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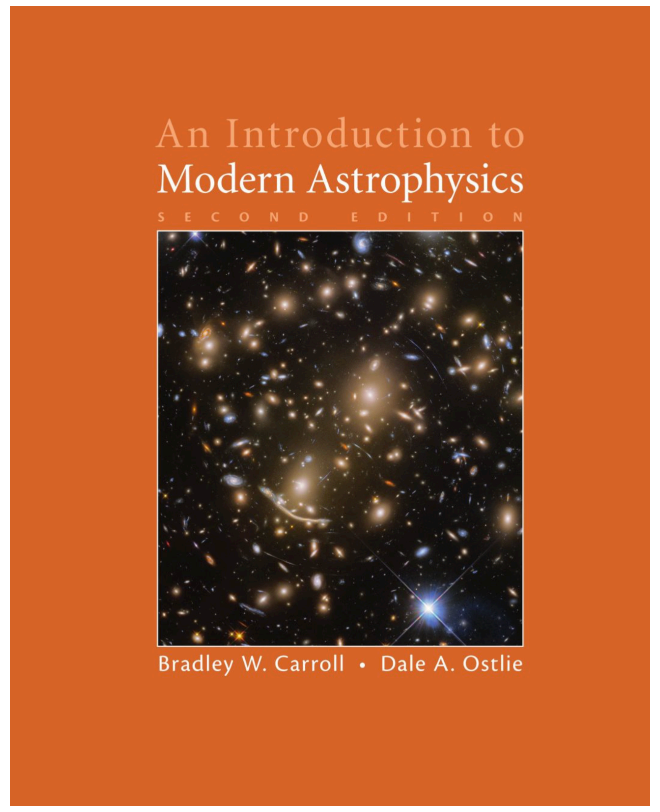
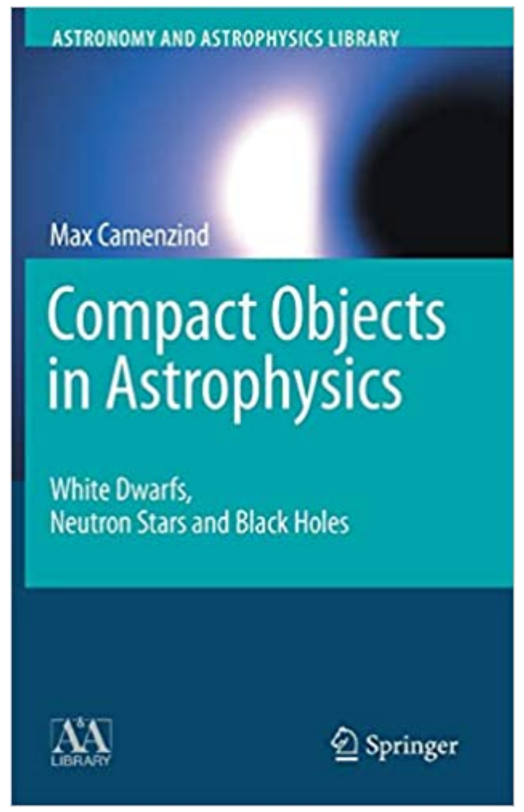
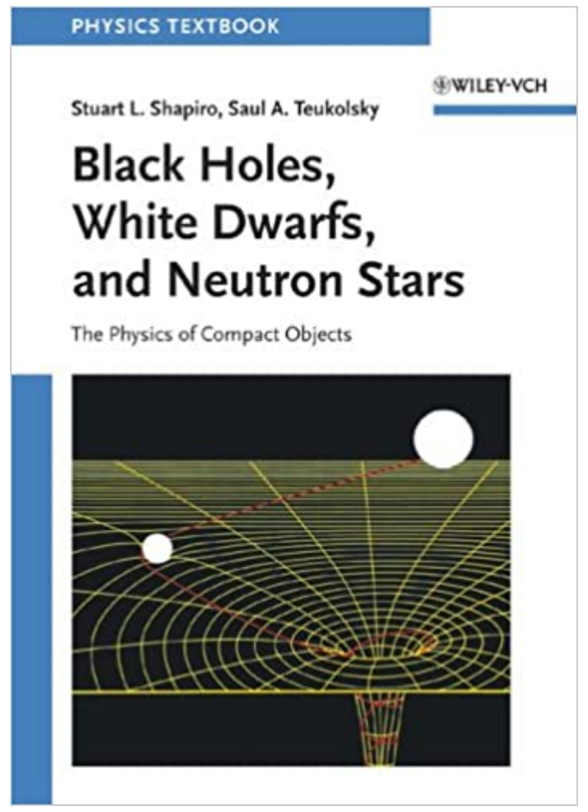
References



An Introduction to Modern Astrophysics by Bradley W. Carroll,Dale A. Ostlie

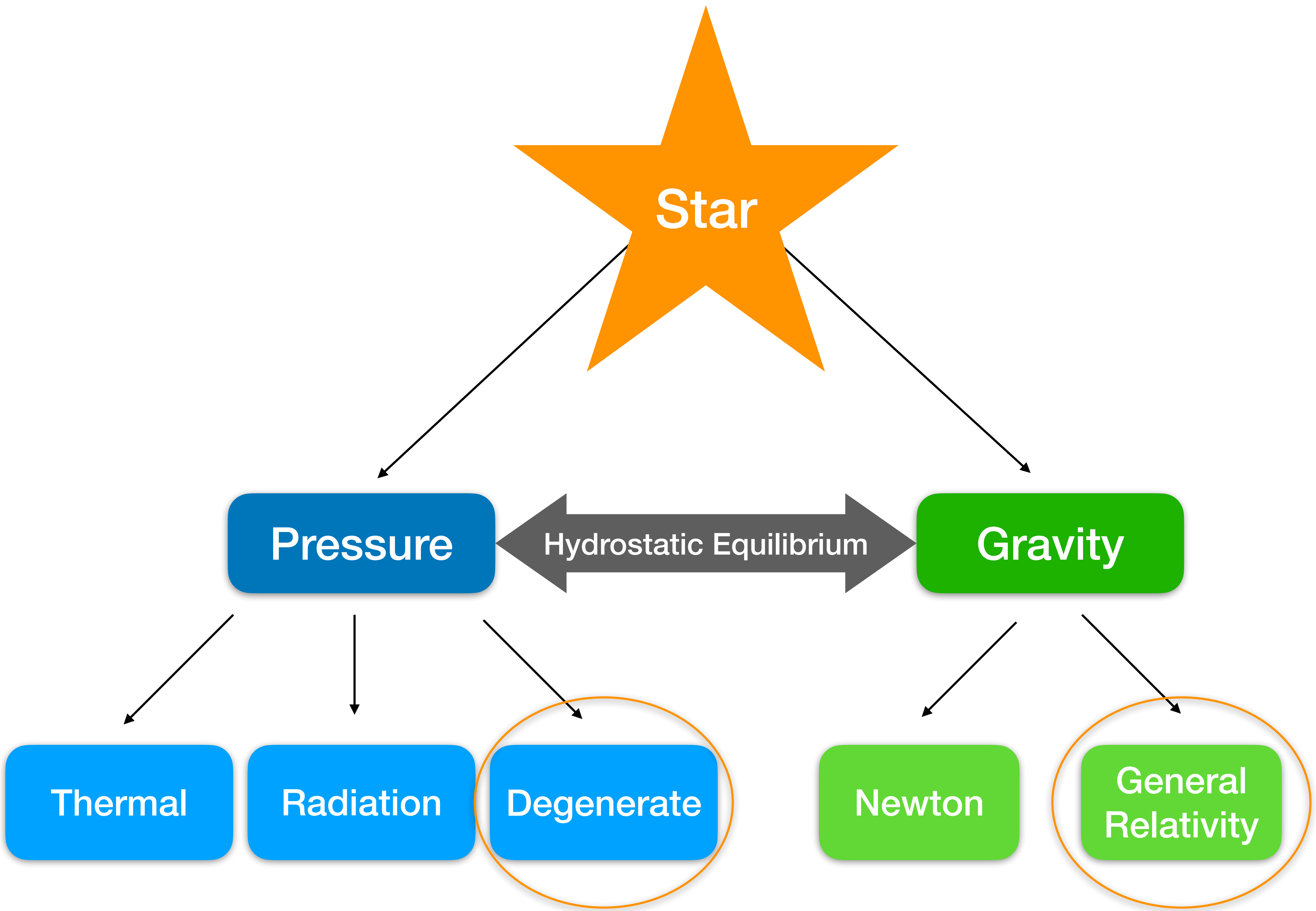


Compact Objects in Astrophysics: White Dwarfs, Neutron Stars and Black Holes by Max Camenzind



Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects by Stuart L. Shapiro

What is a Star?



