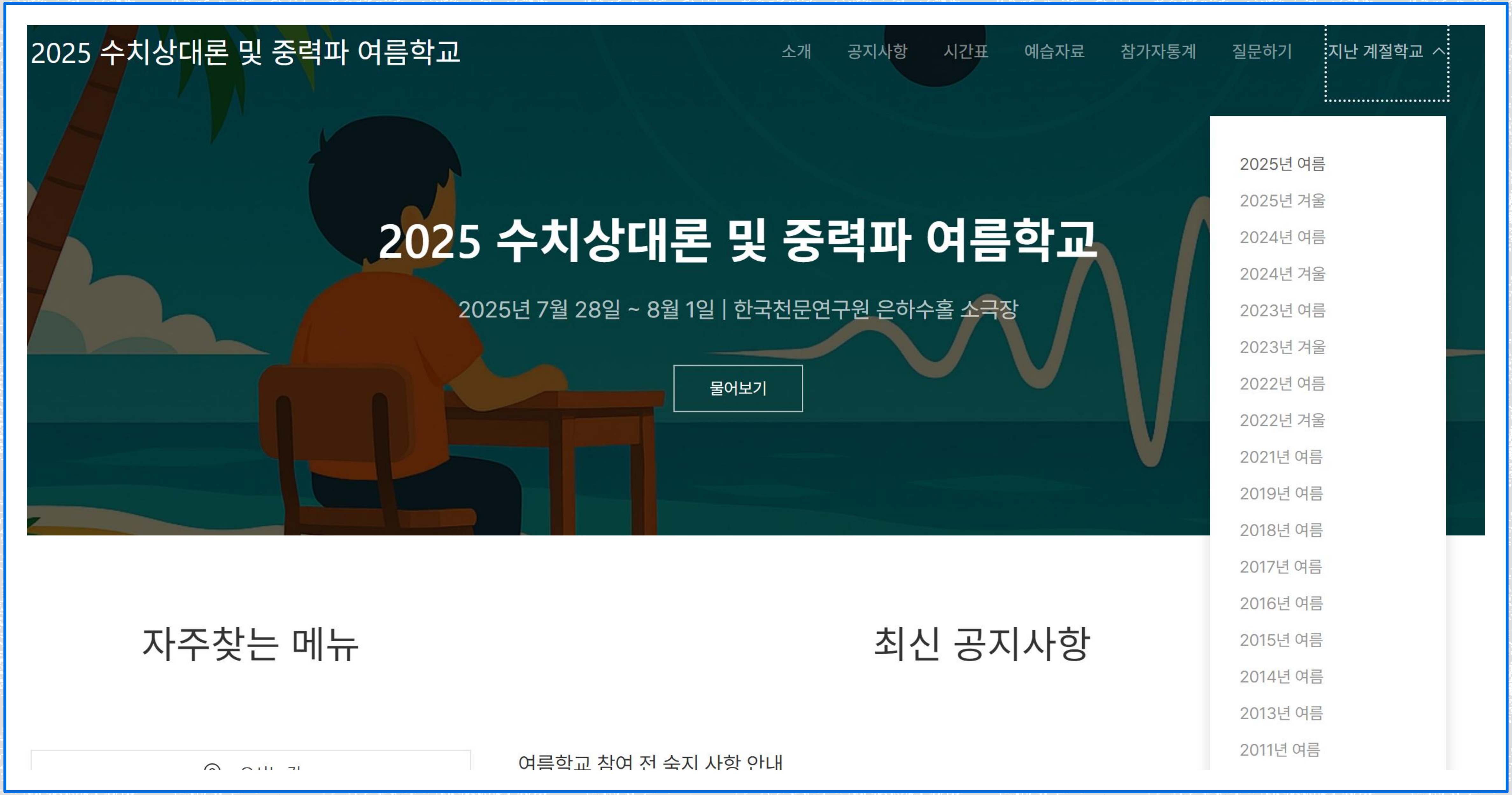
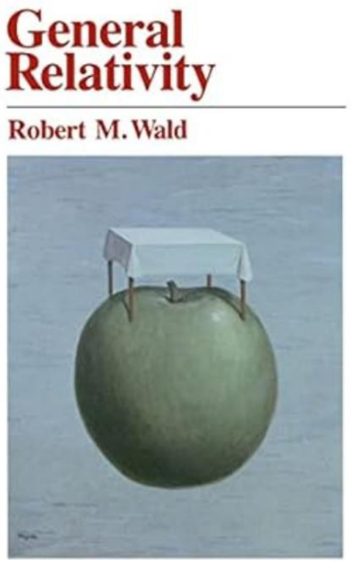
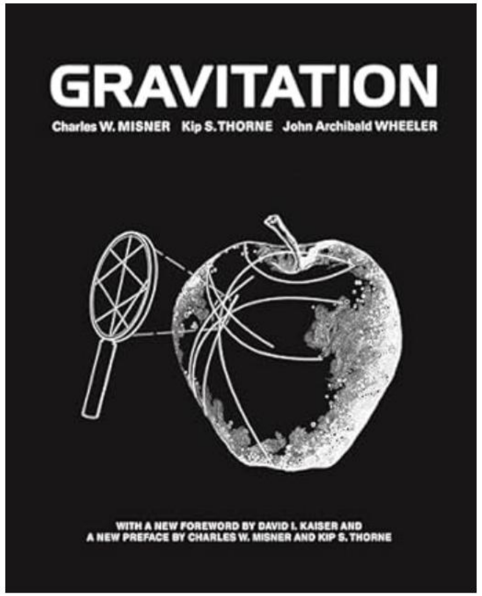
3+1 FORMALISM

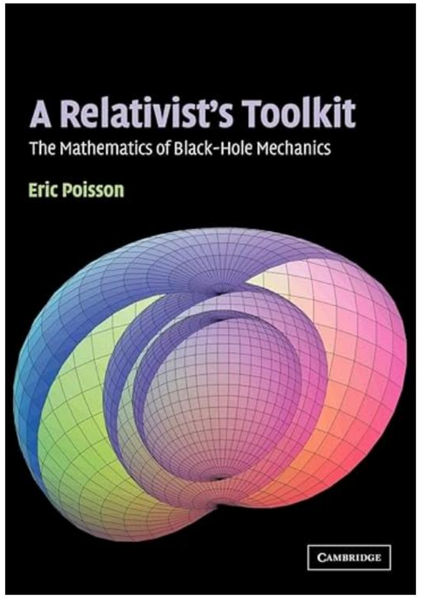
- Young-Hwan Hyun (CAU)
- **2025.07.31. Thursday**
- **2025 GWNR Summer School**
- **@KASI (Daejeon, Korea)**

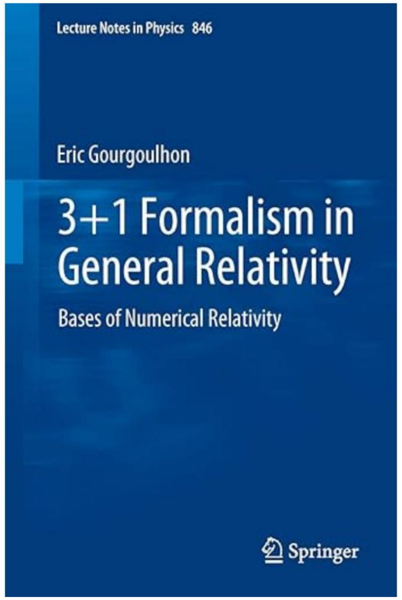

2025 수치상대론 및 중력파 여름학교
소개
공지사항
시간표
예습자료
참가자통계
질문하기
지난 계절학교
2025년 여름
2025년 겨울
2024년 여름
2024년 겨울
2023년 여름
2023년 겨울
2022년 여름
2022년 겨울
2021년 여름
2019년 여름
2018년 여름
2017년 여름
2016년 여름
2015년 여름
2014년 여름
2013년 여름
2011년 여름
2025 수치상대론 및 중력파 여름학교
2025년 7월 28일 ~ 8월 1일 | 한국천문연구원 은하수홀 소극장
물어보기
자주찾는 메뉴
최신 공지사항
여름학교 참여 전 숙지 사항 안내

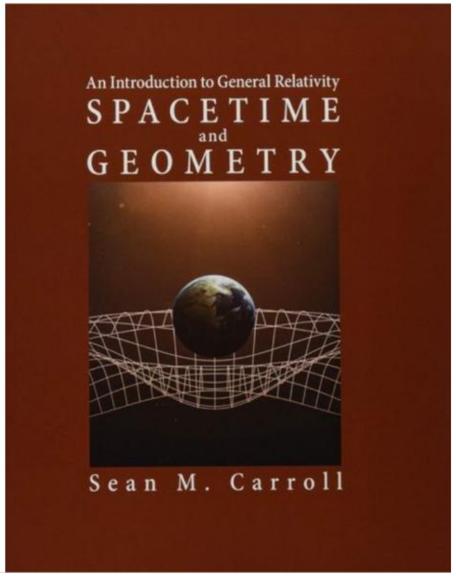
REFERENCES


General Relativity
Robert M. Wald

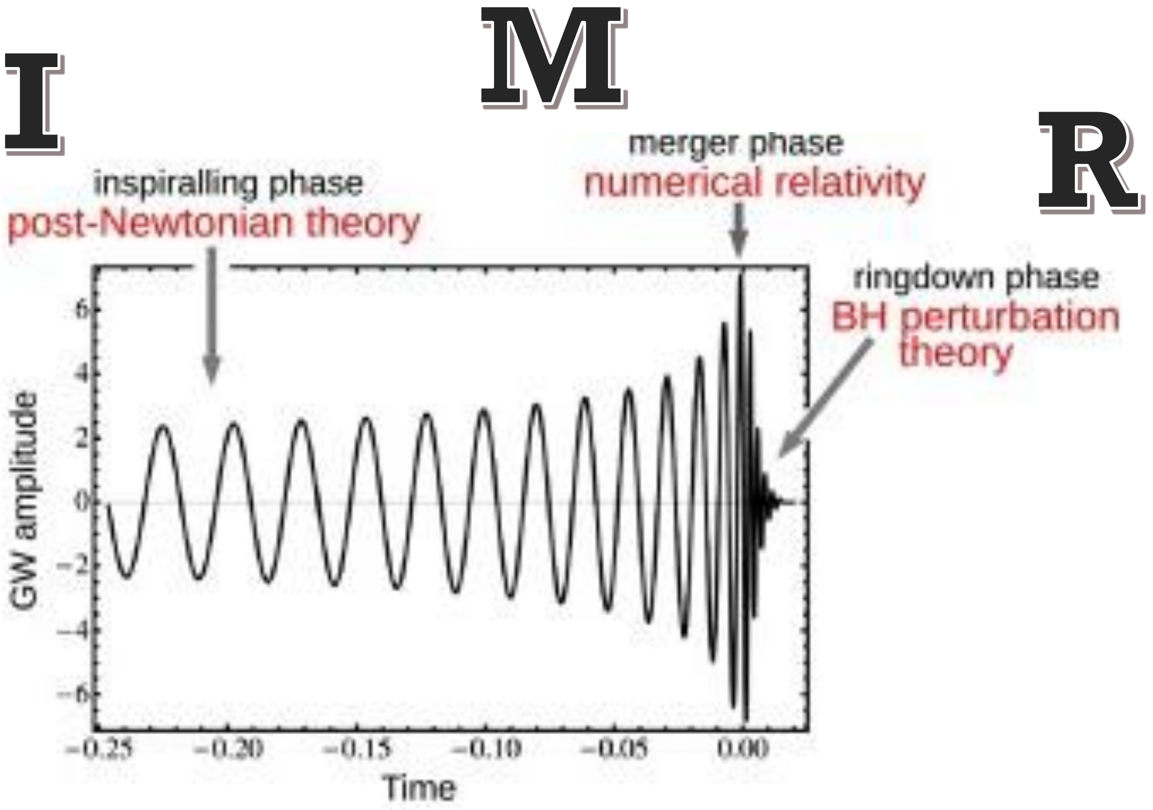

GRAVITATION
Charles W. MISNER Kip S. THORNE John Archibald WHEELER
WITH A NEW FOREWORD BY DAVID I. KAISER AND
A NEW PREFACE BY CHARLES W. MISNER AND KIP S. THORNE

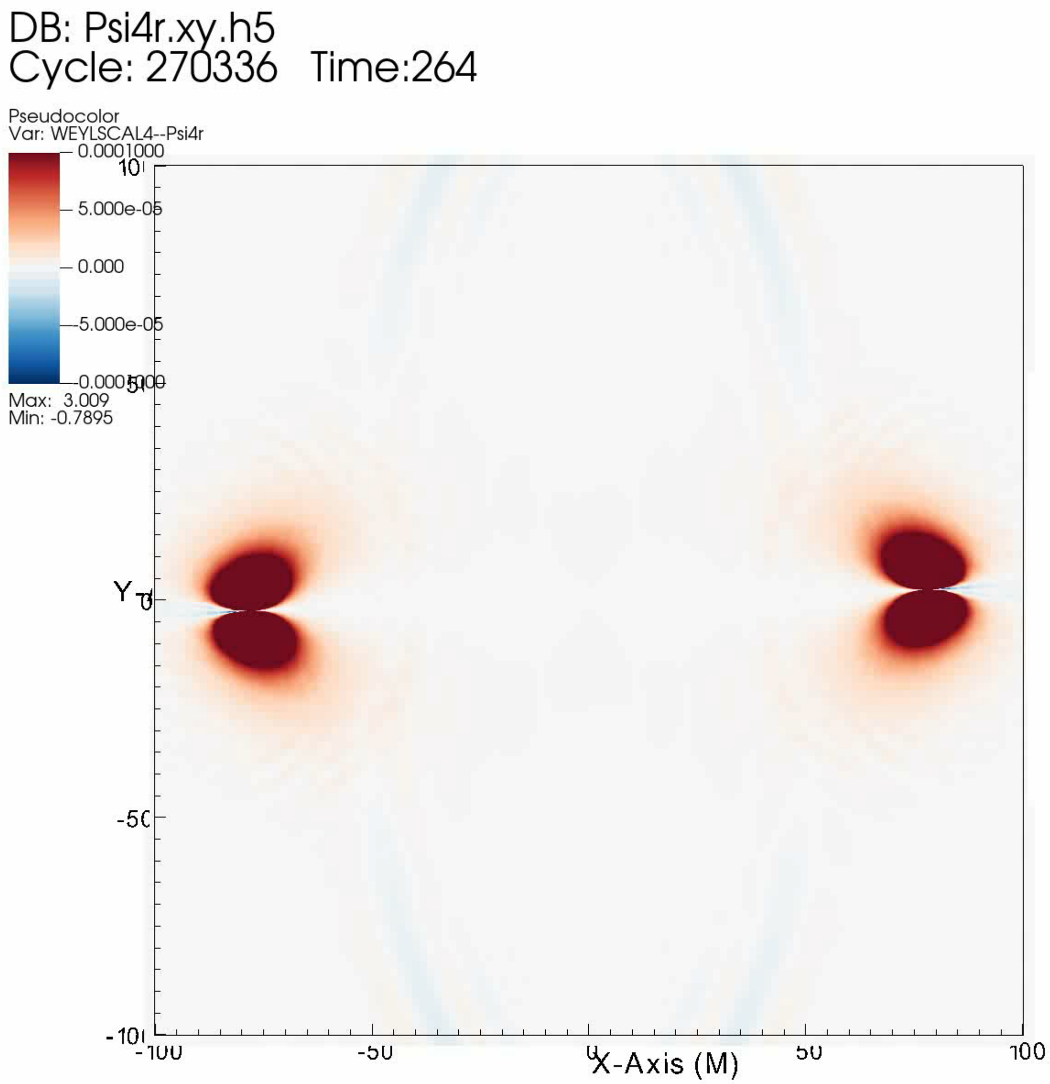

A Relativist's Toolkit
The Mathematics of Black-Hole Mechanics
Eric Poisson
CAMBRIDGE


Lecture Notes in Physics 846
Eric Gourgoulhon
3+1 Formalism in General Relativity
Bases of Numerical Relativity
Springer


An Introduction to General Relativity
SPACETIME
and
GEOMETRY
Sean M. Carroll

NR GW SUMMER SCHOOL





Produced by Dongchan Kim

user: axis
Tue May 21 19:37:53 2024

EINSTEIN EQ.

ELEMENTS IN EINSTEIN'S EQ.

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein tensor

Curvature of spacetime

Rank-2 symmetric tensor

4x4 matrix components

Divergence-free

Newtonian g'l constant

General relativity

Accelerated objects

Related to gravity

$6.674 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$

Speed of light

- Special relativity
- Locally non-gravitational physics
- Causal structure of spacetime

$c = 299,792,458\,\mathrm{m/s}$

Energy-momentum tensor

Matter(Energy) distribution

Energy, momentum, pressure

Stars, galaxies, …

Standard model particle,

Dark matter, dark energy,

vacuum, radiation….

COUNTING # OF EQNS IN EINSTEIN'S EQ.

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- **Rank-2 tensor > 2 dimensional matrix**
- **Index range:** $\mu = 0, 1, 2, 3 \leftrightarrow x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z)$
  > 4x4 matrix
- **Symmetric tensor:** $G_{\mu\nu} = G_{\nu\mu} \quad G_{01} = G_{10}, \; G_{02} = G_{20}, \cdots$
- **We have 10 eqns.:**

$$\begin{pmatrix} G_{00} & G_{01} & G_{02} & G_{03} \\ G_{01} & G_{11} & G_{12} & G_{13} \\ G_{02} & G_{12} & G_{22} & G_{23} \\ G_{03} & G_{13} & G_{23} & G_{33} \end{pmatrix}$$

3+1?

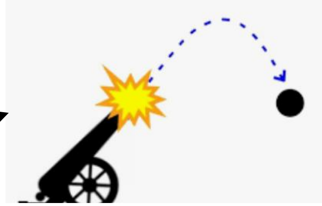
$$G_{00} = \frac{8\pi G}{c^4} T_{00}$$
$$G_{01} = \frac{8\pi G}{c^4} T_{01}$$
$$G_{11} = \frac{8\pi G}{c^4} T_{11}$$
$$\vdots$$

VARIABLES IN EINSTEIN'S EQ.

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

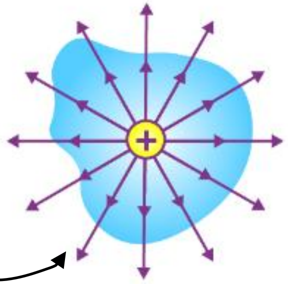
- Metric tensor
  - Unknowns to solve in Einstein's eq. such as x in f(x)=y.
  - Field variable, not x(t) but $g_{\mu\nu}(t,x)$
  - Related to gravitational field **g** in Newtonian gravity
  - Describes the spacetime structure
  - Fundamental quantity: 4x4 symmetric tensor, 10 dof's

    >> What is its value? 10 is real dofs? Why not just 3+1?



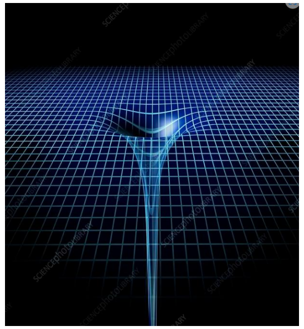
$$\mathbf{F} = m\mathbf{a} \rightarrow \mathbf{x}(t)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \rightarrow \mathbf{E}(x)$$



3+1?

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}$$



3+1 DECOMP. OF EINSTEIN EQ.?

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{2} g_{\mu\nu} R(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- We don't know the exact functional form of G in terms of g yet.

Riemann tensor :

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} = \boxed{2\partial_{[\mu}\Gamma^\rho_{\nu]\sigma} + 2\Gamma^\rho_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\sigma}}$$

Weyl tensor (Not Ricci in Riemann) :

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{4}{(n-2)} g_{[\rho[\mu} R_{\nu]\sigma]} + \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R$$

Ricci tensor :

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu} = \boxed{2\partial_{[\rho}\Gamma^\rho_{\nu]\mu} + 2\Gamma^\rho_{[\rho|\lambda|}\Gamma^\lambda_{\nu]\mu}}$$

Ricci scalar :

$$R \equiv R^{\mu\nu}{}_{\mu\nu} = 2\partial_{[\mu}\Gamma^\mu_{\nu]\rho} g^{\rho\nu} + 2\Gamma^\mu_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\rho} g^{\rho\nu}$$

Levi-Civita(affine) connection (Christoffel symbol) :

$$\boxed{\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})}$$

?

3+1 DECOMP. OF EINSTEIN EQ.?

$$G_{\mu\nu}(g_{\mu\nu})$$

```
EinsteinCD[-μ, -ν] // ToRicci // RiemannToChristoffel // NoScalar // ChristoffelToGradMetric[#, g] & // Expand //
ToCanonical
```

확장

$$-\frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\gamma\delta} - \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\alpha g_{\mu\nu} + \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\mu g_{\nu\alpha} + \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\nu\delta} \partial_\gamma g_{\mu\alpha} +$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\delta g_{\beta\gamma} + \frac{1}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_\delta g_{\delta1\delta2} + \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\gamma g_{\mu\alpha} \partial_\delta g_{\nu\beta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta \partial_\beta g_{\alpha\gamma} +$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta \partial_\gamma g_{\alpha\beta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\delta g_{\beta\delta2} \partial_{\delta1} g_{\alpha\gamma} - \frac{3}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_{\delta1} g_{\alpha\gamma} \partial_{\delta2} g_{\beta\delta} +$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\beta g_{\alpha\gamma} \partial_{\delta2} g_{\delta\delta1} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_{\delta2} g_{\delta\delta1} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\mu g_{\nu\alpha} -$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\mu g_{\nu\alpha} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\mu g_{\alpha\gamma} \partial_\nu g_{\beta\delta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} \partial_\nu \partial_\mu g_{\alpha\beta}$$

$$\frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\alpha\partial_\mu g_{\beta\nu} = \frac{1}{2}g^{00}\partial_0\partial_\mu g_{0\nu} + \frac{1}{2}g^{01}\partial_0\partial_\mu g_{1\nu} + \frac{1}{2}g^{02}\partial_0\partial_\mu g_{2\nu} + \frac{1}{2}g^{03}\partial_0\partial_\mu g_{3\nu}$$
$$+\frac{1}{2}g^{10}\partial_1\partial_\mu g_{0\nu} + \frac{1}{2}g^{11}\partial_1\partial_\mu g_{1\nu} + \frac{1}{2}g^{12}\partial_1\partial_\mu g_{2\nu} + \frac{1}{2}g^{13}\partial_1\partial_\mu g_{3\nu} + \frac{1}{2}g^{20}\partial_2\partial_\mu g_{0\nu} + \frac{1}{2}g^{21}\partial_2\partial_\mu g_{1\nu}$$
$$+\frac{1}{2}g^{22}\partial_2\partial_\mu g_{2\nu} + \frac{1}{2}g^{23}\partial_2\partial_\mu g_{3\nu} + \frac{1}{2}g^{30}\partial_3\partial_\mu g_{0\nu} + \frac{1}{2}g^{31}\partial_3\partial_\mu g_{1\nu} + \frac{1}{2}g^{32}\partial_3\partial_\mu g_{2\nu} + \frac{1}{2}g^{33}\partial_3\partial_\mu g_{3\nu}$$

- 2nd order PDE of $g_{\mu\nu}$

3+1 DECOMP. OF EINSTEIN EQ.

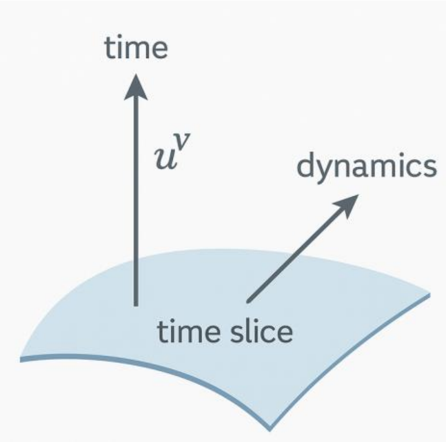
$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1) \ ^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow \ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2) \ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow \ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3) \ ^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \ \rightarrow \ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\ \qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ \qquad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow \ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$

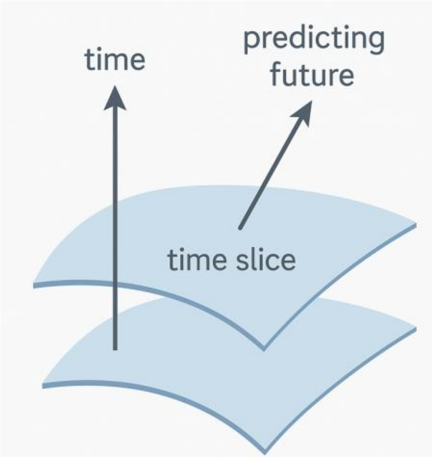
2 QUESTIONS

TWO BIG QUESTIONS BEFORE 3+1 FORMALISM

1. **How Do We Define Physical Quantities in GR?**

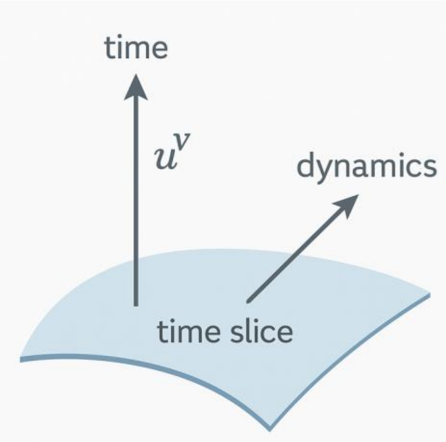


2. **Can We Predict the Future in GR from the Present?**

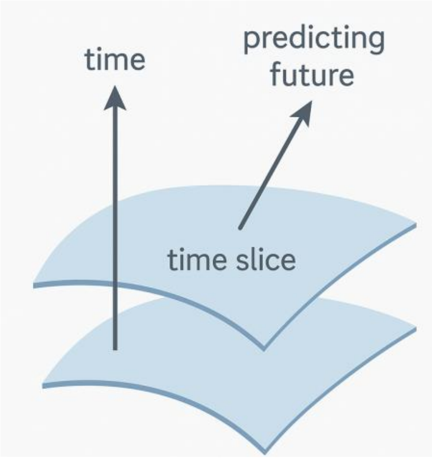


TWO BIG QUESTIONS BEFORE 3+1 FORMALISM

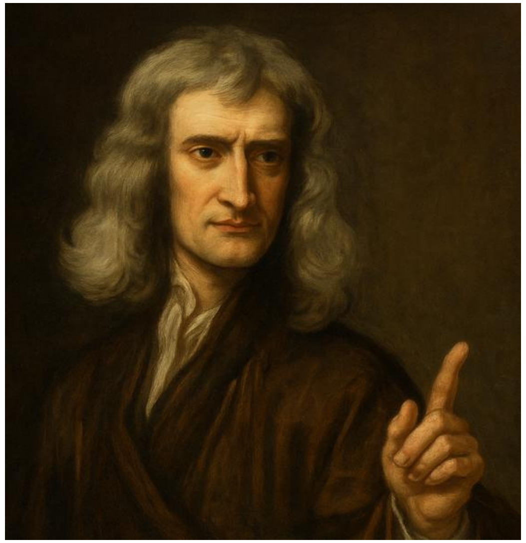
1. **How Do We Define Physical Quantities in GR?**

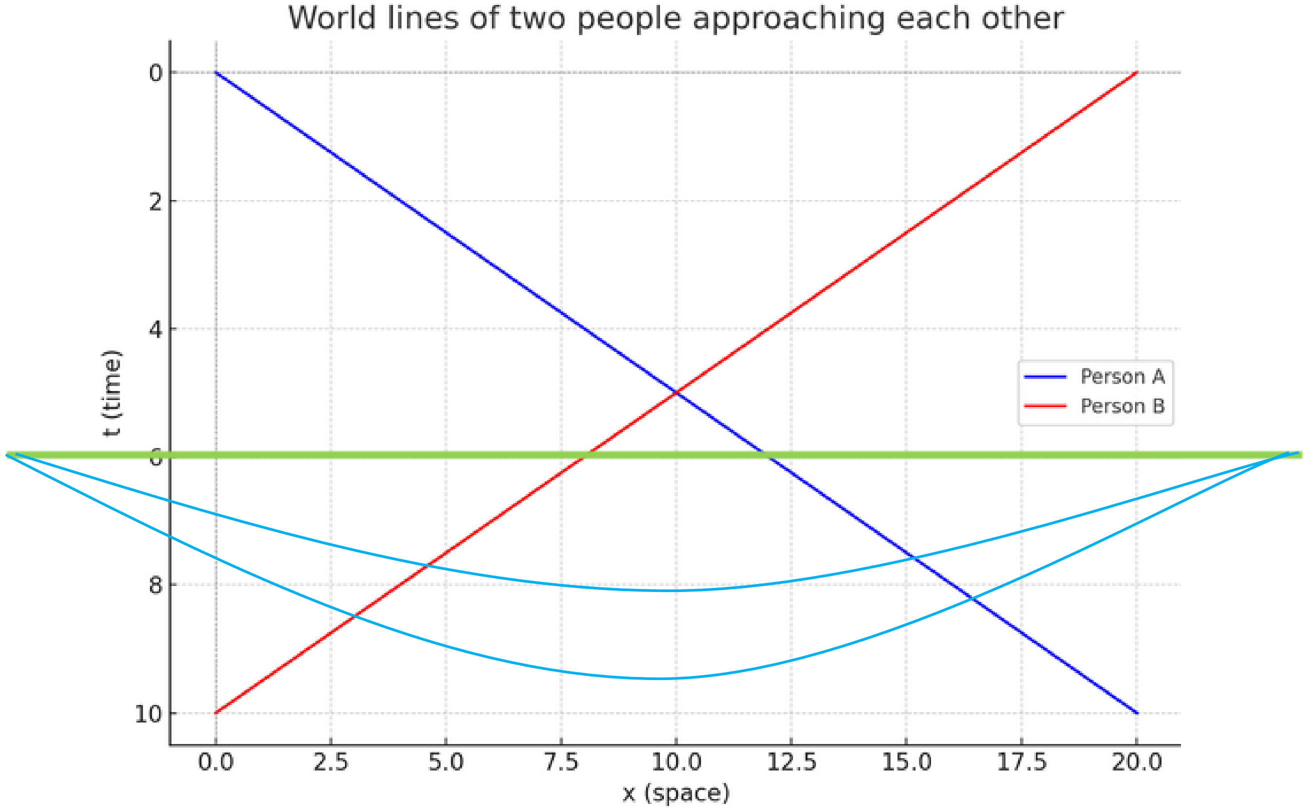


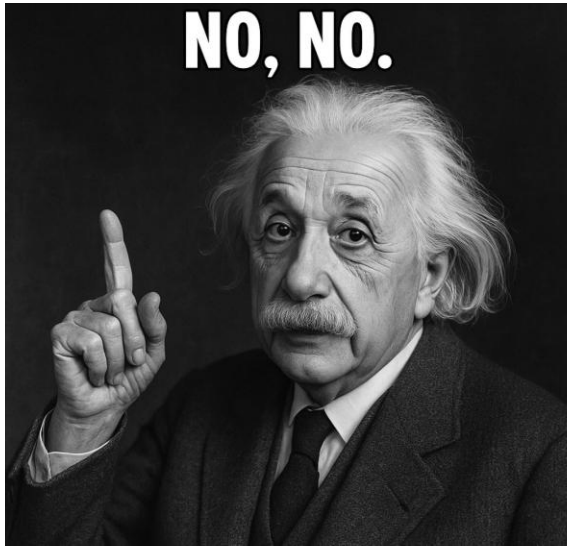
2. **Can We Predict the Future in GR from the Present?**



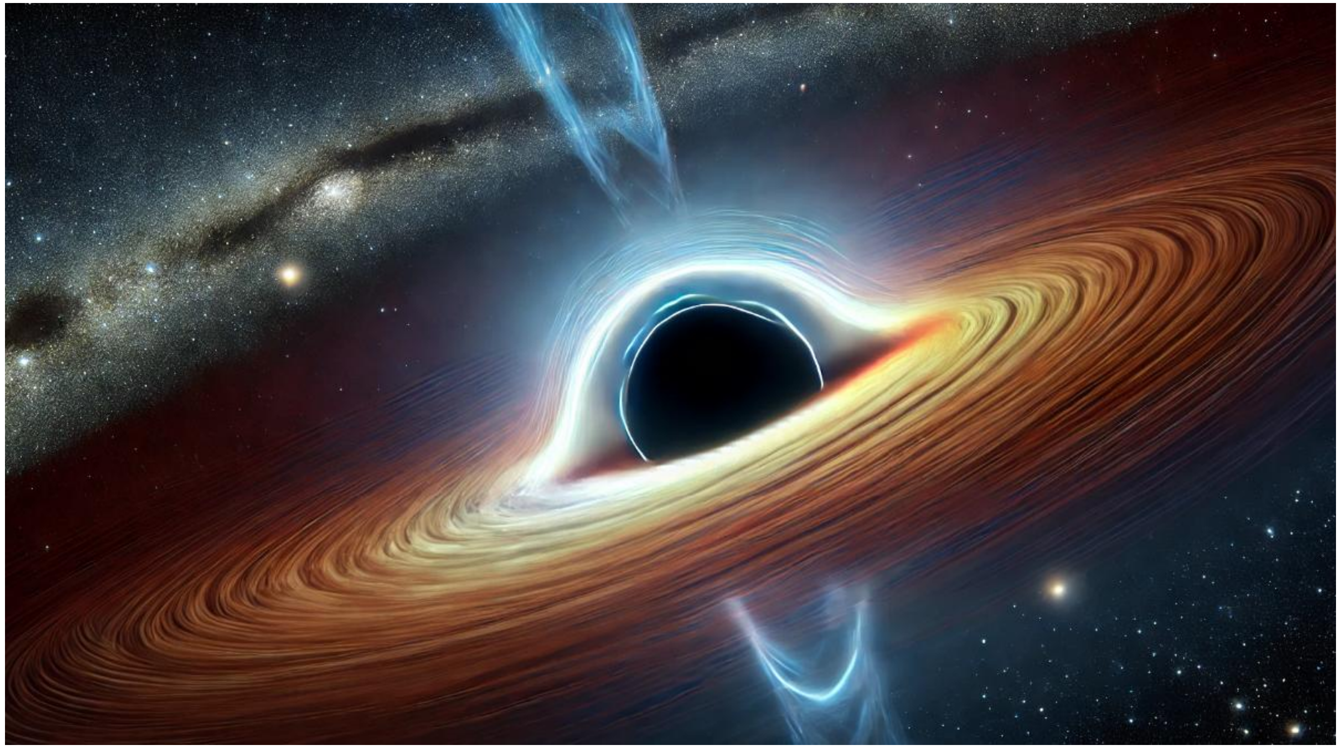
HOW DO WE DEFINE PHYSICAL QUANTITIES IN GR?







MASS OF BLACK HOLE?



ADM FORMALISM

PHYSICAL REVIEW VOLUME 116, NUMBER 5 DECEMBER 1, 1959

Dynamical Structure and Definition of Energy in General Relativity*†

R. Arnowitt, *Department of Physics, Syracuse University, Syracuse, New York*
S. Deser, *Department of Physics, Brandeis University, Waltham, Massachusetts*

AND

C. W. Misner‡ *Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*

(Received July 6, 1959)

The problem of the dynamical structure and definition of energy for the classical general theory of relativity is considered on a formal level. As in a previous paper, the technique used is the Schwinger action principle. Starting with the full Einstein Lagrangian in first order Palatini form, an action integral is derived in which the algebraic constraint variables have been eliminated. This action possesses a "Hamiltonian" density which, however, vanishes due to the differential constraints. If the differential constraints are then substituted into the action, the true, nonvanishing Hamiltonian of the theory emerges. From an analysis of the equations of motion and the constraint equations, the two pairs of dynamical variables which represent the two independent degrees of freedom of the gravitational field are explicitly exhibited. Four other variables remain in theory; these may be arbitrarily specified, any such specification representing a choice of coordinate frame. It is shown that it is possible to obtain truly canonical pairs of variables in terms of the dynamical and arbitrary variables. Thus a statement of the dynamics is meaningful only after a set of coordinate conditions have been chosen. In general, the true Hamiltonian will be time dependent even for an isolated gravitational field. There thus arises the notion of a preferred coordinate frame, i.e., that frame in which the Hamiltonian is conserved. In this special frame, on physical grounds, the Hamiltonian may be taken to define the energy of the field. In these respects the situation in general relativity is analogous to the parametric form of Hamilton's principle in particle mechanics.

Richard Arnowitt(-14,86), Stanley Deser(-23,92) and Charles Misner(-23,91) at the *ADM-50: A Celebration of Current GR Innovation* conference held in November 2009 to honor the 50th anniversary of their paper.

