^2}{\partial t^2}\left(\int \rho y^2 dV\right). \qquad (23b)$$

g_{ab}



$= \eta_{ab} + h_{ab}$

with $|h_{ab}| \ll 1$

$$G_{ab}[\eta_{cd} + h_{cd}] = 8\pi G T_{ab}[\text{Matter}, \eta_{cd} + h_{cd}]$$

- **선형 근사:** "편평한 시공간의 미약한 섭동"

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi G}{c^4} T_{ab}$$

$$\rightarrow \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right) h_{ab} = 0$$

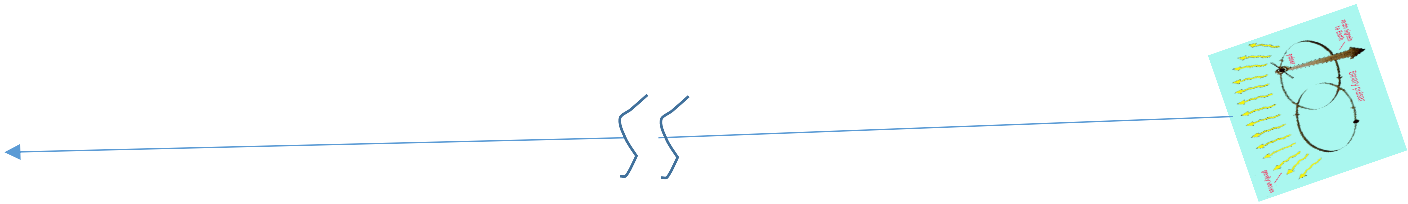
in vacuum with TT- gauge

- **우주론적 중력파:**
  $ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j$

- In general, $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$

✓ Expansion around flat space

$$g_{ab} = \eta_{ab} + h_{ab}$$



$$\text{with } |h_{ab}| \ll 1 \qquad \overset{\eta_{ab}}{\longrightarrow} \quad 0$$

- Linearized gravity:

$$G_{ab}[\eta_{cd} + h_{cd}] = 8\pi G T_{ab}[\text{Matter}, \eta_{cd} + h_{cd}]$$

  – Expand it up to the linear order.

$$\delta R_{ab} = \partial_c \delta\Gamma^c_{ab} - \partial_a \delta\Gamma^c_{cb} + \delta\Gamma\Gamma + \Gamma\delta\Gamma - \delta\Gamma\Gamma - \Gamma\delta\Gamma$$

$$\begin{aligned} \delta\Gamma^c_{ab} &= \frac{1}{2}\delta g^{cd}\left(\partial_c \eta_{bd} + \cdots\right) + \frac{1}{2}\eta^{cd}\left(\partial_a \delta g_{bd} + \partial_b h_{ad} - \partial_d h_{ab}\right) + \cdots \\ &= \frac{1}{2}\eta^{cd}\left(\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}\right) + \vartheta\left(h^2_{cd}\right) \end{aligned}$$

$$\delta G_{ab} = -\frac{1}{2}\partial^2 \bar{h}_{ab} + \frac{1}{2}\partial_a \partial^c \bar{h}_{bc} + \frac{1}{2}\partial_b \partial^c \bar{h}_{ac} - \frac{1}{2}\eta_{ab}\partial^c \partial^d \bar{h}_{cd}$$

with $\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab} h$

- Gauge condition: $x^a \to x'^a = x^a + \xi^a(x)$

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}(\eta_{cd} + h_{cd})$$
$$= \eta_{ab} + h_{ab} - \partial_a \xi_b - \partial_b \xi_a + \vartheta\left(\xi^2\right)$$

$$\bar{h}'_{ab} = \bar{h}_{ab} - \partial_a \xi_b - \partial_b \xi_a + \eta_{ab}\partial^c \xi_c$$

Even if $\partial^a \bar{h}_{ab} \neq 0$, $\quad \partial'^a \bar{h}'_{ab} = \partial^a \bar{h}_{ab} - \partial^2 \xi_b - \partial_b \partial^a \xi_a + \partial_b \partial^c \xi_c = \partial^a \bar{h}_{ab} - \partial^2 \xi_b = 0$

provided that $\xi^a(x)$ is chosen such that $\partial^2 \xi_b = \partial^a \bar{h}_{ab}$

So, in the Transverse gauge condition, i.e., $\partial^a \bar{h}_{ab} = 0 \quad \left(\partial^a h_{ab} = \frac{1}{2}\partial_b h\right)$

$$\delta G_{ab} = -\frac{1}{2}\partial^2 \bar{h}_{ab}$$

Finally, we have $\boxed{\partial^2 \bar{h}_{ab} = -16\pi G T_{ab} \quad \text{w/} \quad \partial^a \bar{h}_{ab} = 0}$

In vacuum, i.e., $T_{ab} = 0$,

$$\partial^2 \bar{h}_{ab} = \left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 \right) \bar{h}_{ab} = 0 \qquad \text{with} \qquad \partial^a \bar{h}_{ab} = 0$$

- Massless spin-2 field propagating in the flat spacetime: Fierz & Pauli (1939)
- Propagation speed: c the speed of light!
- Residual gauge in the case of vacuum: h=0, Traceless gauge
- Transverse gauge: $\partial^a \bar{h}_{ab} = \partial^a h_{ab} = 0$
- Transverse-Traceless (TT) gauge: $\partial^a h_{ab} = 0$ & $h = 0$

$$\therefore \quad h^{00} = h^{0i} = 0, \qquad h^i_i = 0, \qquad \partial^j h_{ij} = 0, \qquad \& \quad \partial^2 h_{ij} = 0$$

h_{ij}: 6 variables, 1 condition 3 conditions

➔ 6-(1+3) = 2 independent components, or degrees of freedom!!

✓ Plane wave solution:

$$h_{ij}^{TT}(x) = e_{ij}(\vec{k})e^{ik\cdot x}, \quad e_{ij}\text{: Polarization tensor}$$

$$\partial^2 h_{ij} = 0 \ \rightarrow -(k\cdot k)e_{ij}e^{ik\cdot x} = 0 \ \rightarrow k\cdot k = 0 \ \rightarrow k^\mu\text{: Null} - \text{like vector}$$

$$\rightarrow k^\mu = (\omega/c, \vec{k}) \ \& \ \omega/c = |\vec{k}|$$

$$\partial^j h_{ij} = 0 \ \rightarrow ik^j e_{ij}e^{ik\cdot x} = 0 \ \rightarrow k^j e_{ij} = 0 :$$ $\vec{k}$ & $\boldsymbol{e}$ are orthogonal or transverse

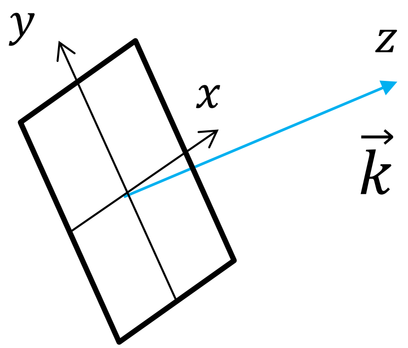
Choosing the z-direction along $\vec{k}$, we have $\vec{k} = (0,0,|\vec{k}|) = |\vec{k}|\hat{z}$, and so

$$k^j e_{ij} = k^z e_{iz} = 0 \ \rightarrow \ e_{iz} = e_{zi} = 0.$$

$$h_i^i = 0 \ \rightarrow (e_{xx} + e_{yy} + e_{zz})e^{ik\cdot x} = 0 \rightarrow e_{xx} + e_{yy} + e_{zz} = e_{xx} + e_{yy} = 0$$

$$\rightarrow e_{yy} = -e_{xx}$$

Thus, $$e_{ij} = \begin{pmatrix} e_{xx} & e_{xy} & 0 \\ e_{xy} & -e_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\vec{k} = |\vec{k}|\hat{z}$$

$$\therefore\ h^{TT}_{\mu\nu} = \mathcal{R}e\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(-\frac{\omega}{c}ct + k^z z)} \right\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos\omega(t - z/c)$$

$$\therefore\ h^{TT}_{\alpha\beta} = h_+(t-z) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + h_\times(t-z) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Arbitrary function of (t-z).

+ polarization (plus)

× polarization (cross)

- Two degrees of freedom
- Superposition of these two polarizations in general

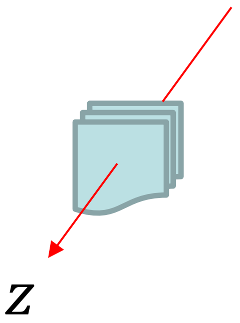
Or, $ds^2 = -dt^2 + [1 + h_+(t-z)]dx^2 + [1 - h_+(t-z)]dy^2 + 2h_\times(t-z)dxdy + dz^2$

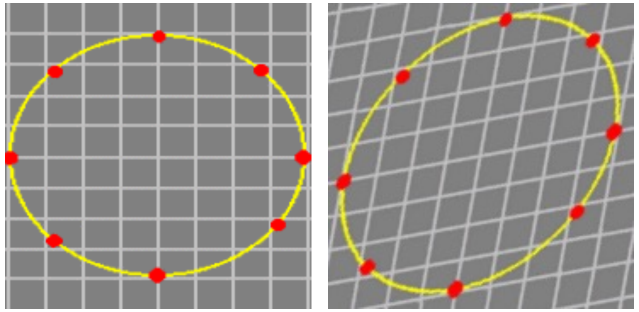
✓ 중력파가 지나가면 어떻게 되는가?