Gravity

Relativistic Approach

To quantify how relativistic the object is, we can consider two dimensionless quantities as follows:

$$\xi = \frac{GM}{Rc^2}, \qquad \beta = \frac{v}{c}.$$

Relativistic Objects

- Black hole: $R = 2GM/c^2$ (Schwarzschild BH), $R = GM/c^2$ (Extreme Kerr BH) -> $\xi = 0.5 \sim 1$.
- Neutron Star: $M \sim 1.4M_{\odot}$, $R \sim 10\text{km}$ -> $\xi \sim 0.2$.

  $P_{\text{rot}} \sim 1\text{ms}$ -> $\beta \sim 0.2$.
- Jet: $\beta > 0.99$.

Newtonian Objects

- White Dwarf: $M \sim M_{\oplus}$, $R \sim R_{\oplus}$ -> $\xi \sim 0.0003$, $\beta \sim 0.0003$.
- Sun: $M \sim 1M_{\odot}$, $R \sim 1.4 \times 10^6\text{km}$ -> $\xi \sim 10^{-6} \ll 1$.

General Relativity

Space-time tells matter how to move, matter tells space-time how to curve.

by John Wheeler

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}$$

3+1 ADM formalism

BSSN

Z4C

$$(\partial_t - \beta^l \partial_l)\tilde{A}_{ij} = \alpha e^{-4\phi}\left(R_{ij} - \frac{1}{3}\gamma_{ij}R\right) - e^{-4\phi}\left(D_i D_j \alpha - \frac{1}{3}\gamma_{ij}\Delta\alpha\right)$$
$$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) + \tilde{A}_{il}\partial_j\beta^l + \tilde{A}_{jl}\partial_i\beta^l - \frac{2}{3}\beta^l{}_{,l}\tilde{A}_{ij}$$
$$-8\pi\alpha e^{-4\phi}T_{\mu\nu}\left[\gamma^\mu_i\gamma^\nu_j - \frac{1}{3}\gamma^{\mu\nu}\gamma_{ij}\right]$$

현영환 박사님 강의 참고

General Relativity

Space-time tells matter how to move, matter tells space-time how to curve.

by John Wheeler

Geodesic Motion

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$$

$1.4M_\odot$ neutron star

1.67×10^{57} neutrons

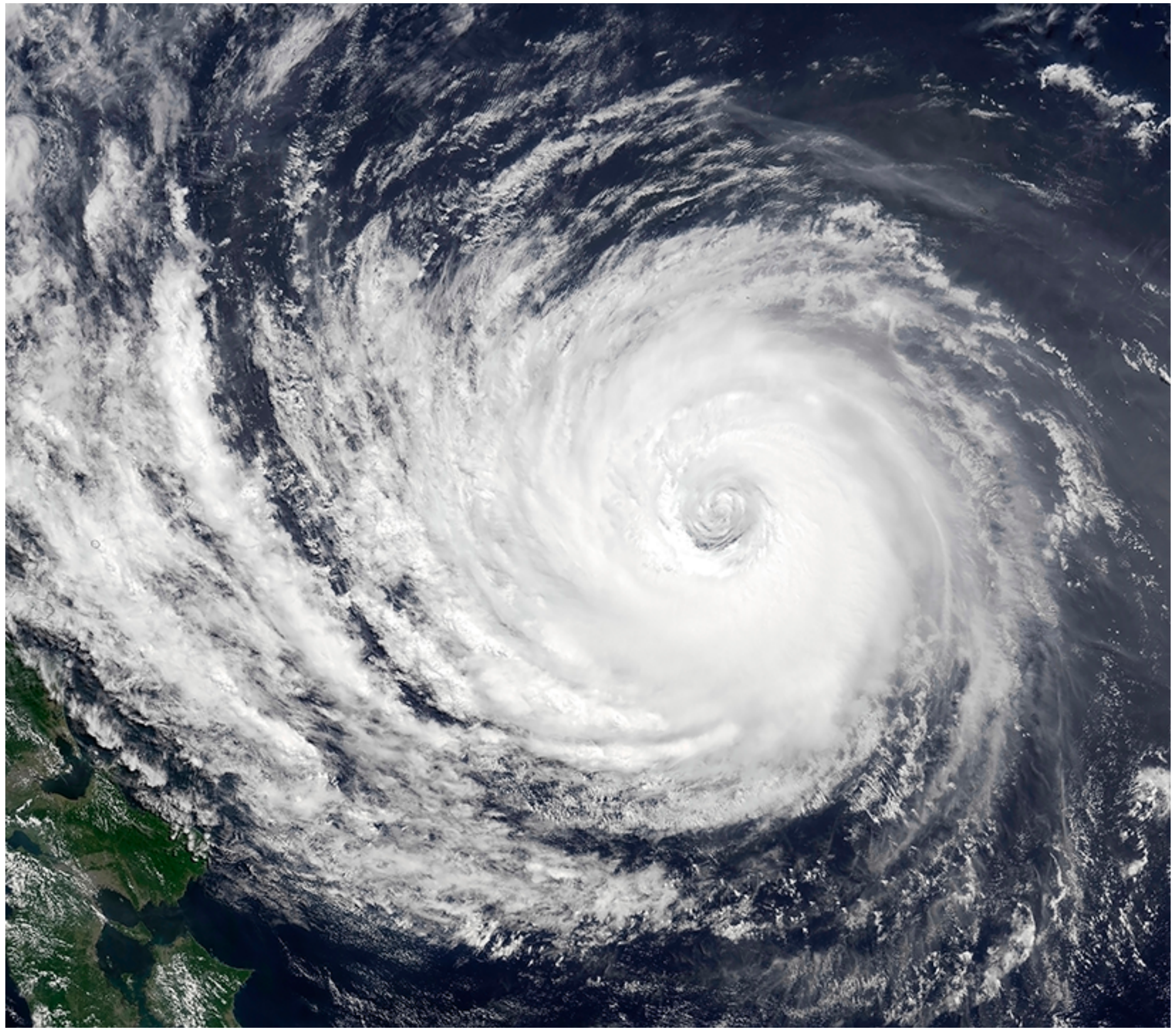
1 second for geodesic motion

5.302×10^{49} years $\gg$

Age of Universe

1.38×10^9 years

Fluid



General Relativity

Space-time tells matter how to move, matter tells space-time how to curve.