ADM FORMALISM (HILBERT ACTION DECOMPOSITION)

$$S_{\mathrm{H}} = \int_{\mathcal{V}} R\sqrt{-g}\, d^4x, \ \mathcal{V} := \cup_{t=t_1}^{t_2} \Sigma_t$$

$$\varsigma \ \ R = \widehat{R} + [-K^2 + K_{\mu\nu}K^{\mu\nu}] + 2\nabla_\mu(Kn^\mu) - \frac{2}{N}(\widehat{\nabla}_\mu \widehat{\nabla}^\mu N)]$$

$$\varsigma \ \ \sqrt{-g} = N\sqrt{\gamma}$$

$$\Rightarrow \boxed{\begin{aligned} S_{\mathrm{H}} &= S_{\mathrm{H,vol.}} + S_{\mathrm{H,surf.}} \\ &= \int_{t_1}^{t_2} \left\{ \int_{\Sigma_t} N(\widehat{R} - K^2 + K_{ij}K^{ij})\sqrt{\gamma}\, d^3x \right\} dt \\ &\quad - \int_{\mathcal{V}} 2(\widehat{\nabla}_i \widehat{\nabla}^i N)\sqrt{\gamma}\, d^4x + \int_{\mathcal{V}} 2\nabla_\mu(Kn^\mu)\sqrt{-g}\, d^4x \end{aligned}}$$

The gravitational Lagrangian density for $q = (\gamma_{ij}, N, \beta^i)$ is given by

$$\boxed{\mathcal{L}_{\mathrm{H,vol.}}(q, \dot{q}) = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) = N\sqrt{\gamma}[\widehat{R} + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{ij}K_{kl}]}$$

ADM FORMALISM (CANONICAL VARIABLE)

The gravitational Lagrangian density for $q = (\gamma_{ij}, N, \beta^i)$ is given by

$$\boxed{\mathcal{L}_{\mathrm{H,vol.}}(q,\dot{q}) = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) = N\sqrt{\gamma}[\widehat{R} + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{ij}K_{kl}]}$$

$$\begin{aligned}
K_{ij} &= \frac{1}{2}\mathcal{L}_{\boldsymbol{n}}P_{ij} = \frac{1}{2N}\mathcal{L}_{\boldsymbol{m}}P_{ij} = \frac{1}{2N}\mathcal{L}_{\boldsymbol{\partial_t}-\boldsymbol{\beta}}P_{ij} \\
&= \frac{1}{2N}(\mathcal{L}_{\boldsymbol{\partial_t}}P_{ij} - \mathcal{L}_{\boldsymbol{\beta}}P_{ij}) \\
&\varsigma \;\; \mathcal{L}_{\boldsymbol{\partial_t}}P_{ij} = (\partial_t)^\alpha\partial_\alpha P_{ij} + P_{\alpha j}\partial_i \underbrace{\cancel{(\partial_t)^\alpha}}_{=\delta_t^\alpha} + K_{i\alpha}\partial_j \underbrace{\cancel{(\partial_t)^\alpha}}_{=\delta_t^\alpha} = \frac{\partial P_{ij}}{\partial t} = \frac{\partial \gamma_{ij}}{\partial t} \\
&\varsigma \;\; \mathcal{L}_{\boldsymbol{\beta}}P_{ij} = \mathcal{L}_{\boldsymbol{\beta}}\gamma_{ij} = \beta^k\cancel{\widehat{\nabla}_k\gamma_{ij}} + \gamma_{kj}\widehat{\nabla}_i\beta^k + \gamma_{ik}\widehat{\nabla}_j\beta^k \\
&= \frac{1}{2N}(\dot{\gamma}_{ij} - \gamma_{kj}\widehat{\nabla}_i\beta^k - \gamma_{ik}\widehat{\nabla}_j\beta^k)
\end{aligned}$$

ADM FORMALISM (CONJUGATE MOMENTUM)

The gravitational Lagrangian density for $q = (\gamma_{ij}, N, \beta^i)$ is given by

$$\boxed{\mathcal{L}_{\text{H,vol.}}(q, \dot{q}) = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) = N\sqrt{\gamma}[\widehat{R} + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{ij}K_{kl}]}$$

$$\pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = \frac{\partial}{\partial \dot{\gamma}_{ij}}[N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij})]$$

$$\curvearrowleft \quad K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - \gamma_{kj}\widehat{\nabla}_i\beta^k - \gamma_{ik}\widehat{\nabla}_j\beta^k)$$

$$= -2N\sqrt{\gamma}\, K \underbrace{\frac{\partial K}{\partial \dot{\gamma}_{ij}}}_{=\gamma^{ij}\frac{1}{2N}} + 2N\sqrt{\gamma}\, \underbrace{K^{kl}\frac{\partial K_{kl}}{\partial \dot{\gamma}_{ij}}}_{K^{ij}\frac{1}{2N}}$$

$$= \sqrt{\gamma}(-K\gamma^{ij} + K^{ij})$$

$$= \sqrt{\gamma}(K^{ij} - K\gamma^{ij})$$

$$\boxed{\pi^{ij} = \sqrt{\gamma}(K^{ij} - K\gamma^{ij})}$$

ADM FORMALISM (ADM HAMILTONIAN)

$$\begin{aligned}
\mathcal{H}_{\text{H,vol.}} &= \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}_{\text{H,vol.}} \\
&= \sqrt{\gamma}(K^{ij} - K\gamma^{ij})\dot{\gamma}_{ij} - N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij})
\end{aligned}$$

$$\curvearrowleft\ K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - \gamma_{kj}\widehat{\nabla}_i\beta^k - \gamma_{ik}\widehat{\nabla}_j\beta^k)$$

$$\hookrightarrow\ \dot{\gamma}_{ij} = 2NK_{ij} + \gamma_{kj}\widehat{\nabla}_i\beta^k + \gamma_{ik}\widehat{\nabla}_j\beta^k$$

$$\begin{aligned}
&= \sqrt{\gamma}\underbrace{(K^{ij} - K\gamma^{ij})(2NK_{ij}}_{\to 2N(K_{ij}K^{ij}-K^2)} + \gamma_{kj}\widehat{\nabla}_i\beta^k + \gamma_{ik}\widehat{\nabla}_j\beta^k) - N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij}) \\
&= -\sqrt{\gamma}[N(\widehat{R} + K^2 - K_{ij}K^{ij}) - 2(K^i{}_j - K\gamma^i{}_j)\widehat{\nabla}_i\beta^j] \\
&= -\sqrt{\gamma}[N(\widehat{R} + K^2 - K_{ij}K^{ij}) + 2\beta^j(\widehat{\nabla}_i K^i{}_j - \nabla_j K) - 2\widehat{\nabla}_i(K^i{}_j\beta^j - K\beta^i) \\
&= -\sqrt{\gamma}[N\underbrace{(\widehat{R} + K^2 - K_{ij}K^{ij})}_{=C_0} - 2\beta^j\underbrace{(\widehat{\nabla}_j K - \widehat{\nabla}_i K^i{}_j)}_{=C_j}] + 2\sqrt{\gamma}\,\widehat{\nabla}_i(K^i{}_j\beta^j - K\beta^i)
\end{aligned}$$

$$\Rightarrow\ \boxed{\mathcal{L}_{\text{H,vol.}} = N\sqrt{\gamma}(\widehat{R} - K^2 + K_{ij}K^{ij})}$$

$$\hookrightarrow\ \boxed{\mathcal{H}_{\text{H,vol.}} = -\sqrt{\gamma}(NC_0 - 2\beta^i C_i) + 2\sqrt{\gamma}\,\widehat{\nabla}_i(K^i{}_j\beta^j - K\beta^i)}$$

$$\boxed{H_{\text{H,vol.flat}} = -\int_{\Sigma_t}(NC_0 - 2\beta^i C_i)\sqrt{\gamma}\,d^3x}$$

$$\begin{cases} C_0 = \widehat{R} + K^2 - K_{ij}K^{ij} \\ C_i = \widehat{\nabla}_i K - \widehat{\nabla}_j K^j{}_i \end{cases}$$

ADM FORMALISM (BOUNDARY TERM)

$$\delta S_{\mathrm{H}} = \delta\left(\int d^n x\sqrt{-g}\,\frac{1}{16\pi G}R\right)$$

$$= \int d^n x\frac{1}{16\pi G}\left[(\delta\sqrt{-g})R + \underbrace{\sqrt{-g}(\delta R_{\mu\nu})g^{\mu\nu}}_{\text{will be vanished by a surface integral}} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}\right]$$

$$= \int d^n x\sqrt{-g}\frac{1}{16\pi G}\left(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\right)\delta g^{\mu\nu} + \int d^n x\sqrt{-g}\frac{1}{16\pi G}(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu}$$

$$= \int d^n x\sqrt{-g}\frac{1}{16\pi G}G_{\mu\nu}\delta g^{\mu\nu} + \underbrace{\int d^n x\sqrt{-g}\frac{1}{16\pi G}(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu}}_{\rightarrow\text{boundary term}}$$

$$= \delta S_{\text{Ei.eq.}} + \delta S_{\text{v.b.}}$$

ς when the variation boundary term is canceled by an additional term,

$$\rightarrow \int d^n x\sqrt{-g}\frac{1}{16\pi G}G_{\mu\nu}\delta g^{\mu\nu} \qquad (12.19)$$

ADM FORMALISM (GIBBONS-HAWKING TERM)

$$\boxed{S_{\rm H} = \int d^n x \sqrt{-g}\, \frac{1}{16\pi G} R}$$

$$\rightarrow \boxed{S_{\rm G} \equiv S_{\rm H} + S_{\text{G-H}} = \int_{\mathcal{M}} d^n x \sqrt{-g}\, \frac{1}{16\pi G} R + \int_{\partial\mathcal{M}} d^{n-1} \sqrt{\gamma} \frac{1}{16\pi G} \sigma_{\mathcal{K}} 2\mathcal{K}}$$

$\circlearrowleft$ To have a finite action even in the flat case

$$\Rsh \boxed{S_{\rm G}^{\rm reg.} \equiv S_{\rm H} + S_{\text{G-H}}^{\text{regularized}} = \int_{\mathcal{M}} d^n x \sqrt{-g}\, \frac{1}{16\pi G} R + \int_{\partial\mathcal{M}} d^{n-1} \sqrt{\gamma} \frac{1}{16\pi G} \sigma_{\mathcal{K}} 2(\mathcal{K} - \mathcal{K}_0)}$$

ADM FORMALISM (GRAVITATIONAL HAMILTONIAN INCLUDING GH TERM)

$$H_{\mathrm{G}}^{\mathrm{reg.}} = H_{\mathrm{H,vol.}} + H_{\mathrm{G,surf.}}^{\mathrm{reg.}}$$

$$= -\frac{1}{16\pi G}\int_{\Sigma_t} d^3x\sqrt{\gamma}\,(NC_0 - 2\beta^i C_i)$$

$$+ \frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\ \cdot 2[r_i(K^i{}_j\beta^j - K\beta^i) - N(\kappa-\kappa_0)|_{S_t}]$$

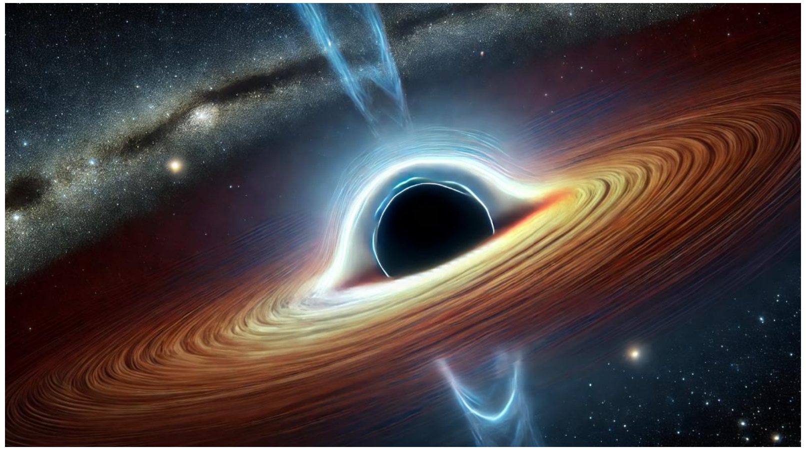
$$\circlearrowleft\ H \xrightarrow[C_0\,=\,0\,=\,C_i]{\text{solution}} H_{\text{solution}}$$

$$\boxed{H_{\text{solution}} = \frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\ \cdot 2[r_i(K^i{}_j\beta^j - K\beta^i) - N(\kappa-\kappa_0)|_{S_t}]}$$

where K_{ij} : extrinsic curvature on Σ_t,

κ : extrinsic curvature scalar on $\Sigma_t \cap S$

ADM FORMALISM



$$M_{\text{ADM}} = -\frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\;\cdot 2(\kappa-\kappa_0)$$

$$M_{\text{ADM}} = \frac{1}{16\pi G}\int_{S_t,r\to\infty} d^2\theta\sqrt{\sigma}\,(\gamma_{ij,j}-\gamma_{jj,i})r_0^i \quad \text{(in Cartesian)}$$

ADM FORMALISM (ADM MASS)

$$(1)\ ds^2 = -f(r)dt^2 + \underbrace{\frac{1}{f(r)}dr^2 + \overbrace{r^2 d\theta^2 + r^2\sin^2\theta d\phi^2}^{S_t \to \sigma_{ab}}}_{\Sigma \to \gamma_{ij}},\quad f(r) = 1 - \frac{2GM}{r}$$

$$(2)\ r_i = \frac{1}{\sqrt{\gamma^{rr}}}\delta_i^r = \sqrt{\gamma_{rr}}\delta_i^r = (f^{-1/2}, 0, 0),$$

$$r^i = \gamma^{ij} r_j = \left(\sqrt{1 - \frac{2GM}{r}}, 0, 0\right) = (f^{1/2}, 0, 0)$$

$$(3)\ M_{\text{ADM}} = -\frac{1}{16\pi G}\int_{S_t, r\to\infty} d^2\theta\sqrt{\sigma}\cdot 2(\kappa - \kappa_0)$$

$$(4)\ \kappa = \nabla_i r^i = \partial_i r^i + \Gamma^i_{ij} r^j = \partial_i r^i + \sum_i \frac{\gamma_{ii,j}}{2\gamma_{ii}} r^j = \partial_r r^r + \sum_i \frac{\gamma_{ii,r}}{2\gamma_{ii}} r^r$$

$$\circlearrowleft\ r^r = \sqrt{f}$$

$$= \partial_r\sqrt{f} + \frac{1}{2}\left(\frac{\gamma_{rr,r}}{\gamma_{rr}} + \frac{\gamma_{\theta\theta,r}}{\gamma_{\theta\theta}} + \frac{\gamma_{\phi\phi,r}}{\gamma_{\phi\phi}}\right)\sqrt{f}$$

$$\circlearrowleft\ (\gamma_{rr}, \gamma_{\theta\theta}, \gamma_{\phi\phi}) = (f^{-1}, r^2, r^2\sin^2\theta)$$

$$= \frac{f_{,r}}{2\sqrt{f}} + \frac{1}{2}\left[\frac{f^{-1}_{,r}}{f^{-1}} + \underbrace{\frac{(r^2)_{,r}}{r^2} + \frac{(r^2\sin^2\theta)_{,r}}{r^2\sin^2\theta}}_{=\frac{4}{r}}\right]\sqrt{f}$$

$$= \frac{2}{r}\sqrt{f} = \frac{2}{r}\sqrt{1 - \frac{2GM}{r}}$$

$$\xrightarrow{\sqrt{1-2\epsilon}\approx 1-\epsilon} \frac{2}{r}\left(1 - \frac{GM}{r} + \mathcal{O}(r^{-2})\right) = \frac{2}{r} - \frac{2GM}{r^2} + \mathcal{O}(r^{-3})$$

$$(5)\ \kappa - \kappa_0 = -\frac{2GM}{r^2} + \mathcal{O}(r^{-3})$$

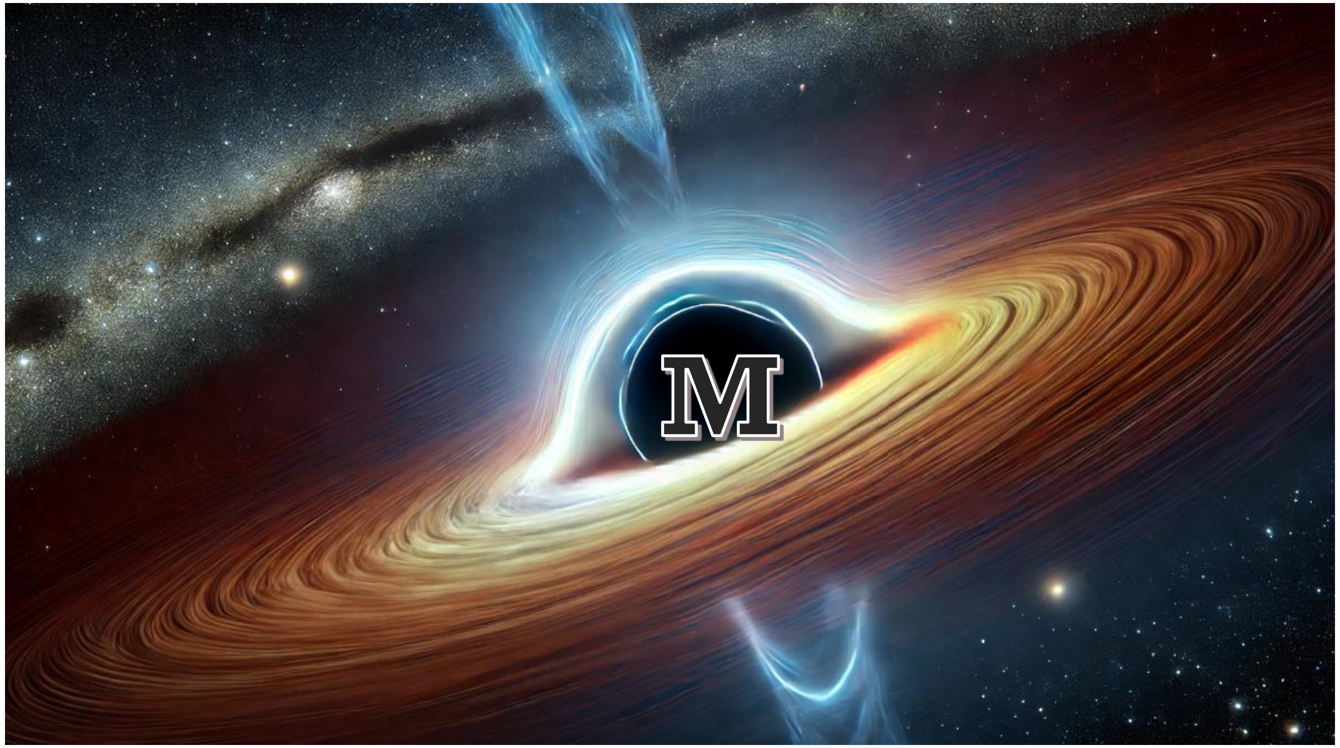
$$(6)\ M_{\text{ADM}} = -\frac{1}{16\pi G}\int_{S_t, r\to\infty} d^2\theta\sqrt{\sigma}\cdot 2(\kappa - \kappa_0)$$

$$= -\frac{1}{16\pi G}\int_{S_t, r\to\infty} d^2\theta\sqrt{\sigma}\left(-\frac{4GM}{r^2} + \mathcal{O}(r^{-3})\right)$$

$$= -\frac{1}{16\pi G}\lim_{r\to\infty} 4\pi r^2 \cdot \left(-\frac{4GM}{r^2} + \mathcal{O}(r^{-3})\right)$$

$$= M$$

ADM FORMALISM TO ADM MASS



$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

WITH ADM FORMALISM

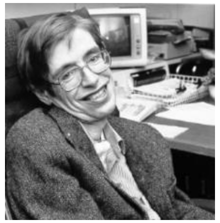
The Four Laws of Black Hole Mechanics

J. M. Bardeen*

Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking

Institute of Astronomy, University of Cambridge, England



Received January 24, 1973

Abstract. Expressions are derived for the mass of a stationary axisymmetric solution of the Einstein equations containing a black hole surrounded by matter and for the difference in mass between two neighboring such solutions. Two of the quantities which appear in these expressions, namely the area A of the event horizon and the "surface gravity" κ of the black hole, have a close analogy with entropy and temperature respectively. This analogy suggests the formulation of four laws of black hole mechanics which correspond to and in some ways transcend the four laws of thermodynamics.

To evaluate δM, we express the mass formula derived in the previous section in the form

$$M = \int_S \left(2T_a{}^b + \frac{1}{8\pi} R\delta_a^b\right) K^a d\Sigma_b + 2\Omega_H J_H + \frac{\kappa}{4\pi} A . \tag{28}$$

The variation of the term involving the scalar curvature, R, gives

$$-\frac{1}{8\pi}\int_S \left\{\left(R_{cd} - \frac{1}{2} g_{cd} R\right) h^{cd} + 2h^c_{[c;d]}{}^{;d}\right\} K^a d\Sigma_a . \tag{29}$$

But

$$2h^c_{[c;d]}{}^{;d} K^a = 2\left(K^a h_c^{[c;d]} - K^d h_c^{[c;a]}\right)_{;d} , \tag{30}$$

using $h_{cd;a}K^a + h_{ad}K^a{}_{;c} + h_{ac}K^a{}_{;d} = 0$. One can therefore transform the last term in (29) into the 2-surface integral

$$-\frac{1}{4\pi}\int_{\partial S} \left(K^a h_c^{[c;d]} - K^d h_c^{[c;a]}\right) d\Sigma_{ad} . \tag{31}$$

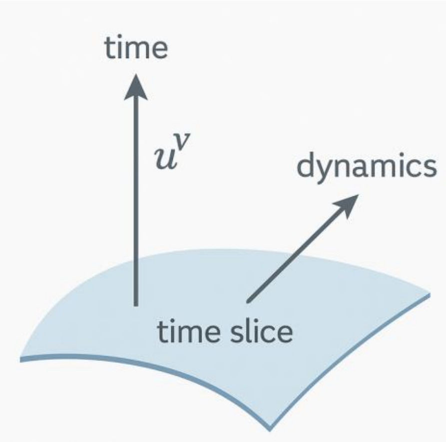
The integral over ∂S_∞ gives $-\delta M$ and, by Eq. (27), the integral over ∂B gives $-\frac{\delta\kappa}{4\pi} A - 2\delta\Omega_H J_H$.

The variation of the energy-momentum tensor term in (28) is

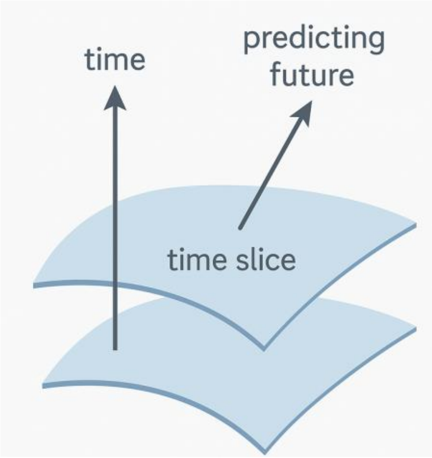
$$\begin{aligned} 2\delta \int T_a{}^b K^a d\Sigma_b = & -2\int \Omega\delta\{T_a{}^b \tilde{K}^a d\Sigma_b\} + 2\delta\int pK^a d\Sigma_a \\ & + 2\int u^a \delta\{(\varepsilon + p)(-u^c u^d g_{cd})^{-1} u_a K^b d\Sigma_b\} . \end{aligned} \tag{32}$$

TWO BIG QUESTIONS BEFORE 3+1 FORMALISM

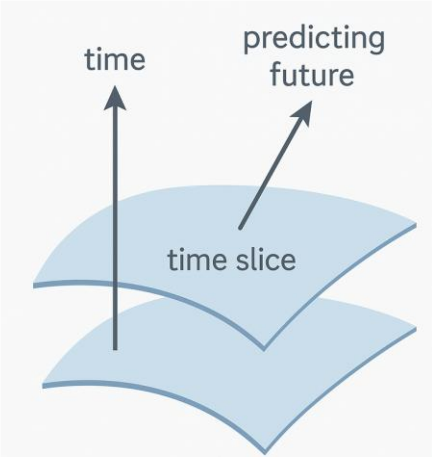
1. **How Do We Define Physical Quantities in GR?**



2. **Can We Predict the Future in GR from the Present?**



CAN WE PREDICT THE FUTURE IN GR FROM THE PRESENT?



$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = x \;\rightarrow\; f(x) = \frac{1}{2}x^2 + C \xrightarrow{f(0)=1} \frac{1}{2}x^2 + 1$$

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} = f(x) \;\rightarrow\; f(x) = Ae^{x} + Be^{-x} \xrightarrow[]{\substack{f(0)=2\\ f'(0)=0}} e^{x} + e^{-x}$$

EI. EQ.: 2ND ORDER PDE

$G_{\mu\nu}(g_{\mu\nu})$

```
EinsteinCD[-μ, -ν] // ToRicci // RiemannToChristoffel // NoScalar // ChristoffelToGradMetric[#, g] & // Expand //
ToCanonical
```

확장

$$-\frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\gamma\delta} - \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\alpha g_{\mu\nu} + \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\mu g_{\nu\alpha} + \frac{1}{2} g^{\alpha\beta} \partial_\beta \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\nu\delta} \partial_\gamma g_{\mu\alpha} +$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha g_{\mu\nu} \partial_\delta g_{\beta\gamma} + \frac{1}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_\delta g_{\delta1\delta2} + \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\gamma g_{\mu\alpha} \partial_\delta g_{\nu\beta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta \partial_\beta g_{\alpha\gamma} +$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \partial_\delta \partial_\gamma g_{\alpha\beta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\delta g_{\beta\delta2} \partial_{\delta1} g_{\alpha\gamma} - \frac{3}{8} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_{\delta1} g_{\alpha\gamma} \partial_{\delta2} g_{\beta\delta} +$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\beta g_{\alpha\gamma} \partial_{\delta2} g_{\delta\delta1} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\delta1\delta2} g_{\mu\nu} \partial_\gamma g_{\alpha\beta} \partial_{\delta2} g_{\delta\delta1} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\mu g_{\nu\alpha} -$$
$$\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\mu g_{\nu\alpha} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\mu g_{\alpha\gamma} \partial_\nu g_{\beta\delta} + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \partial_\beta g_{\gamma\delta} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\beta\gamma} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\alpha\beta} \partial_\nu \partial_\mu g_{\alpha\beta}$$

$$\frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\alpha\partial_\mu g_{\beta\nu} = \frac{1}{2}g^{00}\partial_0\partial_\mu g_{0\nu} + \frac{1}{2}g^{01}\partial_0\partial_\mu g_{1\nu} + \frac{1}{2}g^{02}\partial_0\partial_\mu g_{2\nu} + \frac{1}{2}g^{03}\partial_0\partial_\mu g_{3\nu}$$
$$+ \frac{1}{2}g^{10}\partial_1\partial_\mu g_{0\nu} + \frac{1}{2}g^{11}\partial_1\partial_\mu g_{1\nu} + \frac{1}{2}g^{12}\partial_1\partial_\mu g_{2\nu} + \frac{1}{2}g^{13}\partial_1\partial_\mu g_{3\nu} + \frac{1}{2}g^{20}\partial_2\partial_\mu g_{0\nu} + \frac{1}{2}g^{21}\partial_2\partial_\mu g_{1\nu}$$
$$+ \frac{1}{2}g^{22}\partial_2\partial_\mu g_{2\nu} + \frac{1}{2}g^{23}\partial_2\partial_\mu g_{3\nu} + \frac{1}{2}g^{30}\partial_3\partial_\mu g_{0\nu} + \frac{1}{2}g^{31}\partial_3\partial_\mu g_{1\nu} + \frac{1}{2}g^{32}\partial_3\partial_\mu g_{2\nu} + \frac{1}{2}g^{33}\partial_3\partial_\mu g_{3\nu}$$

- 2nd order PDE of $g_{\mu\nu}$

INITIAL VALUE PROBLEM / WELL-POSEDNESS

- **Initial value formulation:**
  - **appropriate initial data** > **subsequent uniquely determined dynamical evolution**

- **Appropriate initial data:**
  - **small changes in initial data** > **small change in solution**
    > **predictable physics law**
  - **Any changes in initial data can not change solutions outside causal future.**

  > **"Initial value formulation" is well-posed.**

INITIAL VALUE PROBLEM / WELL-POSEDNESS

$$g^{\mu\nu}(x;\phi_\alpha;\nabla_\sigma\phi_\alpha)\nabla_\mu\nabla_\nu\phi_\beta = F_\beta(x;\phi_\alpha;\nabla_\sigma\phi_\alpha)$$

THEOREM 10.1.3. *Let $(\phi_0)_1, \ldots, (\phi_0)_n$ be any solution of the quasilinear hyperbolic system (10.1.21) on a manifold M and let $(g_0)^{ab} = g^{ab}(x; (\phi_0)_j; \nabla_c(\phi_0)_j)$. Suppose $(M, (g_0)_{ab})$ is globally hyperbolic (or, alternatively, consider a globally hyperbolic region of this spacetime). Let Σ be a smooth spacelike Cauchy surface for $(M, (g_0)_{ab})$. Then, the initial value formulation of equation (10.1.21) is well posed on Σ in the following sense: For initial data on Σ sufficiently close to the initial data for $(\phi_0)_1, \ldots, (\phi_0)_n$, there exists an open neighborhood O of Σ such that equation (10.1.21) has a solution, $\phi_1, \ldots, \phi_n$, in O and $(O, g_{ab}(x; \phi_j; \nabla_c\phi_j))$ is globally hyperbolic. The solution is unique in O and propagates causally in the sense that if the initial data for $\phi'_1, \ldots, \phi'_n$ agree with that of $\phi_1, \ldots, \phi_n$ on a subset, S, of Σ, then the solutions agree on $O \cap D^+(S)$. Finally, the solutions depend continuously on the initial data in the sense described above for the Klein-Gordon field.*

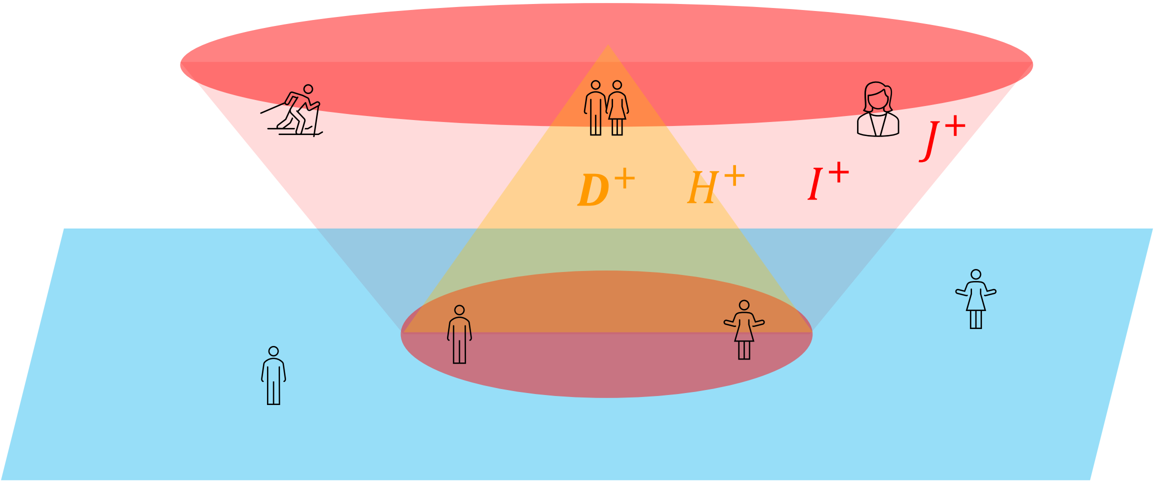
[Wald (1984) p.251]

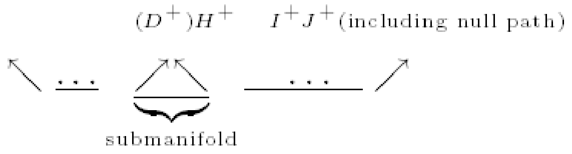
INITIAL VALUE PROBLEM / WELL-POSEDNESS

THEOREM 10.2.2. *Let Σ be a three-dimensional C^∞ manifold, let h_{ab} be a smooth Riemannian metric on Σ, and let K_{ab} be a smooth symmetric tensor field on Σ. Suppose h_{ab} and K_{ab} satisfy the constraint equations (10.2.28) and (10.2.30). Then there exists a unique C^∞ spacetime, (M, g_{ab}), called the maximal Cauchy development of (Σ, h_{ab}, K_{ab}), satisfying the following four properties:* (i) *(M, g_{ab}) is a solution of Einstein's equation.* (ii) *(M, g_{ab}) is globally hyperbolic with Cauchy surface Σ.* (iii) *The induced metric and extrinsic curvature of Σ are, respectively, h_{ab} and K_{ab}.* (iv) *Every other spacetime satisfying* (i)–(iii) *can be mapped isometrically into a subset of (M, g_{ab}). Furthermore, (M, g_{ab}) satisfies the desired domain of dependence property in the following sense. Suppose (Σ, h_{ab}, K_{ab}) and $(\Sigma', h'_{ab}, K'_{ab})$ are initial data sets with maximal developments (M, g_{ab}) and (M', g'_{ab}). Suppose there is a diffeomorphism between $S \subset \Sigma$ and $S' \subset \Sigma'$ which carries (h_{ab}, K_{ab}) on S into (h'_{ab}, K'_{ab}) on S'. Then $D(S)$ in the spacetime (M, g_{ab}) is isometric to $D(S')$ in the spacetime (M', g'_{ab}). Finally, the solution g_{ab} on M depends continuously on the initial data (h_{ab}, K_{ab}) on Σ.* (A precise definition of the topologies on initial data and solutions which makes this map continuous is given in Hawking and Ellis 1973.)

[Wald (1984) p.264]

CAUSAL STRUCTURE





$D^{\pm}(S)$: future/past **domain** of dependence (determined region)

$H^{\pm}(S)$: future/past **Cauchy horizon** (determined region limit)

$I^{\pm}(S)$: chronological **future/past** (massive-influenced region)

$J^{\pm}(S)$: causal **future/past** (everything-influenced region)

3. GLOBALLY HYPERBOLIC SPACETIME

- **Cauchy surface**

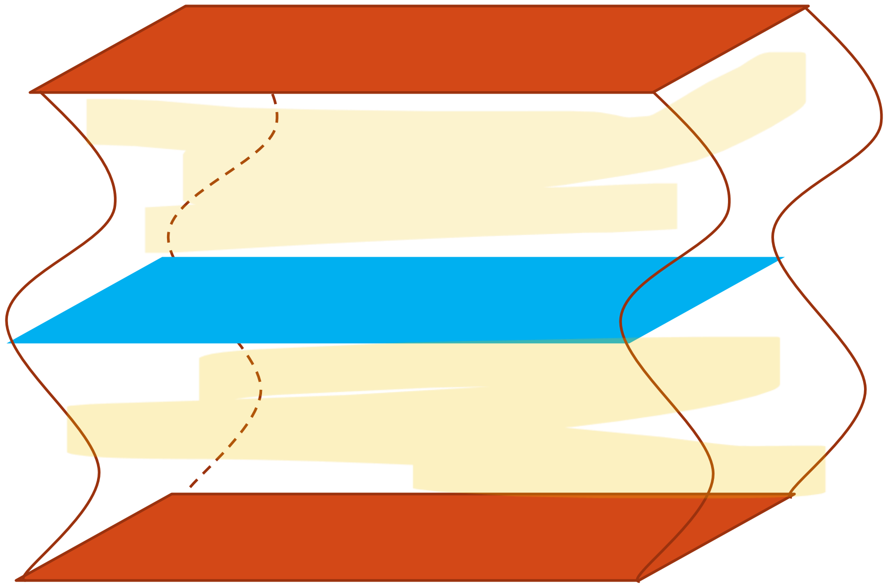
  Σ in $\mathcal{M}$ of one-time intersections with each causal curve

  $(\Sigma|_{\text{d.o.d of } \Sigma = \mathcal{M}})$

- **Globally hyperbolic spacetime**

  $(\mathcal{M}, g)$ which has Σ_{Cauchy}

  $\xrightarrow{\text{topology}} (\mathcal{M}, g)|_{\mathcal{M} = \Sigma \times \mathbb{R}}$

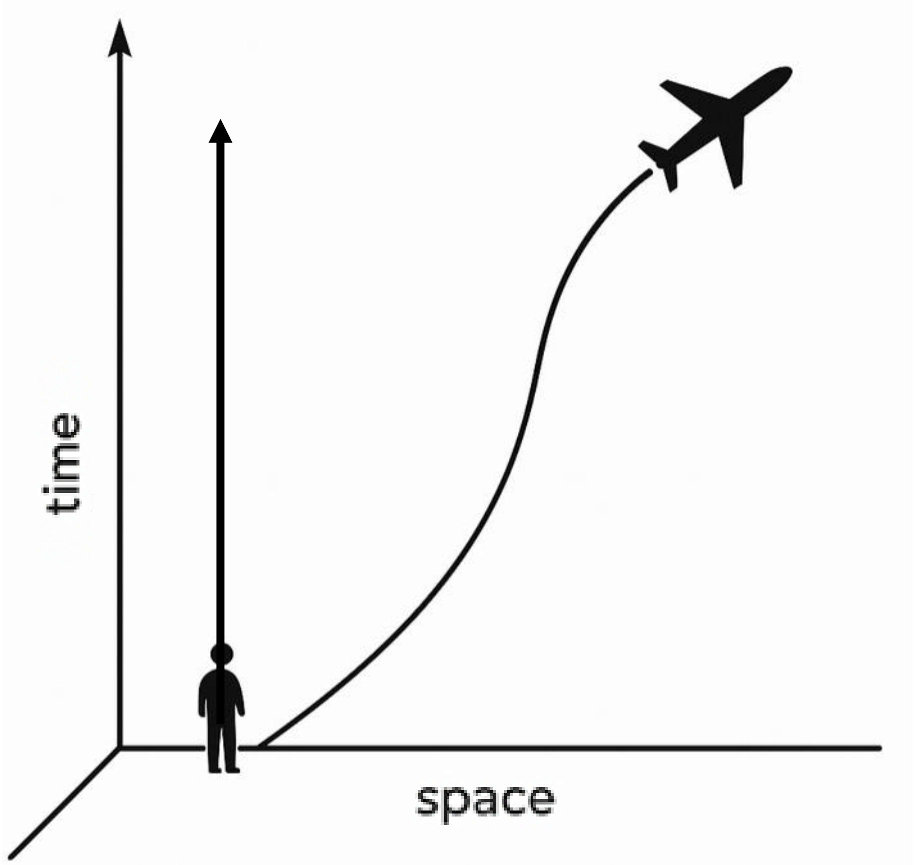


5 QUESTIONS

Q.'S BEFORE 3+1 FORMALISM

1. Why All Measurements Need a Frame ?
   → No Observer, No Physics
2. Why We Need a Spacetime Slice?
   → To Observe Anything, Need to Define "Now" and "Here"
3. How Do We Slice Spacetime?
   → Foliation, Lapse, and Shift, (Gauge Fixing)
4. Given a Spacetime Slice, Can We Specify Any Metric?
   → No, It Might Be Unphysical! (No Match With E-p Distribution.)
   → Physical Meaning of the Constraints
5. Once We fix a Slice's Geometry of the Spacetime,
   How Does it Change Over Time?
   → Through the Evolution eq.