- 시공간 자체가 변함: $ds^2 = -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + 2h_\times dxdy + dz^2$
- 물체의 변형 및 길이 변화 일으킴

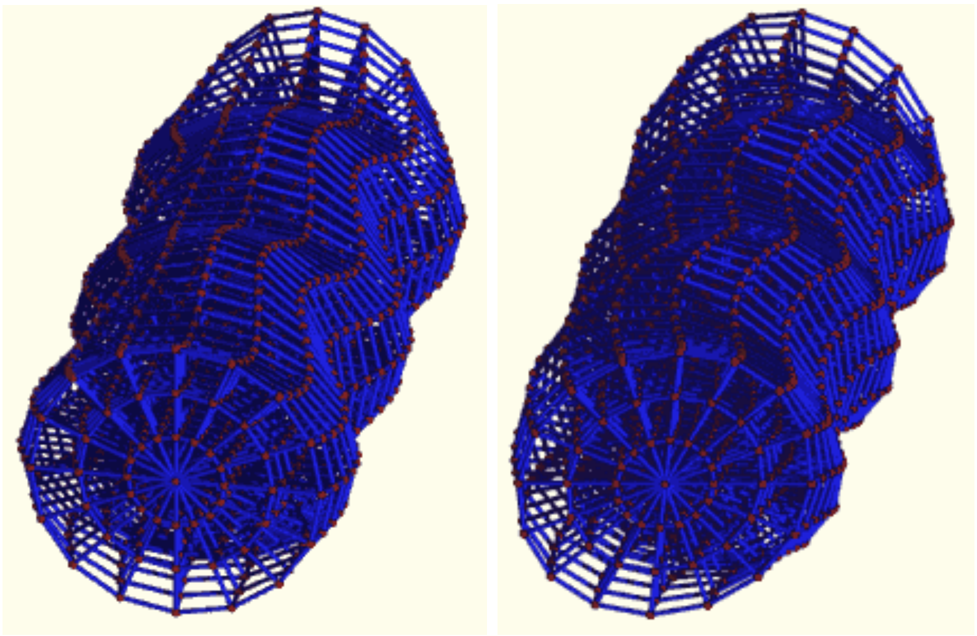
$$\begin{aligned} L'_x &= \int \sqrt{1 + h_{xx}}dx \\ &\sim \int (1 + \frac{1}{2}h_{xx})dx \\ &= (1 + \frac{1}{2}h\sin(\omega t)) \int dx \\ &= L + \frac{1}{2}Lh\sin(\omega t) \end{aligned}$$

✓ **Strain:** $\frac{\Delta L}{L} \cong \frac{1}{2}h_{GW}(t)$





플러스(좌), 크로스(우) 편극된 중력파



Credit: Ravikumar Kopparapu

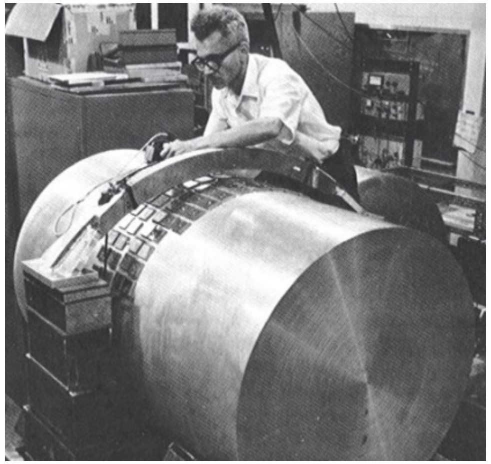
- Lots of confusions and contraversals for the reality of GWs till 1950s
- **"Sticky bead" argument:** Chapel Hill meeting in 1957

$$\frac{d^2\eta^i}{d\tau^2} = -R_{0j0}{}^i\eta^j = \frac{1}{2}\frac{d^2h_{ij}^{\mathrm{TT}}}{dt^2}\xi^j.$$

(Pirani '57)

- It was **J. Weber ('63)** who tried for the first time the detection experiment by using a resonant-mass cylindrical bar **at 1660Hz**.
- It was very sensitive $\boldsymbol{h \sim 10^{-16}}$, but still far from $\sim 10^{-22}$.

UMD



➔ ALEGRO: $h \sim 10^{-19}$ in 1990s

LSU ('09)

✓ Generation of gravitational waves

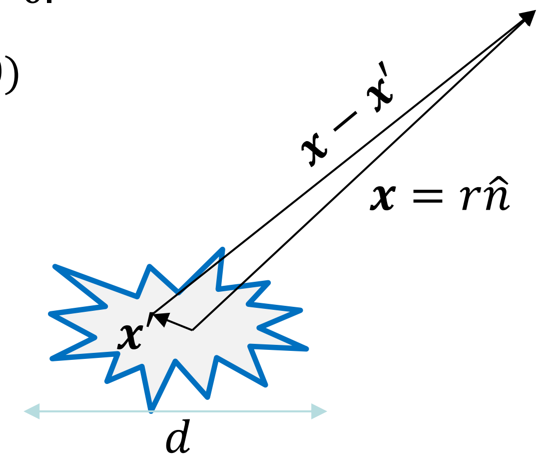
In the linearized approximation, *e.g.*, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu} \ll 1|$,

$$\partial^2 \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} + \vartheta(h^2) \quad \text{in the Transverse (Lorentz) gauge } \partial^\mu \bar{h}_{\mu\nu} = 0.$$

Outside the source, one can additionally impose the traceless gauge (i.e., $\bar{h} = h = 0$)

$$h_{ij}^{TT}(t, \boldsymbol{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{n}) \int d^3x' \frac{T_{kl}(t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}, \boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

$$\cong \frac{1}{r}\frac{4G}{c^4} \Lambda_{ij,kl}(\hat{n}) \int d^3x' \, T_{kl}\left(t - \frac{r}{c} + \frac{\boldsymbol{x}' \cdot \hat{n}}{c}, \boldsymbol{x}'\right) \quad \text{at } r \gg d.$$



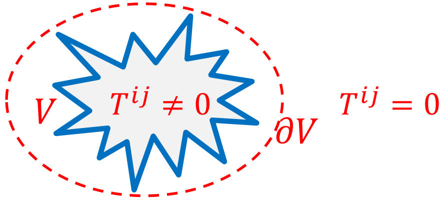
In the low-velocity expansion for source motions, i.e., $\omega \frac{\boldsymbol{x}' \cdot \hat{n}}{c} \leq \frac{\omega_s d}{c} \sim \frac{v}{c} \ll 1$,

$$h_{ij}^{TT}(t, \boldsymbol{x}) = \frac{1}{r}\frac{4G}{c^4} \Lambda_{ij,kl}(\hat{n}) \int d^3x' \left[T_{kl}\left(t - \frac{r}{c}, \boldsymbol{x}'\right) + \frac{x'^i n^i}{c} \partial_t T_{kl} + \frac{1}{2c^2} x'^i x'^j n^i n^j \partial_t^2 T_{kl} + \cdots \right]$$

$$= \frac{1}{r}\frac{4G}{c^4} \Lambda_{ij,kl}(\hat{n}) \left[S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \frac{1}{2c^2} n_m n_p \ddot{S}^{kl,mp} + \cdots \right]\Bigg|_{t \to t - r/c} .$$

Here,

$$S^{ij}(t) = \int d^3x \, T^{ij}(t, \boldsymbol{x}), \; S^{ij,k}(t) = \int d^3x T^{ij}(t, \boldsymbol{x}) x^k, S^{ij,kl}(t) = \int d^3x T^{ij}(t, \boldsymbol{x}) x^k x^l, \cdots .$$



- Note: $\partial_\mu T^{\mu\nu} = 0 \;\to\; \partial_0 T^{00} + \partial_i T^{i0} = 0 \;\&\; \partial_0 T^{0j} + \partial_i T^{ij} = 0$

$$\to \partial_0^2\left(T^{00}x^i x^j\right) = \left(\partial_k \partial_l T^{kl}\right)\left(x^i x^j\right) = 2T^{ij} + \partial_k\left(\partial_l T^{kl} x^i x^j - T^{ik}x^j - T^{jk}x^i\right),$$

0 since $T^{ij} = 0$ outside

$$\text{Or, } T^{ij} = \frac{1}{2}\partial_0^2\left(T^{00}x^i x^j\right) + \partial_k(\cdots) \to \int T^{ij} = \frac{1}{2}\frac{\partial^2}{c^2 \partial t^2}\int_V T^{00}x^i x^j + \oint_{\partial V}\cdots \sim \partial_t^2\left(\int T^{00}x^i x^j\right).$$

- Thus, defining the momenta of the energy density (T^{00}) and the momentum density (T^{0i}) for the source, respectively as follows,

  $M(t) = \frac{1}{c^2}\int d^3x\, T^{00}(t,\boldsymbol{x})$: Monopole of the source,

  $M^i = \frac{1}{c^2}\int d^3x\, T^{00}(t,\boldsymbol{x})x^i$: Dipole moment,

  $M^{ij} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\boldsymbol{x})x^i x^j = \frac{1}{c^2}\int d^3x\, \rho\,(t,\boldsymbol{x})x^i x^j$: Quadrupole moment, $\cdots$,

  $P^i(t) = \frac{1}{c}\int d^3x\, T^{0i}(t,\boldsymbol{x})$, $P^{i,j} = \frac{1}{c}\int d^3x\, T^{0i}(t,\boldsymbol{x})x^j$, $P^{i,jk} = \frac{1}{c}\int d^3x\, T^{0i}(t,\boldsymbol{x})x^j x^k$, $\cdots$,

we have

$$S^{ij} = \frac{1}{2}\ddot{M}^{ij}, \dot{S}^{ij,k} = \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\left(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}\right), \ddot{S}^{ij,kl} = \cdots, \cdots.$$

- Note also that

  $\dot{M} = 0 + \vartheta(h^2)$: Energy conservation of the source at the leading order,

  $\dot{P}^i = 0 + \vartheta(h^2)$: Momentum conservation at the leading order.