by John Wheeler

$$T^{ab} = \rho_0 h u^a u^b + P g^{ab},\ J^a = \rho_0 u^a$$

$$\nabla_a T^{ab} = 0,\ \nabla_a J^a = 0.$$

ρ : rest mass fluid density

P : gas pressure

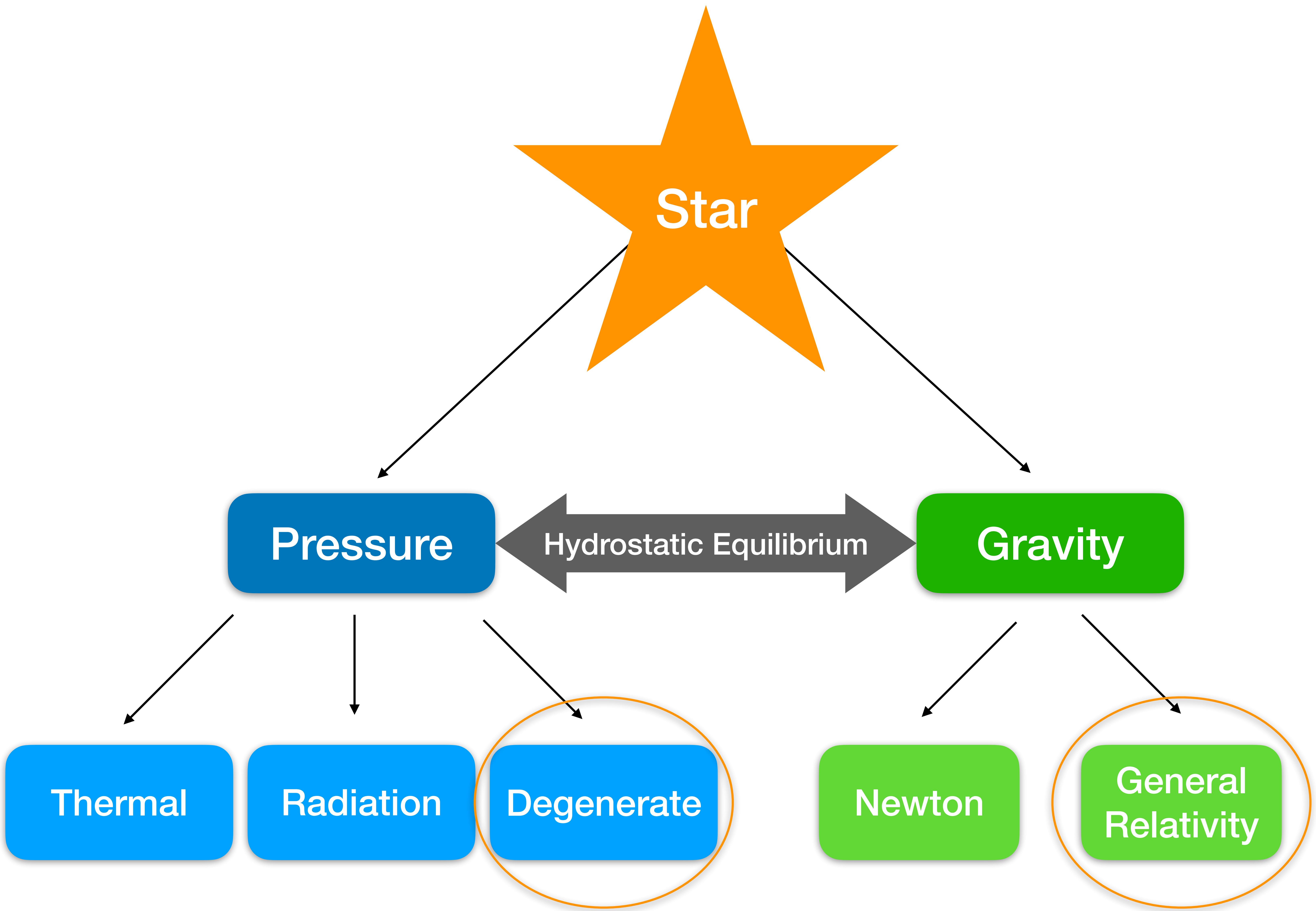
h : specific enthalpy

u^a : four-velocity of the fluid

Conservation of Energy/Momentum

Conservation of Mass

What is a Star?



Pressure

Thermal vs Degenerate Pressure

Thermal Pressure

characterized by kinetic motion

Degenerate Pressure

characterized by quantum states

Fermi energy state

In statistical physics, the distribution function of and ideal gas in equilibrium

$$f(E) = \frac{1}{e^{(E-\mu)/kT} \pm 1}$$

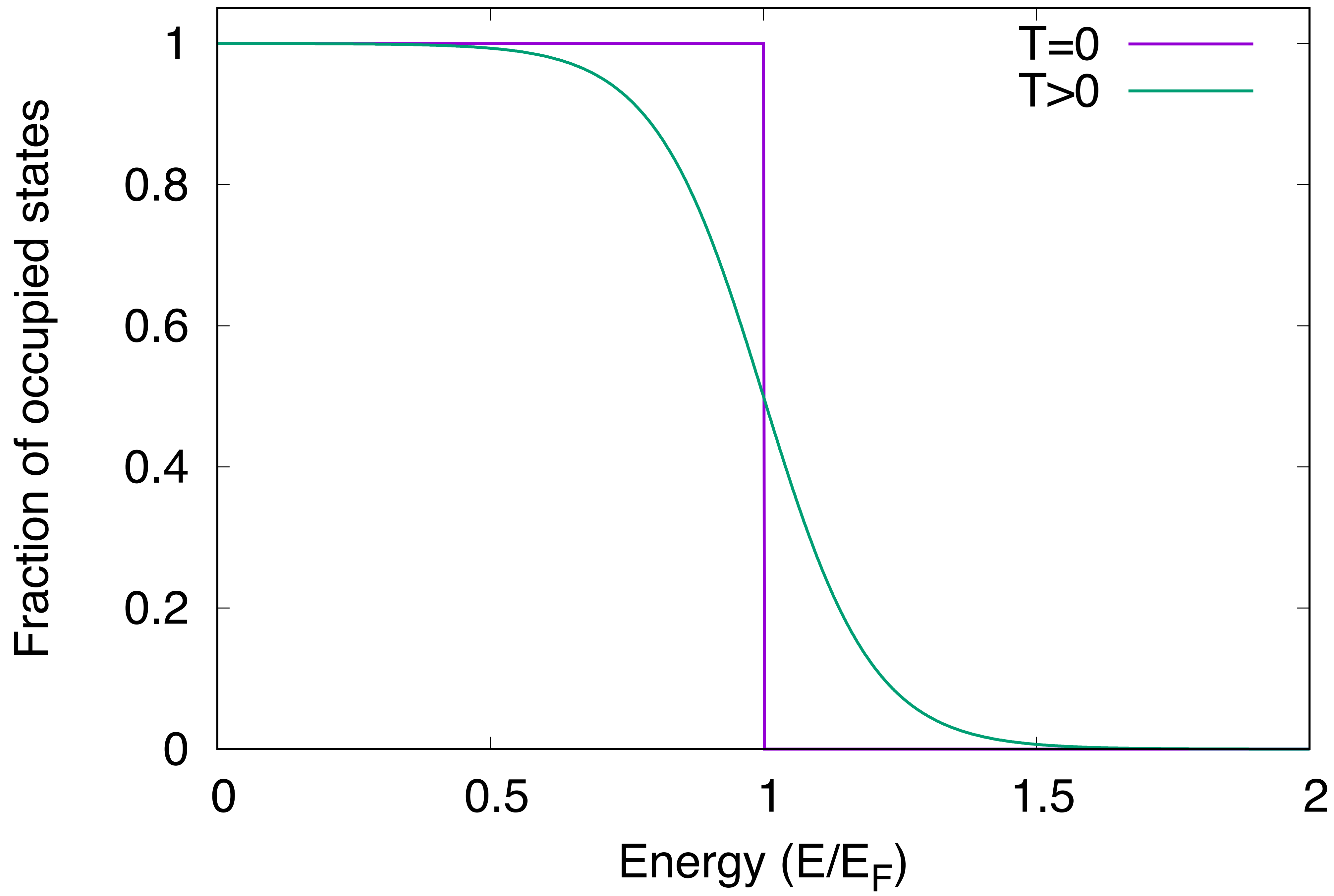
Fermi – Dirac statistics (+) Bose – Einstein statistics (−)