WHY ALL MEASUREMENTS NEED A FRAME ?



WHY ALL MEASUREMENTS NEED A FRAME ?

1. **Four-velocity**

$$U^{\mu}_{\mathrm{A/B}} \equiv \gamma_{\mathrm{A/B}}(c, \boldsymbol{v}_{\mathrm{A/B}}) = \frac{dx^{\mu}_{\mathrm{A/B}}}{d\tau_{\mathrm{A}}}$$

$$U_{\mu}U^{\mu} = \gamma^2(-c^2 + v^2) = -c^2 \xrightarrow{\text{natural unit}} = -1$$

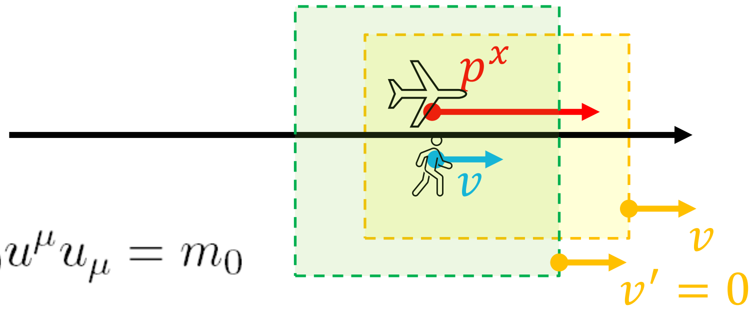
2. **Four-momentum**

$$p^{\mu} = m_0 U^{\mu} = m_0 \gamma(c, \boldsymbol{v}) = (E/c, \boldsymbol{p})$$

3. **Observed energy**

$$E_{\text{measured}} = -p^{\mu} v_{\mu}$$

$$E = -p^{\mu} u_{\mu} = -m_0 u^{\mu} u_{\mu} = m_0$$



$$p^{\mu} = (E, p^x, 0, 0)$$

$$v^{\mu} = \gamma(1, v, 0, 0)$$

$$-p^{\mu} v_{\mu} = \gamma(E - v p^x) = E'$$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

$$p'^{\mu} = \Lambda^{\mu}{}_{\nu} p^{\nu}, \qquad v'^{\mu} = \Lambda^{\mu}{}_{\nu} v^{\nu}$$

$$v'^{\mu} = (1, 0, 0, 0)$$

$$-p'^{\mu} v'_{\mu} = p'^{0} = \Lambda^{0}{}_{0} p^{0} + \Lambda^{0}{}_{1} p^{1} = \gamma p^{0} - \gamma v p^{1} = \gamma(E - v p^x)$$

WHY ALL MEASUREMENTS NEED A FRAME ?

4. **Energy-Momentum Tensor**

(1) In the absence of pressure (i.e., for a single particle), the energy-momentum tensor reduces to:

$$\boxed{T^{\mu\nu}|_{\text{particle}} = \rho_{m_0} U^\mu U^\nu}$$

(2) perfect fluid, which includes isotropic pressure but no viscosity or heat conduction.

$$\boxed{T^{\mu\nu}|_{\text{perfect fluid}} = \rho_{m_0} U^\mu U^\nu + \mathcal{P} P^{\mu\nu}}$$

where $P^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu$ is the projection tensor onto the spatial hypersurface orthogonal to U^μ.

WHY ALL MEASUREMENTS NEED A FRAME ?

4. **Energy-Momentum Tensor**

(3) The most general covariant decomposition of the energy-momentum tensor, especially useful in numerical relativity and the ADM/BSSN formalism:

$$\boxed{T^{\mu\nu}|_{3+1\text{ split}} = \rho\, n^\mu n^\nu + j^\mu n^\nu + j^\nu n^\mu + S^{\mu\nu}}$$

$$\hookrightarrow \begin{cases} \rho = T^{\mu\nu} n_\mu n_\nu & : \text{ energy density as measured by the observer } n^\mu \\ j^\mu = -T^{\alpha\beta} n_\alpha \gamma^\mu{}_\beta & : \text{ spatial momentum density projected orthogonal to } n^\mu \\ S^{\mu\nu} = T^{\alpha\beta} \gamma^\mu{}_\alpha \gamma^\nu{}_\beta & : \text{ spatial stress tensor (purely projected)} \end{cases}$$

where the projection tensor is given by $\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu$,
which satisfies $\gamma^{\mu\nu} n_\nu = 0$. This projects tensors onto the 3D slice.

WHY ALL MEASUREMENTS NEED A FRAME ?

4. **Energy-Momentum Tensor**

The Lorentz factor between the particle 4-velocity U^μ and the observer n^μ is:

$$\boxed{\gamma = -U^\mu n_\mu}$$

Then, by contracting the energy-momentum tensor with n^μ, we obtain:

$$\boxed{\rho = T^{\mu\nu} n_\mu n_\nu = \rho_{m_0} U^\mu U^\nu n_\mu n_\nu \xrightarrow{U^\mu n_\mu = -\gamma} \rho_{m_0} \gamma^2 = \rho_m}$$: total energy density observed by n^μ

$$\boxed{j^\mu = -T^{\mu'\nu} n_\nu \gamma^\mu{}_{\mu'} = -\rho_{m_0} U^\mu U^\nu n_\nu \gamma^\mu{}_{\mu'} \xrightarrow[\frac{\gamma^\mu{}_\nu U^\nu}{\gamma} \equiv v^\mu]{U^\mu n_\mu = -\gamma} \rho_{m_0} \gamma^2 v^\mu = \rho_m v^\mu}$$

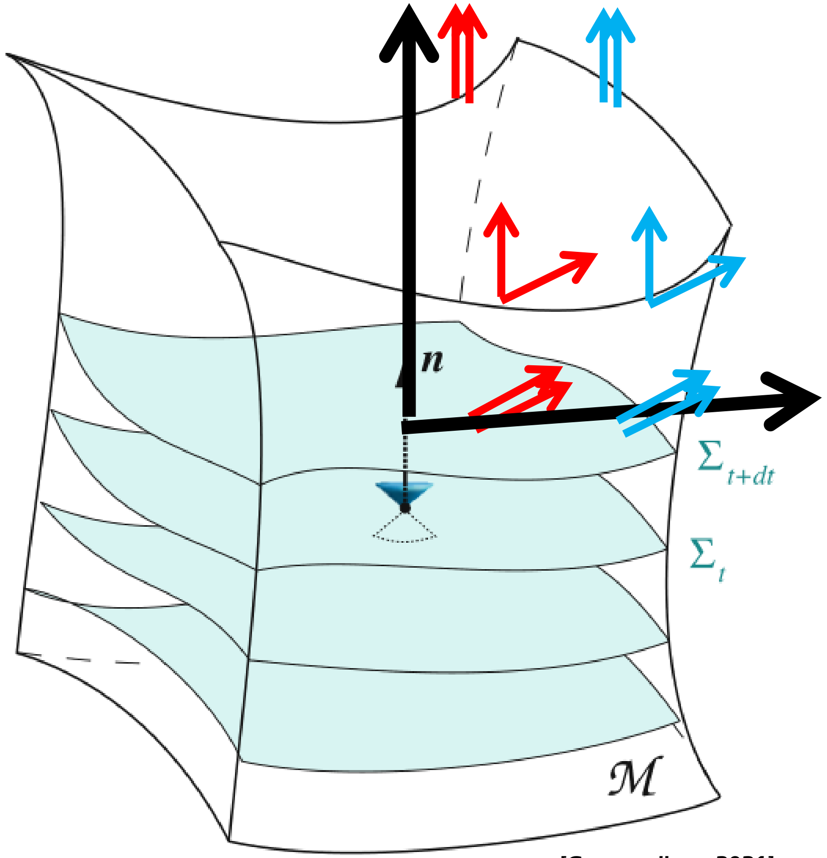
with $v^\mu = \dfrac{\gamma^\mu{}_\nu U^\nu}{\gamma}$ the 3-velocity relative to n^μ

Therefore, ρ includes Lorentz boosting of rest mass energy,

and j^μ gives the momentum density.

(EX) 3+1 Decomposition

$$\begin{cases} \text{(energy density)} & \rho_e = T_{nn} \\ \text{(momentum density)} & p_\alpha = -T_{n\hat{\alpha}} \\ \text{(stress tensor)} & S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}} \end{cases}$$



[Gourgoulhon, 2021]

Q.'S BEFORE 3+1 FORMALISM

1. **Why All Measurements Need a Frame ?**
   → No Observer, No Physics
2. **Why We Need a Spacetime Slice?**
   → To Observe Anything, Need to Define "Now" and "Here"
3. **How Do We Slice Spacetime?**
   → Foliation, Lapse, and Shift, (Gauge Fixing)
4. **Given a Spacetime Slice, Can We Specify Any Metric?**
   → No, It Might Be Unphysical! (No Match With E-p Distribution.)
   → Physical Meaning of the Constraints
5. **Once We fix a Slice's Geometry of the Spacetime,**
   How Does it Change Over Time?
   → Through the Evolution eq.

WHY WE NEED A SPACETIME SLICE?

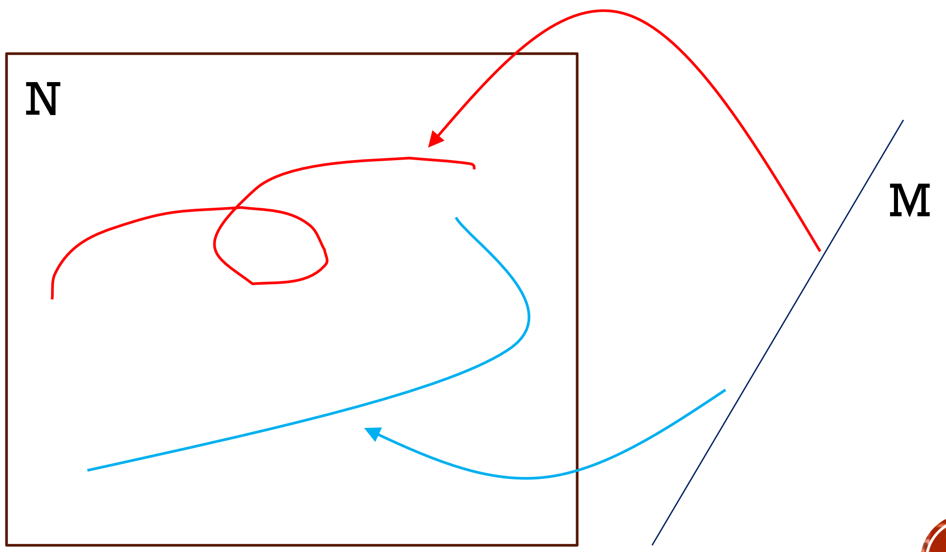
- **Manifold (topological → differential(smooth) → Riemannian)**

WHY WE NEED A SPACETIME SLICE?

- **Manifold (topological → differential(smooth) → Riemannian)**
- **Immersion/embedding**

$$(\text{immersion})\begin{cases}(\text{derivative one-to-one})\\(\text{points no need to be one-to-one} \rightarrow \text{self-interaction possible})\end{cases}$$

$$(\text{embedding})\begin{cases}(\text{immersion} + \text{topologically same (homeomorphic)})\\(\text{Locally } N \rightarrow M \text{ is homeomorphic, that is, locally immersion is embedding.})\\\quad\hookrightarrow (\text{Local neighbourhood of a point } x \text{ on } N\\\qquad \text{can not be mapped to a self-interacted image.})\end{cases}$$



WHY WE NEED A SPACETIME SLICE?

- **Manifold (topological → differential(smooth) → Riemannian)**
- **Immersion/embedding**

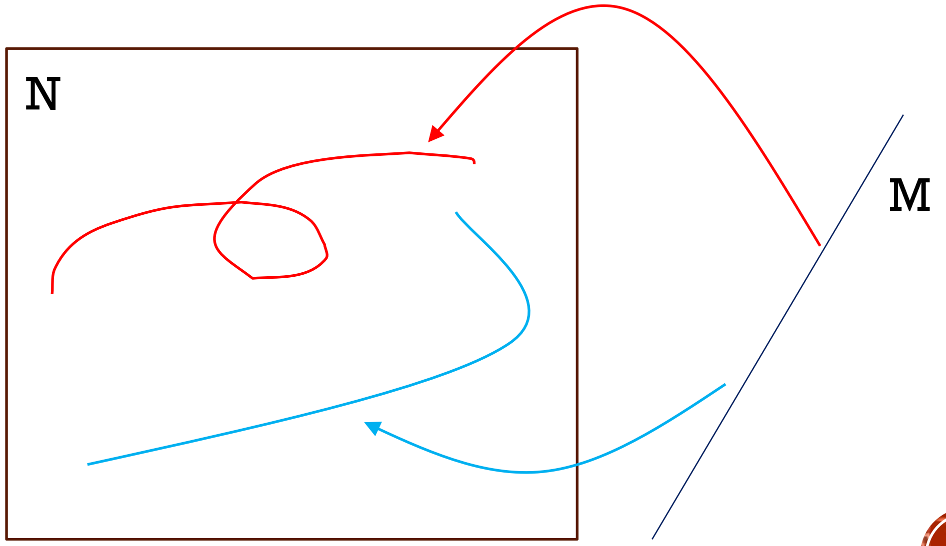
$$\text{(immersion)}\begin{cases}\text{(derivative one-to-one)}\\ \text{(points no need to be one-to-one} \rightarrow \text{self-interaction possible)}\end{cases}$$

$$\text{(embedding)}\begin{cases}\text{(immersion + topologically same (homeomorphic))}\\ \text{(Locally } N \rightarrow M \text{ is homeomorphic, that is, locally immersion is embedding.)}\\ \quad \Lsh \text{ (Local neighbourhood of a point } x \text{ on } N\\ \qquad \text{can not be mapped to a self-interacted image.)}\end{cases}$$

- **Codimension/submanifold**

$$\text{(immersion, } M \rightarrow N) \rightarrow [\ \dim(N) - \dim(M)\ :\ \text{codimension}\ ]$$

$$\text{(embedding, } M \rightarrow N) \ \rightarrow \ [\ M \text{ is a submanifold of } N\ ]$$



WHY WE NEED A SPACETIME SLICE?

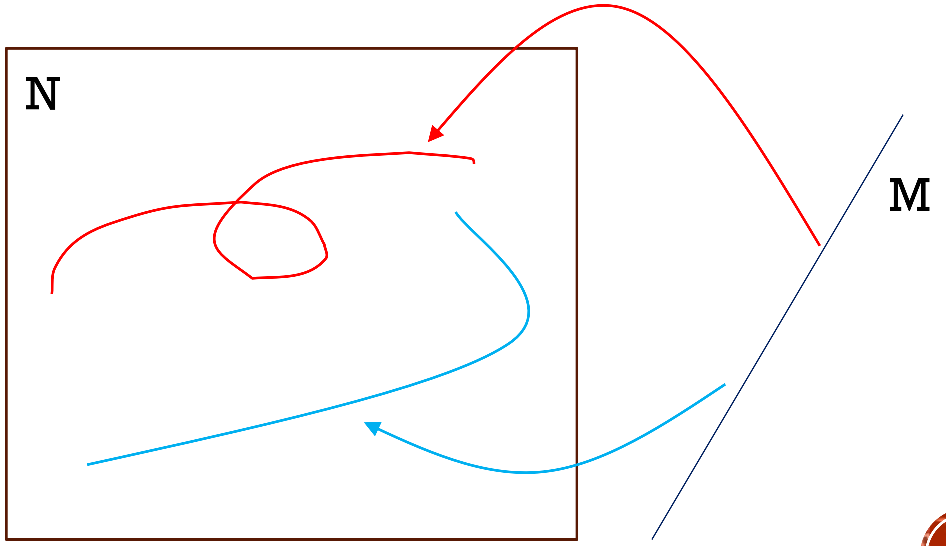
- **Manifold (topological → differential(smooth) → Riemannian)**
- **Immersion/embedding**

$$(\text{immersion})\begin{cases}(\text{derivative one-to-one})\\(\text{points no need to be one-to-one}\rightarrow\text{self-interaction possible})\end{cases}$$

$$(\text{embedding})\begin{cases}(\text{immersion + topologically same (homeomorphic)})\\(\text{Locally } N\rightarrow M \text{ is homeomorphic, that is, locally immersion is embedding.})\\\quad\text{↳ (Local neighbourhood of a point } x \text{ on } N\\\qquad\text{can not be mapped to a self-interacted image.})\end{cases}$$

- **Codimension/submanifold**

$$(\text{immersion}, M\rightarrow N)\rightarrow[\ \dim(N)-\dim(M)\ :\ \text{codimension}\ ]$$
$$(\text{embedding}, M\rightarrow N)\ \rightarrow\ [\ M \text{ is a submanifold of } N\ ]$$

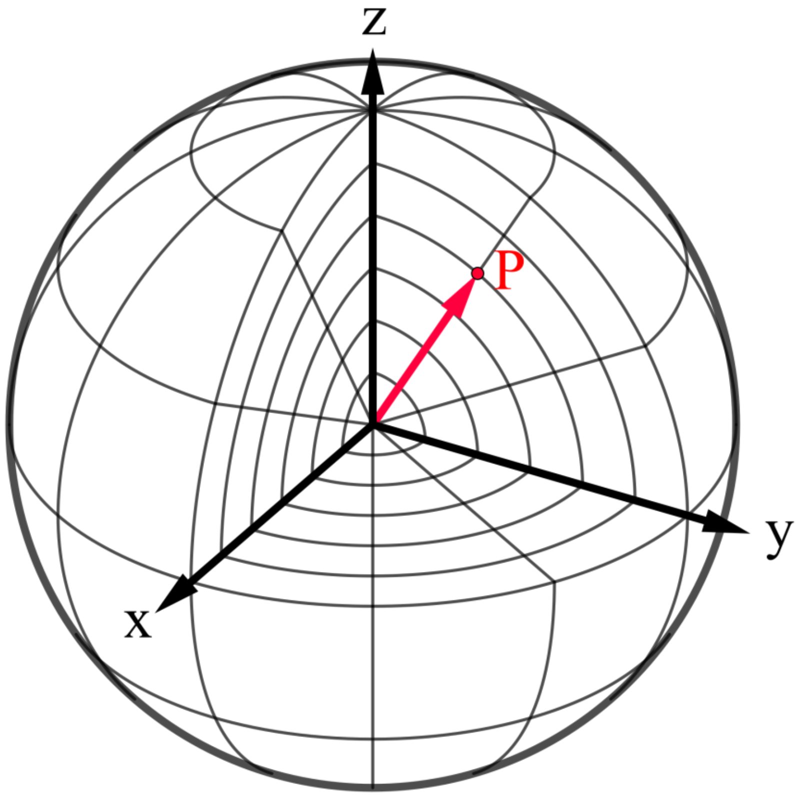


- **hypersurface**

codimension-1 submanifold

WHY WE NEED A SPACETIME SLICE?

- **Various hypersurfaces and foliation**



Q.'S BEFORE 3+1 FORMALISM

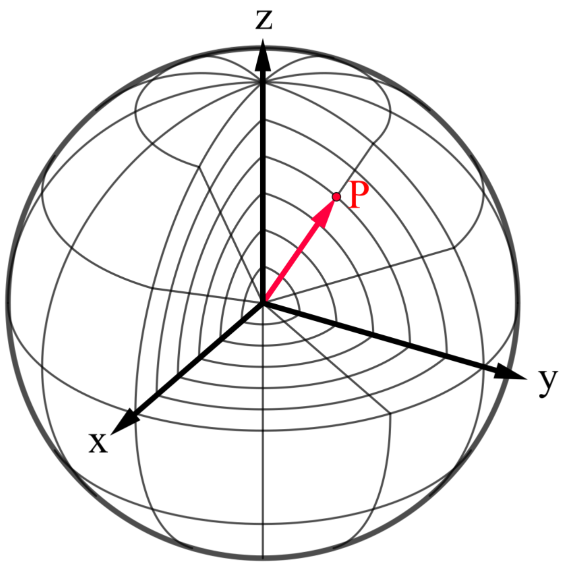
1. **Why All Measurements Need a Frame ?**
   → No Observer, No Physics
2. **Why We Need a Spacetime Slice?**
   → To Observe Anything, Need to Define "Now" and "Here"
3. **How Do We Slice Spacetime?**
   → Foliation, Lapse, and Shift, (Gauge Fixing)
4. **Given a Spacetime Slice, Can We Specify Any Metric?**
   → No, It Might Be Unphysical! (No Match With E-p Distribution.)
   → Physical Meaning of the Constraints
5. **Once We fix a Slice's Geometry of the Spacetime,**
   How Does it Change Over Time?
   → Through the Evolution eq.

HOW DO WE SLICE SPACETIME?

- **Non-degenerate scalar fields labels submanifolds. > foliation**

$$\begin{cases} (f^i\text{: exterior coordinates}) \leftarrow (\text{foliation}) \leftarrow \begin{bmatrix} (M \to N)\text{'s Codimension number's} \\ \text{non-degenerated } f^i(x) \text{ labels submanifold.} \end{bmatrix} \\ (y^a\text{: coordinates on the submanifold } M) \end{cases}$$

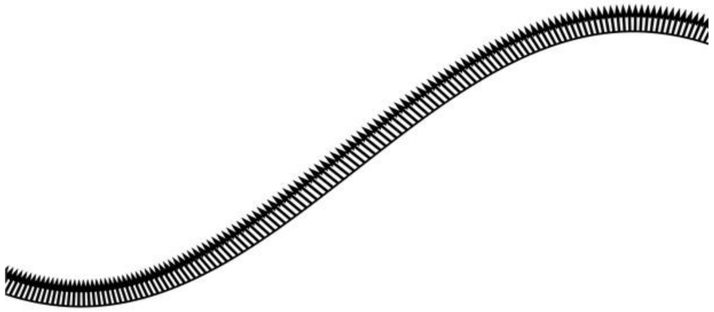
$\Lsh$ [in a neighborhood of M, coordinates: $x^\mu = (f^i, y^a)$]

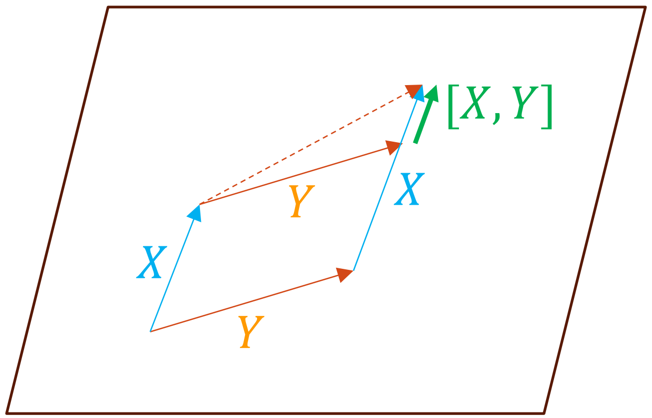


HOW DO WE SLICE SPACETIME?

- **Surface forming one-form > foliation (Frobenius theorem)**

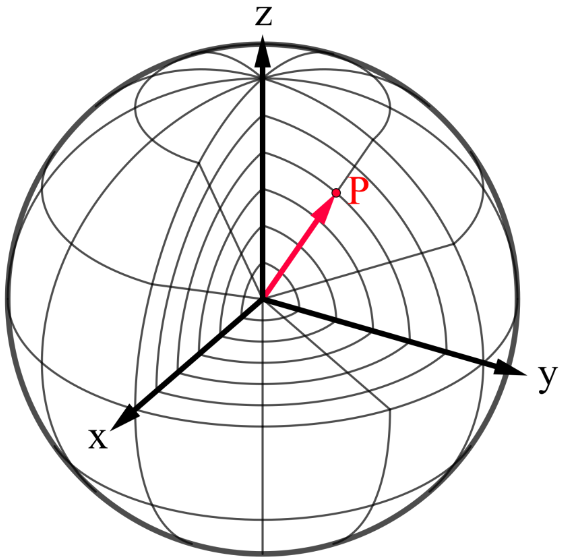
$\mathcal{D}_p = T_pS$ for some submanifold $S \ni p$ (Distribution $\mathcal{D} = \ker \omega$ arises as tangent spaces to a foliation of immersed submanifolds.)
$[X, Y] \in \Gamma(\mathcal{D})$ for all $X, Y \in \Gamma(\mathcal{D})$.
$d\omega(X, Y) = 0$ for all $X, Y \in \ker \omega$.
$\omega \wedge d\omega = 0$.
For $\omega = \xi_\mu dx^\mu$: $\xi_{[\mu} \nabla_\nu \xi_{\sigma]} = 0$
$\nabla_{[\mu} \xi_{\nu]} X^\mu Y^\nu = 0$ for all $X, Y \in \ker \omega$.
$[V_{(a)}, V_{(b)}] \in \text{span}(V_{(c)})$.
For 1-forms $n^{(a)}$: $\nabla_{[\mu} n^{(a)}_{\nu]} V^\mu W^\nu = 0$ for $V, W \in \ker n^{(a)}$.

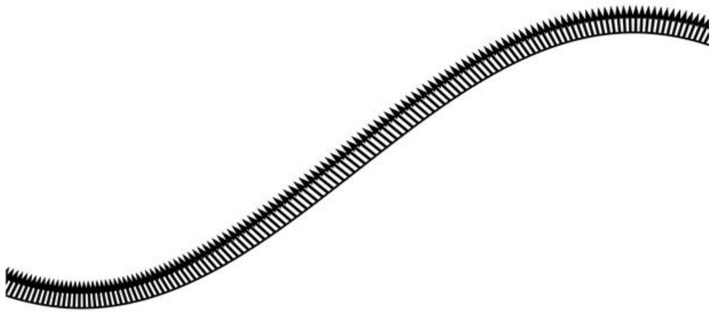


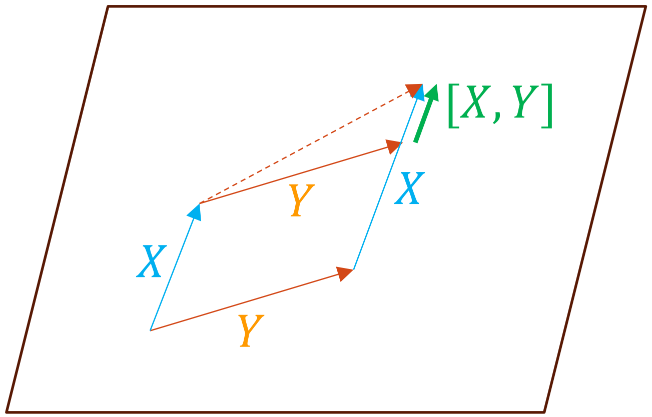


HOW DO WE SLICE SPACETIME?

- **Foliation (scalar field, normal vector, vectors)**







HOW DO WE SLICE SPACETIME?

- **Scalar field > Submanifold** Σ_t
- **Gradient of Scalar field > normal vector (one-form basis)**

$$\mathrm{d}t = \boldsymbol{\nabla} t \qquad \langle V \text{ on } \Sigma_t,\ \boldsymbol{\nabla} t\rangle = 0$$

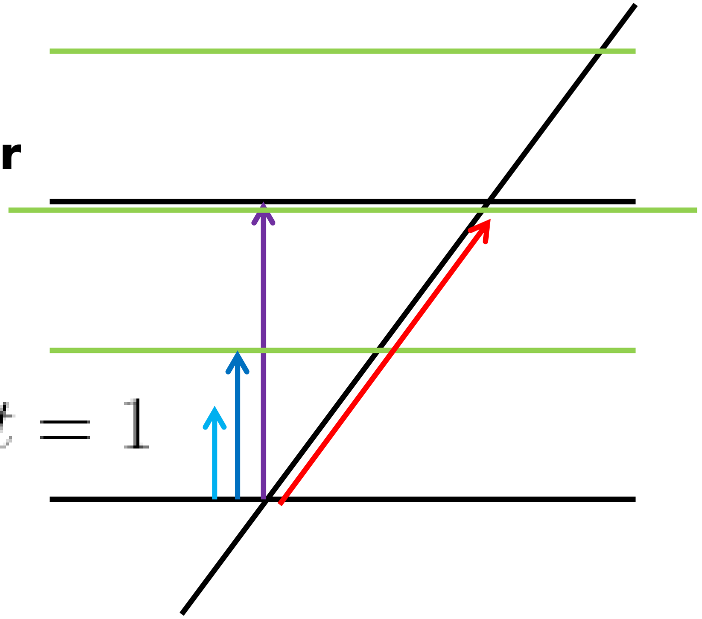
- **A curve intersecting the hypersurface > tangent basis vector**

$$\gamma(t) \text{ by } \Sigma_t :\ \boldsymbol{t} = \partial_t$$

- **Tangent basis and one-form**

$$\gamma(t) \text{ by } \Sigma_t :\ \langle \boldsymbol{t}, \boldsymbol{\nabla} t\rangle = t^\alpha \nabla_\alpha t = t^\alpha \partial_\alpha t = \partial_t t = 1$$

- **A congruence of curves > coordinates > vector components**



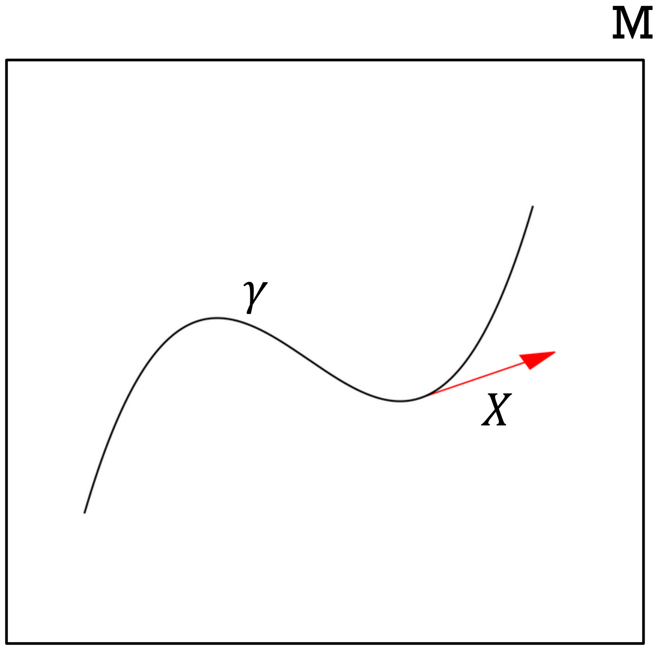
$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} \equiv (\partial_t)^\alpha \xrightarrow{\text{when } x^\alpha \equiv (t, y^a)} \delta_t{}^\alpha = (1,0,0,0)$$

$$(y_a)^\alpha = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t \equiv (\partial_{y^a})^\alpha \xrightarrow{\text{when } x^\alpha \equiv (t, y^a)} \delta_a{}^\alpha \xrightarrow{\alpha = 1} (0,1,0,0)$$

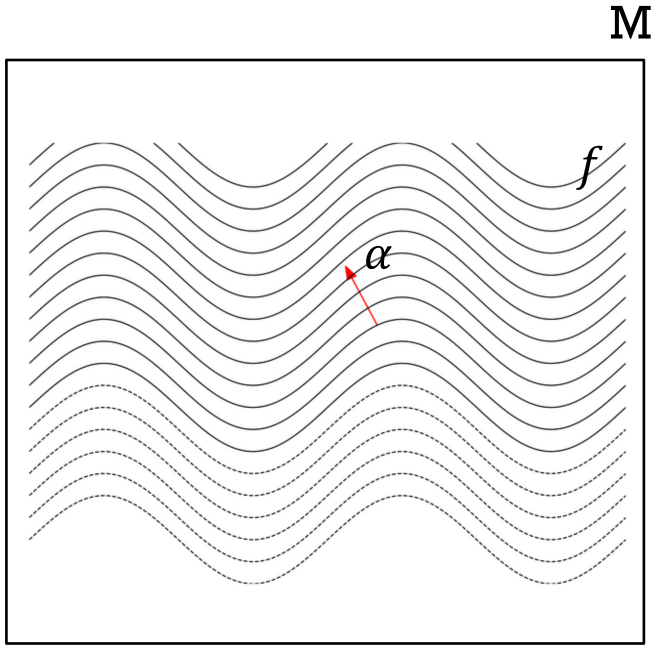
$$\nabla_\alpha t = \left(\frac{\partial t}{\partial x^\alpha}\right) \equiv (\mathrm{d}t)_\alpha \xrightarrow{\text{when } x^\alpha \equiv (t, y^a)} \delta^t{}_\alpha = (1,0,0,0)$$