- **Order of magnitude (amendment):**

$$h_{ij}^{TT}(t,\boldsymbol{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n})\left[S^{kl} + \frac{1}{c}n_m\dot{S}^{kl,m} + \frac{1}{2c^2}n_m n_p\ddot{S}^{kl,mp} + \cdots\right]\Big|_{t\to t-r/c}.$$

Here, $\sim\vartheta(1)$ $\sim\vartheta\left(\frac{v}{c}\right)$ $\sim\vartheta\left(\frac{v^2}{c^2}\right)$

$$S^{ij}(t) = \int d^3x\, T^{ij}(t,\boldsymbol{x}) = \frac{1}{2}\partial_t^2\left(\frac{1}{c^2}\int d^3x\, T^{00}x^i x^j\right) = \frac{1}{2}\ddot{M}^{ij} \quad \leftarrow \quad M^{ij} = \frac{1}{c^2}\int d^3x\, T^{00}x^i x^j,$$

$$S^{ij,k}(t) = \int d^3x T^{ij}(t,\boldsymbol{x})x^k = \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\left(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}\right)$$

$$\leftarrow \quad M^{ijk} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\boldsymbol{x})x^i x^j x^k \quad \& \quad P^{i,jk} = \frac{1}{c}\int d^3x\, T^{0i}(t,\boldsymbol{x})x^j x^k.$$

- What orders are these quantities of T^{00}, T^{0i} & T^{ij} in v/c?
- For simplicity, consider a free point-like particle moving on a trajectory $x_0(t)$ in flat ST. Then,

$$T^{\mu\nu}(t,x) = \frac{p^\mu p^\nu}{\gamma m}\,\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_0(t)), \quad \text{with} \quad \gamma = 1/\sqrt{1 - v^2/c^2}, \; p^\mu = \gamma m\frac{dx_0^\mu}{dt} = (E/c, \boldsymbol{p}).$$

➔ $T^{\mu\nu}(t,x) = \sum_A \frac{p_A^\mu p_A^\nu}{\gamma_A m_A}\,\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A(t))$ for a set of free point particles.

- For a two-body system **with interactions** in the non-relativistic limit,

$$T^{\mu\nu}(t,x) = \sum_{A=1,2}\gamma_A m_A\frac{dx_A^\mu}{dt}\frac{dx_A^\nu}{dt}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A(t)) + V(r) + \cdots = \sum_{A=1,2} m_A\frac{dx_A^\mu}{dt}\frac{dx_A^\nu}{dt}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A(t)) + \vartheta\left(\frac{v^2}{c^2}\right).$$

Here, the gravitational binding energy $\mathrm{V(r)} \sim -Gm_1m_2/r$ is the order of v^2/c^2. Thus, leading orders are

$$T^{00} = \sum_A m_A c^2\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A(t)) \sim \vartheta(1),$$

$$T^{0i} = \sum_A m_A c\dot{x}^i(t)\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A(t)) \sim \vartheta\left(\frac{v}{c}\right),$$

$$T^{ij} = \sum_A m_A\dot{x}^i(t)\dot{x}^j(t)\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_A(t)) \sim \vartheta\left(\frac{v^2}{c^2}\right).$$

✓ **Mass quadrupole radiation:** $\sim \partial_t^2\left(\int d^3x\, T^{00}(t,\boldsymbol{x})x^i x^j\right)$: "Mass quadrupole rad."
$\sim \ddot{M}^{ij}$

$$h_{ij}^{TT}(t,\boldsymbol{x}) \cong \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n})\left[S^{kl} + \frac{1}{c}n_m \dot{S}^{kl,m} + \frac{1}{2c^2}n_m n_p \ddot{S}^{kl,mp} + \cdots\right]\Bigg|_{t\to t-\frac{r}{c}}$$

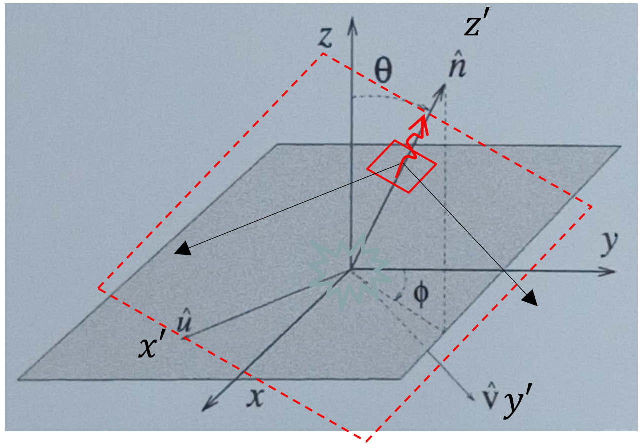
$$= \left(h_{ij}^{TT}\right)_{\text{quad}} + \left(h_{ij}^{TT}\right)_{\text{next-to-leading}} + \cdots$$

$\sim \{\dddot{M}^{ijk}, \ddot{P}^{i,jk}\}$
$\sim \{\partial_t^3\left(\int d^3x\, T^{00}(t,\boldsymbol{x})x^i x^j x^k\right), \partial_t^2\left(\int d^3x\, T^{0i}(t,\boldsymbol{x})x^j x^k\right)\}$
"Mass octupole rad.", "Current quadrupole rad."

- $\left[h_{ij}^{TT}(t,\boldsymbol{x})\right]_{\text{quad}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}(t-r/c) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\left[\left(\ddot{M}^{kl} - \frac{1}{3}\delta^{kl}\ddot{M}\right) + \frac{1}{3}\delta^{kl}\ddot{M}\right]_{t-r/c} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{TT}(t-r/c)$

➔ $Q^{ij} \equiv \int d^3x'\, \rho(t,\boldsymbol{x}')\left(x'^i x'^j - \frac{1}{3}r'^2\delta^{ij}\right)$: Quadrupole moment

$Q_{\rm ij}^{TT}(t,\boldsymbol{x}) = \Lambda_{ij,kl}(\hat{n})Q_{ij}(t)$



$$(\mathcal{R})_{ij} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

✓ **Angular distribution:** $n_i = \mathcal{R}_{ij}n'_j$

$M_{ij} = (\mathcal{R}M'\mathcal{R}^T)_{ij} \to M' = \mathcal{R}^T M \mathcal{R}$

➔ $h'_+(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}'_{11} - \ddot{M}'_{22}\right) = \frac{1}{r}\frac{G}{c^4}[\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta)$
$+\ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) - \ddot{M}_{33}\sin^2\theta - \ddot{M}_{12}\sin 2\phi\,(1+\cos^2\theta)$
$+\ddot{M}_{13}\sin\phi\sin 2\theta + \ddot{M}_{23}\cos\phi\sin 2\theta]$,

$h'_\times(t;\theta,\phi) = \frac{2}{r}\frac{G}{c^4}\ddot{M}'_{12} = \frac{1}{r}\frac{G}{c^4}[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta + 2\ddot{M}_{12}\cos 2\phi\cos\theta$
$-2\ddot{M}_{13}\cos\phi\sin\theta + 2\ddot{M}_{23}\sin\phi\sin\theta]$,

✓ **Radiated energy:**

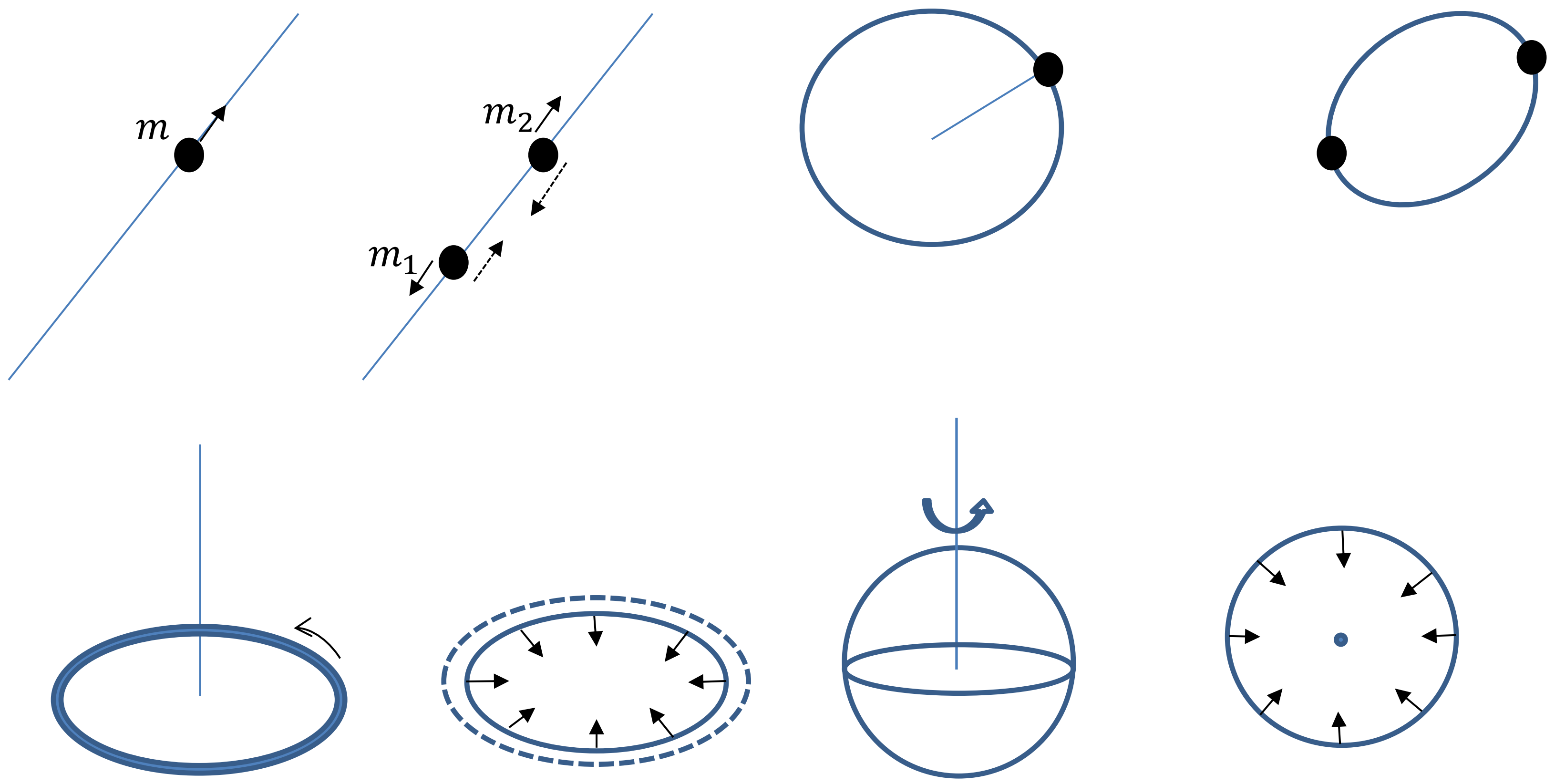
- Power per unit solid angle: $\left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \left(\frac{dE}{dt d\Omega}\right)_{\text{quad}} = \frac{r^2 c^3}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle = \frac{G}{8\pi c^5}\Lambda_{ij,kl}(\hat{n})\langle \dddot{Q}_{ij}\dddot{Q}_{kl}\rangle.$
- Power: $P_{\text{quad}} = \left(\frac{dE}{dt}\right)_{\text{quad}} = \int d\Omega \left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle = \frac{G}{5c^5}\left\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}\left(\dddot{M}_{kk}\right)^2\right\rangle.$
- Energy radiated per unit solid angle: $\left(\frac{dE}{d\Omega}\right)_{\text{quad}} = \int dt \left(\frac{dP}{d\Omega}\right)_{\text{quad}} =$
  $\frac{G}{8\pi^2 c^5}\Lambda_{ij,kl}(\hat{n})\int_0^\infty d\omega\, \omega^6 \tilde{Q}_{ij}(\omega)\tilde{Q}_{kl}^*(\omega),$
  where $Q_{ij}(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{Q}_{ij}(\omega)e^{-i\omega t}$ and $\tilde{Q}_{ij}(-\omega) = \tilde{Q}_{ij}^*(\omega)$ is used.
- Total radiated energy: $E_{\text{quad}} = \int d\Omega \left(\frac{dE}{d\Omega}\right)_{\text{quad}} = \frac{G}{5\pi c^5}\int_0^\infty d\omega\, \omega^6 \tilde{Q}_{ij}(\omega)\tilde{Q}_{kl}^*(\omega).$
- Energy spectrum: $\left(\frac{dE}{d\omega}\right)_{\text{quad}} = \frac{G}{5\pi c^5}\omega^6 \tilde{Q}_{ij}(\omega)\tilde{Q}_{kl}^*(\omega).$
- For a monochromatic source, radiating at ω_0, $\tilde{Q}_{ij}(\omega) = q_{ij}2\pi\delta(\omega-\omega_0)$. Using $2\pi\delta(\omega=0) = T$,