Completely degenerate fermion i.e., $T \to 0$

μ is called the Fermi energy.

$$f(E) = \frac{1}{e^{(E-E_F)/kT} + 1}$$

$$f(E) = \begin{cases} 1, & E \le E_F \\ 0, & E > E_F \end{cases}$$



Equation of State by Electron Degeneracy

$$P = \frac{\pi m_e^4 c^5}{h^3}\left[x_F\left(1+x_F^2\right)^{\frac{1}{2}}\left(\frac{2}{3}x_F^2-1\right)+\ln\left[x_F+\left(1+x_F^2\right)^{\frac{1}{2}}\right]\right]$$

$$= \frac{8\pi m_e^4 c^5}{15h^3}\left[x_F^5-\frac{5}{14}x_F^7+\frac{5}{24}x_F^9+\cdots\right] \quad \text{for} \quad x_F \ll 1$$: non-relativistic limit

$$= \frac{2\pi m_e^4 c^5}{3h^3}\left[x_F^4-x_F^2+\frac{3}{2}\ln\left(2x_F\right)+\cdots\right] \quad \text{for} \quad x_F \gg 1$$: ultra-relativistic limit

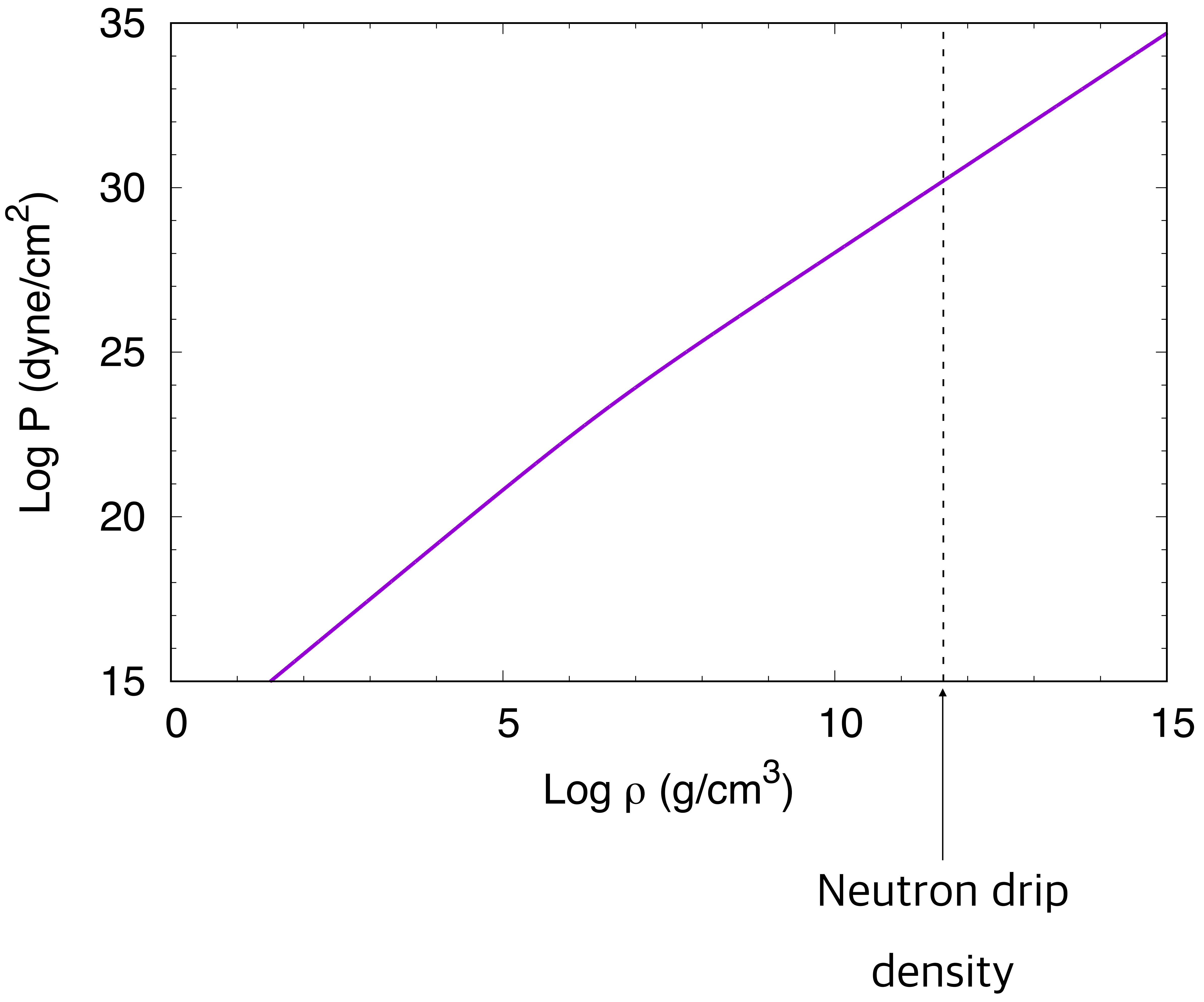
Recall that $p_F = m_e c x \sim n_e^{1/3} \sim \rho^{1/3}$.