> Not unit vectors

TANGENT /DUAL VECTOR BASIS



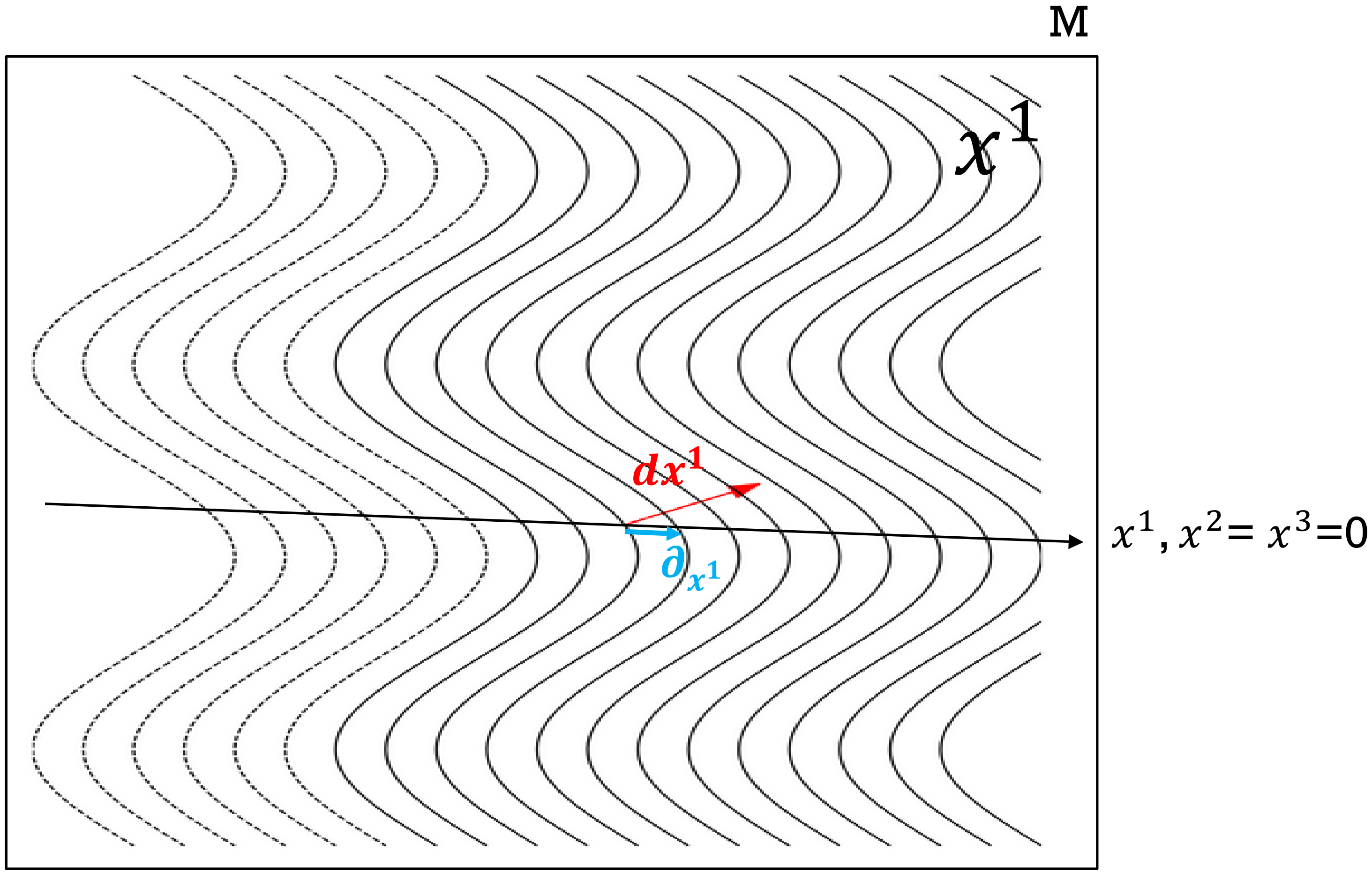
$$\mathbf{X}_p : C^\infty(M) \to \mathbb{R}, \qquad f \mapsto \mathbf{X}_p[f] = \frac{\mathrm{d}}{\mathrm{d}\lambda} f(\gamma(\lambda))\Big|_{\lambda=0}$$



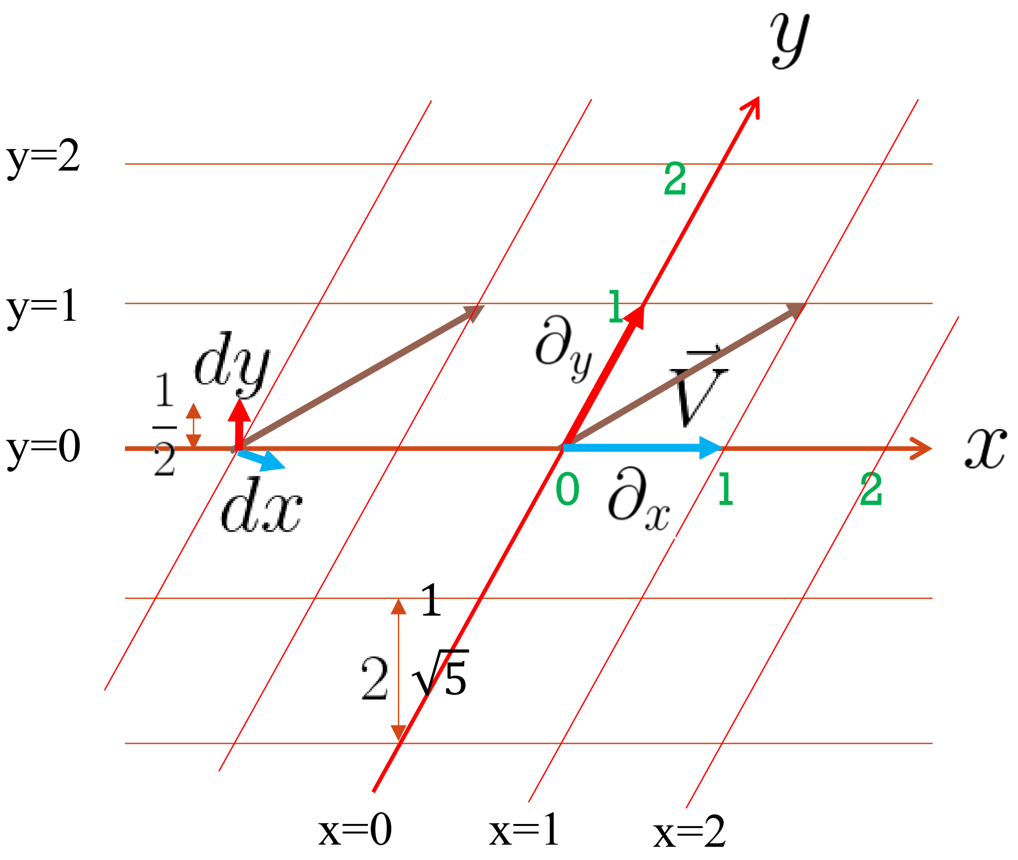
$$\alpha : T_pM \to \mathbb{R}, \quad v \mapsto \langle \alpha, v \rangle$$

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} \equiv \mathrm{d}f\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)$$

TANGENT /DUAL VECTOR FROM SCALAR FIELD



TANGENT /DUAL VECTOR BASIS



$$\frac{\mathrm{d}}{\mathrm{d}x} = \boldsymbol{\partial}_x \text{ (tangent basis vector} \rightarrow \text{vector: covariant basis)}$$

$$\vec{V} = \boldsymbol{V} = V_x\ \boldsymbol{\partial_x} + V_y\ \boldsymbol{\partial_y}$$

$$\boldsymbol{\nabla} x = \mathrm{d}x\ (\underbrace{\text{one-form}}_{\text{(normal)}} \text{ basis vector} \rightarrow \text{dual-vector: contravariant basis})$$

$$\boldsymbol{\partial_x} \cdot \boldsymbol{dx} = \frac{\partial x}{\partial x} = 1,\ \boldsymbol{\partial_y} \cdot \boldsymbol{dy} = \frac{\partial y}{\partial y} = 1,$$

$$\boldsymbol{\partial_x} \cdot \boldsymbol{dy} = \frac{\partial y}{\partial x} = 0,\ \boldsymbol{\partial_y} \cdot \boldsymbol{dx} = \frac{\partial x}{\partial y} = 0$$

$$\vec{V} = (\vec{V} \cdot \partial_x)\mathrm{d}x + (\vec{V} \cdot \partial_y)\mathrm{d}y = V_x \mathrm{d}x + V_y \mathrm{d}y = V_i \mathrm{d}x^i$$

$$\vec{V} = (\vec{V} \cdot \mathrm{d}x)\partial_x + (\vec{V} \cdot \mathrm{d}y)\partial_y = V^x \partial_x + V^y \partial_y = V^i \partial_i$$

(covariant/contravariant)-(component/basis)

HOW DO WE SLICE SPACETIME?

- **Foliation of the spacetime**
  - **Constant time hypersurface**

    > **normal vector: dt (not unit length)**

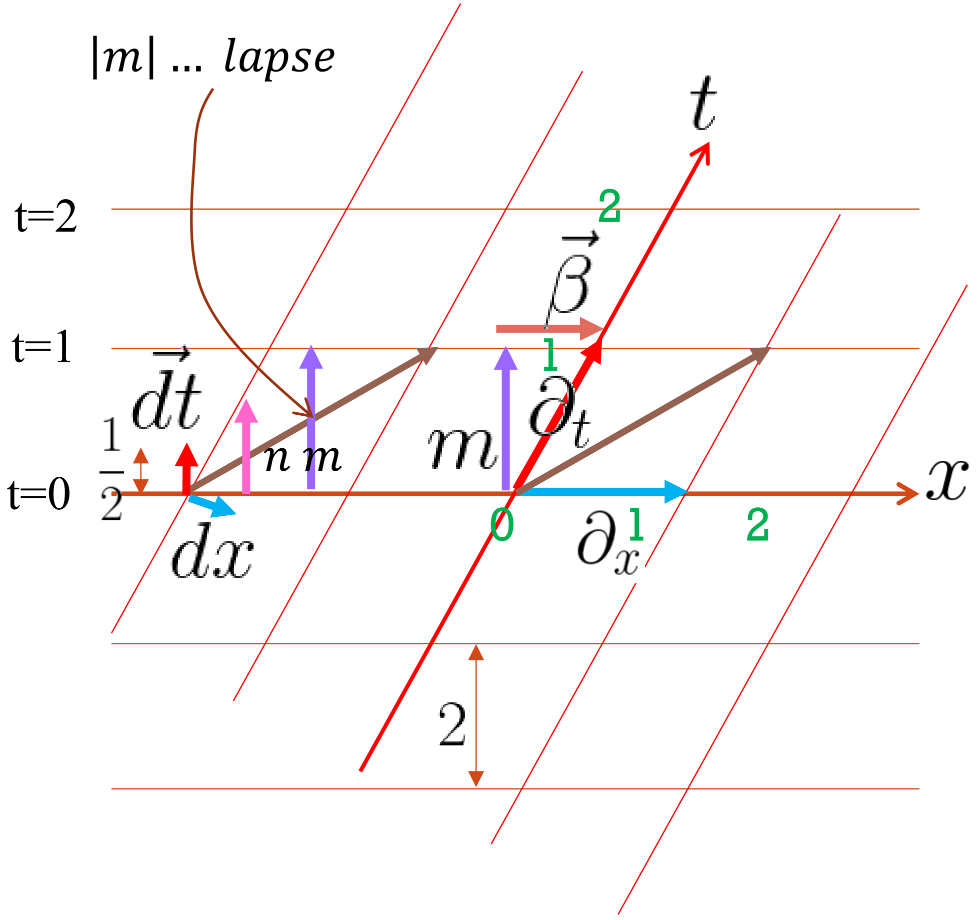
    > **unit normal vector: n**

    > **normal vector with length to next t=1 hypersurface: m (|m|=lapse N)**
  - **Draw constant x=0 curve**

    > **tangent basis until next grid: ∂_t**

    > **difference of ∂_t from m = shift vector: β**
  - **Information of bases on the hypersurface**

    > γ_{ij}



HOW DO WE SLICE SPACETIME?

- **Foliation of the spacetime**
  - **Constant time hypersurface**

    > **normal vector: dt (not unit length)**

    > **unit normal vector: n**

    > **normal vector with length to next t=1 hypersurface: m (|m|=lapse N)**
  - **Draw constant x=0 curve**

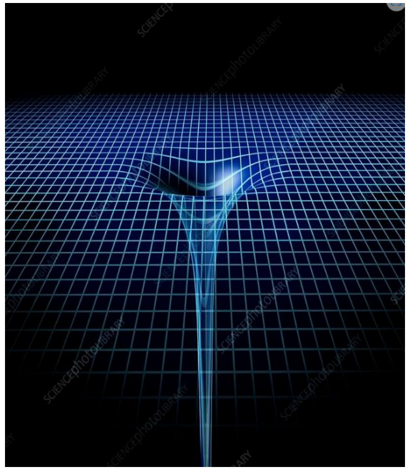
    > **tangent basis until next grid: ∂_t**

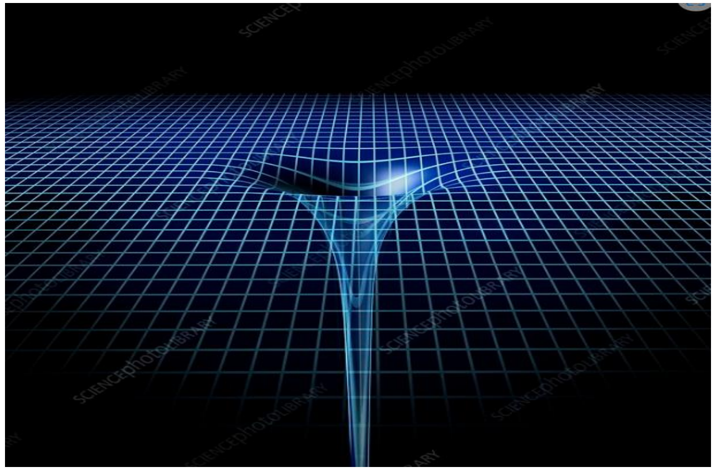
    > **difference of ∂_t from m = shift vector: β**
  - **Information of bases on the hypersurface**

    > γ_{ij}

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$G_{\mu'\nu'}(g_{\mu'\nu'}) = \frac{8\pi G}{c^4} T_{\mu'\nu'} \qquad G_{\mu\nu}(\tilde{g}_{\mu\nu}) = \frac{8\pi G}{c^4} \tilde{T}_{\mu\nu}$$





Q.'S BEFORE 3+1 FORMALISM

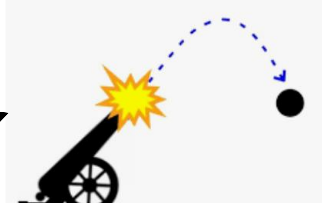
1. Why All Measurements Need a Frame ?
   → No Observer, No Physics
2. Why We Need a Spacetime Slice?
   → To Observe Anything, Need to Define "Now" and "Here"
3. How Do We Slice Spacetime?
   → Foliation, Lapse, and Shift, (Gauge Fixing)
4. Given a Spacetime Slice, Can We Specify Any Metric?
   → No, It Might Be Unphysical! (No Match With E-p Distribution.)
   → Physical Meaning of the Constraints
5. Once We fix a Slice's Geometry of the Spacetime,
   How Does it Change Over Time?
   → Through the Evolution eq.

VARIABLES IN EINSTEIN'S EQ.

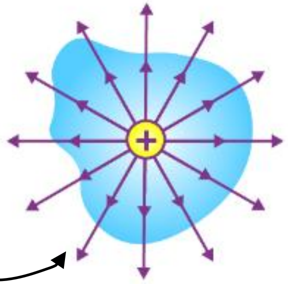
$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Metric tensor
  - Unknowns to solve in Einstein's eq. such as x in f(x)=y.
  - Field variable, not x(t) but $g_{\mu\nu}(t,x)$
  - Related to gravitational field **g** in Newtonian gravity
  - Describes the spacetime structure
  - Fundamental quantity: 4x4 symmetric tensor, 10 dof's

    >> What is its value? 10 is real dofs? Why not just 3+1?



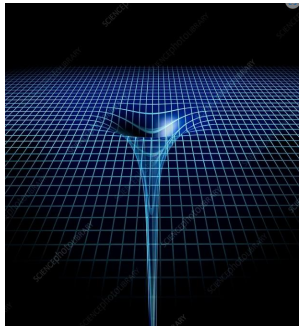
$$\mathbf{F} = m\mathbf{a} \rightarrow \mathbf{x}(t)$$



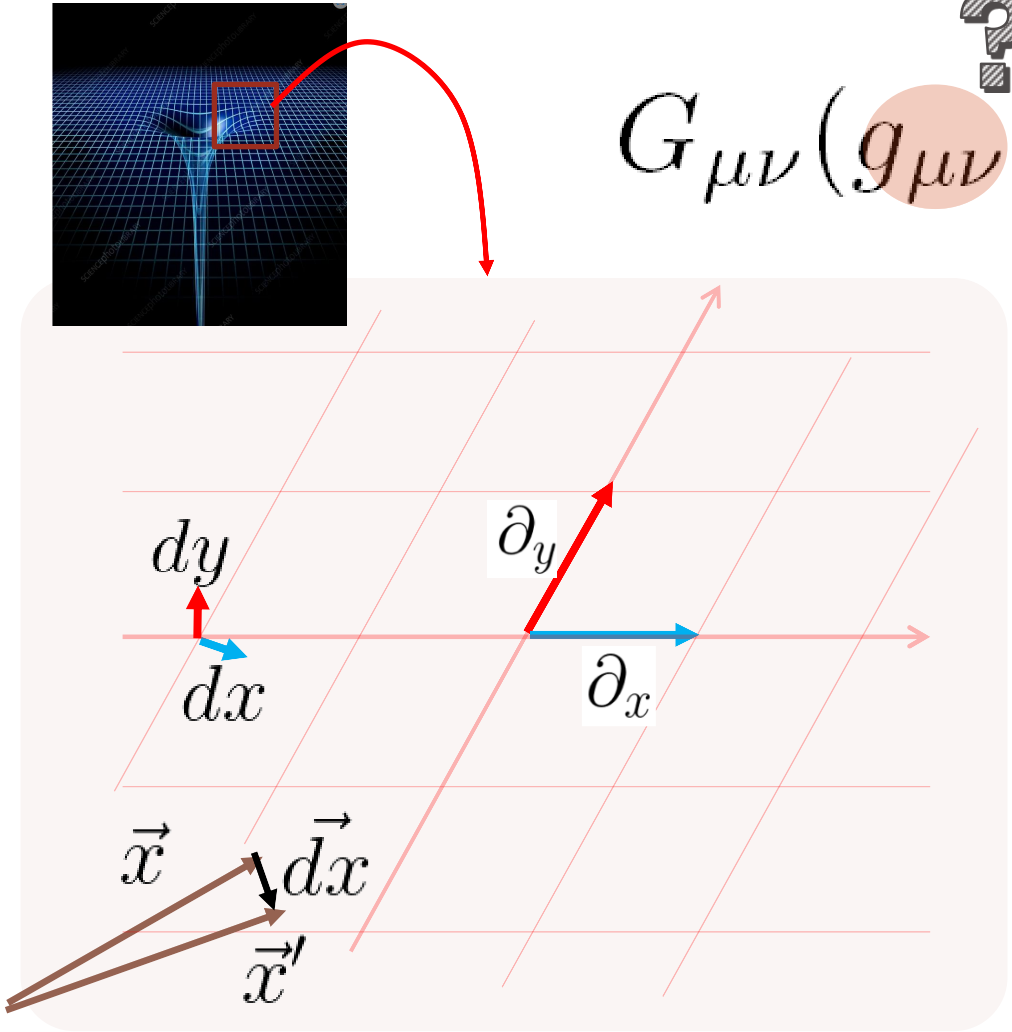
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \rightarrow \mathbf{E}(x)$$

3+1?

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}$$



METRIC TENSOR IN EINSTEIN EQ.



$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Metric tensor components from coordinate basis
  - Information about the lengths and angles of the basis vectors

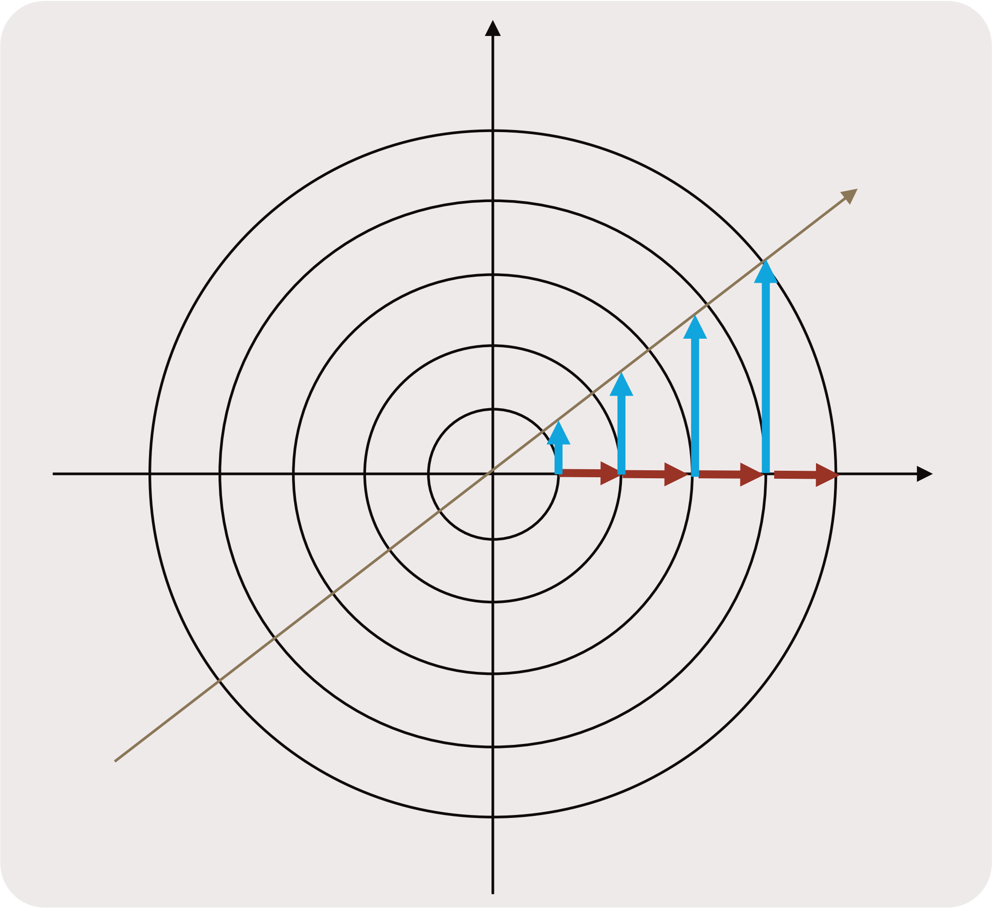
$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu = \boldsymbol{g}(\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu) = (g_{\mu'\nu'} \mathbf{d}x^{\mu'} \otimes \mathbf{d}x^{\nu'})(\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu)$$
$$g^{\mu\nu} = \mathbf{d}x^\mu \cdot \mathbf{d}x^\nu = \boldsymbol{g}(\mathbf{d}x^\mu, \mathbf{d}x^\nu) = (g^{\mu'\nu'} \boldsymbol{\partial}_{\mu'} \otimes \boldsymbol{\partial}_{\nu'})(\mathbf{d}x^\mu, \mathbf{d}x^\nu)$$

$$\begin{cases} g_{xx} = \boldsymbol{\partial}_x \cdot \boldsymbol{\partial}_x = |\boldsymbol{\partial}_x|^2 & \rightarrow \text{length}^2 \\ g_{xy} = \boldsymbol{\partial}_x \cdot \boldsymbol{\partial}_y = |\boldsymbol{\partial}_x||\boldsymbol{\partial}_y| \cos\theta & \rightarrow \text{angle info.} \end{cases} \rightarrow \text{grid info.}$$

$$ds^2 = \boldsymbol{g}(\mathbf{d}x, \mathbf{d}x) = \boldsymbol{g}(\mathrm{d}x^\mu \boldsymbol{\partial}_\mu, \mathrm{d}x^\nu \boldsymbol{\partial}_\nu) = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \rightarrow \text{invariant length}$$

METRIC TENSOR EXAMPLE

- 2-D Polar coordinates



$$
\begin{aligned}
ds^2 &= g_{rr} dr^2 + g_{\theta\theta} d\theta^2 \\
&= (\boldsymbol{\partial_r} \cdot \boldsymbol{\partial_r}) dr^2 + (\boldsymbol{\partial_\theta} \cdot \boldsymbol{\partial_\theta}) d\theta^2 \\
&= \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2
\end{aligned}
$$

$$
g_{ij} = \begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}
$$

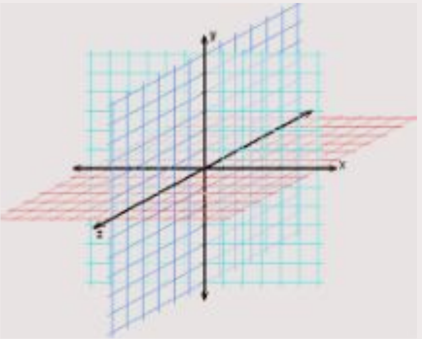
- Alternative coordinate system of flat 2-D space
- Orthogonal coordinates
- Symmetric tensor > metric tensor property
- Diagonal components indicates lengths of the basis

METRIC TENSOR EXAMPLES

- 3-D flat space

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

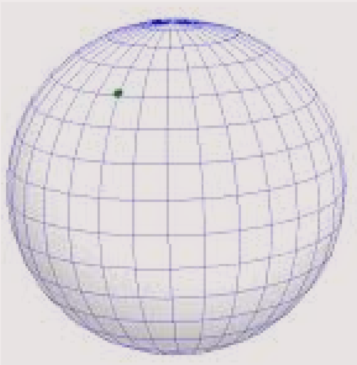
$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



2-D sphere surface

$$ds^2 = r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}$$

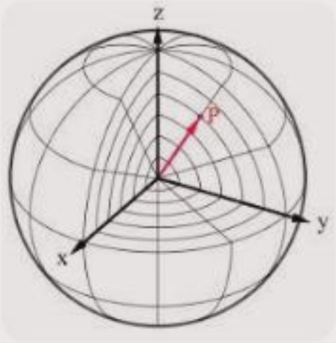


3-D spherical coordinates

$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$(= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

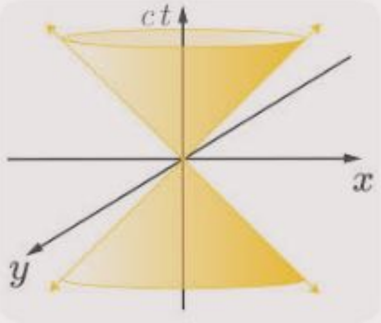


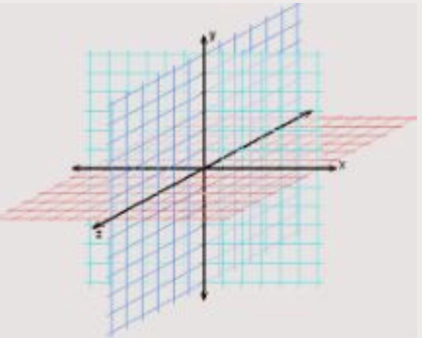
METRIC TENSOR EXAMPLES (PSEUDO-RIEMANNIAN, FLAT)

- 4-D flat Minkowski spacetime (Cartesian) 4-D flat Minkowski spacetime (Spherical)

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

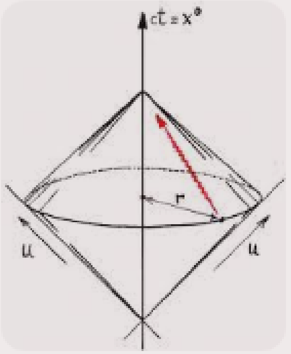
$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

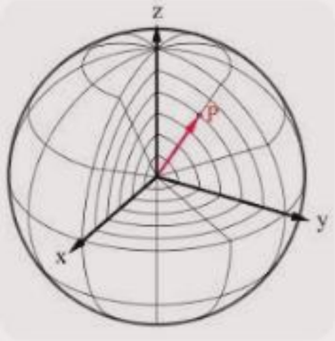




$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$



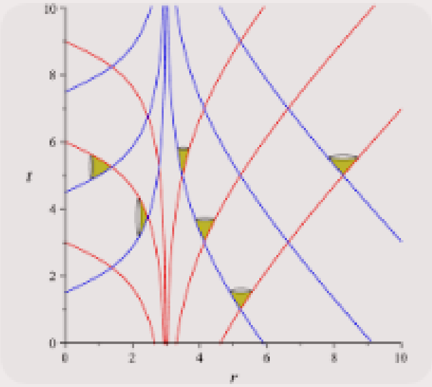


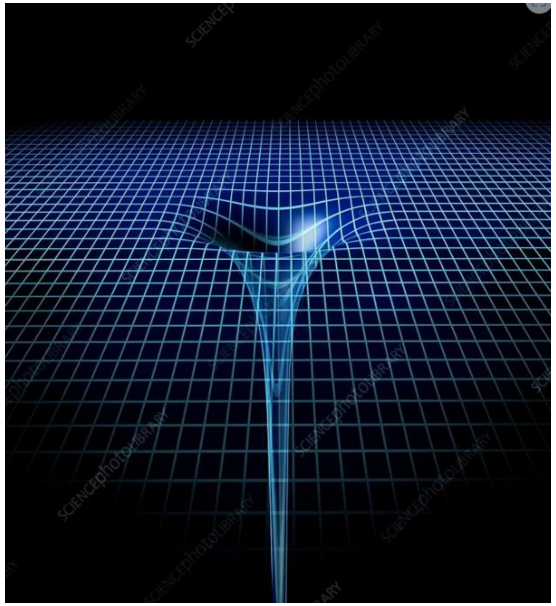
METRIC TENSOR EXAMPLES (PSEUDO-RIEMANNIAN, CURVED)

- 4-D curved spacetime

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{2GM}{r}}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta\mathrm{d}\phi^2$$

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$





Q.'S BEFORE 3+1 FORMALISM

1. **Why All Measurements Need a Frame ?**
   → No Observer, No Physics
2. **Why We Need a Spacetime Slice?**
   → To Observe Anything, Need to Define "Now" and "Here"
3. **How Do We Slice Spacetime?**
   → Foliation, Lapse, and Shift, (Gauge Fixing)
4. **Given a Spacetime Slice, Can We Specify Any Metric?**
   → No, It Might Be Unphysical! (No Match With E-p Distribution.)
   → Physical Meaning of the Constraints
5. **Once We fix a Slice's Geometry of the Spacetime,**
   How Does it Change Over Time?
   → Through the Evolution eq.

PHYSICAL DOFS OF METRIC TENSOR

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Considering the gauge fixing, we have 6 physical dof's in the metric.

$$\begin{pmatrix} G_{00} & G_{01} & G_{02} & G_{03} \\ G_{01} & G_{11} & G_{12} & G_{13} \\ G_{02} & G_{12} & G_{22} & G_{23} \\ G_{03} & G_{13} & G_{23} & G_{33} \end{pmatrix}$$

10 → 6

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

10 → 6

$g_{\mu\nu},\ g_{\bar\mu\bar\nu},\ g_{\tilde\mu\tilde\nu}$ describe the same physics.

$t' = f_0(t,x,y,z),\ \ x' = f_1(t,x,y,z), \cdots$

We can reduce 4 $g_{\mu\nu}$ components by fixing the gauge.

3+1 DECOMPOSITION (METRIC DECOMPOSITION)

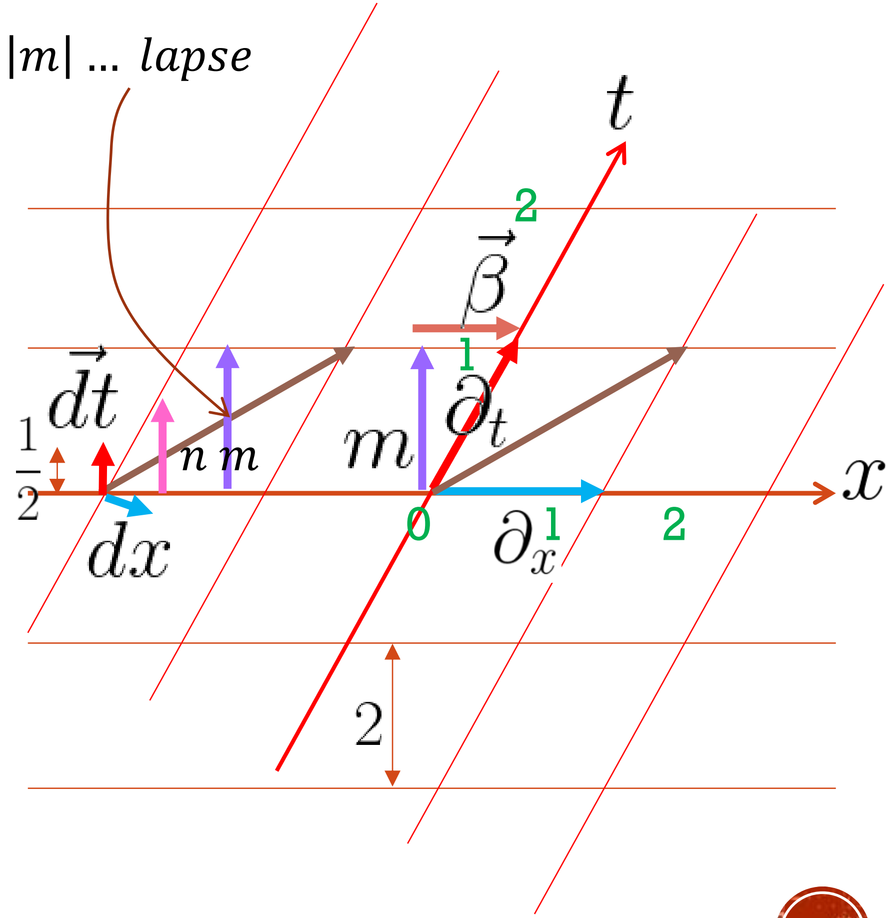
$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu,\ g^{\mu\nu} = (\mathbf{d}x^\mu)\cdot(\mathbf{d}x^\nu)$$

$$\downarrow \begin{cases} \boldsymbol{t} = \boldsymbol{m} + \boldsymbol{\beta} & \\ N \equiv \alpha & \text{(lapse function)} \\ N^\alpha \equiv \beta^\alpha = (0, \vec{\beta}) & \text{(shift vector)} \\ m^\alpha = N n^\alpha = (1, -\vec{\beta}) & \text{(evolution vector)} \end{cases}$$

$$\begin{aligned} ds^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -(N^2 - \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ &= -N^2\mathrm{d}t^2 + \gamma_{ij}\underbrace{(\mathrm{d}x^i + \beta^i\mathrm{d}t)}_{\mathrm{d}\hat{x}^i}\underbrace{(\mathrm{d}x^j + \beta^j\mathrm{d}t)}_{\mathrm{d}\hat{x}^j} \end{aligned}$$

Note that $\mathrm{d}x^i$ is not on Σ_t when $\vec{\beta} \neq 0$, but $\mathrm{d}\hat{x}^i$ is on Σ_t.

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\partial_t}\cdot\boldsymbol{\partial_t} & \boldsymbol{\partial_t}\cdot\boldsymbol{\partial_x} & \cdots & \cdots \\ \cdots & & & \\ \cdots & & \gamma_{ij} & \\ \cdots & & & \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{m}\cdot\boldsymbol{m} + \boldsymbol{\beta}\cdot\boldsymbol{\beta} & \boldsymbol{\beta}\cdot\boldsymbol{\partial_x} & \cdots & \cdots \\ \cdots & & & \\ \cdots & & \gamma_{ij} & \\ \cdots & & & \end{pmatrix} \\ &= \begin{pmatrix} -N^2 + \beta_k\beta^k & \cdots & \beta_i & \cdots \\ \cdots & & & \\ \beta_i & & \gamma_{ij} & \\ \cdots & & & \end{pmatrix} \end{aligned}$$



3+1 DECOMPOSITION (PROJECTION TENSOR)

$$
\begin{aligned}
ds^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\
&= -(N^2 - \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \\
&= -N^2\mathrm{d}t^2 + \gamma_{ij}\underbrace{(\mathrm{d}x^i + \beta^i\mathrm{d}t)}_{\mathrm{d}\hat{x}^i}\underbrace{(\mathrm{d}x^j + \beta^j\mathrm{d}t)}_{\mathrm{d}\hat{x}^j}
\end{aligned}
$$

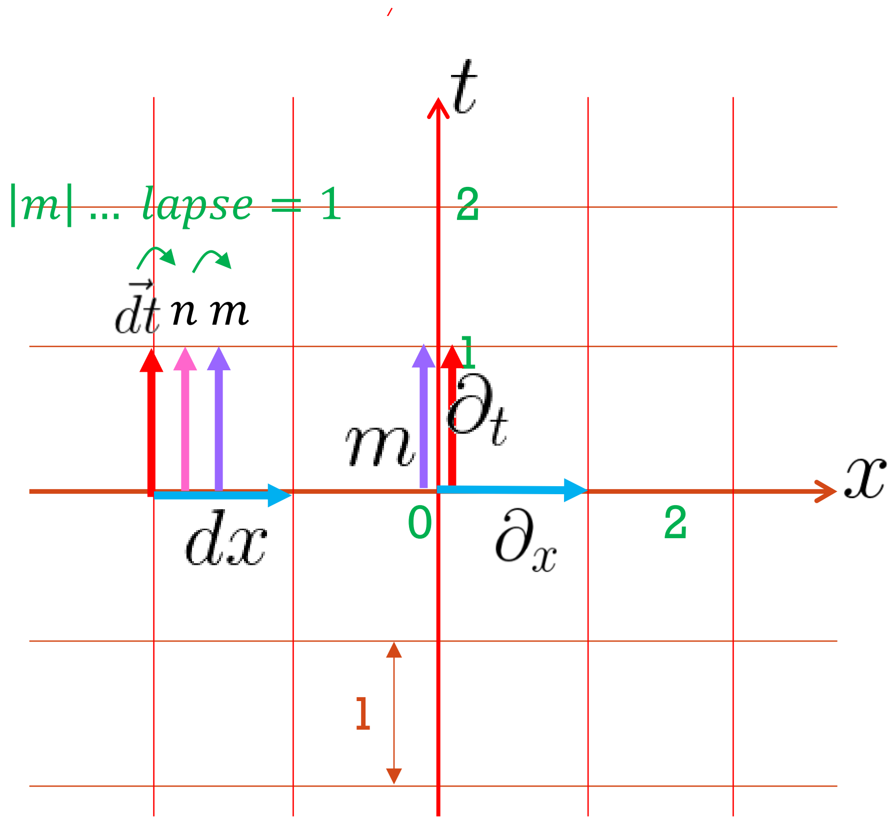
Note that $\mathrm{d}x^i$ is not on Σ_t when $\vec{\beta} \neq 0$, but $\mathrm{d}\hat{x}^i$ is on Σ_t.