  ➔ $\left(\frac{dP}{d\Omega d\omega}\right)_{\text{quad}} = \frac{1}{T}\times\left[\frac{G}{8\pi^2 c^5}\Lambda_{ij,kl}(\hat{n})\omega^6\tilde{Q}_{ij}(\omega)\tilde{Q}_{kl}^*(\omega)\right] = \frac{G\omega_0^6}{4\pi c^5}\Lambda_{ij,kl}(\hat{n})q_{ij}q_{kl}^*\,\delta(\omega-\omega_0)$

✓ **Momentum flux:**

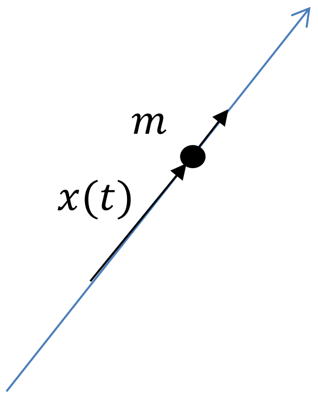
- $\frac{dP^i}{dt} = -\frac{c^3}{32\pi G}r^2\int d\Omega\,\langle \dot{h}_{kl}^{TT}\partial^i h_{kl}^{TT}\rangle$
- Momentum flux per unit solid angle: $\left(\frac{dP^i}{dt d\Omega}\right)_{\text{quad}} = -\frac{G}{8\pi c^5}\dot{Q}_{kl}^{TT}\partial^i Q_{kl}^{TT}.$
- BUT, the total momentum flux: $\left(\frac{dP^i}{dt}\right)_{\text{quad}} = -\frac{G}{8\pi c^5}\int d\Omega\,\dot{Q}_{kl}^{TT}\partial^i Q_{kl}^{TT} = 0$ since the integrand is odd under reflection, $\boldsymbol{x}\to-\boldsymbol{x}$. Note that $Q_{kl}(-\boldsymbol{x}) = Q_{kl}(\boldsymbol{x})$ & $\partial^i \to -\partial^i$.
- This vanishing of the total momentum flux is not true any more if higher order contributions to GWs are taken into account.

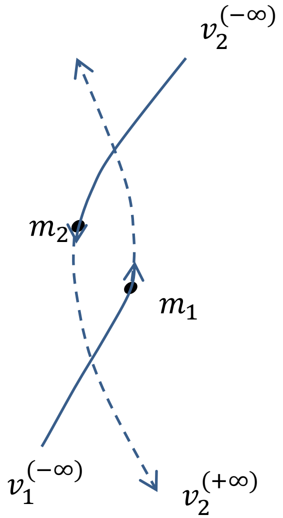
✓ Various time-varying sources:

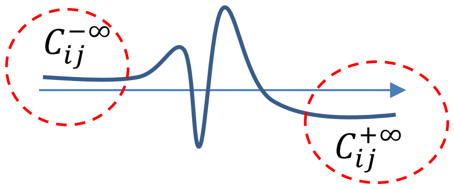


…… ……

• Ex1) Point object moving on a strait line:







- Guess: No radiation in a constant motion, but emits GWs in an acceleration?
- Non-vanishing quadrupole moment:

$$M_{xx} = mx^2 \rightarrow \dot{M}_{xx} = 2mx\dot{x} \rightarrow \ddot{M}_{xx} = 2mx\ddot{x} + 2m\dot{x}^2$$

i) Uniform motion, $\dot{x} = v =$ Constnat., $or\ \ a = \ddot{x} = 0$;

$\ddot{M}_{xx} = 2mv^2 =$ Constant.

$\rightarrow\ \ \ddot{Q}_{ij} = \ddot{M}_{ij} - \frac{1}{3}\delta_{ij}(2mv^2) = \frac{2}{3}mv^2\times\mathrm{diag}(2,-1,-1)$

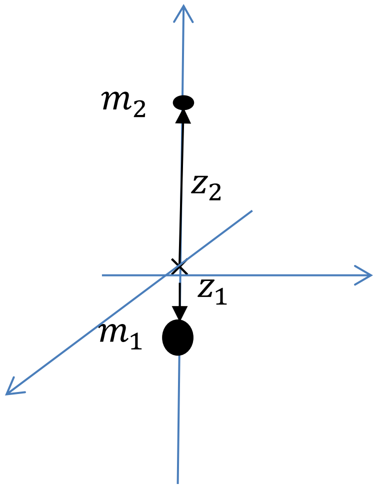
$\rightarrow h_{ij} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij} = C_{ij} =$ Constant. $(\neq 0)$. Note that $ds^2 = -dt^2 + (\delta_{ij} + C_{ij})dx^i dx^j = -dt^2 + \delta_{ij}dx'^i dx'^j$ with $x'^i \equiv x^i/\sqrt{1 + C_{ii}}$ (No sum in i). Thus, the spacetime is still flat, e.g., Minkowski metric. Note also that adding a constant tensor is also a solution of $\partial^2\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}$ and $\partial^\mu\bar{h}_{\mu\nu} = 0$. Actually, there would be no radiation since $\frac{dP}{d\Omega} \sim \left(\frac{d\ddot{Q}_{ij}}{dt}\right)^2 = 0$. For a hyperbolic encounter, however, one may have $h_{ij}(t = -\infty) = C_{ij}^{-\infty}$ & $h_{ij}(t = +\infty) = C_{ij}^{+\infty}$ determined by the asymptotic velocities. These different constancies give a non-vanishing physical effect although both initial and final spacetimes are all flat. This effect is so-called "Gravitational Memory".

ii) Accelerating motion, i.e., $\ddot{x} \neq 0$;

$$h_{ij} = \frac{1}{r}\frac{2G}{c^4}\mathrm{diag}(2,-1,-1)\times\frac{2}{3}m[xa(t) + v^2(t)] \ \text{ and so }\ \frac{d\ddot{Q}_{ij}}{dt} \neq 0$$

Thus, there will be a GW radiation. $h_+(t;\theta,\phi) = \frac{G}{rc^4}\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta) \xrightarrow{\hat{n}=\hat{x}} 0, \xrightarrow{\hat{n}=\hat{y},\hat{z}} \frac{G}{rc^4}\ddot{M}_{11} = \frac{2mG}{rc^4}(x\ddot{x} + \dot{x}^2)$, and $h_\times = \frac{G}{rc^4}\ddot{M}_{11}\sin 2\phi\cos\theta$.

- **Ex2) Harmonic oscillator:**



- For a two-body system,

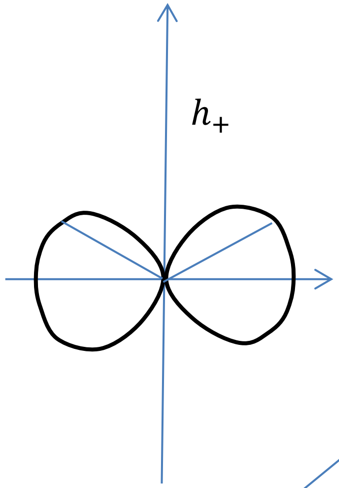
$$M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m_1\left(x_{CM}^i - \frac{m_2}{m}x^i\right)\left(x_{CM}^j - \frac{m_2}{m}x^j\right)$$

$$+m_2\left(x_{CM}^i + \frac{m_1}{m}x^i\right)\left(x_{CM}^j + \frac{m_1}{m}x^j\right) = m_1 x_{CM}^i x_{CM}^j - \frac{m_1 m_2}{m}\left(x_{CM}^i x^j + x^i x_{CM}^j\right)$$

$$+\frac{m_1 m_2^2}{m}x^i x^j + m_2 x_{CM}^i x_{CM}^j + \frac{m_2 m_1}{m}\left(x_{CM}^i x^j + x^i x_{CM}^j\right) + \frac{m_2 m_1^2}{m}x^i x^j$$

$$= (m_1+m_2)x_{CM}^i x_{CM}^j + \frac{m_1 m_2}{m}(m_2+m_1)x^i x^j$$

$\rightarrow \therefore$ $\boxed{M^{ij} = m x_{CM}^i x_{CM}^j + \mu x^i x^j}$ with the total mass $m = m_1 + m_2$, the reduced mass $\mu = m_1 m_2/m$, and $\vec{x} = \vec{x}_2 - \vec{x}_1$. Thus, for an isolated system, $\vec{x}_{CM}$ does not change and so the center-of-mass term does not contribute to radiation.