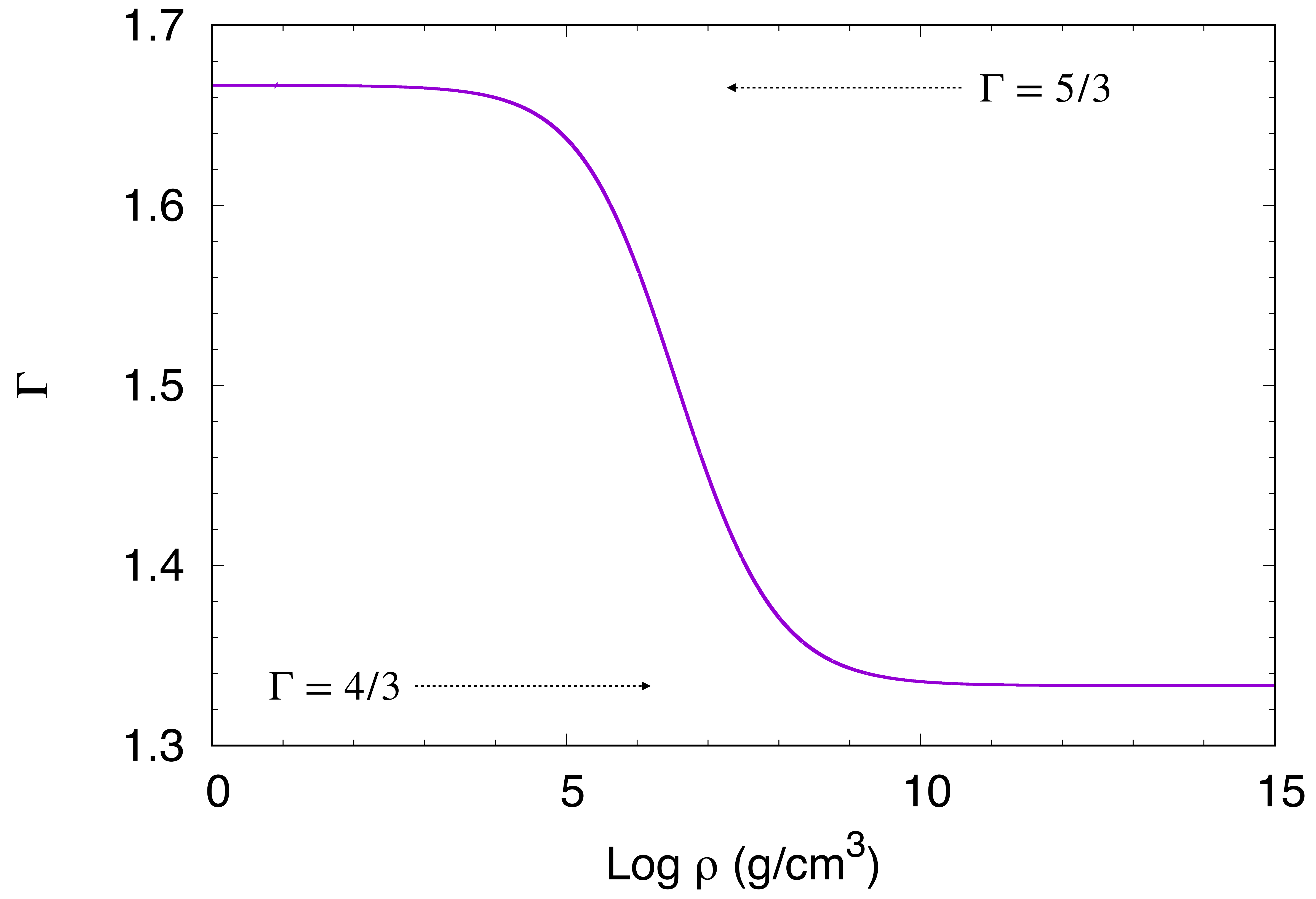
Above asymptotic limit of the pressure gives polytropic equation of state i.e., $P = K\rho^\Gamma = K\rho^{1+1/N}$.

$\Gamma = \frac{5}{3}$: non-relativistic

$\Gamma = \frac{4}{3}$: relativistic

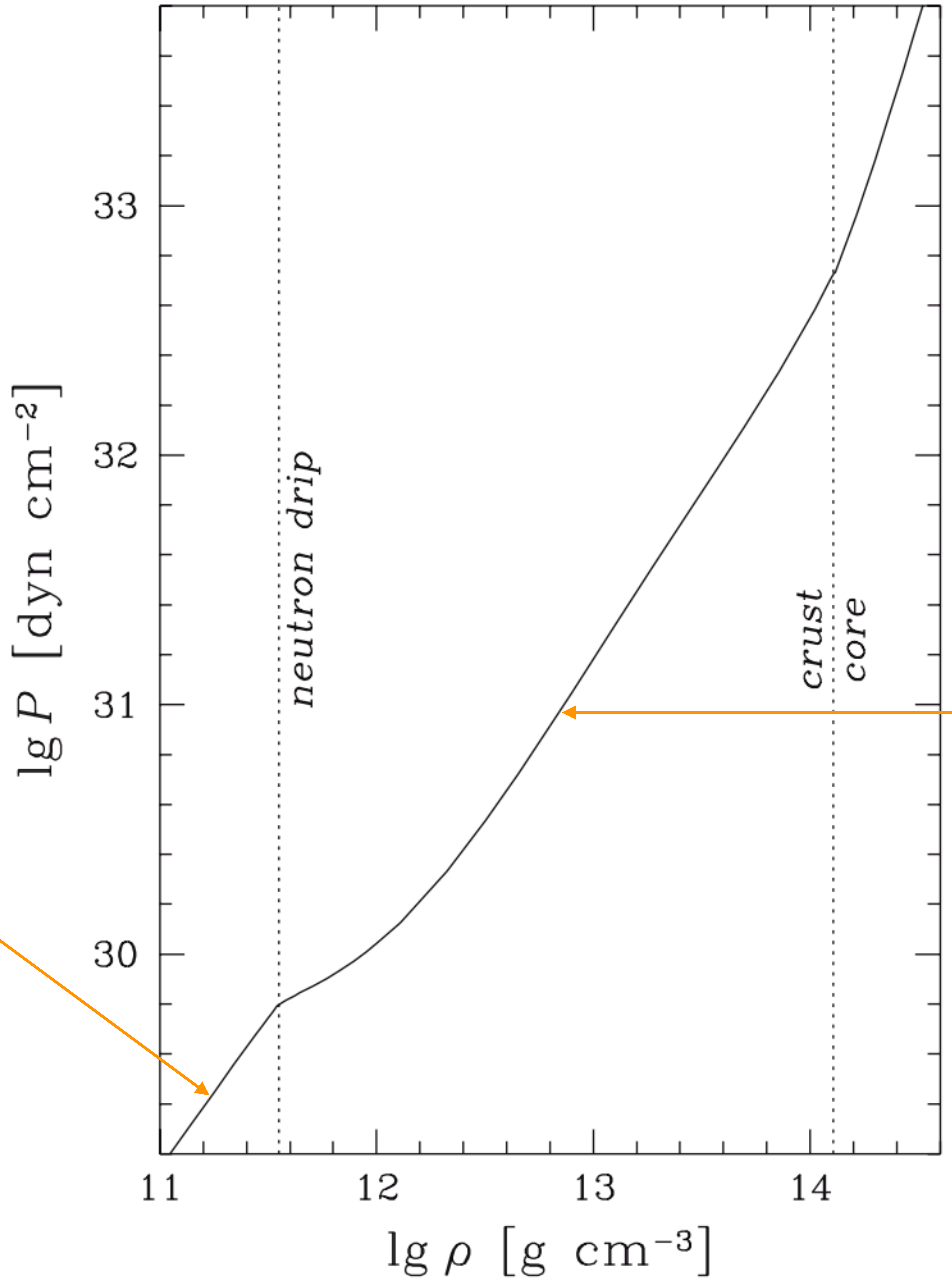
Equation of State by Electron Degeneracy





Equation of State beyond Neutron Drip

Up to neutron drip density:
Relativistic electron degeneracy pressure ($P \sim \rho^{4/3}$)