Gaussian normal coordinates : $ds^2 = \sigma\mathrm{d}z^2 + \gamma_{ij}\mathrm{d}y^i\mathrm{d}y^j$

$$
\begin{aligned}
\Lsh\ ds^2 &= g_{\mu\nu}(\mathbf{d}x^\mu)\otimes(\mathbf{d}x^\nu) \\
&= -\mathrm{d}t^2 + \gamma_{ij}\mathrm{d}y^i\mathrm{d}y^j \Leftrightarrow -(N^2 - \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i\mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \\
&\circlearrowleft\ \mathbf{d}t = -N^{-1}\boldsymbol{n} \xrightarrow{N=1} -\boldsymbol{n} = -n_\mu(\mathbf{d}x^\mu) \qquad \Lsh\ N = 1,\ \beta^i = 0 \\
&= -n_\mu n_\nu \mathrm{d}x^\mu\mathrm{d}x^\nu + \gamma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \\
&= (-n_\mu n_\nu + \gamma_{\mu\nu})\mathrm{d}x^\mu\mathrm{d}x^\nu
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \gamma_{\mu\nu} &= g_{\mu\nu} - \sigma n_\mu n_\nu \\
&= P_{\mu\nu}\quad \text{(projection tensor)}
\end{aligned}
$$

> Projection tensor = metric on hypersurface



3+1 DECOMPOSITION (PROJECTION TENSOR (2))

- **Definition:** $$P_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu$$

- **Projected vectors are tangent to the hypersurface**

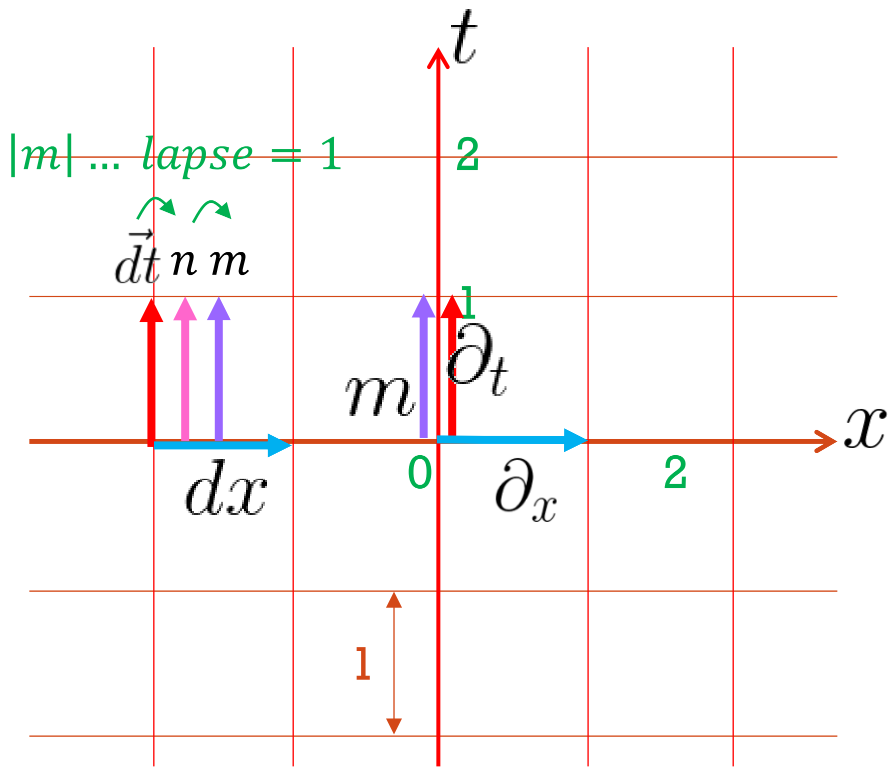
$$(P_{\mu\nu}V^\mu)n^\nu = g_{\mu\nu}V^\mu n^\nu - \sigma n_\mu \overbrace{n_\nu V^\mu n^\nu}^{=\sigma V^\mu,\ \sigma^2=1}$$
$$= 0$$

- **Act like the metric for tangent vectors**

$$P_{\mu\nu}V^\mu W^\nu = g_{\mu\nu}V^\mu W^\nu - \sigma n_\mu n_\nu V^\mu W^\nu \rightarrow 0$$
$$= g_{\mu\nu}V^\mu W^\nu$$

- **Idemp**

$$P^\mu{}_\lambda P^\lambda{}_\nu = \ldots = P^\mu{}_\nu$$



3+1 DECOMPOSITION (INVERSE METRIC)

$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu,\ g^{\mu\nu} = (\mathbf{d}x^\mu)\cdot(\mathbf{d}x^\nu)$$

$$\downarrow \begin{cases} \boldsymbol{t} = \boldsymbol{m} + \boldsymbol{\beta} & \\ N \equiv \alpha & \text{(lapse function)} \\ N^\alpha \equiv \beta^\alpha = (0, \vec{\beta}) & \text{(shift vector)} \\ m^\alpha = N n^\alpha = (1, -\vec{\beta}) & \text{(evolution vector)} \end{cases}$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{\beta^j}{N^2} \\ \frac{\beta^i}{N^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{N^2} \end{pmatrix}$$

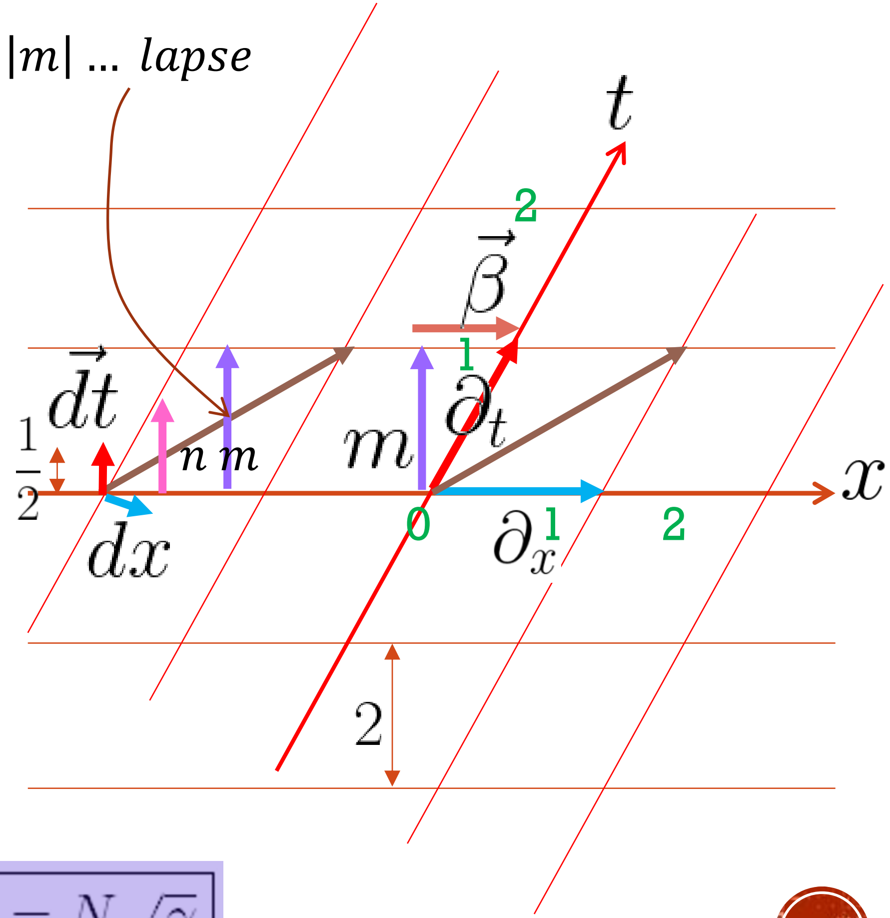
Note that $g^{ij} \neq \gamma^{ij}$

$$g^{00} = (\mathbf{d}t)\cdot(\mathbf{d}t) = \left(-\frac{1}{N}n^\mu\right)\left(-\frac{1}{N}n_\mu\right) = \sigma\frac{1}{N^2} = -\frac{1}{N^2}$$

$$g^{0\mu} = (\mathbf{d}t)\cdot(\mathbf{d}x^\mu) = \delta^\beta_{\ \alpha}\left(-\frac{1}{N}n^\alpha\right)\underbrace{(\mathbf{d}x^\mu)_\beta}_{=\delta^\mu_{\ \beta}} = -\frac{1}{N}n^\mu = \frac{1}{N^2}(-1, \vec{\beta})$$

$$g^{ij} = (\mathbf{d}x^i)\cdot(\mathbf{d}x^j) = g^{\mu\nu}(\mathbf{d}x^i)_\mu(\mathbf{d}x^j)_\nu = (\sigma n^\mu n^\nu + P^{\mu\nu})\underbrace{(\mathbf{d}x^i)_\mu}_{=\delta^i_\beta}\underbrace{(\mathbf{d}x^j)_\nu}_{=\delta^j_\nu}$$

$$= \sigma n^i n^j + \gamma^{ij} = -\frac{\beta^i\beta^j}{N^2} + \gamma^{ij}$$



$$\Rightarrow \boxed{\sqrt{-g} = N\sqrt{\gamma}}$$

Q.'S BEFORE 3+1 FORMALISM

1. **Why All Measurements Need a Frame ?**
   → No Observer, No Physics
2. **Why We Need a Spacetime Slice?**
   → To Observe Anything, Need to Define "Now" and "Here"
3. **How Do We Slice Spacetime?**
   → Foliation, Lapse, and Shift, (Gauge Fixing)
4. **Given a Spacetime Slice, Can We Specify Any Metric?**
   → No, It Might Be Unphysical! (No Match With E-p Distribution.)
   → Physical Meaning of the Constraints
5. **Once We fix a Slice's Geometry of the Spacetime,**
   How Does it Change Over Time?
   → Through the Evolution eq.

GIVEN A SPACETIME SLICE, CAN WE SPECIFY ANY METRIC?

No, It Might Be Unphysical! (No Match With E-p Distribution.)
→ Physical Meaning of the Constraints

$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow \ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow \ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ ^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \ \rightarrow \ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) \\ \qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ \qquad - D_i D_j \alpha - 8\pi G\alpha[S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow \ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

CONSTRAINTS AND EVOLUTION EQ.

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

(1) $G_{\mu\nu} = 0$ (vacuum Einstein eq. $\rightarrow$ $10 g_{\mu\nu}$ dof.)

(2) $\nabla_\alpha G^{\alpha\beta} = 0$ (Bianchi id. 4 eq.)

$$\hookrightarrow \partial_0 \underbrace{G^{0\beta}}_{} = \underbrace{-(\partial_i G^{i\beta} + \Gamma^{\alpha}{}_{\alpha\sigma} G^{\sigma\beta} + \Gamma^{\beta}{}_{\alpha\sigma} G^{\alpha\sigma})}_{}$$

no 2nd time derivative. $\leftarrow$ $G_{\mu\nu}$: at most 2nd time derivative.

$\hookrightarrow$ $G_{0\beta} = 0$ $\rightarrow$ no evolution eq. just constraint on initial data.

$\hookrightarrow$ $g_{\mu\nu}$ evolved by $G_{ij} = 0$ from $G_{0\beta} = 0|_{\Sigma_t}$

satisfies constraints in M by $\nabla_\alpha G^{\alpha\beta} = 0$

Q.'S BEFORE 3+1 FORMALISM

1. **Why All Measurements Need a Frame ?**
   → No Observer, No Physics
2. **Why We Need a Spacetime Slice?**
   → To Observe Anything, Need to Define "Now" and "Here"
3. **How Do We Slice Spacetime?**
   → Foliation, Lapse, and Shift, (Gauge Fixing)
4. **Given a Spacetime Slice, Can We Specify Any Metric?**
   → No, It Might Be Unphysical! (No Match With E-p Distribution.)
   → Physical Meaning of the Constraints
5. **Once We fix a Slice's Geometry of the Spacetime,**
   How Does it Change Over Time?
   → Through the Evolution eq.

ONCE WE FIX A SPACETIME SLICE, HOW DOES IT CHANGE OVER TIME?

Through the Evolution eq.

$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ ^{(4)}G_{n\widehat{\mu}} = 8\pi G T_{n\widehat{\mu}} \ \rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ ^{(4)}G_{\widehat{\mu\nu}} = 8\pi G T_{\widehat{\mu\nu}} \ \rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\ \qquad\qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ \qquad\qquad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

PHYSICAL DOFS OF METRIC TENSOR

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Considering the gauge fixing, we have 6 physical dof's in the metric.

$$\begin{pmatrix} G_{00} & G_{01} & G_{02} & G_{03} \\ G_{01} & G_{11} & G_{12} & G_{13} \\ G_{02} & G_{12} & G_{22} & G_{23} \\ G_{03} & G_{13} & G_{23} & G_{33} \end{pmatrix}$$

10 → 6

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

10 → 6

$g_{\mu\nu}$, $g_{\bar{\mu}\bar{\nu}}$, $g_{\tilde{\mu}\tilde{\nu}}$ describe the same physics.

$t' = f_0(t, x, y, z)$, $x' = f_1(t, x, y, z)$, $\cdots$

We can reduce 4 $g_{\mu\nu}$ components by fixing the gauge.

EINSTEIN EQUATION DECOMPOSITION

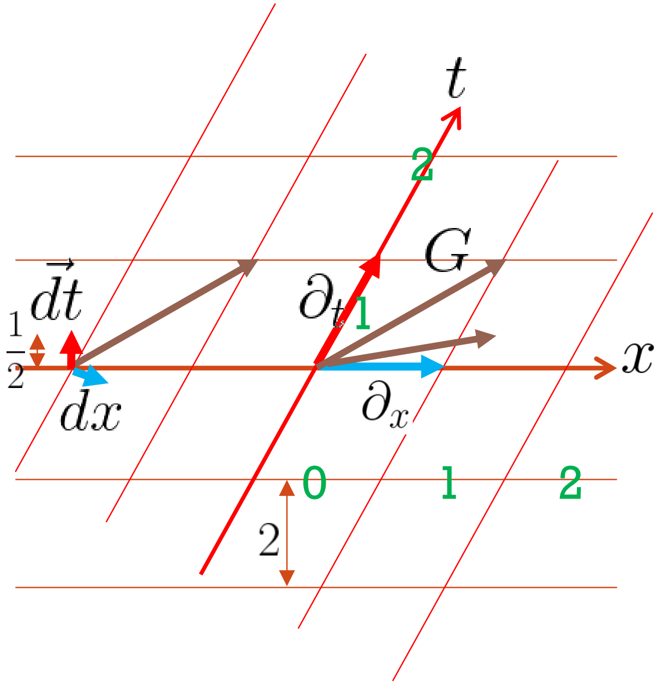
3+1 DEPENDS ON COORDINATE CHOICE.

- Problems in the direct use of g00, g01, ….
  - Not gauge invariant quantities
  - Decompose Einstein eq. in a coordinate independent way
  - We can deal with a specific slice of the spacetime and evolve it.
    - Initial value problem, numerical relativity, ADM formalism, …

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

⇩

$$\begin{cases} G_{0\nu} = \frac{8\pi G}{c^4} T_{0\nu} \\ G_{ij} = \frac{8\pi G}{c^4} T_{ij} \end{cases}$$



3+1 DECOMPOSITION (EINSTEIN TENSOR)

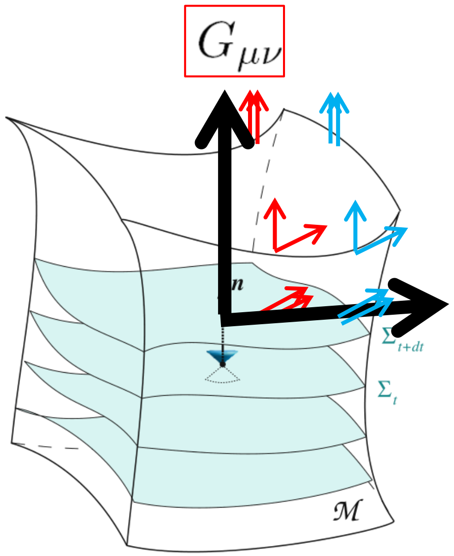
- Vector decomposition:

$$V^\mu = V^\nu \delta^\mu_\nu = V^\mu(-n^\mu n_\nu + P^\mu_\nu) = -(V^\nu n_\nu)n^\mu + P^\mu_\nu V^\nu$$

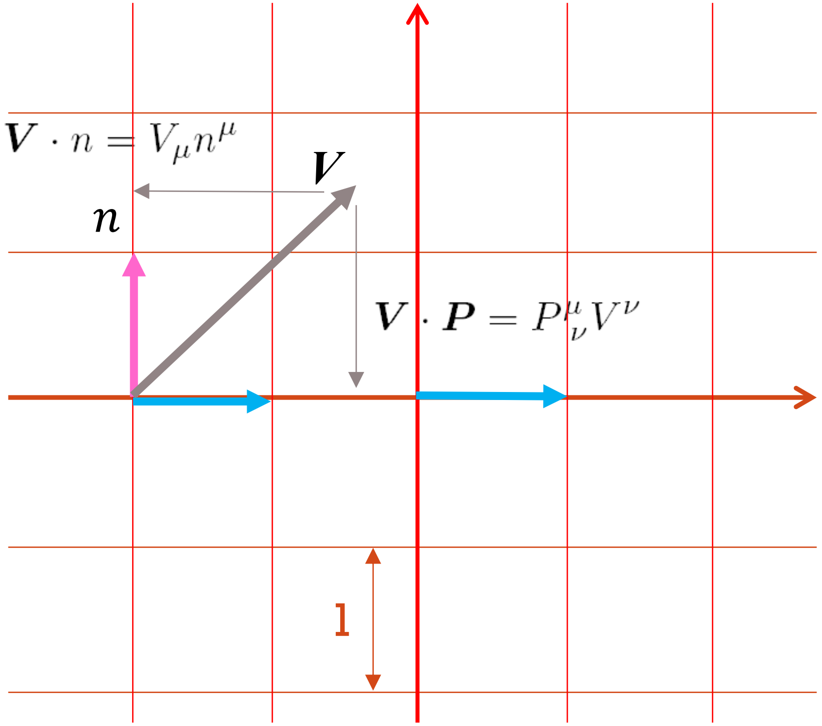
- Einstein tensor decomposition:

$$\begin{cases} X_{nn} \equiv X_{\mu\nu} n^\mu n^\nu \\ X_{n\hat{\nu}} (= X_{nj}) = X_{\mu\nu} n^\mu P^\nu_{\hat{\nu}} \\ X_{\hat{\mu}\hat{\nu}} (= X_{ij}) = X_{\mu\nu} P^\mu_{\hat{\mu}} P^\nu_{\hat{\nu}} \end{cases}$$

$$\text{then we have} \begin{cases} G_{nn} = 8\pi G T_{nn} \\ G_{ni} = 8\pi G T_{ni} \\ G_{ij} = 8\pi G T_{ij} \end{cases}$$



[Gourgoulhon, 2021]



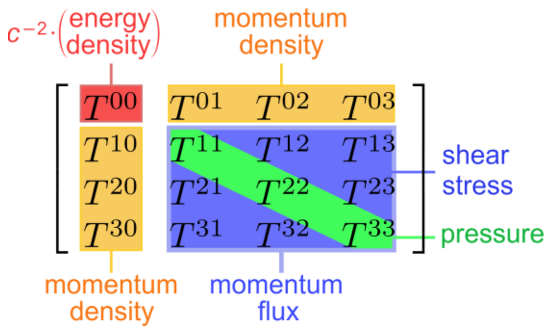
3+1 DECOMPOSITION (ENERGY-MOMENTUM TENSOR)

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

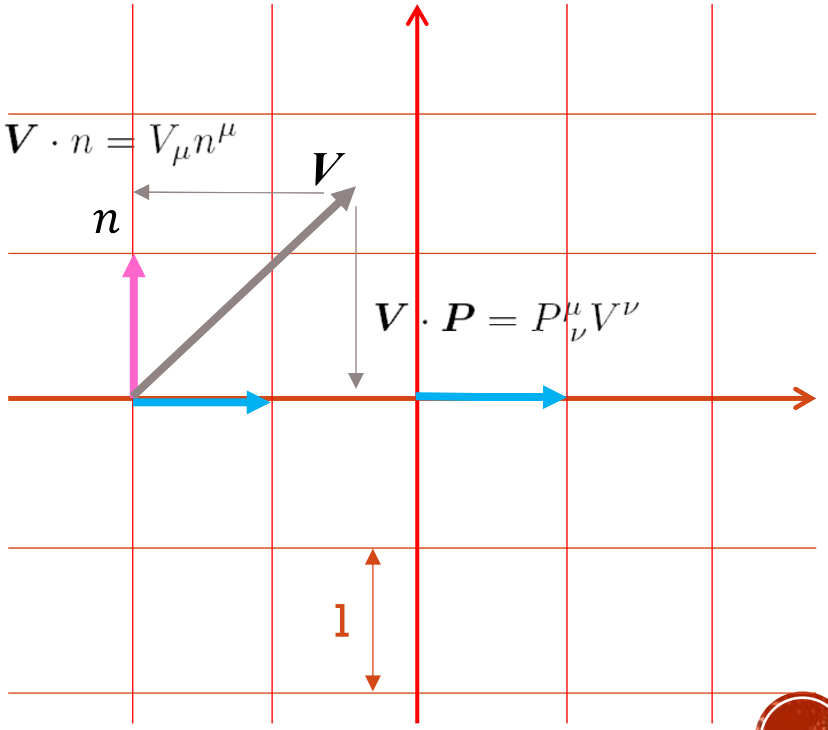
Eulerian observer (observer moving along a normal vector)

$$\boldsymbol{T} = E\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{p} + \boldsymbol{p}\otimes\boldsymbol{n} + \boldsymbol{S}$$

$$\begin{aligned} T &= T^{\mu}_{\ \mu} \\ &= g^{\mu\nu}T_{\mu\nu} \\ &= (\sigma n^{\mu}n^{\nu} + P^{\mu\nu})T_{\mu\nu} \\ &= -n^{\mu}n^{\nu}T_{\mu\nu} + P^{\mu\nu}T_{\mu\nu} \\ &= -E + S \end{aligned}$$

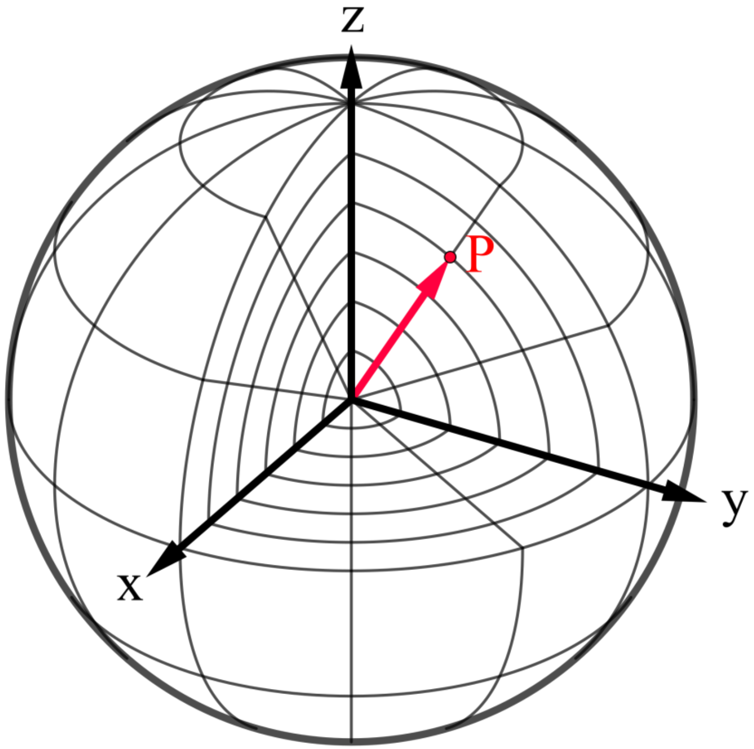


$$\begin{cases} \text{(energy density)} & \rho_e = T_{nn} \\ \text{(momentum density)} & p_{\alpha} = -T_{n\hat{\alpha}} \\ \text{(stress tensor)} & S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}} \end{cases}$$



3+1 DECOMPOSITION TO HYPERSURFACE QUANTITIES

$$G_{\mu\nu} \rightarrow \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu}\widehat{\nu}} \end{cases} \xrightarrow[\substack{{}^{(4)}\nabla \rightarrow {}^{(3)}\nabla \\ g_{\mu\nu} \rightarrow \gamma_{ij}}]{{}^{(4)}R \rightarrow {}^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)$$



$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$\Lsh\ x^i = (r,\theta,\phi) \ \rightarrow\ \gamma_{ij} \ \rightarrow\ {}^{(3)}\nabla \ \rightarrow\ {}^{(3)}R$$

$$ds^2 = r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

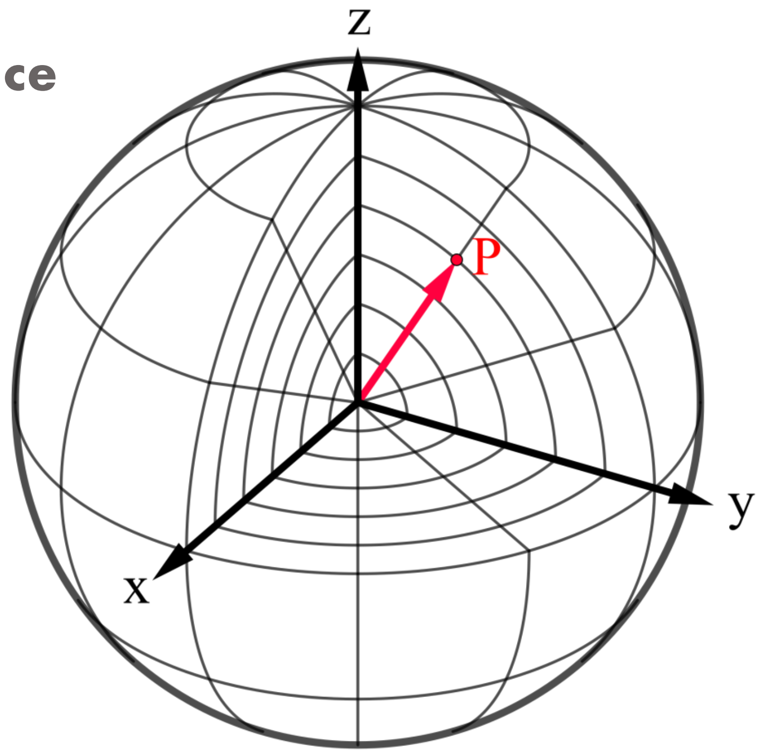
$$\Lsh\ x^a = (\theta,\phi) \ \rightarrow\ \gamma_{ab} \ \rightarrow\ {}^{(2)}\nabla \ \rightarrow\ {}^{(2)}R$$

We need to understand the quantities of the hypersurface.

FUNDAMENTAL FORM OF HYPERSURFACE

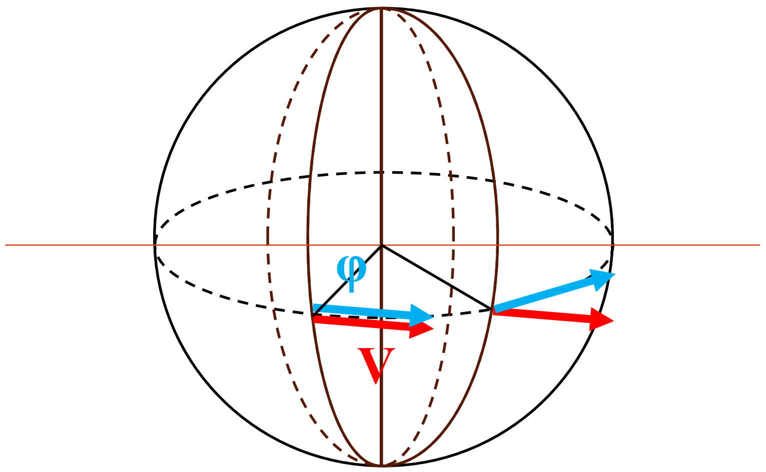
- **1st fundamental form of the hypersurface**:
  Projection tensor > project all tensors on to the hyper surface
  $$\boxed{P_{\mu\nu}} = g_{\mu\nu} - \sigma n_\mu n_\nu$$
- **2nd fundamental form of the hypersurface**:
  Change of projection tensor along the normal direction
  > bended hypersurface
  >> **Extrinsic curvature**
  $$\boxed{K_{\mu\nu}} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu}$$
  $$= \frac{1}{2}P^\alpha_{\ \mu}P^\beta_{\ \nu}\mathcal{L}_n g_{\alpha\beta}$$
  $$= \nabla_\mu n_\nu - \sigma n_\mu a_\nu$$

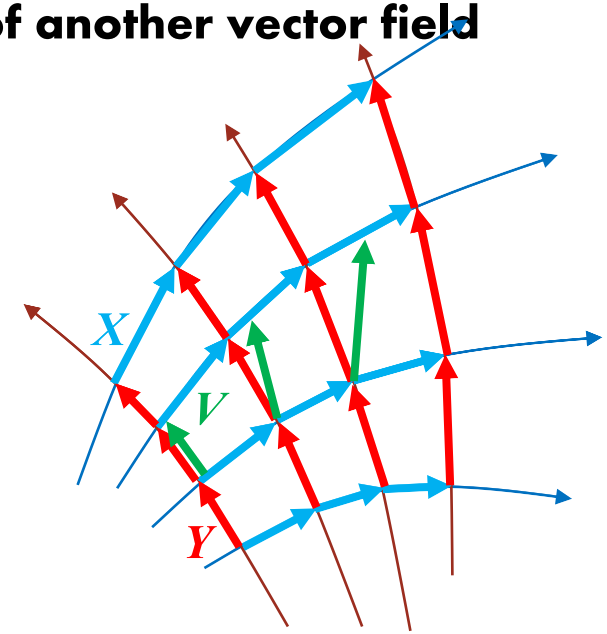
$$ds^2 = \gamma_{ij}d\hat{x}^i d\hat{x}^j = \hat{r}^2 d\hat{\theta}^2 + \hat{r}^2 \sin^2\hat{\theta} d\hat{\phi}^2, \ \ \hat{x}^i = (\hat{\theta}, \hat{\phi})$$



DERIVATIVES IN DEFINITION OF K_{MN}

- **Vector derivatives describe how a vector has changed relative to a reference vector.**
  - **Covariant derivative ∇_μ: Vector change relative to a parallelly transported vector**
  - **Lie derivative L_V : Vector change along the flow of another vector field**



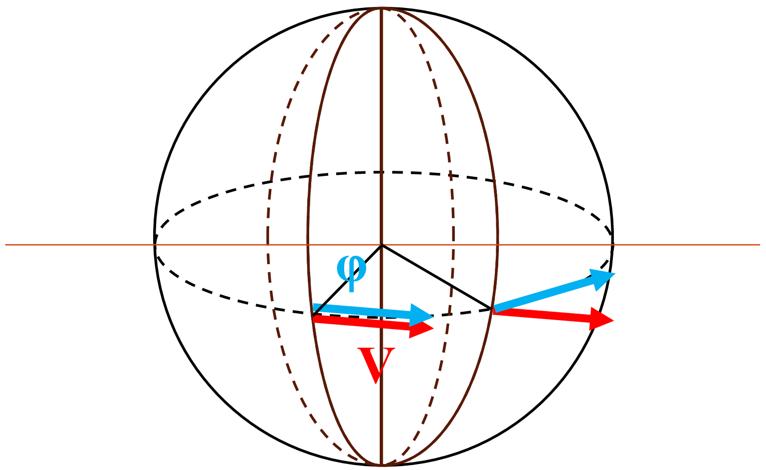


$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$
$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda$$

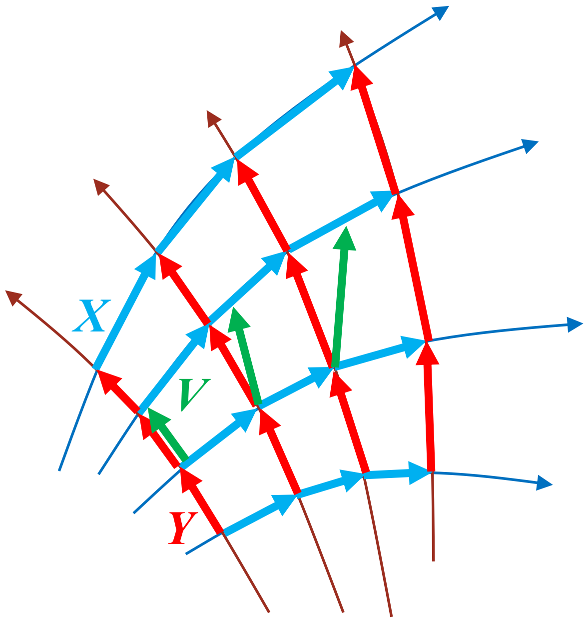
$$\mathcal{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + (\partial_\mu V^\nu)\omega_\nu$$
$$\mathcal{L}_V U^\mu = [V, U]^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu$$

DERIVATIVES IN DEFINITION OF K_{MN}

- Vector derivatives describe how a vector has changed relative to a reference vector.
  - Covariant derivative ∇_μ: Vector change relative to a parallelly transported vector
  - Lie derivative $\mathcal{L}_V$: Vector change along the flow of another vector field

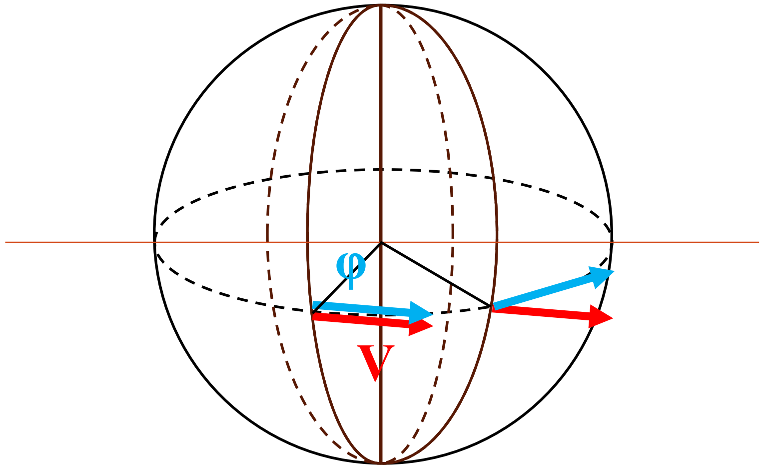


$$^{(2)}\nabla_\phi(\partial_\phi)^\mu = 0 \;\leftrightarrow\; {}^{(3)}\nabla_\phi(\partial_\phi)^\mu \neq 0$$
$$^{(2)}\nabla_\phi V^\mu \neq 0 \;\leftrightarrow\; {}^{(3)}\nabla_\phi V^\mu = 0$$



$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y} = [\boldsymbol{X}, \boldsymbol{Y}] = 0$$
$$\mathcal{L}_{\boldsymbol{X}}V = [\boldsymbol{X}, \boldsymbol{V}] \neq 0$$

Derivatives in definition of Kμν



$$^{(2)}\nabla_\phi(\partial_\phi)^\mu = 0 \;\leftrightarrow\; {}^{(3)}\nabla_\phi(\partial_\phi)^\mu \neq 0$$

$$^{(2)}\nabla_\phi V^\mu \neq 0 \;\leftrightarrow\; {}^{(3)}\nabla_\phi V^\mu = 0$$