- Defining $z_0(t) = z_2 - z_1 = L + a\cos\omega_s t$,

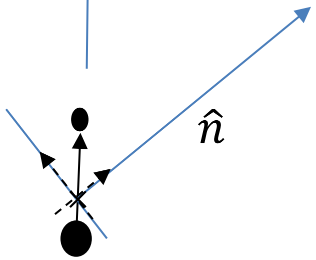
$$M^{ij}(t) = \mu z_0^2(t)\delta^{i3}\delta^{j3} = \mu\delta^{i3}\delta^{j3}(L^2 + 2La\cos\omega_s t + a^2\cos^2\omega_s t)$$

$$= \mu\delta^{i3}\delta^{j3}\left(L^2 + 2La\cos\omega_s t + a^2\frac{1+\cos 2\omega_s t}{2}\right)$$

$$= \mu\delta^{i3}\delta^{j3}\left(\frac{a^2}{2}\cos 2\omega_s t + 2La\cos\omega_s t + Const.\right)$$

➔ $h_+(t;\theta,\phi) = -\frac{G}{rc^4}\ddot{M}_{33}(t_{ret})\sin^2\theta$ & $h_\times(t;\theta,\phi) = 0.$

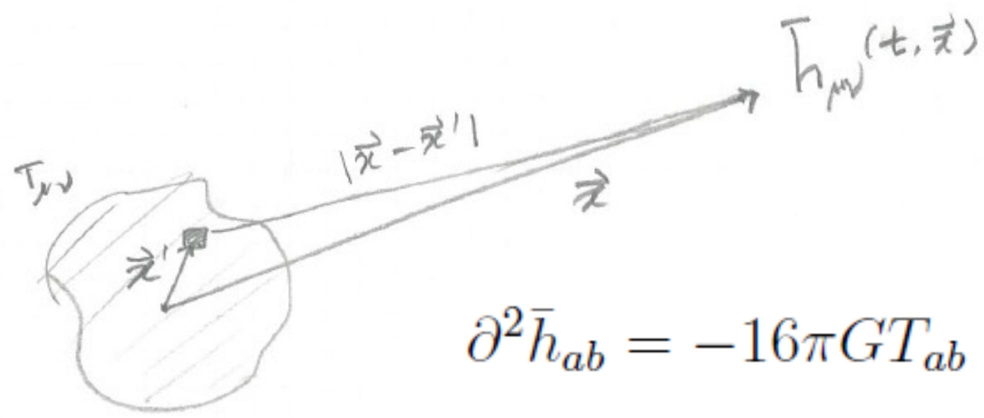
$$\therefore \quad h_+(t;\theta,\phi) = \frac{2G\mu\omega_s^2}{rc^4}\sin^2\theta\,[a^2\cos 2\omega_s(t - r/c) + La\cos\omega_s(t - r/c)]$$



- Note:
  - Frequencies of GWs radiated: $\omega_{GW} = \omega_s, 2\omega_s$
  - No radiation along the oscillation axis: $h_{+,\times}(\theta = 0, \pi) = 0.$ ➔ General result: The component of source motion to the line-of-sight does not contribute since $M^{ij} \sim n^i$ $\rightarrow h_{ij}^{TT} \sim \Lambda_{ij,kl}\, n^k = 0$
  - Symmetric with respect to the perpendicular plane: $h_{+,\times}(\theta) = h_{+,\times}(\pi - \theta)$

✓ 중력파의 발생원과 세기는?

$$I^{ij} = \int \rho x^i x^j d^3x$$



$$\partial^2 \bar{h}_{ab} = -16\pi G T_{ab}$$

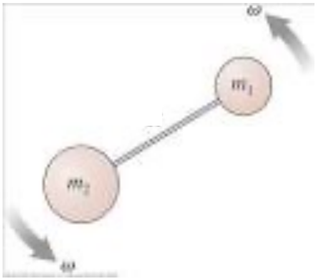
$$h_{ij}(t,\vec{x}) = \frac{2G}{c^4}\frac{1}{r}\ddot{I}_{ij}\left(t - \frac{r}{c}\right)$$

$$\sim \frac{1}{r} \times \omega^2 \times M_{source} \times R_{source}^2 \times \vartheta(1)$$

$$\sim 1.6\times10^{-44}\,\frac{s^2}{kg\,m}$$

$M_{\odot} \sim 10^{30}\ kg$!!

- 주먹을 흔들어도 발생하나 지극히 약함:



$1\ ton \times 2,\ 2\ m\ \&\ 1\ kHz$

➔ $\boldsymbol{h_{GW} \sim 9\times10^{-39}}$

at $r \sim \lambda = 300\ km$

$\sim 10^{11}$ protons at $v \sim 0.999999991c$

➔ $\boldsymbol{h_{GW} \sim 10^{-43}}$

✓ 중력파에 의한 지구 직경의 변화: $h_{GW} \sim \Delta L/L$

$$\frac{\Delta L}{L} = h$$

$$= \frac{1}{1000000000000000000000}$$

$\Delta L \sim h\ L \sim 10^{(-21)} \times 6{,}400\text{km} \times 2$

$\sim 10^{(-14)}\ \text{m}$

~ Size of a proton

➔ 극도로 약한 효과!!

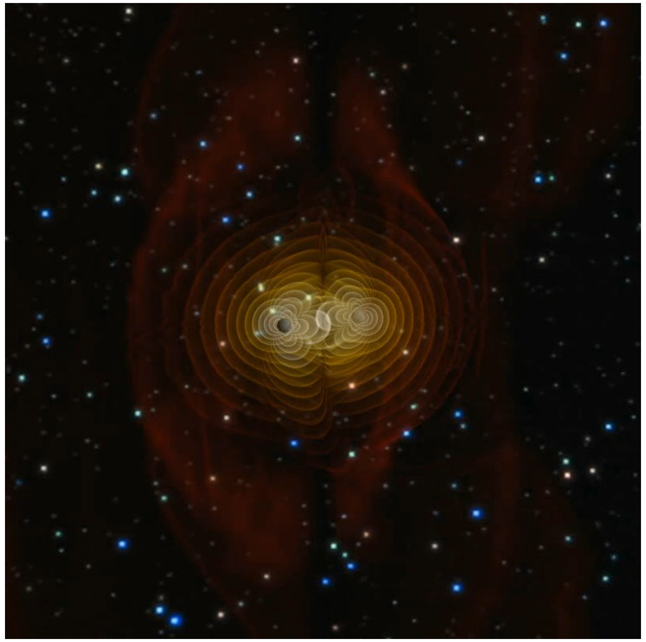
- 격렬한 천체현상에서 강력한 중력파 발생:

$$h_{\mathrm{BM}} \sim 10^{-20} \frac{\mathrm{Mpc}}{r} \frac{M}{\mathrm{M}_\odot} \left(\frac{M}{\mathrm{M}_\odot} \frac{f}{\mathrm{kHz}}\right)^{2/3}$$

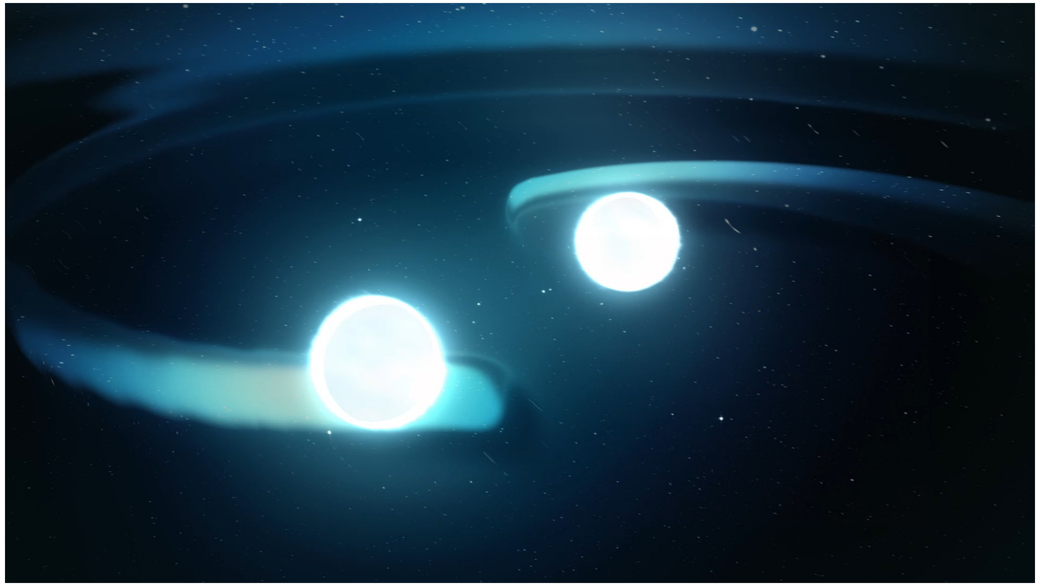
Ex) Black hole binary of $10M_\odot$

➔ $h \sim 5 \times 10^{-21}$

($\gg 10^{-39}$)