Neutron degeneracy pressure

Douchin & Haensel (2001)
a.k.a. SLy EoS

Neutron Star Equation of State

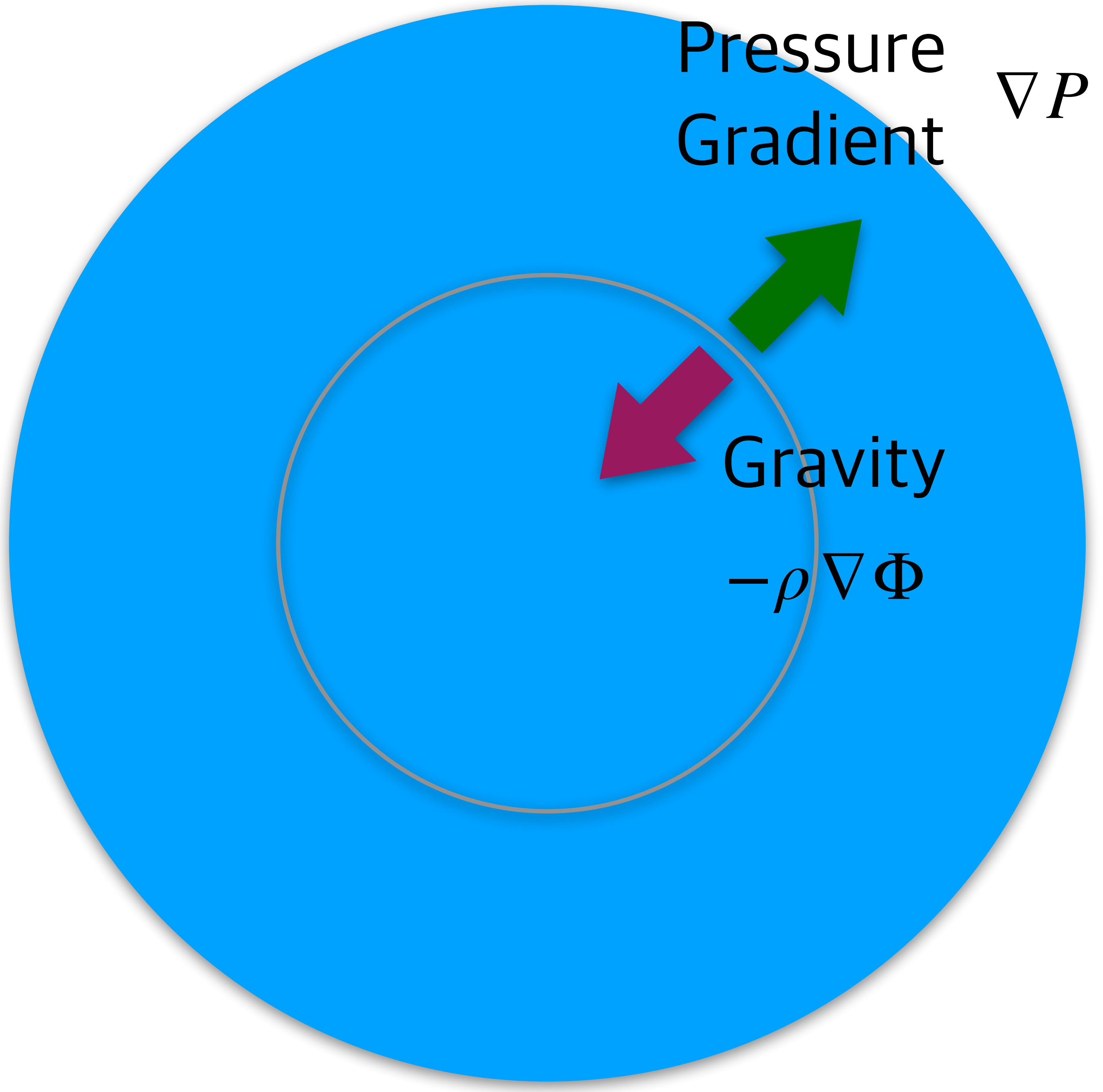
EoS	Composition and model	Reference
BPAL12	$npe\mu$, effective nucleon energy functional	Bombaci et al. 1995 [82]
BGN1H1	$np\Sigma\Lambda\Xi e\mu$, effective baryon energy functional	Balberg et al. [47]
BBB1	$npe\mu$, Brueckner theory, Argonne NN plus Urbana NNN potentials	Baldo et al. [52]
FPS	$npe\mu$, effective nucleon energy functional	Pandharipande and Ravenhall [323]
BGN2H1	$np\Sigma\Lambda\Xi e\mu$, effective baryon energy functional	Balberg et al. [47]
BBB2	$npe\mu$, Brueckner theory, Paris NN plus Urbana NNN potentials	Baldo et al. [52]
SLy	$npe\mu$, effective nucleon energy functional	Douchin and Haensel [136]
BGN1	$npe\mu$, effective baryon energy functional	Balberg et al. [47]
APR	$npe\mu$, variational theory, Nijmegen NN plus Urbana NNN potentials	Akmal et al. [26]
BGN2	$npe\mu$, effective nucleon energy functional	Balberg et al. [47]

Equation of states based on the non-relativistic modeling.

Neutron Star Equation of State

Stellar Structure

Equilibrium Structure of Star



Fluid equation (Euler equation).

1. Continuity Equation: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

2. Momentum Equation: $\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} + \nabla P = -\rho \nabla \Phi$

3. Energy Equation: $\rho \frac{\partial e}{\partial t} + \vec{v} \cdot \nabla e + \frac{P}{\rho} \nabla \cdot \vec{v} = 0$

For spherical star (non-rotating, non-magnetized), LHS of 1. Continuity and 3. Energy equation is 0. The remaining equation is 2. Momentum equation and can be rewritten in spherical coordinates as

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \quad \text{where } M_r = 4\pi \int_0^r \rho(r) r^2 dr \quad \Rightarrow \quad \frac{1}{r^2} \frac{d}{dr}\left(\frac{r^2}{\rho} \frac{dP}{dr}\right) = -4\pi G \rho$$

Lane-Emden Equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Polytropic equation of state: $P = K\rho^{\Gamma} = K\rho^{1+1/N}$

This equation can further be reduced to dimensionless form by writing

$$\rho = \rho_c\theta^N,\ \ r = a\xi,\ \ a = \left[\frac{(N+1)\,K\rho_c^{\frac{1}{N}-1}}{4\pi G}\right]^{1/2}$$

, where ρ_c is the central density.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^N$$