$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

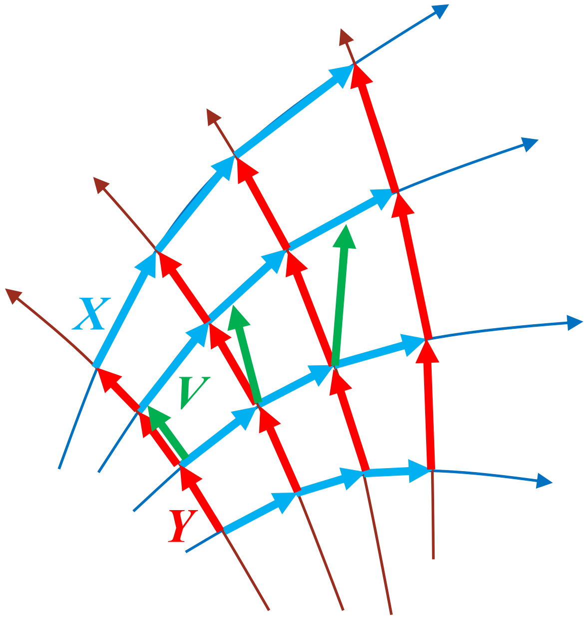
$$\Lsh\; x^i = (r,\theta,\phi) \;\rightarrow\; \gamma_{ij} \;\rightarrow\; {}^{(3)}\nabla \;\rightarrow\; {}^{(3)}R$$

$$\Lsh\; \nabla_\phi(\partial_\phi)^i = \cancel{\partial_\phi\delta^i_\phi} + \Gamma^i_{\phi j}\delta^j_\phi = \Gamma^i_{\phi\phi} = -\frac{g_{\phi\phi,i}}{2g_{ii}}$$

$$ds^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$$

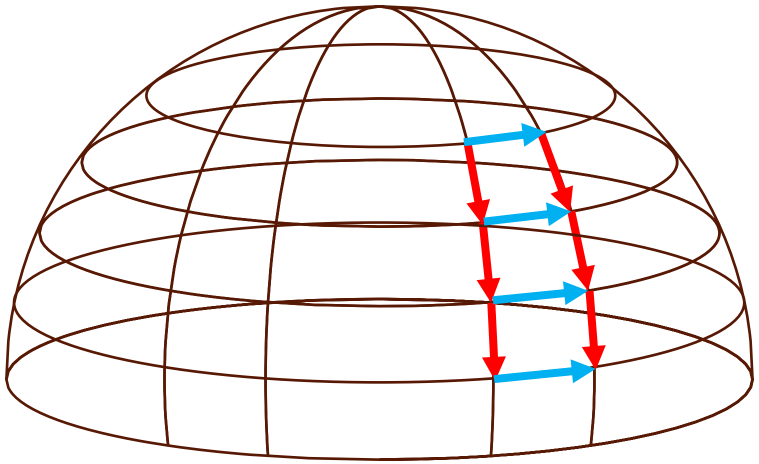
$$\Lsh\; x^a = (\theta,\phi) \;\rightarrow\; \gamma_{ab} \;\rightarrow\; {}^{(2)}\nabla \;\rightarrow\; {}^{(2)}R$$

$$\Lsh\; \nabla_\phi(\partial_\phi)^a = \cancel{\partial_\phi\delta^a_\phi} + \Gamma^a_{\phi b}\delta^b_\phi = \Gamma^a_{\phi\phi} = -\frac{g_{\phi\phi,a}}{2g_{aa}}$$



$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y} = [\boldsymbol{X},\boldsymbol{Y}] = 0$$

$$\mathcal{L}_{\boldsymbol{X}}V = [\boldsymbol{X},\boldsymbol{V}] \neq 0$$



$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\partial_\theta}}(\partial_\phi)^\mu &= [\boldsymbol{\partial_\theta},\boldsymbol{\partial_\phi}]^\mu \\
&= (\partial_\theta)^\nu\partial_\nu(\partial_\phi)^\mu - (\partial_\phi)^\nu\partial_\nu(\partial_\theta)^\mu \\
&= \delta^\nu_\theta\cancel{\partial_\nu\delta^\mu_\phi} - \delta^\nu_\phi\cancel{\partial_\nu\delta^\mu_\theta} \\
&= 0
\end{aligned}$$

Definitions of Kμν

$$P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\left(K_{\mu\nu}=\frac{1}{2}\mathcal{L}_nP_{\mu\nu}\right)$$

$$\Rightarrow K_{\alpha\beta}=\frac{1}{2}P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\mathcal{L}_n(g_{\mu\nu}-\sigma n_{\mu}n_{\nu})$$

$$=\frac{1}{2}P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\mathcal{L}_n g_{\mu\nu}-\cancel{\frac{1}{2}P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\mathcal{L}_n(\sigma n_{\mu}n_{\nu})}\rightarrow =0 \because P\cdot n=0$$

$$\therefore \boxed{K_{\mu\nu}=\frac{1}{2}\mathcal{L}_nP_{\mu\nu}\rightarrow\frac{1}{2}P^{\alpha}{}_{\mu}P^{\beta}{}_{\nu}\mathcal{L}_n g_{\alpha\beta}}$$

$$K_{\mu\nu}=\pm\frac{1}{2}\mathcal{L}_n\gamma_{\mu\nu}\rightarrow\begin{cases}\text{"+": sphere's positive } K\\ \text{"−": bowl's positive } K\end{cases}$$

$$K_{\mu\nu}=\frac{1}{2}P^{\alpha}{}_{\mu}P^{\beta}{}_{\nu}\underbrace{\mathcal{L}_n g_{\alpha\beta}}_{2\nabla_{(\alpha}n_{\beta)}}$$

$$=\frac{1}{2}P^{\alpha}{}_{\mu}P^{\beta}{}_{\nu}(2\nabla_{\beta}n_{\alpha}+\underbrace{\cancel{2\nabla_{[\alpha}n_{\beta]}}}_{\text{by the Frobenius theorem}})$$

$$=\frac{1}{2}P^{\alpha}{}_{\mu}P^{\beta}{}_{\nu}\cdot 2\nabla_{\alpha}n_{\beta}\quad\text{(by the symmetric property of } K_{\mu\nu})$$

$$=(\delta^{\alpha}_{\mu}-\sigma n^{\alpha}n_{\mu})(\delta^{\beta}_{\nu}-\sigma n^{\beta}n_{\nu})\nabla_{\alpha}n_{\beta}$$

$$=\nabla_{\mu}n_{\nu}-\cancel{\sigma n^{\beta}n_{\nu}\nabla_{\mu}n_{\beta}}-\sigma n^{\alpha}n_{\mu}\nabla_{\alpha}n_{\nu}+\cancel{n^{\alpha}n^{\beta}n_{\mu}n_{\nu}\nabla_{\alpha}n_{\beta}}$$

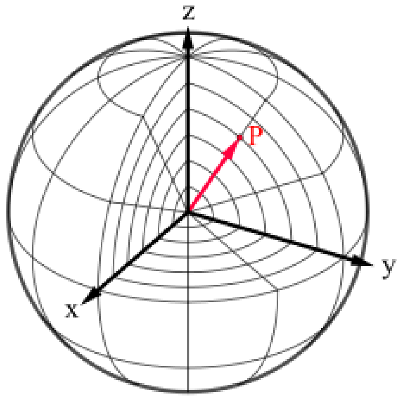
$$\text{(where } \nabla_{\mu}\underbrace{(n_{\nu}n^{\nu})}_{\text{scalar}}=\partial_{\mu}\underbrace{(n_{\nu}n^{\nu})}_{=\sigma}=0$$

$$(\nabla_{\mu}n_{\nu})n^{\nu}=(\nabla_{\mu}n^{\nu})n_{\nu}\ \text{(by metric compatibility)}$$

$$\therefore n^{\nu}(\nabla_{\mu}n_{\nu})=0\ )$$

$$\therefore \boxed{K_{\mu\nu}=\frac{1}{2}\mathcal{L}_nP_{\mu\nu}\rightarrow\nabla_{\mu}n_{\nu}-\sigma n_{\mu}a_{\nu}}\ ,\ \text{where } a^{\mu}=n^{\nu}\nabla_{\nu}n^{\mu}$$

EXAMPLE OF K_{MN} .



(1) 3-dimensional manifold M

(2) foliation of M with $\Sigma_r's$

(3) coordinates: $x^i = (r, \theta, \phi)$ (r along normal dir., (θ, ϕ) on Σ_r)

(4) metric: $ds^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + \underbrace{r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2}_{=\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b}$

(5) Christoffel symbol: $\Gamma^r_{\theta\theta} = \dfrac{g_{\theta\theta,r}}{-2g_{rr}} = \dfrac{2r}{-2} = -r,\ \Gamma^\theta_{\theta r} = \cdots,\ \cdots$

(6) curvature: $M \to {}^{(3)}R = 0, \quad \Sigma_r \to {}^{(2)}R = \dfrac{2}{r^2}$

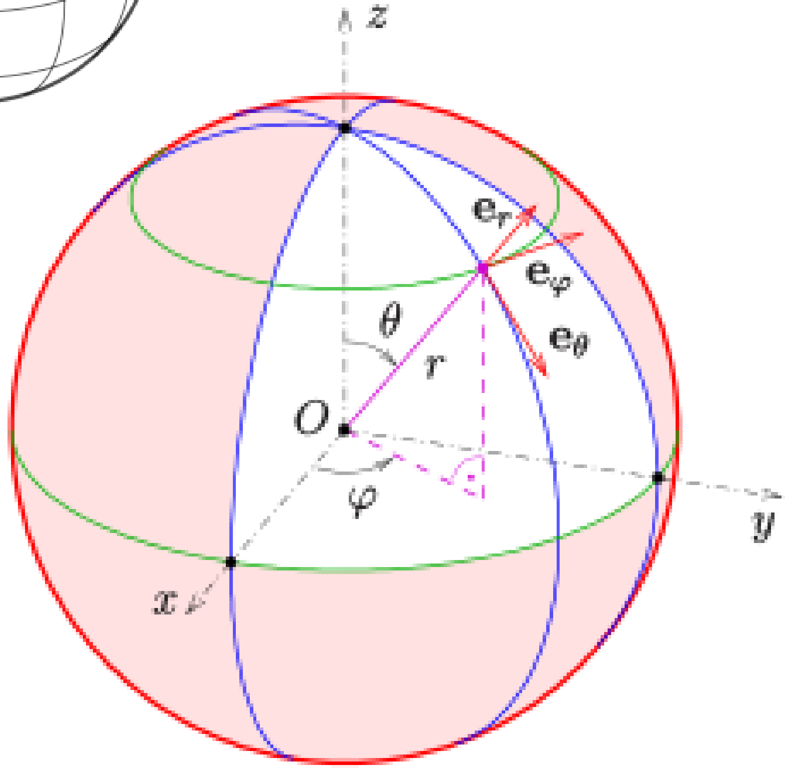
(7) normal vector: $(\mathbf{d}r)_i = \nabla_i r = \partial_i r = \delta^r_i = (1, 0, 0) \equiv r_i$

projection tensor: $\gamma_{ij} = g_{ij} - r_i r_j$

(8) extrinsic curvature: $K_{\theta\theta} = \begin{cases} = \dfrac{1}{2}\mathcal{L}_{\boldsymbol{r}}\gamma_{\theta\theta} = \dfrac{1}{2}(r^i\cancel{\nabla_i\gamma_{\theta\theta}} + 2(\nabla_\theta r^\mu)\gamma_{\mu\theta})\ \text{(wrong)} \\ \qquad = \dfrac{1}{2}(\underbrace{r^i\partial_i}_{\partial_r}\underbrace{\gamma_{\theta\theta}}_{=r^2} + \underbrace{\cancel{2(\partial_\theta r^i)\gamma_{i\theta}}}_{=2\partial_\theta r^\theta\gamma_{\theta\theta}=0}) = r \\ = \nabla_\theta r_\theta - \sigma_r r_\theta \cancel{a_\theta} = \nabla_\theta r_\theta = \nabla_\theta \delta^r_\theta = -\Gamma^i_{\theta\theta}\delta^r_i = -\Gamma^r_{\theta\theta} = r \end{cases}$

$$K_{\phi\phi} = -\Gamma^r_{\phi\phi} = -\frac{g_{\phi\phi,r}}{-2g_{rr}} = r\sin^2\theta$$

$$K = K^\theta{}_\theta + K^\phi{}_\phi = g^{\theta\theta}K_{\theta\theta} + g^{\phi\phi}K_{\phi\phi} = \frac{1}{r^2}\cdot r + \frac{1}{r^2\sin^2\theta}\cdot r\sin^2\theta = \frac{2}{r}$$



Properties of Pμν, Kμν

$$G_{\mu\nu} \;\rightarrow\; \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu}\widehat{\nu}} \end{cases} \xrightarrow[\substack{{}^{(4)}\nabla \,\to\, {}^{(3)}\nabla \\ g_{\mu\nu} \,\to\, \gamma_{ij}}]{{}^{(4)}R \,\to\, {}^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)$$

$$\boxed{P_{\mu\nu}} = g_{\mu\nu} - \sigma n_\mu n_\nu \qquad \begin{cases} n^\mu P_{\mu\nu} = 0,\ n^\nu P_{\mu\nu} = 0 \\ P_{[\mu\nu]} = 0 \\ P_{\mu\nu} V^\mu W^\nu = g_{\mu\nu} V^\mu W^\nu \end{cases} \rightarrow \gamma_{ij}$$

$$\begin{aligned} \boxed{K_{\mu\nu}} &= \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \\ &= \frac{1}{2} P^\alpha{}_\mu P^\beta{}_\nu \mathcal{L}_n g_{\alpha\beta} \\ &= \nabla_\mu n_\nu - \sigma n_\mu a_\nu \end{aligned} \qquad \begin{cases} n^\mu K_{\mu\nu} = 0,\ n^\nu K_{\mu\nu} = 0 \\ K_{[\mu\nu]} = 0 \end{cases} \rightarrow K_{ij}$$

INTRINSIC CURVATURE

$$G_{\mu\nu} \;\rightarrow\; \begin{cases} G_{\boldsymbol{nn}} \\ G_{\widehat{\mu\nu}} \end{cases} \xrightarrow[{}^{(4)}\nabla \;\rightarrow\; {}^{(3)}\nabla]{{}^{(4)}R \;\rightarrow\; {}^{(3)}R} (K_{ij},\ \gamma_{ij},\ N,\ \beta^i,\ \partial_t)$$

$$g_{\mu\nu} \;\rightarrow\; \gamma_{ij}$$

$$M\ :\ g_{\mu\nu},\ \nabla_\mu[\Gamma(g)] \;\rightarrow\; [\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda{}_{\rho\mu\nu}V^\rho \;\rightarrow\; R$$

$$\Lsh\ \nabla_\mu g_{\nu\rho} = 0$$

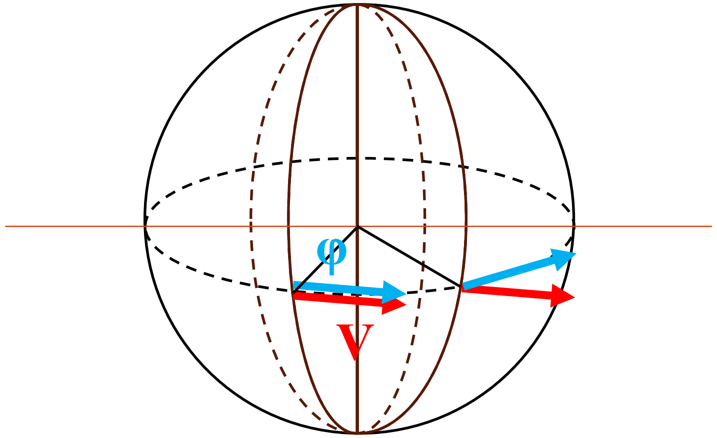
$$\begin{aligned} \circlearrowleft\ \nabla_\mu P_{\nu\rho} &= \nabla_\mu(g_{\nu\rho} + n_\nu n_\rho) \\ &= \nabla_\mu g_{\nu\rho} + (\nabla_\mu n_\nu)n_\rho + n_\nu(\nabla_\mu n_\rho) \\ &= K_{\mu\nu}n_\rho + n_\nu K_{\mu\rho} + n_\mu a_\nu n_\rho + n_\nu a_\mu n_\rho \\ &\neq 0 \end{aligned}$$

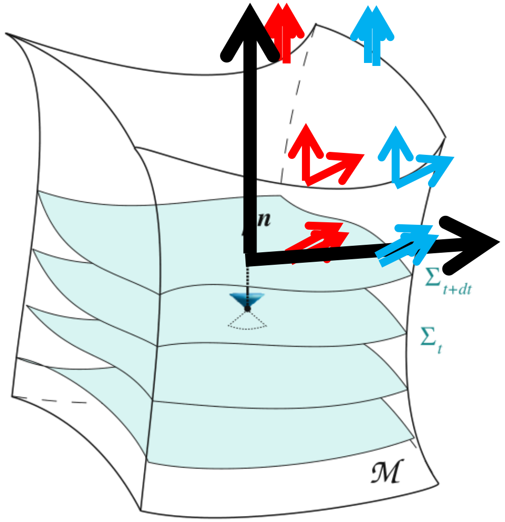
$$\circlearrowleft\ \widehat{\nabla}_\sigma X^{\mu\cdots}_{\nu\cdots} = P^\alpha_\sigma P^\mu_\beta \cdots \nabla_\alpha X^{\beta\cdots}_{\gamma\cdots}$$

$$\widehat{\nabla}_\mu \widehat{\nabla}_\nu X^\rho = P_\mu{}^{\mu'} P_\nu{}^{\nu''} P_{\rho''}{}^{\rho'} \nabla_{\mu'}(P_{\rho'}{}^{\rho''} P_{\nu''}{}^{\nu'} \nabla_{\nu'} X^{\rho'})$$

$$\boldsymbol{\Sigma}_t\ :\ \gamma_{\mu\nu},\ \widehat{\nabla}_\mu[\Gamma(\gamma)] \;\rightarrow\; [\widehat{\nabla}_\mu, \widehat{\nabla}_\nu]V^\lambda = \widehat{R}^\lambda{}_{\rho\mu\nu}V^\rho \;\rightarrow\; \widehat{R}$$

$$\Lsh\ \widehat{\nabla}_\mu \gamma_{\nu\rho} = 0$$





[Gourgoulhon, 2021]

Intrinsic curvature (2. Gauss eq. (1))

$$\widehat{\nabla}_\mu \widehat{\nabla}_\nu V^\rho = P^\alpha{}_\mu P^\beta{}_\nu P^\rho{}_\gamma \underbrace{\nabla_\alpha (P^\delta{}_\beta P^\gamma{}_\lambda \nabla_\delta V^\lambda)}_{= -\sigma n^\delta (\nabla_\alpha n_\beta) P^\gamma{}_\lambda \nabla_\delta V^\lambda - \sigma P^\delta{}_\beta (\nabla_\alpha n^\gamma) n_\lambda \nabla_\delta V^\lambda + P^\delta{}_\beta P^\gamma{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda}$$

$\rightarrow [\widehat{\nabla}_\mu, \widehat{\nabla}_\nu] V^\lambda = \widehat{R}^\lambda{}_{\rho\mu\nu} V^\rho \rightarrow \widehat{R}$

$\hookrightarrow \nabla_\mu g_{\nu\rho} = 0$

(using $\nabla_\mu P^\alpha{}_\beta = \nabla_\mu (\delta^\alpha{}_\beta - \sigma n^\alpha n_\beta) = -\sigma \nabla_\mu (n^\alpha n_\beta)$
and $P^\alpha{}_\beta n_\alpha = 0$)

$$= P^\alpha{}_\mu P^\beta{}_\nu P^\rho{}_\gamma \big(-\sigma n^\delta (\nabla_\alpha n_\beta) P^\gamma{}_\lambda \nabla_\delta V^\lambda - \sigma (\nabla_\alpha n^\gamma) P^\delta{}_\beta \underbrace{n_\lambda \nabla_\delta V^\lambda}_{= -V^\lambda \nabla_\delta n_\lambda \ \because\ \nabla_\delta (n_\lambda V^\lambda) = 0} + P^\delta{}_\beta P^\gamma{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda\big)$$

$$= -\sigma \underbrace{P^\alpha{}_\mu P^\beta{}_\nu (\nabla_\alpha n_\beta)}_{K_{\mu\nu}} P^\rho{}_\gamma P^\gamma{}_\lambda n^\delta \nabla_\delta V^\lambda + \sigma \overbrace{P^\alpha{}_\mu P^\rho{}_\gamma (\nabla_\alpha n^\gamma)}^{K^\rho{}_\mu} \underbrace{P^\beta{}_\nu P^\delta{}_\beta (\nabla_\delta n_\lambda) V^\lambda}_{P^\delta{}_\nu (\nabla_\delta n_\lambda) \delta^\lambda{}_\kappa V^\kappa = P^\delta{}_\nu (\nabla_\delta n_\lambda) P^\lambda{}_\eta P^\eta{}_\kappa V^\kappa = K_{\nu\eta} P^\eta{}_\kappa V^\kappa = K_{\nu\lambda} V^\lambda}$$

$$+ P^\alpha{}_\mu \underbrace{P^\beta{}_\nu P^\delta{}_\beta}_{P^\delta{}_\nu} \underbrace{P^\rho{}_\gamma P^\gamma{}_\lambda}_{P^\rho{}_\lambda} \nabla_\alpha \nabla_\delta V^\lambda$$

$$= -\sigma K_{\mu\nu} P^\rho{}_\lambda n^\delta \nabla_\delta V^\lambda + \sigma K^\rho{}_\mu K_{\nu\lambda} V^\lambda + P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda$$

$$\equiv \nabla_{\widehat{\mu}} \nabla_{\widehat{\nu}} V^{\widehat{\rho}} - \sigma K_{\mu\nu} \nabla_n V^{\widehat{\rho}} + \sigma K^\rho{}_\mu K_{\nu\lambda} V^\lambda$$

(where we defined $\nabla_{\widehat{\mu}} \nabla_{\widehat{\nu}} V^{\widehat{\rho}} \equiv P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda$)

Intrinsic curvature (3. Gauss eq. (2))

$$\hat{\nabla}_\mu \hat{\nabla}_\nu V^\rho = \nabla_{\hat{\mu}}\nabla_{\hat{\nu}} V^{\hat{\rho}} - \sigma K_{\mu\nu}\nabla_n V^{\hat{\rho}} + \sigma K^\rho{}_\mu K_{\nu\lambda} V^\lambda$$

$$(\text{where } \nabla_{\hat{\mu}}\nabla_{\hat{\nu}} V^{\hat{\rho}} \equiv P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda)$$

$$\Rightarrow \; 2\hat{\nabla}_{[\mu}\hat{\nabla}_{\nu]} V^\rho \equiv \hat{R}^\rho{}_{\sigma\mu\nu} V^\sigma$$

$$= 2(-\sigma K_{[\mu\nu]} P^\rho{}_\gamma n^\delta \nabla_\delta V^\lambda + \sigma K^\rho{}_{[\mu} K_{\nu]\lambda} V^\lambda + P^\alpha{}_{[\mu} P^\delta{}_{\nu]} P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda)$$

(where $K_{[\mu\nu]} = 0$, and $P^\alpha{}_{[\mu} P^\delta{}_{\nu]} P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda$ $= P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_{[\alpha}\nabla_{\delta]} V^\lambda = \frac{1}{2} P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda R^\lambda{}_{\sigma\alpha\delta} V^\sigma$)

$$= 2(\sigma K^\rho{}_{[\mu} K_{\nu]\sigma} + \frac{1}{2} P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda R^\lambda{}_{\sigma\alpha\delta}) V^\sigma$$

$$\Rightarrow \; \hat{R}^\rho{}_{\sigma\mu\nu} = P^\rho{}_\alpha P^\beta{}_\sigma P^\gamma{}_\mu P^\delta{}_\nu R^\alpha{}_{\beta\gamma\delta} + \sigma(K^\rho{}_\mu K_{\sigma\nu} - K^\rho{}_\nu K_{\sigma\mu})$$

$$\Rightarrow \; \boxed{\hat{R}_{\mu\nu\rho\sigma} = R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma}} \quad \textbf{(Gauss eq.)}$$

(intrinsic)=(projected R)+(bending)

Intrinsic curvature (4. Gauss eq. (3))

Contracted Gauss' equation :

$$P^{\mu\rho}(\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2K_{\rho[\mu}K_{\nu]\sigma})$$

$$\rightarrow \quad \widehat{R}_{\nu\sigma} = \underbrace{P^{\mu\rho}R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}}}_{=P^{\mu\rho}P^{\alpha}{}_{\mu}P^{\beta}{}_{\nu}P^{\gamma}{}_{\rho}P^{\delta}{}_{\sigma}R_{\alpha\beta\gamma\delta}=P^{\alpha\gamma}P^{\beta}{}_{\nu}P^{\delta}{}_{\sigma}R_{\alpha\beta\gamma\delta}=(g^{\alpha\gamma}-\sigma n^{\alpha}n^{\gamma})P^{\beta}{}_{\nu}P^{\delta}{}_{\sigma}R_{\alpha\beta\gamma\delta}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma})$$

$$\rightarrow \quad \boxed{\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}}$$

Intrinsic curvature (4. Gauss eq. (4))

Scalar Gauss relation :

$$P^{\nu\sigma}(\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma})$$

$$\rightarrow \quad \widehat{R} = \underbrace{P^{\nu\sigma}P^{\alpha}_{\ \nu}P^{\beta}_{\ \sigma}}_{=P^{\alpha\beta}=g^{\alpha\beta}-\sigma n^{\alpha}n^{\beta}}(R_{\alpha\beta} - \sigma R_{n\alpha n\beta}) + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]}^{\ \nu}$$

$$= R - \sigma R_{nn} - \sigma R_{nn} + \sigma^2 \cancel{R_{nnnn}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]}^{\ \nu}$$

$$= R - \sigma 2R_{nn} + \sigma(K^2 - K^{\mu\nu}K_{\mu\nu})$$

$$\Rightarrow \boxed{\widehat{R} = R - \sigma 2R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})}$$

$$\Rightarrow \text{when } \sigma = -1,\ \widehat{R} = R + 2R_{nn} - (K^2 - K_{\mu\nu}K^{\mu\nu})$$

Intrinsic curvature (5. Codazzi eq.)

$$2\nabla_{[\hat{\mu}}\nabla_{\hat{\nu}]}n^{\hat{\rho}} = R^{\hat{\rho}}{}_{\lambda\hat{\mu}\hat{\nu}}n^{\lambda} \equiv \boxed{R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}}}$$

$$\hookrightarrow P_{\mu}{}^{\alpha}P_{\nu}{}^{\beta}P^{\rho}{}_{\gamma}\cdot 2\nabla_{[\alpha}\nabla_{\beta]}n^{\gamma}$$

$$= 2P_{[\mu}{}^{\alpha}P_{\nu]}{}^{\beta}P^{\rho}{}_{\gamma}\nabla_{\alpha}\underbrace{\nabla_{\beta}n^{\gamma}}_{=K_{\beta}{}^{\gamma}+\sigma n_{\beta}a^{\gamma}}$$

$$= 2P_{[\mu}{}^{\alpha}P_{\nu]}{}^{\beta}P^{\rho}{}_{\gamma}(\nabla_{\alpha}K_{\beta}{}^{\gamma} + \sigma\underbrace{\nabla_{\alpha}n_{\beta}}_{=K_{\alpha\beta}+\sigma \cancel{n_{\alpha}}a_{\beta}}a^{\gamma} + \sigma\cancel{n_{\beta}}\nabla_{\alpha}a^{\gamma})$$

$$= 2P_{[\mu}{}^{\alpha}P_{\nu]}{}^{\beta}P^{\rho}{}_{\gamma}(\nabla_{\alpha}K_{\beta}{}^{\gamma} + \sigma K_{\alpha\beta}a^{\gamma})$$

$$= 2\hat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} + 2\sigma\cancel{K_{[\mu\nu]}}a^{\rho}$$

$$= \boxed{2\hat{\nabla}_{[\mu}K_{\nu]}{}^{\rho}}$$

Codazzi's equation : $\boxed{2\hat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} = R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}}}$

Contrated Codazzi's equation : $\boxed{2\hat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\hat{\nu}}}$

Summarry

Scalar Gauss relation : $\widehat{R} = R - \sigma 2 R_{nn} + \sigma (K^2 - K_{\mu\nu} K^{\mu\nu})$

Contracted Gauss' equation : $\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma}$

$P^{\nu\sigma}$

$\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma}$ **(Gauss eq.)**

$P^{\mu\rho}$

$2\widehat{\nabla}_{[\mu} \widehat{\nabla}_{\nu]} V^{\rho} \equiv \widehat{R}^{\rho}{}_{\sigma\mu\nu} V^{\sigma}$

$2\nabla_{[\widehat{\mu}} \nabla_{\widehat{\nu}]} n^{\widehat{\rho}} = R^{\widehat{\rho}}{}_{\lambda\widehat{\mu}\widehat{\nu}} n^{\lambda}$

Codazzi's equation : $2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\rho} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}}$

$P^{\mu\rho}$

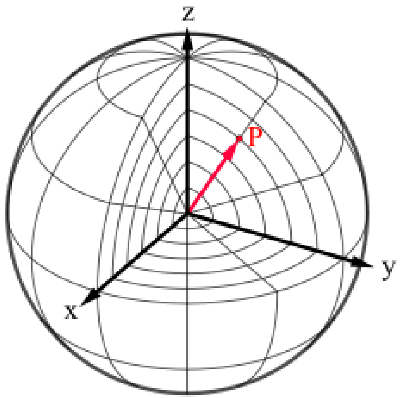
Contracted Codazzi's equation : $2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}}$

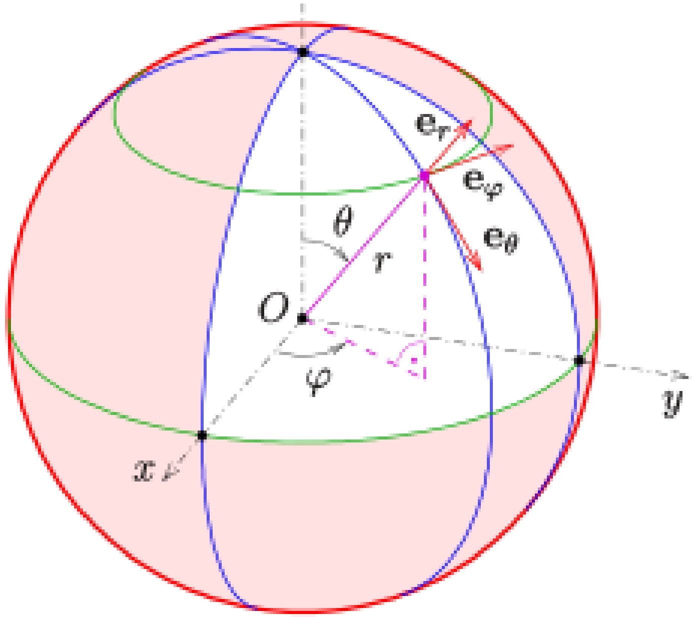
Example of intrinsic curvature

For a sphere of radius r in the flat space, we obtained,

$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$\widehat{R} = \frac{2}{r^2},\ K_{\theta\theta} = r,\ K_{\phi\phi} = r\sin^2\theta,\ K = \frac{2}{r}$$





The contracted Gauss equation in the flat space:

$$\widehat{R} = \cancel{R} - \sigma 2\cancel{R_{nn}} + \overbrace{\sigma}^{=1}(K^2 - K_{\mu\nu}K^{\mu\nu})$$

$$\widehat{R} = K^2 - K_{\mu\nu}K^{\mu\nu} = K^2 - K_{\theta\theta}K^{\theta\theta} - K_{\phi\phi}K^{\phi\phi}$$

$$\hookleftarrow\ K^{\theta\theta} = g^{\theta\theta}g^{\theta\theta}K_{\theta\theta},\ K^{\phi\phi} = g^{\phi\phi}g^{\phi\phi}K_{\phi\phi}$$

$$= K^2 - (g^{\theta\theta})^2(K_{\theta\theta})^2 - (g^{\phi\phi})^2(K_{\phi\phi})^2$$

$$= \left(\frac{2}{r}\right)^2 - \frac{1}{r^4}r^2 - \frac{1}{r^4\sin^4\theta}r^2\sin^4\theta$$

$$= \frac{4}{r^2} - \frac{1}{r^2} - \frac{1}{r^2}$$

$$= \frac{2}{r^2}$$

Dimensional analysis of K and R

3+1 DECOMPOSITION (EINSTEIN EQ. (1-1))

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

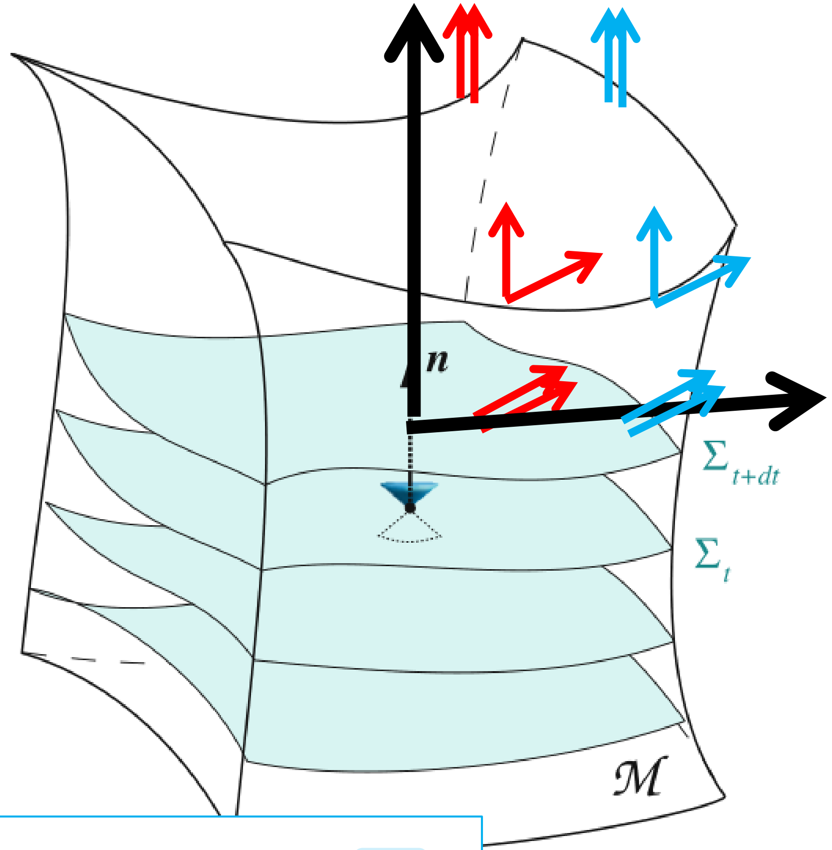
$$\hookrightarrow G_{nn} = 8\pi G T_{nn} \cdots (1)$$

$$\Rightarrow R_{nn} - \frac{1}{2} g_{nn} R = 8\pi G T_{nn}$$

$^{(4)}R \to (R, K)$
$R \to (\widehat{R}, K)$

Now we learned:

$$\begin{cases} \text{(extrinsic curvature)} & K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta} \\ & \quad = \nabla_\mu n_\nu - \sigma n_\mu a_\nu \\ \text{(Gauss eq.)} & \widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma} \\ \quad \text{(Contracted Gauss eq.)} & \widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma} \\ \quad\quad \text{(Gauss scalar eq.)} & \widehat{R} = R - \sigma 2 R_{nn} + \sigma (K^2 - K_{\mu\nu} K^{\mu\nu}) \\ \text{(Gauss-Codazzi eq.)} & 2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\rho} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}} \\ \text{(Contracted Codazzi eq.)} & 2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}} \end{cases}$$



[Gourgoulhon, 2021]

$$\begin{cases} \text{(energy density)} & \rho_e = T_{nn} \\ \text{(momentum density)} & p_\alpha = -T_{n\widehat{\alpha}} \\ \text{(stress tensor)} & S_{\mu\nu} = T_{\widehat{\mu}\widehat{\nu}} \end{cases}$$

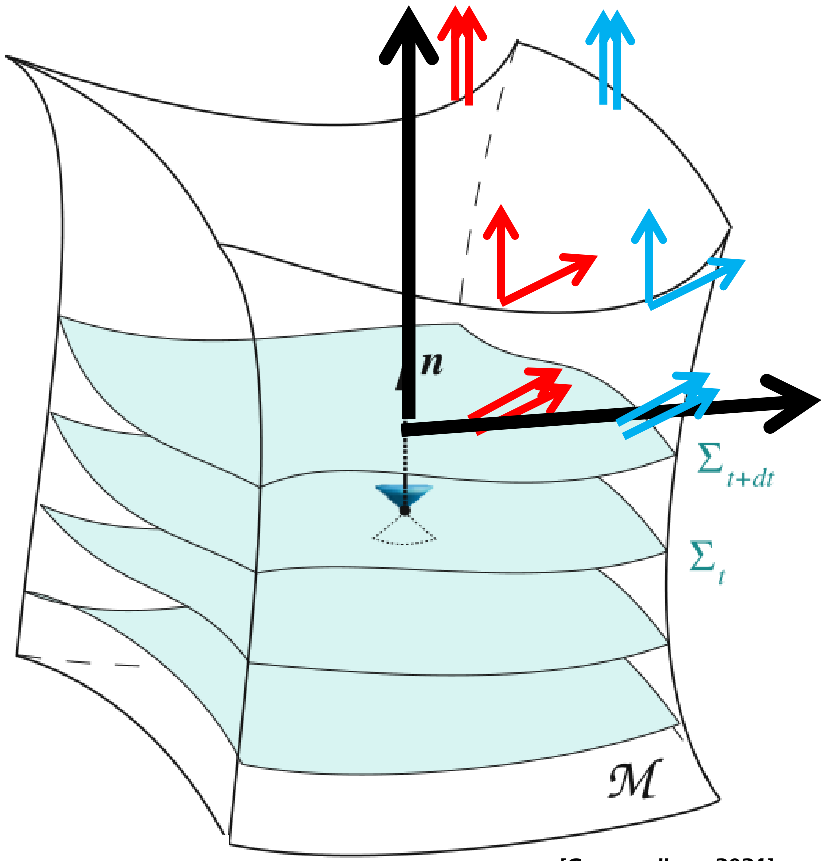
3+1 DECOMPOSITION (EINSTEIN EQ. (1-2))