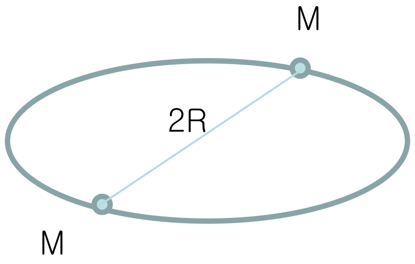


블랙홀 병합과 중력파 발생 (Credit: NASA/C. Henze)



중성자별 병합과 중력파-전자기파 발생
(Credit: LIGO/SXS/R.Hurt and T. Pyle)

Ex) Radiated energy from a compact binary system:



$$\rho(t,\vec{x}) = M\delta(x - R\cos\Omega t)\delta(y - R\sin\Omega t)\delta(z) + M\delta(x + R\cos\Omega t)\delta(y + R\sin\Omega t)\delta(z)$$

$$\begin{aligned} I^{xx} &= \int \rho\, x\, x\, d^3x \\ &= M\,(R\cos\Omega t)^2 \int \delta(y - R\sin\Omega t)\delta(z)\, dydz + M\,(-R\cos\Omega t)^2 \\ &= 2MR^2\cos^2(\Omega t) \\ &= MR^2\,[1 + \cos(2\Omega t)] \end{aligned}$$

$$\begin{aligned} I^{xy} &= \int \rho\, x\, y\, d^3x \\ &= 2MR^2\cos\Omega t\sin\Omega t \\ &= MR^2\sin(2\Omega t) \end{aligned}$$

$$\begin{aligned} I^{yy} &= 2MR^2\sin^2\Omega t \\ &= MR^2\,[1 - \cos(2\Omega t)] \end{aligned}$$

$$\bar{h}_{ij} = \frac{2G}{rc^4}\,\ddot{I}_{ij}$$

$$\bar{h}^{ij}_{\mathrm{TT}} \sim -\frac{G}{c^4}\frac{8\Omega^2 M R^2}{r}\begin{pmatrix} \cos[2\Omega(t-r)] & \sin[2\Omega(t-r)] & 0 \\ \sin[2\Omega(t-r)] & -\cos[2\Omega(t-r)] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= -\frac{8G\Omega^2 M R^2}{rc^4}\cos 2\Omega(t-r)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

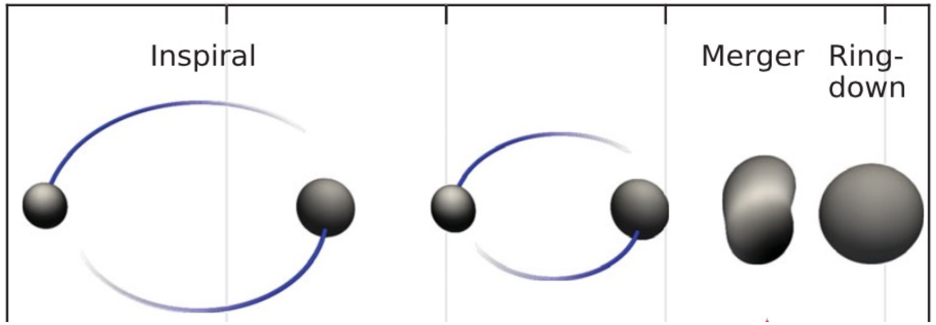
$$-\frac{8G\Omega^2 M R^2}{rc^4}\sin 2\Omega(t-r)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P = \frac{G}{5c^5}\langle \dddot{\mathbb{I}}_{ij}\dddot{\mathbb{I}}^{ij} \rangle$$

$$P = \frac{128G}{5c^5} M^2 R^4 \Omega^6$$

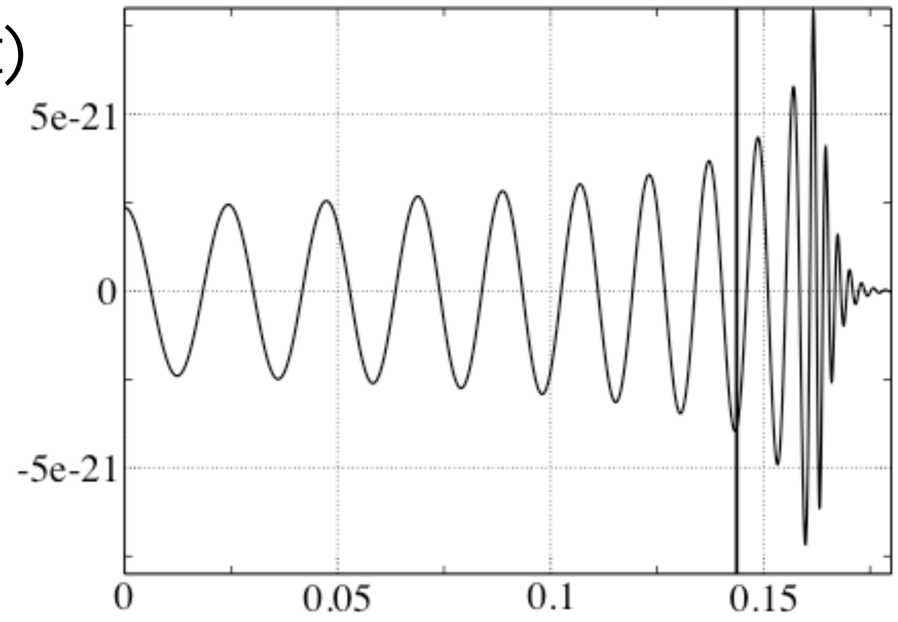
$$= 1.9 \times 10^{33}\left(\frac{M}{M_\odot}\frac{1h}{T}\right)^{10/3}\frac{\mathrm{erg}}{s}$$

Sun's luminosity ~ 3.9×10^{33} erg/s



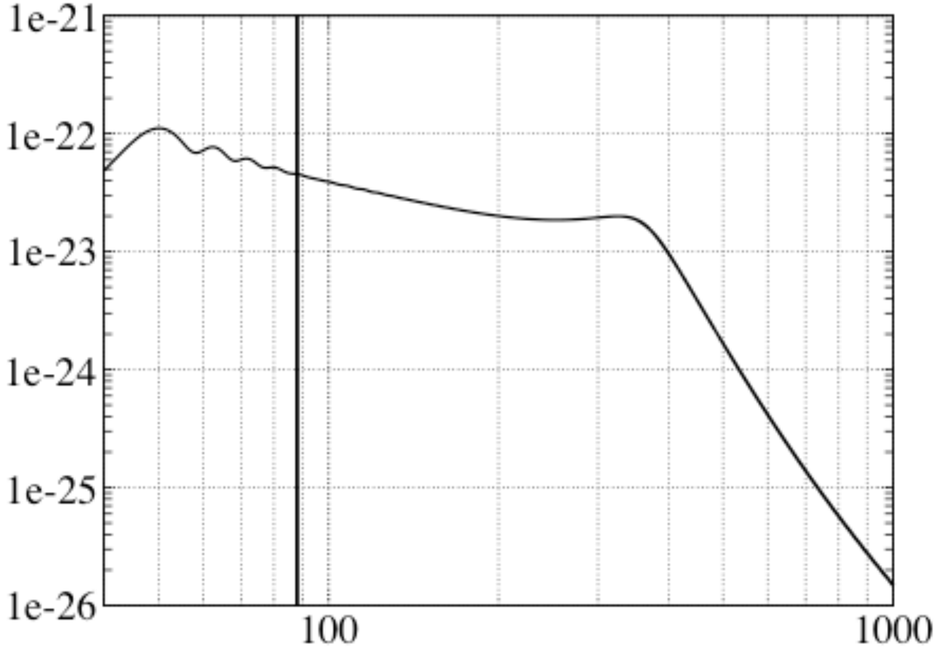
- Binaries emit GWs, resulting in decays of orbit.
- Eventually collide or merge
- Quickly becomes quite, e.g., a stationary single spinning BH which is probably described by the Kerr metric
- PN gives waveforms for inspiral and ringdown phases

h(t)



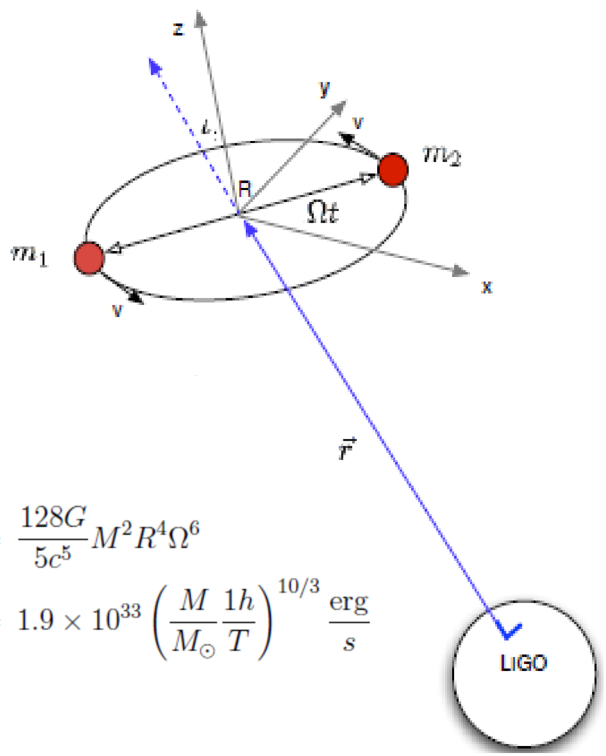
t-ime

h(f)



f-requency

Abadie et al. arXiv:1102.3781



$$P = \frac{128G}{5c^5} M^2 R^4 \Omega^6$$
$$= 1.9 \times 10^{33} \left(\frac{M}{M_\odot}\frac{1h}{T}\right)^{10/3} \frac{\text{erg}}{s}$$

$$\bar{h}^{ij}_{\rm TT} \sim -\frac{G}{c^4}\frac{8\Omega^2 M R^2}{r}\begin{pmatrix} \cos[2\Omega(t-r)] & \sin[2\Omega(t-r)] & 0 \\ \sin[2\Omega(t-r)] & -\cos[2\Omega(t-r)] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h \sim \frac{1}{r} M R^2 f^2 \cos 2\pi f(t-r) \quad \text{with} \quad 2\Omega = \Omega_{GW} = 2\pi f$$

$$f \to f(t):? \qquad \frac{1}{R^2} \sim \frac{v^2}{R} \sim f^2 R \to f^2 \sim \frac{1}{R^3} \to \frac{\dot{f}}{f} = -\frac{3}{2}\frac{\dot{R}}{R}$$

$$E \sim -\frac{G m_1 m_2}{2R} \sim f^{\frac{2}{3}} \to \dot{E} \sim f^{-1/3}\dot{f} \sim -P \sim -f^{10/3}$$