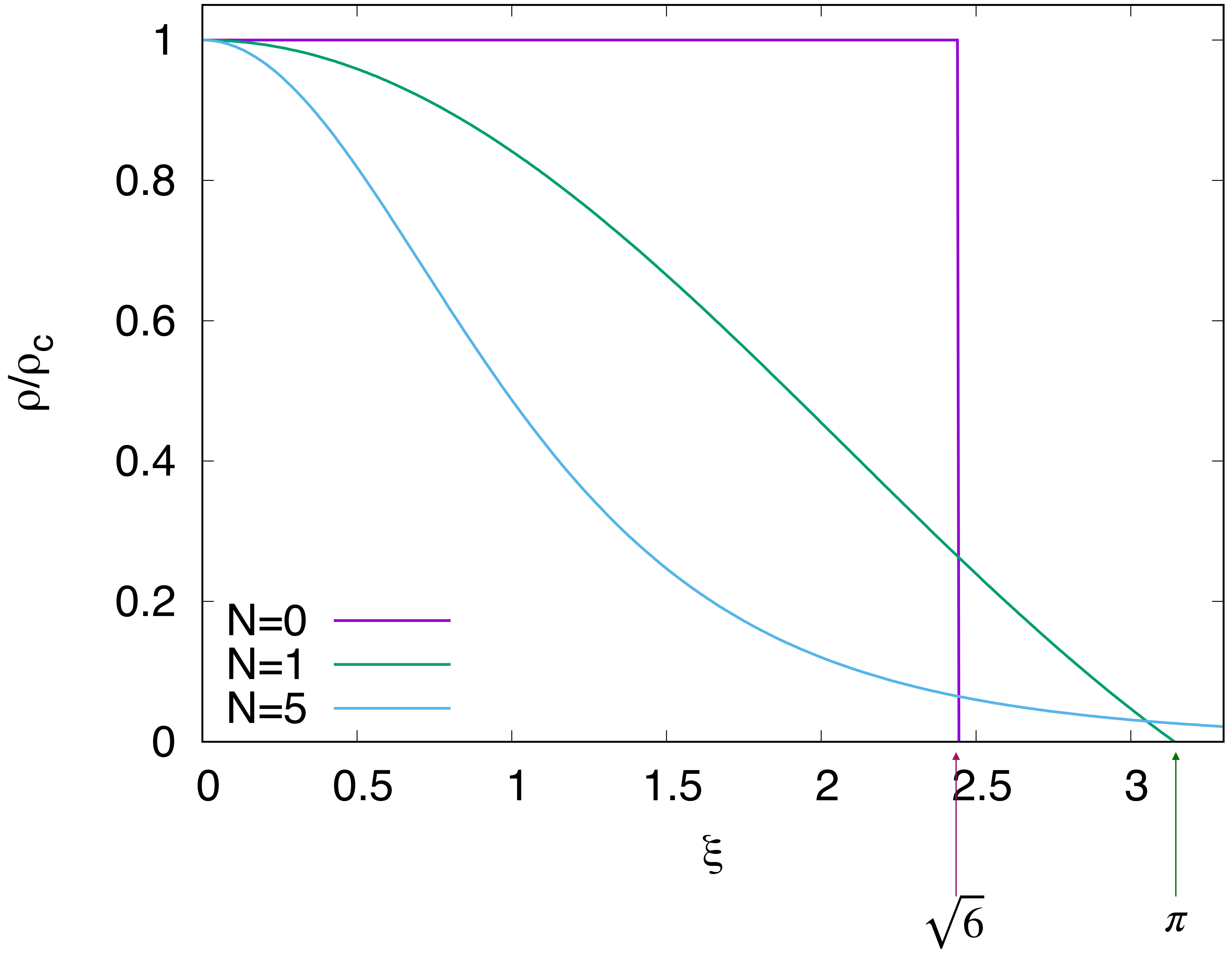
Lane-Emden Equation

Boundary condition: $\theta(0) = 1,\ \dfrac{d\theta(0)}{dr} = 0.$

Analytical solutions are available for particular N values (N=0, 1 and 5)

For N=0, $\theta(\xi) = 1 - \dfrac{1}{6}\xi^2$, For N=1, $\theta(\xi) = \dfrac{\sin\xi}{\xi}$, For N=5, $\theta(\xi) = \dfrac{1}{\sqrt{1+\xi^2/3}}$

Solution of Lane-Emden Equation



1. Conventionally, the equation of state is
   - hard (stiff) when Γ is large or N is small.
   - soft when Γ is small or N is large.
2. Density gradient with respect to ξ is
   - small for hard EoS.
   - large for soft EoS.
3. Recall that the EoS of relativistic degenerate gas (N=3) is softer than that of non-relativistic gas (N=1.5).
   - Relativistic degenerate gas can form more compact star than non-relativistic counterpart.
4. N=5 solution can extend to infinity while the total mass of the solution is finite.

Mass - Radius Relation

$$R = a\xi_s = \sqrt{\frac{(N+1)K}{4\pi G}}\rho_c^{\frac{1-N}{2N}}\xi_s$$

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi a^3 \rho_c \int_0^{\xi_s} \xi^2 \theta^N dr$$

$$= -4\pi a^3 \rho_c \int_0^{\xi_s} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi$$

$$= -4\pi a^3 \rho_c \xi_s^2 \left|\theta'\left(\xi_s\right)\right|$$

$$= 4\pi \left[\frac{(N+1)K}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-N}{2N}} \xi_s^2 \left|\theta'\left(\xi_s\right)\right|$$

$$M(R) = 4\pi R^{\frac{3-N}{1-N}} \left[\frac{(N+1)K}{4\pi G}\right]^{\frac{N}{N-1}} \xi_s^{\frac{5-3N}{1-N}} \left|\theta'\left(\xi_s\right)\right|$$

A star dominated by degenerate pressure becomes smaller as the mass of the star increases. This is the opposite of common sense that we generally know. (N<3).

Stability of Compact Star

1. Turning point method: A star is unstable with respect to any mode of radial oscillation when

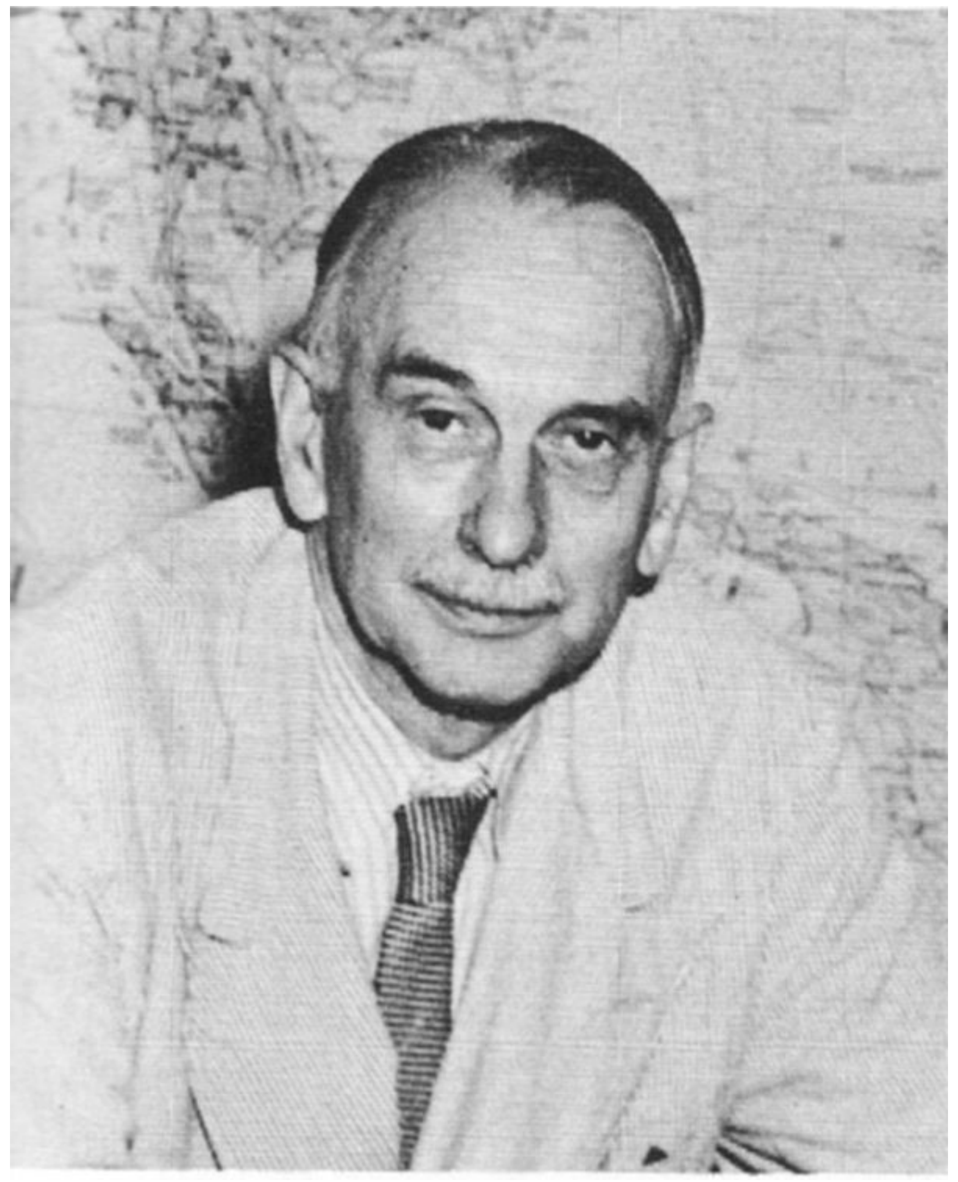
$$\frac{\partial M\left(\rho_c\right)}{\partial \rho_c} < 0. \text{ Or equivalently, } \frac{\partial M\left(R\right)}{\partial R} > 0.$$