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow G_{nn} = 8\pi G T_{nn} \cdots (1)$$

$$\Rightarrow \overbrace{R_{nn}}^{=\frac{1}{2}(\hat{R}-K+K^2-K_{ij}K^{ij})} - \frac{1}{2} R \underbrace{g_{nn}}_{=g_{\mu\nu}n^\mu n^\nu = n_\mu n^\mu = \sigma = -1} = 8\pi G \overbrace{T_{nn}}^{E}$$

$$\Rightarrow \boxed{\hat{R} + K^2 - K_{ij}K^{ij} = 16\pi G E}$$



[Gourgoulhon, 2021]

$$\begin{cases} \text{(Gauss scalar eq.)} & \hat{R} = R - \sigma 2 R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu}) \\ \text{(energy density)} & \rho_e = T_{nn} \end{cases}$$

3+1 Decomposition (Einstein eq. (2-1))

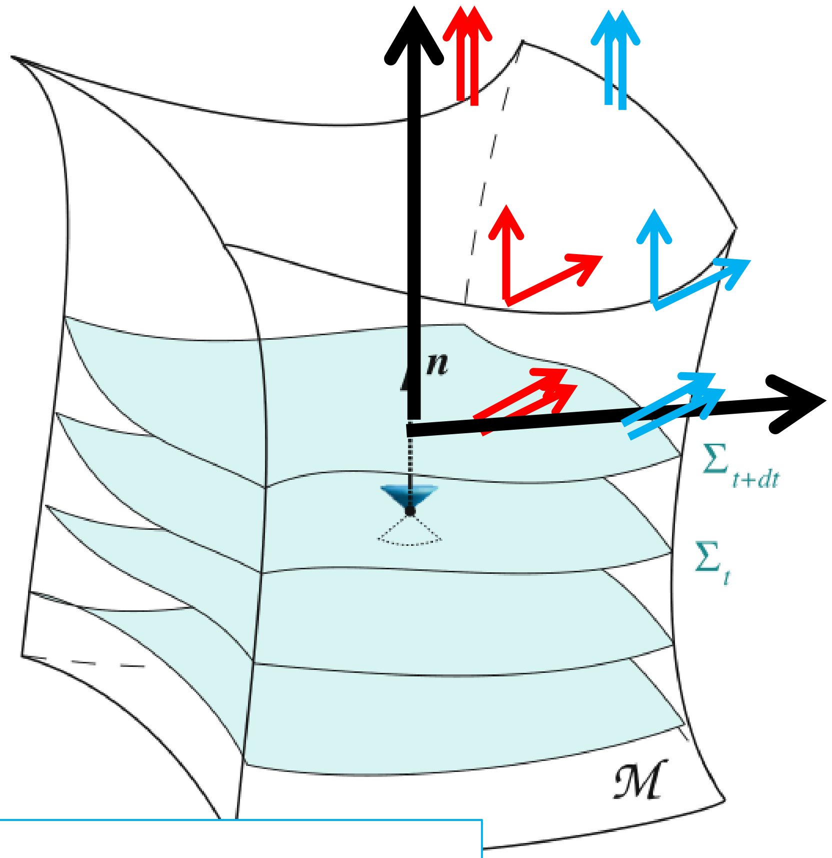
$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \cdots (2)$$

$$\Rightarrow R_{n\hat{\mu}} - \frac{1}{2} g_{n\hat{\mu}} R = 8\pi G T_{n\hat{\mu}}$$

Now we learned:

$$\begin{cases} \text{(extrinsic curvature)} & K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta} \\ & \quad = \nabla_{\mu} n_{\nu} - \sigma n_{\mu} a_{\nu} \\ \text{(Gauss eq.)} & \hat{R}_{\mu\nu\rho\sigma} = R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma} \\ \quad \text{(Contracted Gauss eq.)} & \hat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma R_{n\hat{\nu}n\hat{\sigma}} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma} \\ \quad \text{(Gauss scalar eq.)} & \hat{R} = R - \sigma 2 R_{nn} + \sigma (K^2 - K_{\mu\nu} K^{\mu\nu}) \\ \text{(Gauss-Codazzi eq.)} & 2\hat{\nabla}_{[\mu} K_{\nu]}{}^{\rho} = R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}} \\ \text{(Contracted Codazzi eq.)} & 2\hat{\nabla}_{[\mu} K_{\nu]}{}^{\mu} = R_{n\hat{\nu}} \end{cases}$$



[Gourgoulhon, 2021]

$$\begin{cases} \text{(energy density)} & \rho_e = T_{nn} \\ \text{(momentum density)} & p_{\alpha} = -T_{n\hat{\alpha}} \\ \text{(stress tensor)} & S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}} \end{cases}$$

3+1 Decomposition (Einstein eq. (2-2))

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

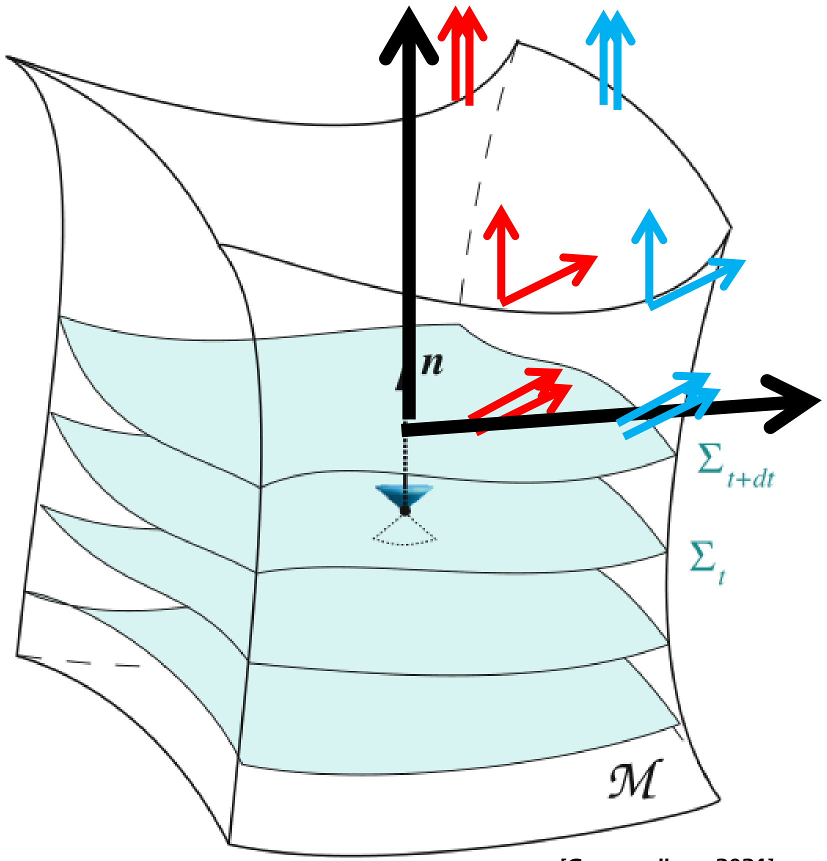
$$\hookrightarrow G_{n\widehat{\mu}} = 8\pi G T_{n\widehat{\mu}} \cdots (2)$$

$$\Rightarrow \overbrace{R_{n\widehat{\mu}}}^{2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu}} - \frac{1}{2} R \underbrace{g_{n\widehat{\mu}}}_{=g_{\alpha\beta}n^{\alpha}P^{\beta}{}_{\mu}=n_{\beta}P^{\beta}{}_{\mu}=0} = 8\pi G \underbrace{T_{n\widehat{\mu}}}_{=-p_{\mu}}$$

$$-2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = 8\pi G p_{\nu}$$

$$\boxed{\widehat{\nabla}_{\nu}K - \widehat{\nabla}_{\mu}K^{\mu}{}_{\nu} = 8\pi G p_{\nu}}$$

$$\begin{cases} \text{(Contracted Codazzi eq.)} & 2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}} \\ \text{(momentum density)} & p_{\alpha} = -T_{n\widehat{\alpha}} \end{cases}$$



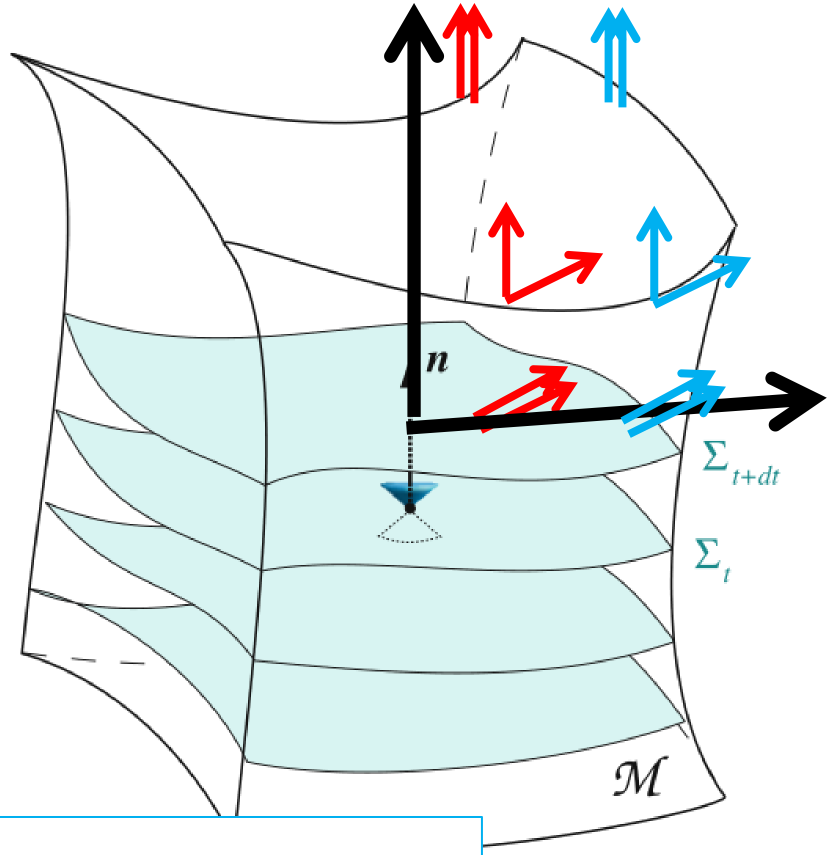
[Gourgoulhon, 2021]

3+1 Decomposition (Einstein eq. (3-1))

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \cdots (3)$$

$$\Rightarrow R_{\hat{\mu}\hat{\nu}} - \frac{1}{2} g_{\hat{\mu}\hat{\nu}} R = 8\pi G T_{\hat{\mu}\hat{\nu}}$$



[Gourgoulhon, 2021]

Now we learned:

(extrinsic curvature)	$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta}$
	$= \nabla_{\mu} n_{\nu} - \sigma n_{\mu} a_{\nu}$
(Gauss eq.)	$\hat{R}_{\mu\nu\rho\sigma} = R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma}$
(Contracted Gauss eq.)	$\hat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma R_{n\hat{\nu}n\hat{\sigma}} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma}$
(Gauss scalar eq.)	$\hat{R} = R - \sigma 2 R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})$
(Gauss-Codazzi eq.)	$2\hat{\nabla}_{[\mu} K_{\nu]}{}^{\rho} = R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}}$
(Contracted Codazzi eq.)	$2\hat{\nabla}_{[\mu} K_{\nu]}{}^{\mu} = R_{n\hat{\nu}}$

(energy density)	$\rho_e = T_{nn}$
(momentum density)	$p_{\alpha} = -T_{n\hat{\alpha}}$
(stress tensor)	$S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}}$

3+1 Decomposition (Einstein eq. (3-2))

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\Lsh\ G_{\widehat{\mu}\widehat{\nu}} = 8\pi G T_{\widehat{\mu}\widehat{\nu}} \cdots (3)$$

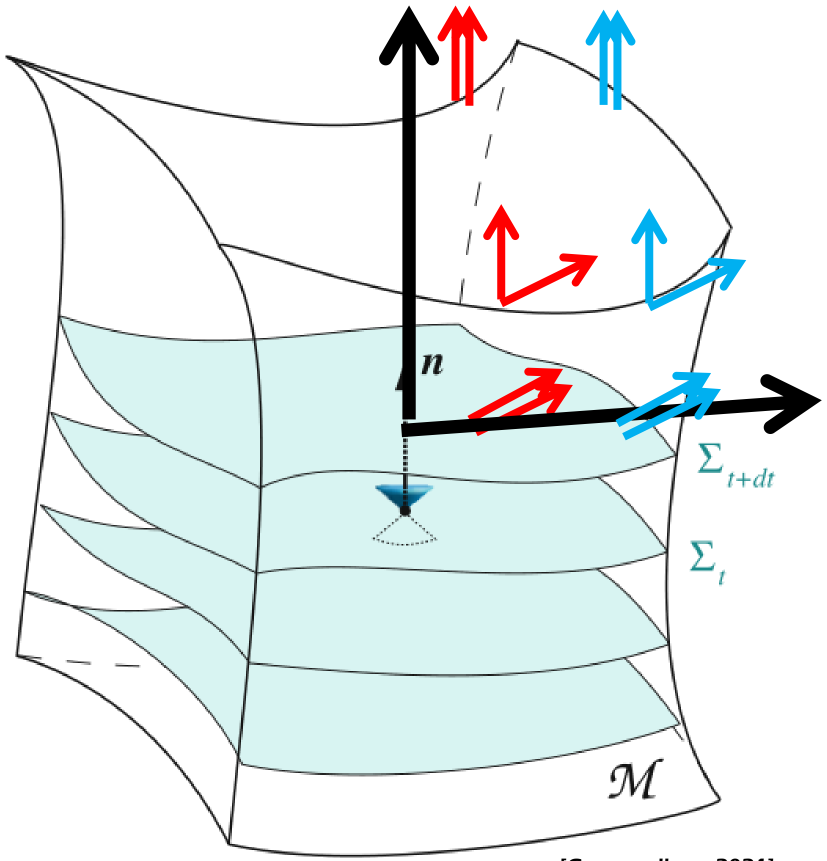
$$\Rightarrow \boxed{R_{\widehat{\mu}\widehat{\nu}} - \frac{1}{2} g_{\widehat{\mu}\widehat{\nu}} R = 8\pi G T_{\widehat{\mu}\widehat{\nu}}}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\Lsh\ G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} G^{\alpha}{}_{\alpha} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha})$$

$$\Lsh\ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} g_{\mu\nu} (R - \frac{1}{2} \underbrace{g^{\alpha}{}_{\alpha} R}_{=2R}) = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha})$$

$$\Rightarrow \boxed{R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha})}$$



[Gourgoulhon, 2021]

3+1 Decomposition (Einstein eq. (3-3))

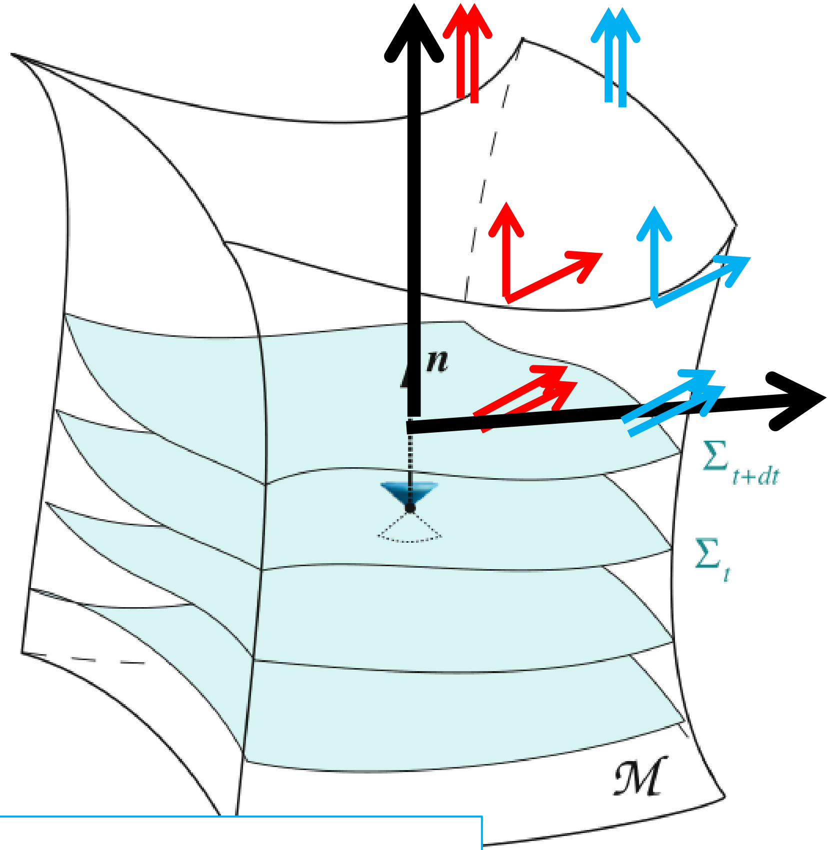
$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \cdots (3)$$

$$R_{\hat{\mu}\hat{\nu}} = 8\pi G\left(T_{\hat{\mu}\hat{\nu}} - \frac{1}{2} g_{\hat{\mu}\hat{\nu}} T^{\alpha}{}_{\alpha}\right)$$

$$\circlearrowleft\ \hat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma R_{n\hat{\nu}n\hat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}$$

$$\hat{R}_{\mu\nu} - R_{n\hat{\mu}n\hat{\nu}} + 2K^{\alpha}{}_{[\alpha}K_{\mu]\nu} = 8\pi G\left[S_{\hat{\mu}\hat{\nu}} - \frac{1}{2} P_{\hat{\mu}\hat{\nu}}(S - E)\right]$$



[Gourgoulhon, 2021]

Now we learned:

(extrinsic curvature)	$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta}$
	$= \nabla_{\mu} n_{\nu} - \sigma n_{\mu} a_{\nu}$
(Gauss eq.)	$\hat{R}_{\mu\nu\rho\sigma} = R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \sigma 2K_{\rho[\mu}K_{\nu]\sigma}$
(Contracted Gauss eq.)	$\hat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma R_{n\hat{\nu}n\hat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}$
(Gauss scalar eq.)	$\hat{R} = R - \sigma 2R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})$
(Gauss-Codazzi eq.)	$2\hat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} = R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}}$
(Contracted Codazzi eq.)	$2\hat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\hat{\nu}}$

(energy density)	$\rho_e = T_{nn}$
(momentum density)	$p_{\alpha} = -T_{n\hat{\alpha}}$
(stress tensor)	$S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}}$

3+1 Decomposition (Einstein eq. (3-4))

$$R_{n\hat{\nu}n\hat{\sigma}} = R_{\hat{\nu}n\hat{\sigma}n} = P_{\nu\alpha}n^{\lambda}P_{\sigma}{}^{\beta}n^{\mu}R^{\alpha}{}_{\lambda\beta\mu}$$

$$= P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}R^{\alpha}{}_{\lambda\beta\mu}n^{\lambda}$$

$$= P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}[\nabla_{\beta},\nabla_{\mu}]n^{\alpha}$$

$$= P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\underbrace{\nabla_{\beta}\nabla_{\mu}n^{\alpha}}_{=K_{\mu}{}^{\alpha}-n_{\mu}a^{\alpha}} - P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\underbrace{\nabla_{\mu}\nabla_{\beta}n^{\alpha}}_{=K_{\beta}{}^{\alpha}-n_{\beta}a^{\alpha}}$$

$$= P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\nabla_{\beta}K_{\mu}{}^{\alpha} - P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\nabla_{\beta}(n_{\mu}a^{\alpha}) - P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\nabla_{\mu}K_{\beta}{}^{\alpha} + P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\nabla_{\mu}(n_{\beta}a^{\alpha})$$

$$= \overbrace{P_{\nu\alpha}P_{\sigma}{}^{\beta}\overbrace{n^{\mu}\nabla_{\beta}K_{\mu}{}^{\alpha}}^{=-K_{\mu}{}^{\alpha}\nabla_{\beta}n^{\mu}=-K_{\mu}{}^{\alpha}(K_{\beta}{}^{\mu}-n_{\beta}a^{\mu})}}^{=-K_{\mu\nu}K_{\sigma}{}^{\mu}} - P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\nabla_{\beta}n_{\mu}a^{\alpha} - P_{\nu\alpha}P_{\sigma}{}^{\beta}\overbrace{n^{\mu}n_{\mu}}^{=-1}\nabla_{\beta}a^{\alpha}$$

$$- P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\nabla_{\mu}K_{\beta}{}^{\alpha} + P_{\nu\alpha}P_{\sigma}{}^{\beta}\underbrace{n^{\mu}\nabla_{\mu}n_{\beta}}_{=a_{\beta}}a^{\alpha} + P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}n_{\beta}\nabla_{\mu}a^{\alpha}$$

$$= -K_{\mu\nu}K_{\sigma}{}^{\mu} - P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\nabla_{\mu}K_{\beta}{}^{\alpha} + P_{\nu\alpha}P_{\sigma}{}^{\beta}\nabla_{\beta}a^{\alpha} + P_{\nu\alpha}P_{\sigma}{}^{\beta}a_{\beta}a^{\alpha}$$

$$= -K_{\mu\nu}K_{\sigma}{}^{\mu} - P_{\nu\alpha}P_{\sigma}{}^{\beta}n^{\mu}\nabla_{\mu}K_{\beta}{}^{\alpha} + \hat{\nabla}_{\sigma}a_{\nu} + a_{\sigma}a_{\nu}$$

3+1 Decomposition (Einstein eq. (3-5))

$$\widehat{R}_{\mu\nu} - R_{n\widehat{\mu}n\widehat{\nu}} + 2K^{\alpha}{}_{[\alpha}K_{\mu]\nu} = 8\pi G[S_{\widehat{\mu}\widehat{\nu}} - \frac{1}{2}P_{\widehat{\mu}\widehat{\nu}}(S-E)] \quad \cdots (3)$$

$$\circlearrowleft \; R_{n\widehat{\mu}n\widehat{\nu}} = K_{\mu\alpha}K^{\alpha}_{\ \nu} - \frac{1}{N}\mathcal{L}_m K_{\mu\nu} + \frac{1}{N}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}N$$

$$R_{\widehat{\mu}\widehat{\nu}} = \widehat{R}_{\mu\nu} - K_{\mu\alpha}K^{\alpha}_{\ \nu} + \frac{1}{N}\mathcal{L}_m K_{\mu\nu} - \frac{1}{N}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}N + \underbrace{2K^{\alpha}{}_{[\alpha}K_{\mu]\nu}}_{=K^{\alpha}{}_{\alpha}K_{\mu\nu}-K^{\alpha}{}_{\mu}K_{\alpha\nu}}$$

$$= \widehat{R}_{\mu\nu} - 2K_{\mu\alpha}K^{\alpha}_{\ \nu} + \frac{1}{N}\mathcal{L}_m K_{\mu\nu} - \frac{1}{N}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}N + KK_{\mu\nu}$$

$$\circlearrowleft \; \mathcal{L}_m K_{\mu\nu} = \mathcal{L}_{(\partial_t-\beta)}K_{\mu\nu} = \partial_t K_{\mu\nu} + \beta^{\alpha}\partial_{\alpha}K_{\mu\nu} + \partial_{\mu}\beta^{\alpha}K_{\alpha\nu} + \partial_{\nu}\beta^{\alpha}K_{\mu\alpha}$$

$$= \widehat{R}_{\mu\nu} - 2K_{\mu\alpha}K^{\alpha}_{\ \nu} + \frac{1}{N}(\partial_t K_{\mu\nu} + \beta^{\alpha}\partial_{\alpha}K_{\mu\nu} + \partial_{\mu}\beta^{\alpha}K_{\alpha\nu} + \partial_{\nu}\beta^{\alpha}K_{\mu\alpha}) - \frac{1}{N}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}N + KK_{\mu\nu}$$

$$\Rightarrow \widehat{R}_{\mu\nu} - 2K_{\mu\alpha}K^{\alpha}_{\ \nu} + \frac{1}{N}(\partial_t K_{\mu\nu} + \beta^{\alpha}\partial_{\alpha}K_{\mu\nu} + \partial_{\mu}\beta^{\alpha}K_{\alpha\nu} + \partial_{\nu}\beta^{\alpha}K_{\mu\alpha}) - \frac{1}{N}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}N + KK_{\mu\nu}$$

$$= 8\pi G[S_{\widehat{\mu}\widehat{\nu}} - \frac{1}{2}P_{\widehat{\mu}\widehat{\nu}}(S-E)]$$

3+1 Decomposition (Einstein eq. (3-6))

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \cdots (3)$$

$$R_{\hat{\mu}\hat{\nu}} = 8\pi G\left(T_{\hat{\mu}\hat{\nu}} - \frac{1}{2} g_{\hat{\mu}\hat{\nu}} T^{\alpha}{}_{\alpha}\right)$$

$$\circlearrowleft \; \widehat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma R_{n\hat{\nu}n\hat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}$$

$$\widehat{R}_{\mu\nu} - R_{n\hat{\mu}n\hat{\nu}} + 2K^{\alpha}{}_{[\alpha}K_{\mu]\nu} = 8\pi G\left[S_{\hat{\mu}\hat{\nu}} - \frac{1}{2}P_{\hat{\mu}\hat{\nu}}(S-E)\right]$$

$$\circlearrowleft \; R_{n\hat{\mu}n\hat{\nu}} = K_{\mu\alpha}K^{\alpha}{}_{\nu} - \frac{1}{N}\mathcal{L}_m K_{\mu\nu} + \frac{1}{N}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}N$$

Now

(ex

$$\widehat{R}_{\mu\nu} - 2K_{\mu\alpha}K^{\alpha}{}_{\nu} + \frac{1}{N}\left(\partial_t K_{\mu\nu} + \beta^{\alpha}\partial_{\alpha}K_{\mu\nu} + \partial_{\mu}\beta^{\alpha}K_{\alpha\nu} + \partial_{\nu}\beta^{\alpha}K_{\mu\alpha}\right) - \frac{1}{N}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}N + KK_{\mu\nu} = 8\pi G\left[S_{\hat{\mu}\hat{\nu}} - \frac{1}{2}P_{\hat{\mu}\hat{\nu}}(S-E)\right]$$

(G

(Contracted Gauss eq.) $R_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma Rn\hat{\nu}n\hat{\sigma} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}$

(Gauss scalar eq.) $\widehat{R} = R - \sigma 2R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})$

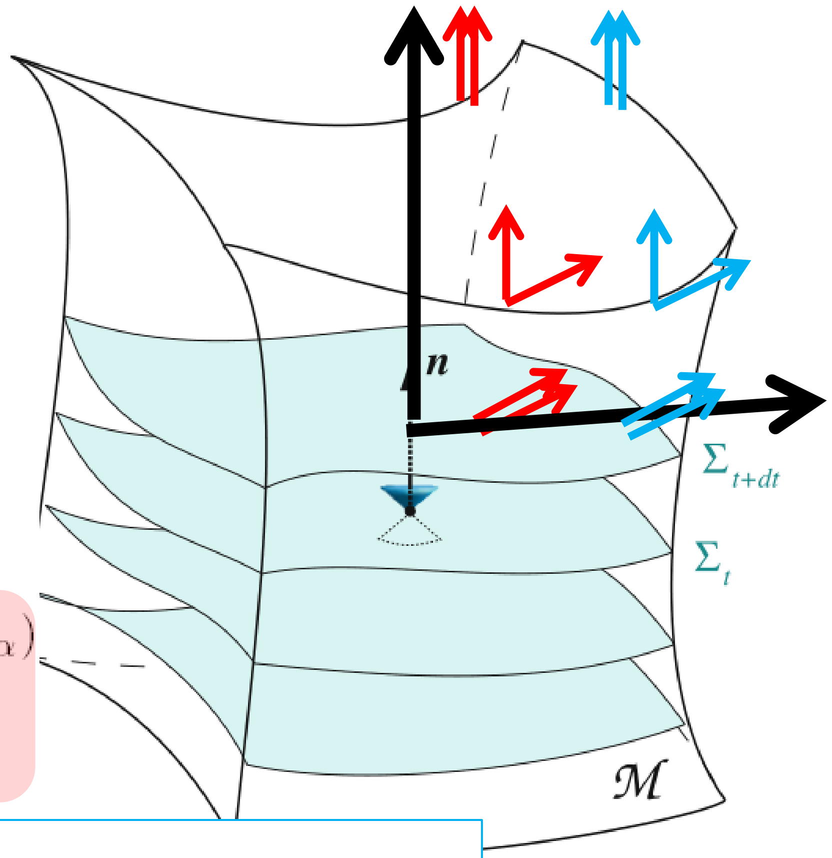
(Gauss-Codazzi eq.) $2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} = R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}}$

(Contracted Codazzi eq.) $2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\hat{\nu}}$

(energy density) $\rho_e = T_{nn}$

(momentum density) $p_{\alpha} = -T_{n\hat{\alpha}}$

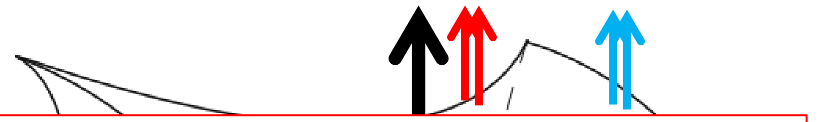
(stress tensor) $S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}}$



[Gourgoulhon, 2021]

3+1 Decomposition (Einstein eq. (4-1))

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$



$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\left\{\begin{array}{lll}
(1)\ G_{nn} = 8\pi G T_{nn} & \to & \widehat{R} + K^2 - K_{ij}K^{ij} = 16\pi G E \\
(2)\ G_{n\widehat{\mu}} = 8\pi G T_{n\widehat{\mu}} & \to & \widehat{\nabla}_\nu K - \widehat{\nabla}_\mu K^\mu{}_\nu = 8\pi G p_\nu \\
(3)\ G_{\widehat{\mu}\widehat{\nu}} = 8\pi G T_{\widehat{\mu}\widehat{\nu}} & \to & \widehat{R}_{\mu\nu} - 2K_{\mu\alpha}K^\alpha{}_\nu \\
 & & + \frac{1}{N}(\partial_t K_{\mu\nu} + \beta^\alpha\partial_\alpha K_{\mu\nu} + \partial_\mu\beta^\alpha K_{\alpha\nu} + \partial_\nu\beta^\alpha K_{\mu\alpha}) \\
 & & - \frac{1}{N}\widehat{\nabla}_\mu\widehat{\nabla}_\nu N + K K_{\mu\nu} = 8\pi G[S_{\widehat{\mu}\widehat{\nu}} - \frac{1}{2}P_{\widehat{\mu}\widehat{\nu}}(S - E)] \\
K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} & & \partial_t P_{\mu\nu} = 2NK_{\mu\nu} + \widehat{\nabla}_\mu\beta_\nu + \widehat{\nabla}_\nu\beta_\mu
\end{array}\right.$$

(Gauss-Codazzi eq.) $2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}}$

(Contracted Codazzi eq.) $2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}}$

(momentum density) $p_\alpha = -T_{n\widehat{\alpha}}$

(stress tensor) $S_{\mu\nu} = T_{\widehat{\mu}\widehat{\nu}}$

3+1 Decomposition (Einstein eq. (4-2))

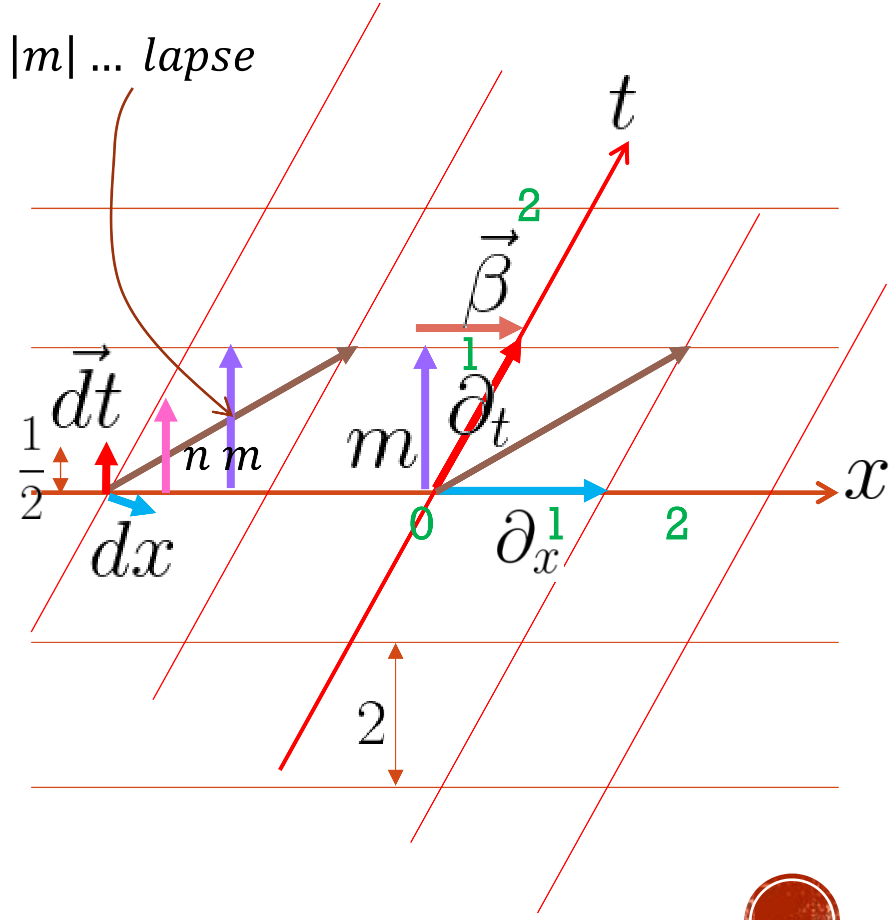
$$K_{ij} = \frac{1}{2}\mathcal{L}_{\boldsymbol{n}} P_{ij} = \frac{1}{2N}\mathcal{L}_{\boldsymbol{m}} P_{ij} = \frac{1}{2N}\mathcal{L}_{\boldsymbol{\partial_t}-\boldsymbol{\beta}} P_{ij}$$

$$= \frac{1}{2N}\left(\mathcal{L}_{\boldsymbol{\partial_t}} P_{ij} - \mathcal{L}_{\boldsymbol{\beta}} P_{ij}\right)$$

$$\hookrightarrow \ \mathcal{L}_{\boldsymbol{\partial_t}} P_{ij} = (\partial_t)^\alpha \partial_\alpha P_{ij} + P_{\alpha j}\partial_i \underbrace{(\partial_t)^\alpha}_{=\delta^\alpha_t} + K_{i\alpha}\partial_j \underbrace{(\partial_t)^\alpha}_{=\delta^\alpha_t} = \frac{\partial P_{ij}}{\partial t} = \frac{\partial \gamma_{ij}}{\partial t}$$

$$\hookrightarrow \ \mathcal{L}_{\boldsymbol{\beta}} P_{ij} = \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij} = \beta^k \widehat{\nabla}_k \gamma_{ij} + \gamma_{kj}\widehat{\nabla}_i \beta^k + \gamma_{ik}\widehat{\nabla}_j \beta^k$$

$$= \frac{1}{2N}\left(\dot{\gamma}_{ij} - \gamma_{kj}\widehat{\nabla}_i \beta^k - \gamma_{ik}\widehat{\nabla}_j \beta^k\right)$$



3+1 Decomposition (Einstein eq. (4-3))

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ G_{nn} = 8\pi G T_{nn} \ \rightarrow\ \hat{R} + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow\ \hat{\nabla}_\nu K - \hat{\nabla}_\mu K^\mu{}_\nu = 8\pi G p_\nu \\ (3)\ G_{\widehat{\mu\nu}} = 8\pi G T_{\widehat{\mu\nu}} \ \rightarrow\ \partial_t K_{\mu\nu} = N(-\hat{R}_{\mu\nu} + 2K_{\mu\alpha}K^\alpha_\nu - KK_{\mu\nu}) \\ \qquad - (\beta^\alpha \partial_\alpha K_{\mu\nu} + \partial_\mu \beta^\alpha K_{\alpha\nu} + \partial_\nu \beta^\alpha K_{\mu\alpha}) \\ \qquad + \hat{\nabla}_\mu \hat{\nabla}_\nu N + 8\pi G N [S_{\widehat{\mu\nu}} - \frac{1}{2} P_{\widehat{\mu\nu}}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow\ \partial_t P_{\mu\nu} = 2NK_{\mu\nu} + \hat{\nabla}_\mu \beta_\nu + \hat{\nabla}_\nu \beta_\mu \end{cases}$$

>>> Intrinsic eq. with coordinates on the hypersurface

3+1 Decomposition (Einstein eq. (4-4))

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ G_{nn} = 8\pi G T_{nn} & \rightarrow\ \widehat{R} + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ G_{n\widehat{\mu}} = 8\pi G T_{n\widehat{\mu}} & \rightarrow\ \widehat{\nabla}_i K - \widehat{\nabla}_j K^j{}_i = 8\pi G p_i \\ (3)\ G_{\widehat{\mu}\widehat{\nu}} = 8\pi G T_{\widehat{\mu}\widehat{\nu}} & \rightarrow\ \partial_t K_{ij} = N(-\widehat{R}_{ij} + 2K_{ik}K^k{}_j - KK_{ij}) \\ & \qquad - (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ & \qquad + \widehat{\nabla}_i \widehat{\nabla}_j N + 8\pi G N[S_{ij} - \frac{1}{2}P_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} & \rightarrow\ \partial_t P_{ij} = 2NK_{ij} + \widehat{\nabla}_i \beta_j + \widehat{\nabla}_j \beta_i \end{cases}$$

>>> Intrinsic eq. with coordinates on the hypersurface

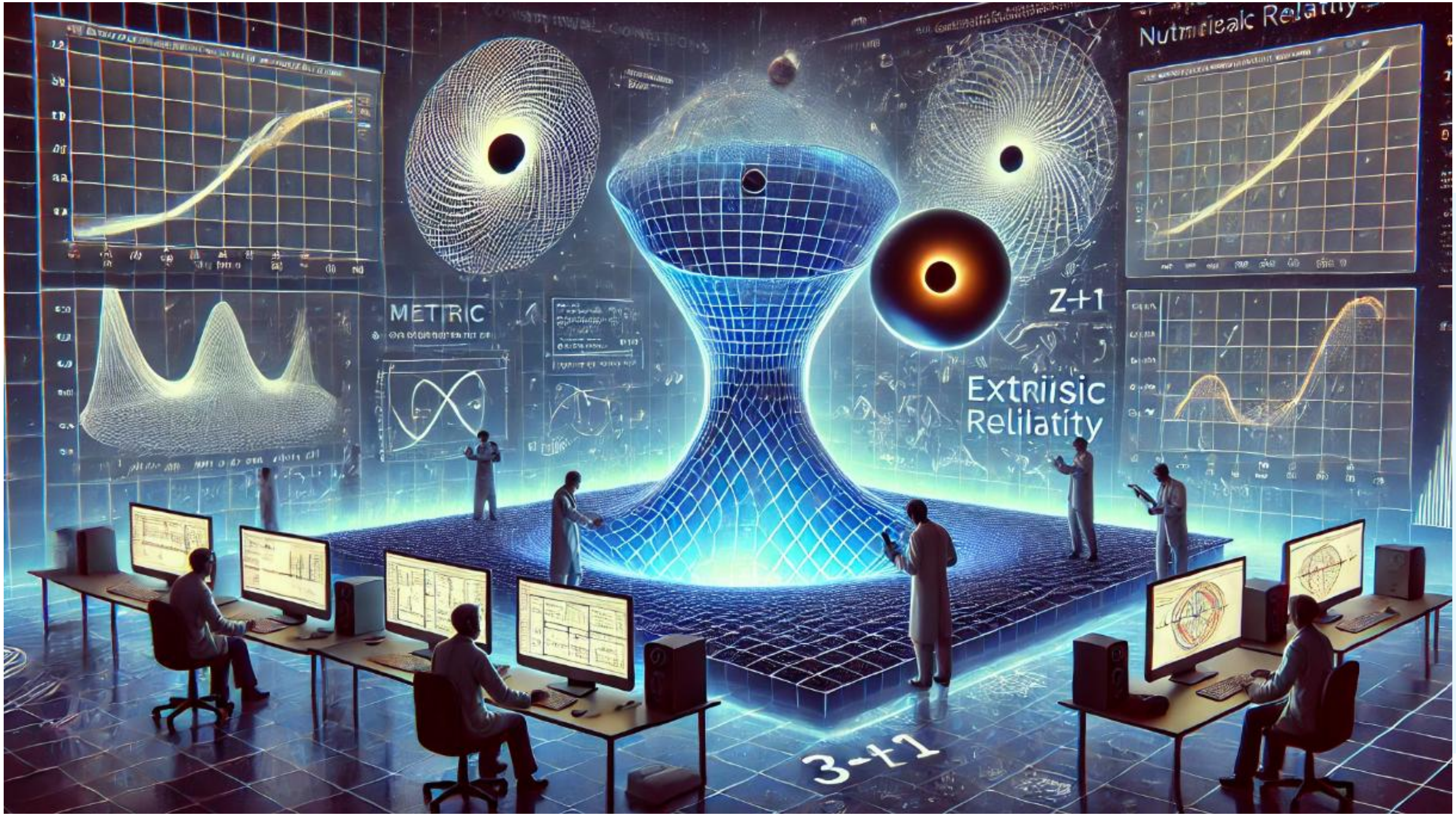
3+1 Decomposition (Einstein eq.)