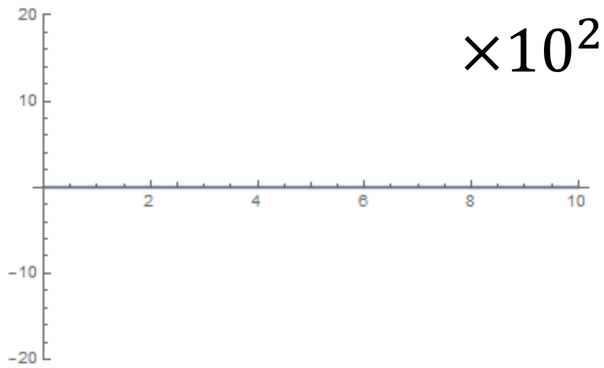
➔ $f(t) \sim (t_{coal} - t)^{-3/8}$

$$h_{+(t)} \sim -\frac{G\mathcal{M}/c^2}{r}\frac{1+\cos^2\iota}{2}\left(\frac{5G\mathcal{M}/c^3}{t_c - t}\right)^{1/4} \cos\left[\left(\frac{t_c - t}{5G\mathcal{M}/c^3}\right)^{5/8} - 2\phi_c\right]$$

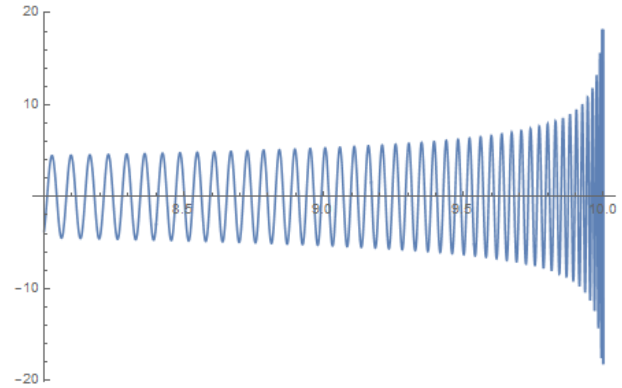
Ex) m1=36Msun, m2=29Msun, r=410Mpc

➔ $M_{Chirp} = \frac{(m1 * m2)^{3/5}}{(m1 + m2)^{1/5}}$ ~ 28 Msun

$$h(f) \sim e^{i\psi(f)} f^{-7/6}$$



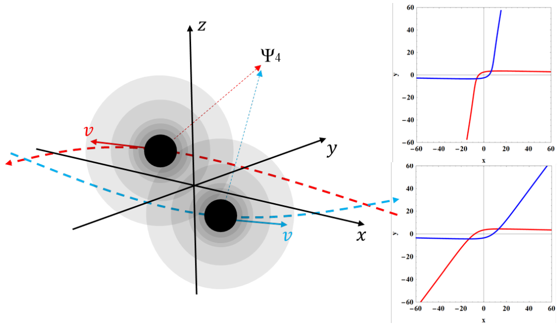
$\times 10^{22}$ ➔

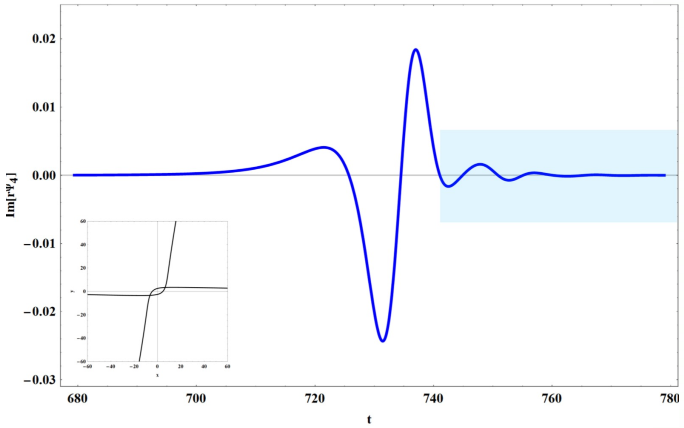


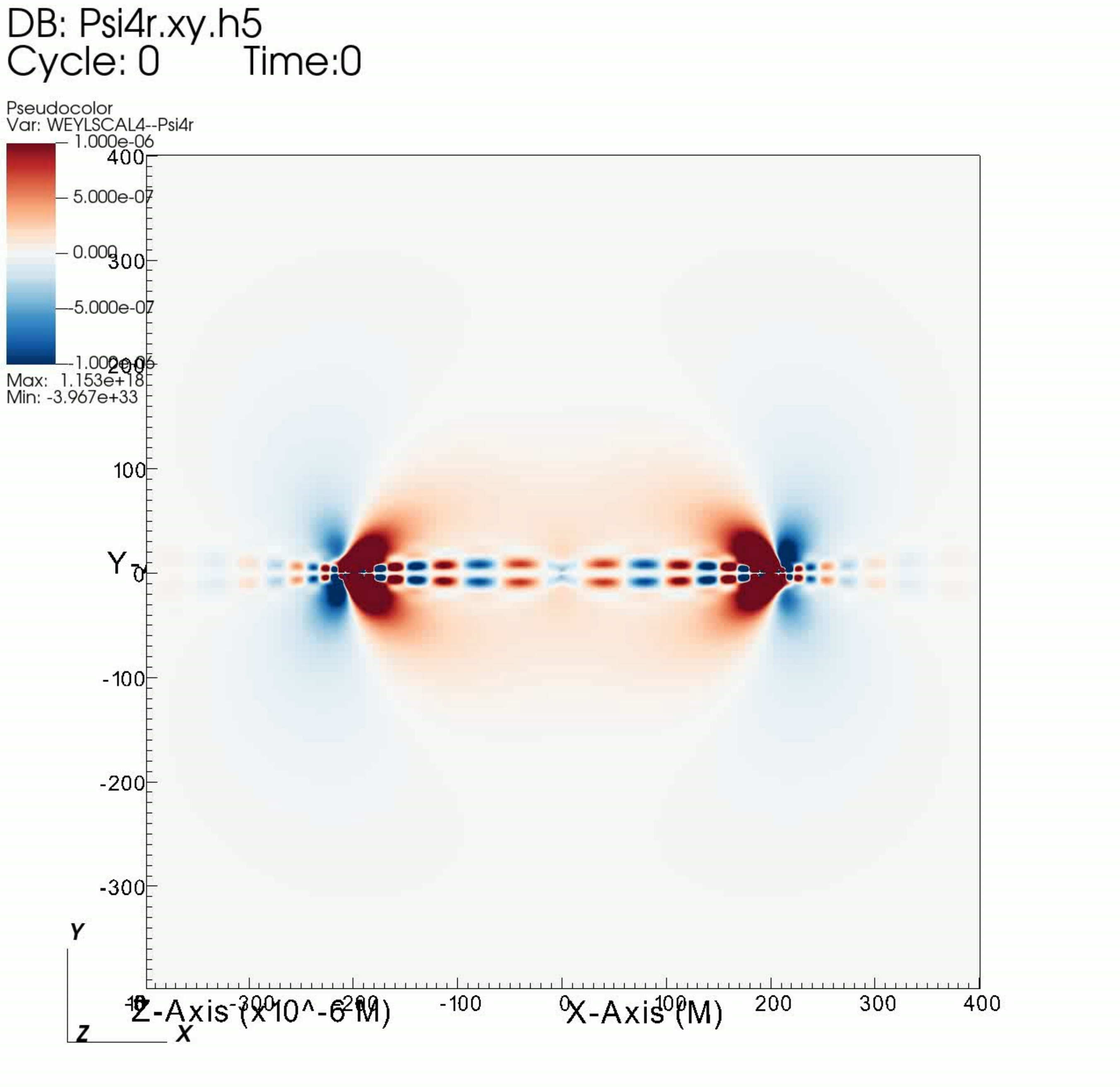
✓ Propagation and interactions of ST wiggles:

$$\psi_4 \equiv R_{abcd} n^a l^b m^c \bar{m}^d$$
$$\cong \ddot{h}_+ - i\ddot{h}_\times$$

Bae, Hyun & Kang (PRL 2024):