$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\Lsh \begin{cases} (1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow \ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow \ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ ^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \ \rightarrow \ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) \\ \qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ \qquad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow \ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$

$$N \rightarrow \alpha,\ K_{\mu\nu} \rightarrow -K_{\mu\nu},\ P_{ij} \rightarrow \gamma_{ij},\ \widehat{\nabla} \rightarrow D,\ R \rightarrow {}^{(4)}R,\ \widehat{R} \rightarrow R$$

TO APPLY 3+1 EQ'S IN NR



INITIAL DATA CONSTRUCTION (CONFORMAL TRANSVERSE TRACELESS DECOMPOSITION)

(1) conformal trf.: $\gamma_{ij} = \gamma^{1/3}\bar{\gamma}_{ij} = \psi^n \bar{\gamma}_{ij}$

($\bar{\gamma} \equiv 1$. $\bar{\gamma}_{ij}$ includes only traceless dofs.)

(2) $K_{ij} \rightarrow$ trace/traceless and conformal:

$$K_{ij} \equiv A_{ij} + \frac{1}{3}\gamma_{ij}K$$

$$A^{ij} \equiv \psi^{-10}\bar{A}^{ij} = \psi^{-4}\tilde{A}^{ij}$$

(3) $\bar{A}^{ij} = \bar{A}^{ij}_{\mathrm{T}}$ traceless $\rightarrow$ transverse/longitudinal:

$$\bar{A}^{ij} \equiv \bar{A}^{ij}_{\mathrm{TT}} + \bar{A}^{ij}_{\mathrm{L}} \ (\bar{D}_j \bar{A}^{ij}_{\mathrm{TT}} = 0, \therefore \ \bar{D}_j \bar{A}^{ij} = \bar{D}_j \bar{A}^{ij}_{\mathrm{L}})$$

$$\bar{A}^{ij}_{\mathrm{L}} = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k \equiv (\bar{L}W)^{ij}$$

$$\bar{D}_j \bar{A}^{ij} = \bar{D}_j \bar{A}^{ij}_{\mathrm{L}} = \bar{D}_j (\bar{L}W)^{ij} \equiv (\bar{\Delta}_{\mathrm{L}} W)^i$$

$$G_{nn}: \ 8\bar{D}^2\psi - \psi\bar{R} - \tfrac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho$$

$$G_{ni}: \ (\bar{\Delta}_{\mathrm{L}} W)^j - \tfrac{2}{3}\psi^6 \bar{\gamma}^{ij}\bar{D}_i K = 8\pi G \psi^{10} p^j$$

$$G_{ij}: \ \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_i D_j \alpha - \alpha 8\pi G(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S-\rho)) + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k$$

Initial data construction (Thin-sandwich formalism)

$$G_{nn} : \ 8\bar{D}^2\psi - \psi\bar{R} - \tfrac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho$$
$$G_{ni} : \ (\bar{\Delta}_{\rm L}W)^j - \tfrac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_i K = 8\pi G\psi^{10}p^j$$

related to remaining $\bar{\gamma}_{ij}, K$ but not appeared in constraint eq.

$$g_{\mu\nu}(\alpha(1), \beta^i(3), \psi(1), \boxed{\bar{\gamma}_{ij}(3+2)})(10)$$
$$+K_{\mu\nu}(\boxed{K(1)}, \boxed{A^{\rm TT}_{ij}(2)}, A^{\rm L}_{ij}(3))(6)$$

GW dof.

ς using $\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$

ς $K_{ij} \to \frac{1}{2\alpha}(-\partial_t\gamma_{ij} + D_i\beta_j + D_j\beta_i)$

$\boxed{?}$ $\to$ We can use a background data of a specific $\dot{\gamma}_{ij}$.

$$8\bar{D}^2\psi - \psi\bar{R} - \tfrac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho \quad \to \quad \psi(1)$$
$$(\bar{\Delta}_{\rm L}\beta)^i - (\bar{L}\beta)^{ij}\bar{D}_j\ln(\alpha\psi^{-6}) = \alpha\psi^{-6}\bar{D}_j(\alpha^{-1}\psi^6\bar{u}^{ij}) + \tfrac{4}{3}\alpha\bar{D}^i K + 16\pi\alpha\psi^4 j^l$$

[conformal thin-sandwich decomposition]

newly added in the constraint eq.

$$g_{\mu\nu}(\boxed{\alpha(1)}, \beta^i(3), \psi(1), \boxed{\bar{\gamma}_{ij}(3+2)})(10)$$
$$+K_{\mu\nu}(\boxed{K(1)}, A^{\rm TT}_{ij}(2), A^{\rm L}_{ij}(3))(6)$$

GW dof.

$\to \boxed{(\dot{\gamma}_{ij} \to)u_{ij}(5),}$ $\alpha,\ \beta^i$

EVOLUTION: BSSN FORMALISM

[evolutions equations]

(1) $\partial_t \phi = -\frac{1}{6}\alpha K + \frac{1}{6}\partial_i\beta^i + \beta^i\partial_i\phi$

(2) $\partial_t K = -D^2\alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) + 4\pi\alpha(\rho + S) + \beta^i D_i K$

[constraint equation]

(1) ${}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho$

(2) ${}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu$

- The modes that violates the momentum constraint propagate with speed zero in the ADM equations.
- These modes lead to instabilities when non-linear source terms are included.
- In the BSSN system, the momentum violating modes propagate with non-zero speed and propagate off the numerical grid, presumably stabilizing the simulation.

$$\partial_t\bar{\Gamma}^i = -2\partial_j\alpha\tilde{A}^{ij} + 2\alpha(\bar{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi G\bar{\gamma}^{ij}p_j + 6\tilde{A}^{ij}\partial_j\phi)$$
$$+\beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{ki}\beta^j{}_{,jk} + \bar{\gamma}^{kj}\beta^i{}_{,kj}$$

(4) $\rightarrow R_{ij} = \bar{R}_{ij} + R^\phi_{ij}$

$$= \underbrace{-\frac{1}{2}\bar{\gamma}^{lm}\bar{\gamma}_{ij,lm}}_{\text{only 2nd deri.}} + \underbrace{\bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k}_{\text{all other mixed derivatives are absorbed to } \bar{\Gamma}^k} + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}(2\bar{\Gamma}^k_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k_{im}\bar{\Gamma}_{klj})$$
$$-2(\bar{D}_i\bar{D}_j\phi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\phi) + 4[(\bar{D}_i\phi)(\bar{D}_j\phi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l\phi)(\bar{D}_m\phi)]$$

MORE ABOUT NR APPLICATIONS

Wikipedia
The Free Encyclopedia

BSSN formalism

Article Talk

From Wikipedia, the free encyclopedia

The **BSSN formalism** (Baumgarte, Shapiro, Shibata, Nakamura formalism) is a formalism of general relativity that was developed by Thomas W. Baumgarte, Stuart L. Shapiro, Masaru Shibata and Takashi Nakamura between 1987 and 1999.[1] It is a modification of the ADM formalism developed during the 1950s.

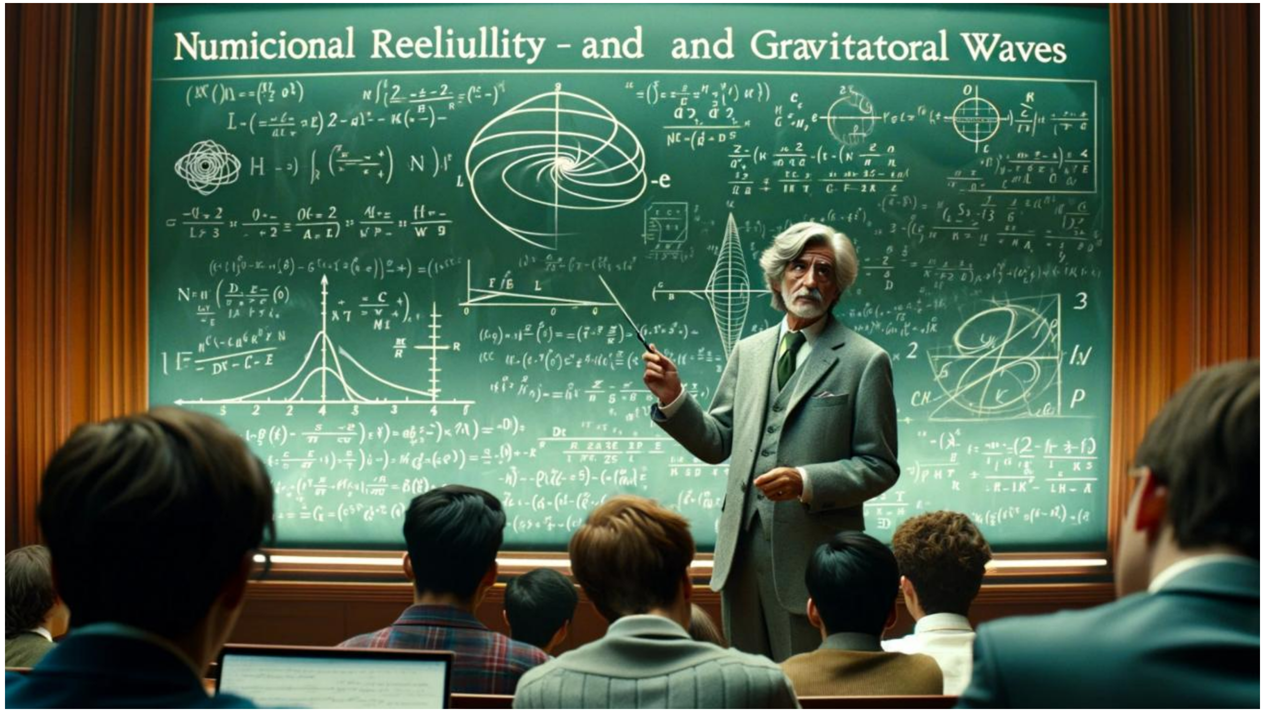
The ADM formalism is a Hamiltonian formalism that does not permit stable and long-term numerical simulations. In the BSSN formalism, the ADM equations are modified by introducing auxiliary variables. The formalism has been tested for a long-term evolution of linear gravitational waves and used for a variety of purposes such as simulating the non-linear evolution of gravitational waves or the evolution and collision of black holes.[2][3]

See also [edit]

- ADM formalism
- Canonical coordinates
- Canonical gravity
- Hamiltonian mechanics

References [edit]

1. ^ Jinho Kim (2008-07-28). "General Relativistic Hydrodynamics Using BSSN formalism" (PDF). Seoul National University. Archived from the original (PDF) on 2012-03-27. Retrieved 2009-10-19.



NR WORKS: SIMULATION OF BBH MERGER

PRL **95,** 121101 (2005) PHYSICAL REVIEW LETTERS week ending 16 SEPTEMBER 2005

Evolution of Binary Black-Hole Spacetimes

Frans Pretorius[1,2,*]

[1]*Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125, USA*
[2]*Department of Physics, University of Alberta, Edmonton, AB T6G 2J1 Canada*

(Received 6 July 2005; published 14 September 2005)

We describe early success in the evolution of binary black-hole spacetimes with a numerical code based on a generalization of harmonic coordinates. Indications are that with sufficient resolution this scheme is capable of evolving binary systems for enough time to extract information about the orbit, merger, and gravitational waves emitted during the event. As an example we show results from the evolution of a binary composed of two equal mass, nonspinning black holes, through a single plunge orbit, merger, and ringdown. The resultant black hole is estimated to be a Kerr black hole with angular momentum parameter $a \approx 0.70$. At present, lack of resolution far from the binary prevents an accurate estimate of the energy emitted, though a rough calculation suggests on the order of 5% of the initial rest mass of the system is radiated as gravitational waves during the final orbit and ringdown.

DOI: 10.1103/PhysRevLett.95.121101 PACS numbers: 04.25.Dm, 04.30.Db, 04.70.Bw

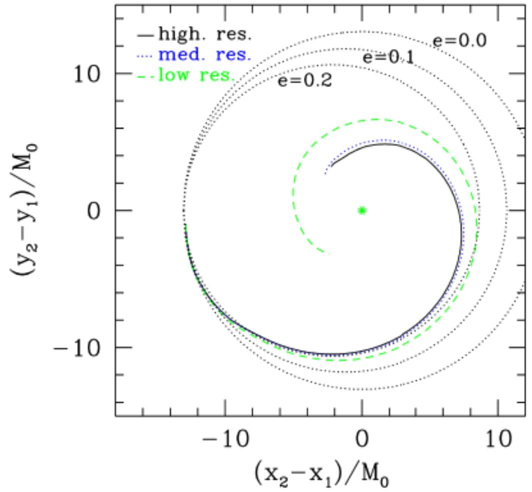


FIG. 1 (color online). A depiction of the orbit for the simulation described in the text (see also Table I). The figure shows the coordinate position of the center of one apparent horizon relative to the other, in the orbital plane $z = 0$. The units have been scaled to the mass M_0 of a single black hole, and curves are shown from simulations with three different resolutions. Overlaid on this figure are reference ellipses of eccentricity 0, 0.1, and 0.2, suggesting that if one were to attribute an initial eccentricity to the orbit it could be in the range 0–0.2.

NR WORKS: SIMULATION OF BBH MERGER

PHYSICAL REVIEW D **74,** 041501(R) (2006)

Spinning-black-hole binaries: The orbital hang-up

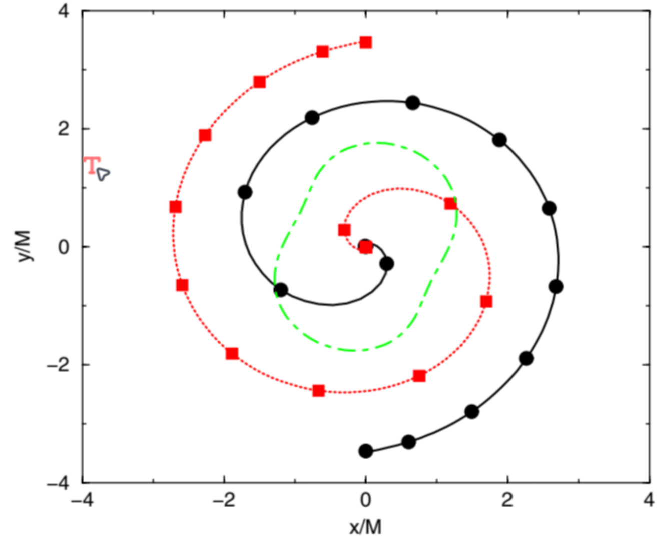
M. Campanelli, C. O. Lousto, and Y. Zlochower

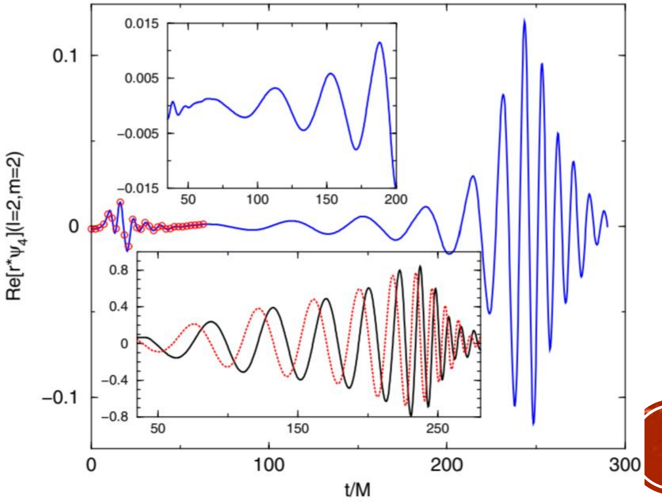
Department of Physics and Astronomy, and Center for Gravitational Wave Astronomy,
The University of Texas at Brownsville, Brownsville, Texas 78520, USA

(Received 4 April 2006; revised manuscript received 22 June 2006; published 16 August 2006)

We present the first fully-nonlinear numerical study of the dynamics of highly-spinning-black-hole binaries. We evolve binaries from quasicircular orbits (as inferred from post-Newtonian theory), and find that the last stages of the orbital motion of black-hole binaries are profoundly affected by their individual spins. In order to cleanly display its effects, we consider two equal-mass holes with individual spin parameters $S/m^2 = 0.757$, both aligned and antialigned with the orbital angular momentum (and compare with the spinless case), and with an initial orbital period of $125M$. We find that the aligned case completes three orbits and merges significantly after the antialigned case, which completes less than one orbit. The total energy radiated for the former case is $\approx 7\%$ while for the latter it is only $\approx 2\%$. The final Kerr hole remnants have rotation parameters $a/M = 0.89$ and $a/M = 0.44$ respectively, showing the unlikeliness of creating a maximally rotating black hole out of the merger two highly spinning holes.

DOI: 10.1103/PhysRevD.74.041501 PACS numbers: 04.25.Dm, 04.25.Nx, 04.30.Db, 04.70.Bw





NR WORKS: SIMULATION OF BBH MERGER

Accurate Evolutions of Orbiting Black-Hole Binaries without Excision

M. Campanelli,[1] C. O. Lousto,[1] P. Marronetti,[2] and Y. Zlochower[1]

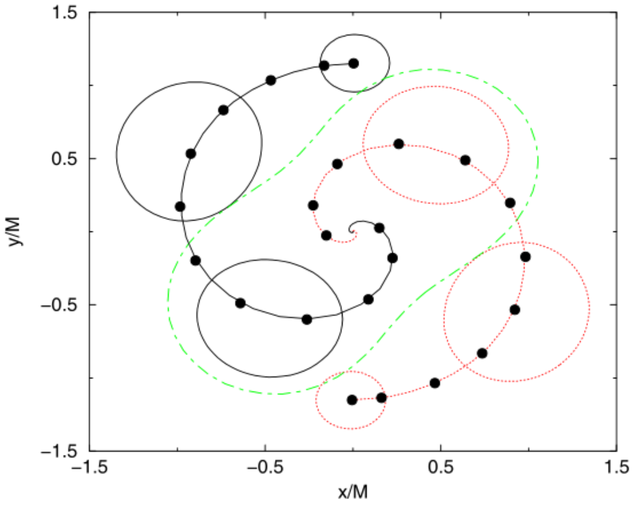
[1]*Department of Physics and Astronomy, and Center for Gravitational Wave Astronomy, The University of Texas at Brownsville, Brownsville, Texas 78520, USA*

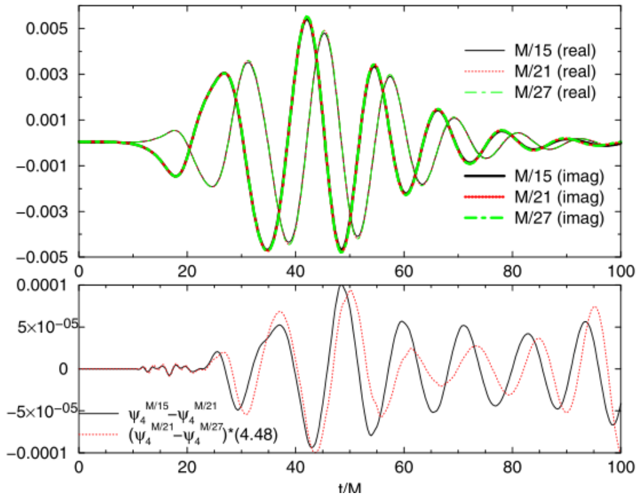
[2]*Department of Physics, Florida Atlantic University, Boca Raton, Florida 33431, USA*

(Received 9 November 2005; published 22 March 2006)

We present a new algorithm for evolving orbiting black-hole binaries that does not require excision or a corotating shift. Our algorithm is based on a novel technique to handle the singular puncture conformal factor. This system, based on the Baumgarte-Shapiro-Shibata-Nakamura formulation of Einstein's equations, when used with a "precollapsed" initial lapse, is nonsingular at the start of the evolution and remains nonsingular and stable provided that a good choice is made for the gauge. As a test case, we use this technique to fully evolve orbiting black-hole binaries from near the innermost stable circular orbit regime. We show fourth-order convergence of waveforms and compute the radiated gravitational energy and angular momentum from the plunge. These results are in good agreement with those predicted by the Lazarus approach.

DOI: 10.1103/PhysRevLett.96.111101 PACS numbers: 04.25.Dm, 04.25.Nx, 04.30.Db, 04.70.Bw





NR WORKS: SIMULATION OF BBH ENCOUNTERS

PHYSICAL REVIEW D **89,** 081503(R) (2014)

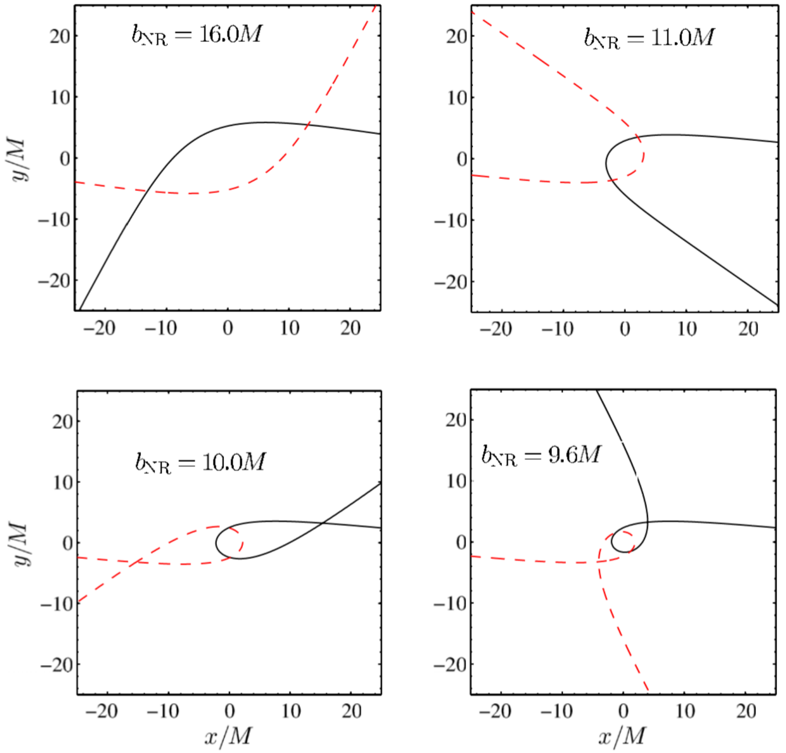
Strong-field scattering of two black holes: Numerics versus analytics

Thibault Damour,[1] Federico Guercilena,[2,3] Ian Hinder,[2] Seth Hopper,[2] Alessandro Nagar,[1] and Luciano Rezzolla[3,2]

[1]*Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette, France*
[2]*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm, Germany*
[3]*Institut für Theoretische Physik, Max-von-Laue-Str. 1, D-60438 Frankfurt am Main, Germany*
(Received 28 February 2014; published 8 April 2014)

We probe the gravitational interaction of two black holes in the strong-field regime by computing the scattering angle χ of hyperboliclike, close binary-black-hole encounters as a function of the impact parameter. The fully general-relativistic result from numerical relativity is compared to two analytic approximations: post-Newtonian theory and the effective-one-body formalism. As the impact parameter decreases, so that black holes pass within a few times their Schwarzschild radii, we find that the post-Newtonian prediction becomes quite inaccurate, while the effective-one-body one keeps showing a good agreement with numerical results. Because we have explored a regime which is very different from the one considered so far with binaries in quasicircular orbits, our results open a new avenue to improve analytic representations of the general-relativistic two-body Hamiltonian.

DOI: 10.1103/PhysRevD.89.081503 PACS numbers: 04.25.D-, 04.30.Db, 95.30.Sf



NR WORKS: SIMULATION OF BBH ENCOUNTERS

PHYSICAL REVIEW D **96,** 084009 (2017)

Gravitational radiation driven capture in unequal mass black hole encounters

Yeong-Bok Bae,[1,2,*] Hyung Mok Lee,[1,3,†] Gungwon Kang,[4,‡] and Jakob Hansen[4,§]

[1] *Astronomy Program Department of Physics and Astronomy, Seoul National University, 1 Gwanak-ro, Gwanak-gu, Seoul 08826, Korea*

[2] *Korea Astronomy and Space Science Institute, 776 Daedeokdae-ro, Yuseong-gu, Daejeon 34055, Korea*

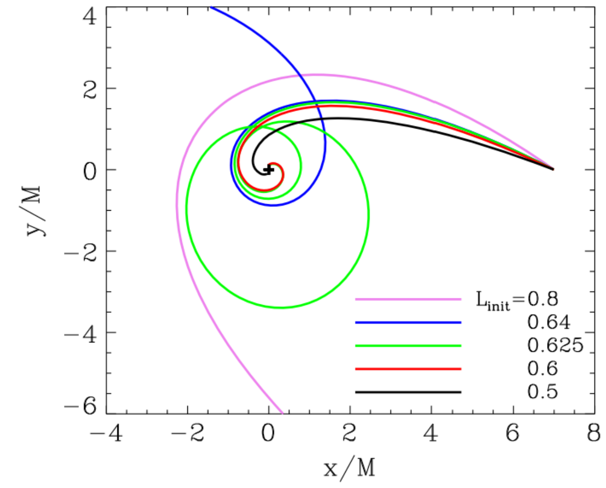
[3] *Center for Theoretical Physics, Seoul National University, 1 Gwanak-ro, Gwanak-gu, Seoul 08826, Korea*

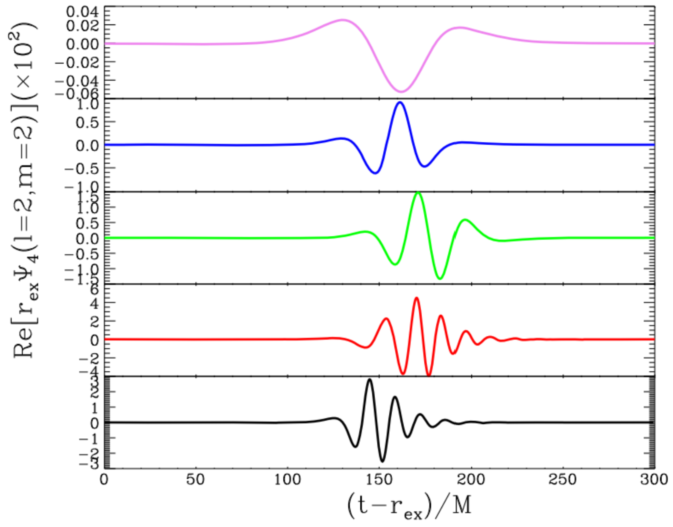
[4] *Supercomputing Center at KISTI, 245 Daehak-ro, Yuseong-gu, Daejeon 34141, Korea*

(Received 6 January 2017; published 4 October 2017)

The gravitational radiation driven capture (GR capture) between unequal mass black holes without spins has been investigated with numerical relativistic simulations. We adopt the parabolic approximation which assumes that the gravitational wave radiation from a weakly hyperbolic orbit is the same as that from the parabolic orbit having the same pericenter distance. Using the radiated energies from the parabolic orbit simulations, we have obtained the critical impact parameter ($b_{\rm crit}$) for the GR capture for weakly hyperbolic orbit as a function of initial energy. The most energetic encounters occur around the boundary between the direct merging and the fly-by orbits, and can emit several percent of total initial energy at the peak. When the total mass is fixed, energy and angular momentum radiated in the case of unequal mass black holes are smaller than those of equal mass black holes having the same initial orbital angular momentum for the fly-by orbits. We have compared our results with two different post-Newtonian (PN) approximations, the exact parabolic orbit (EPO) and PN corrected orbit (PNCO). We find that the agreement between the EPO and the numerical relativity breaks down for very close encounters (e.g., $b_{\rm crit} \lesssim 100$ M), and it becomes worse for higher mass ratios. For instance, the critical impact parameters can differ by more than 50% from those obtained in EPO if the relative velocity at infinity v_∞ is larger than 0.1 for the mass ratio of $m_1/m_2 = 16$. The PNCO gives more consistent results than EPO, but it also underestimates the critical impact parameter for the GR capture at $b_{\rm crit} \lesssim 40$ M.

DOI: 10.1103/PhysRevD.96.084009





NR WORKS: SIMULATION OF BBH ENCOUNTERS

PHYSICAL REVIEW LETTERS **132,** 261401 (2024)

Ringdown Gravitational Waves from Close Scattering of Two Black Holes

Yeong-Bok Bae,[1,3,*,‡] Young-Hwan Hyun,[2,*,§] and Gungwon Kang[3,†]

[1]*Particle Theory and Cosmology Group, Center for Theoretical Physics of the Universe, Institute for Basic Science (IBS), Daejeon 34126, Republic of Korea*

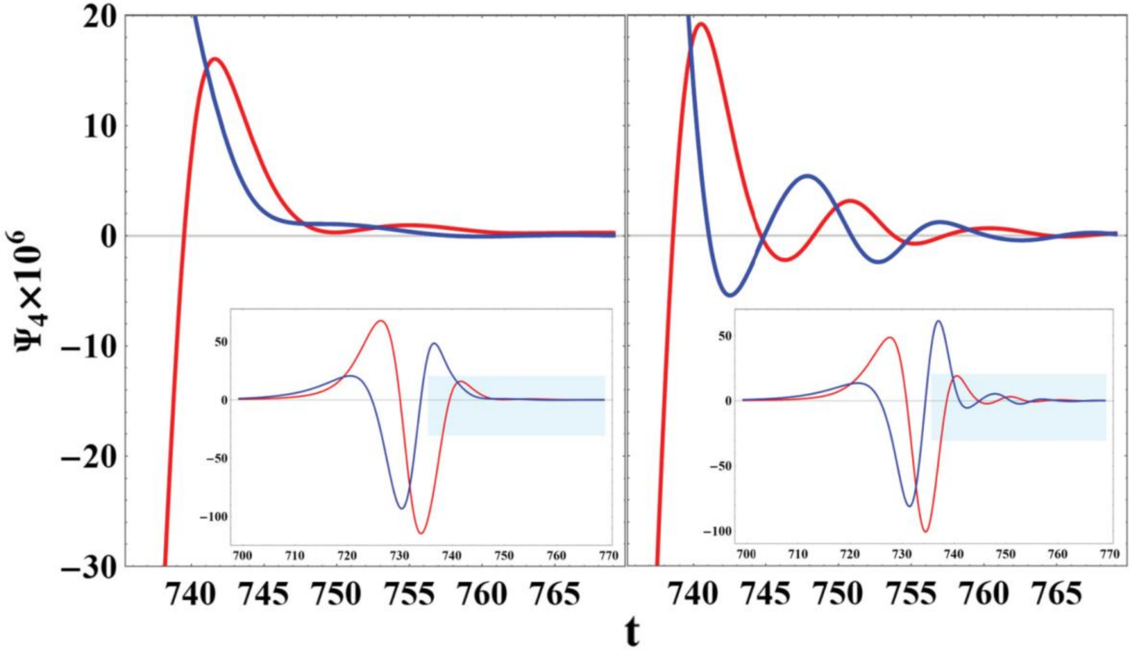
[2]*Korea Astronomy and Space Science Institute (KASI), Daejeon 34055, Republic of Korea*

[3]*Department of Physics, Chung-Ang University, Seoul 06974, Republic of Korea*

(Received 28 October 2023; revised 31 January 2024; accepted 14 May 2024; published 26 June 2024)

We have numerically investigated close scattering processes of two black holes (BHs). Our careful analysis shows for the first time a nonmerging ringdown gravitational wave induced by dynamical tidal deformations of individual BHs during their close encounter. The ringdown wave frequencies turn out to agree well with the quasinormal ones of a single BH in perturbation theory, despite its distinctive physical context from the merging case. Our study shows a new type of gravitational waveform and opens up a new exploration of strong gravitational interactions using BH encounters.

DOI: 10.1103/PhysRevLett.132.261401




ENJOY SPACETIME
TRAVEL!
2025 수치상대론 및 중력파 여름학교