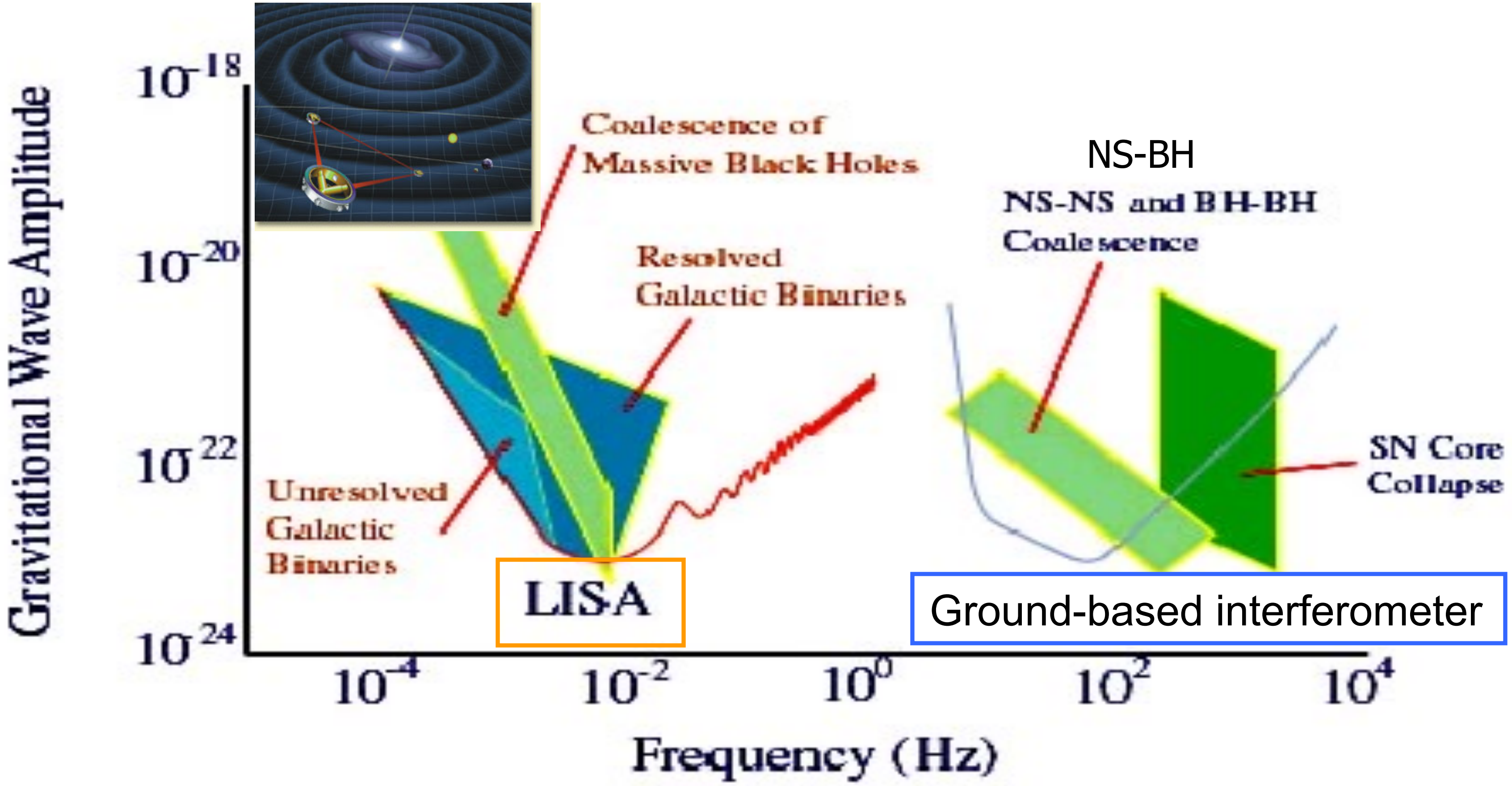


(Credit: Dongchan Kim)

- Astronomical sources:
  - BH-BH, BH-NS, NS-NS coalescences
  - Supernova explosions: GW+Neutrino+…
  - Stochastic signals
  - Cosmic string kinks
  - Etc.

  - Galaxies ~1,000억 개/Universe. Stars ~1,000억 개/Galaxy.

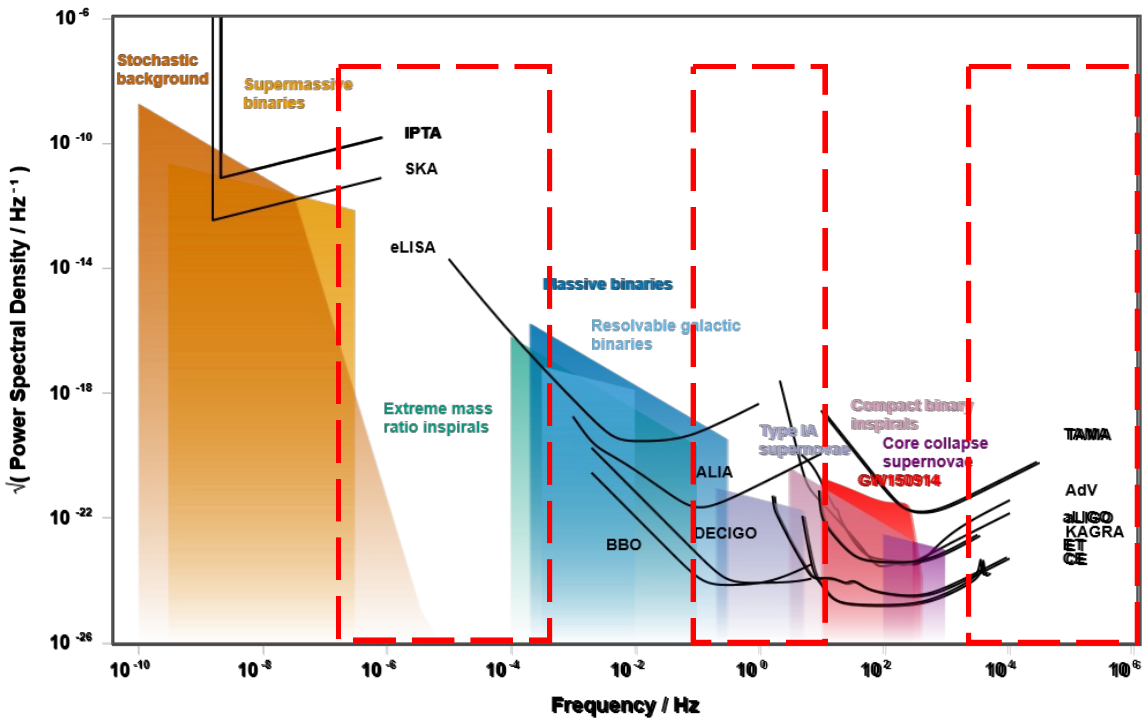
GW astronomy



NS-WD
space-based
f_{gw}~0.01-100 mHz

NS-NS (DNS)
ground-based
(f_{gw}~a few x10-10^3 Hz)

✓Gravitational wave spectrum and detectors:



Based on http://rhcole.com/apps/GWplotter/ by Moore, Cole & Berry

- Interferometer-type detectors:
  - aLIGO, aVirgo, KAGRA, eLISA, INDIGO
- Plans to improve sensitivity:
  - A+ (~22), Voyager (~25), Einstein Telescope (~23), Cosmic Explorer (~27), AAGO, TAIJI, TianQin, etc.
- **Covers 10~1000 Hz only!**
- Multi-wavelength GW astronomy
- Mid-Frequencies (0.1~10 Hz):
  - Space: DECIGO ('01), BBO, ALIA, AMIGO
  - Ground: TOBA ('10), MIGA, ZAIGA

+ "SOGRO" (~'13)

- <u>Bandwidths and significances of sources</u>: (Cutler & Thorne '02)
  - Extremely Low Freq. band (ELF, $\mathbf{10^{-15}\sim10^{-18}}\ \boldsymbol{Hz}$):
    - Primordial GWs
    - Imprint on the polarization of CMB radiations
    - Quantum origin at big bang subsequently amplified by inflation
    - Great potential for probing the physics of inflation
  - Very Low Freq. band (VLF, $\mathbf{10^{-7}\sim10^{-9}}\ \boldsymbol{Hz}$):
    - Emitted by pulsars (e.g., Hulse-Taylor '75)
    - via pulsar timing array, or indirectly by pulses at earth
    - Extremely massive BH binary or violent processes in 0.1 second of the early universe
  - Low Freq. band (LF, $\mathbf{10^{-4}\sim0.1}\ \boldsymbol{Hz}$):
    - From massive (10^5~10^7M⊙) BH binaries out to cosmological distances (CD)
    - From small BHs, NSs and WDs spiraling into massive BHs out to CDs
    - From orbital motions of WDB, NSB, and stellar-mass BHB in our own galaxy
    - And possibly from violent processes in the very early universe
    - To be observed by the space-based detector, LISA
  - **High Freq. band (HF, $\mathbf{10\sim10^{3}}\ \boldsymbol{Hz}$):**
    - From a spinning slightly deformed NS in our Milky Way galaxy
    - From a variety of sources in the more distance:
      - Final inspiral and collisions of NSB and stellar-mass BHB (up to ~100M⊙)
      - Tearing apart of a NS by a companion BH
      - Supernovae, Triggers of GRBs, etc.
    - **<u>To be measured by earth-based detectors such as LIGO, Virgo, KAGRA, and resonant-mass bar</u>**

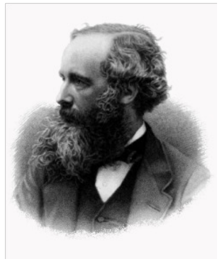
결 론

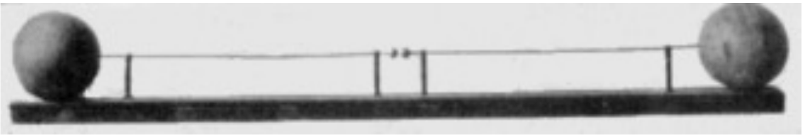
- 특수상대론, 일반상대론, 중력파에 대한 개요

- 중력파는 우주를 탐색하는 매우 독특하고 강력한 도구
- 장기적인 과학사적 관점에서 보면 여전히 태동기
- 과학적 성취의 잠재력이 매우 높고 도전할 미개척지 많음
- 다중신호 천문학 등 기존 관측방법과의 협력으로 상호보완 및 종합적 규명과 이해 가능

EM Waves

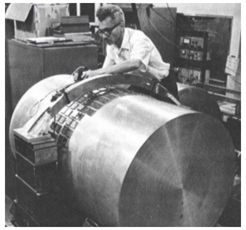
- Theory: Maxwell (1864)
- Detection: H. Hertz (1886)



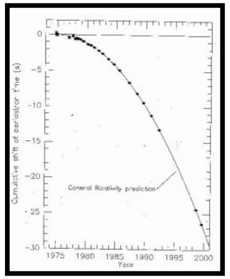


Gravitational Waves

- Theory: Einstein (1916)
- Detection: Not yet (??)



Weber (1960)



Hulse & Taylor (1975)