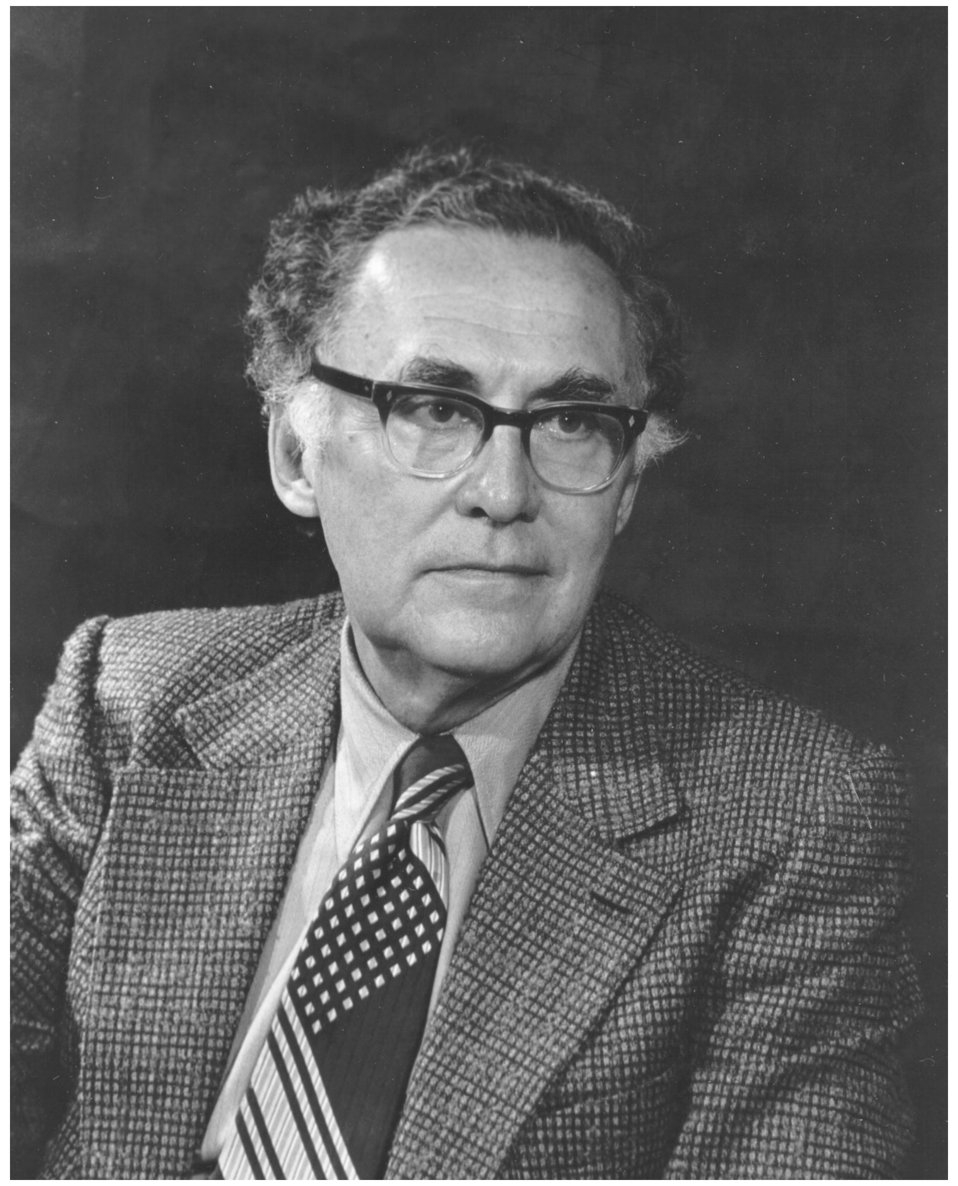
2. Linear stability analysis: This method linearizes the hydrodynamic equations and finds out oscillation frequency (ω^2). A star is unstable when

$$\omega^2 < 0,$$

where all the perturbation can be written as $\delta f(t, r) = \delta f(r) e^{i\omega t}$.

Equilibrium Structure in GR





T

Richard
Tolman

O

Robert
Oppenheimer

V

George
Volkoff

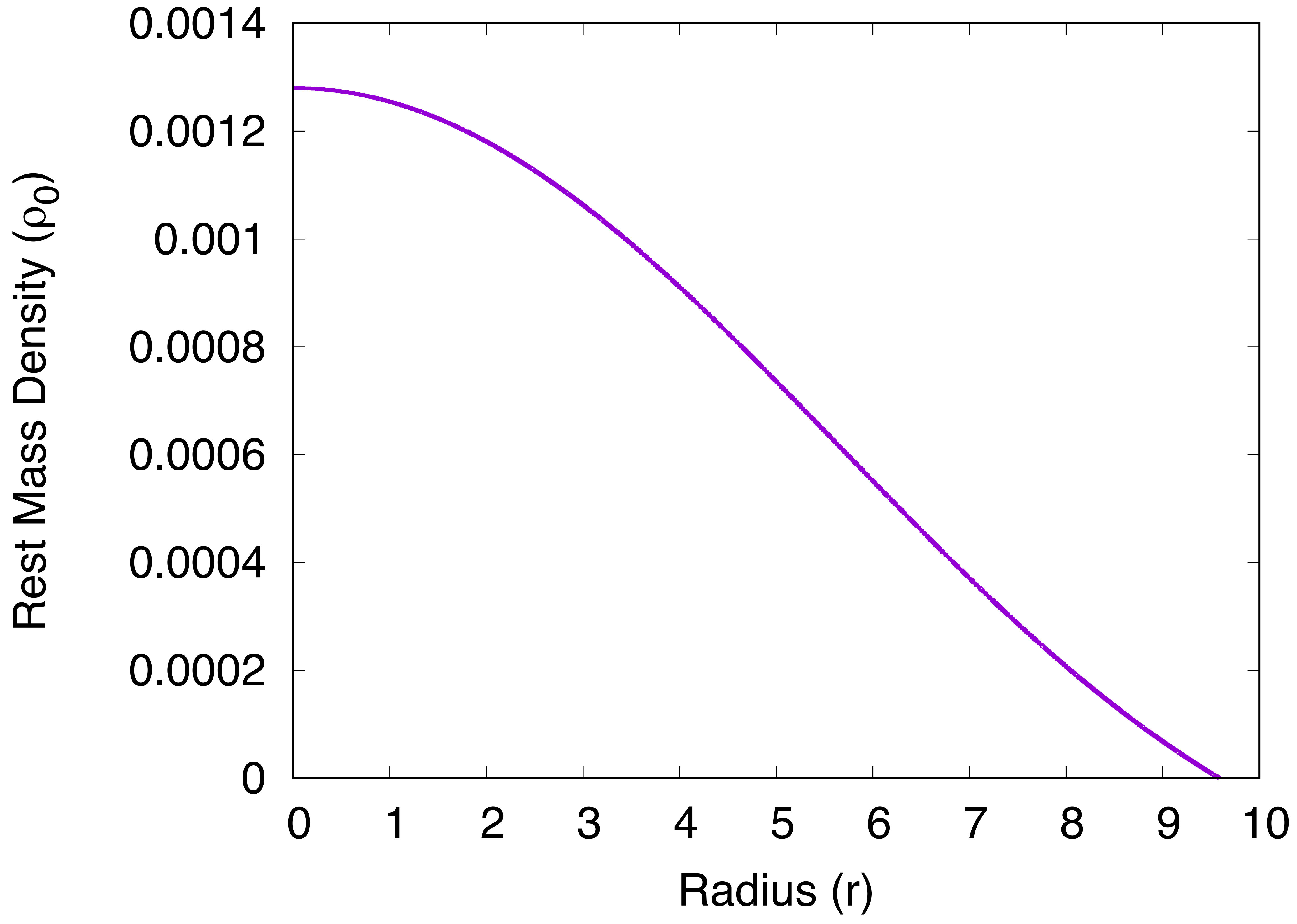
TOV Equation

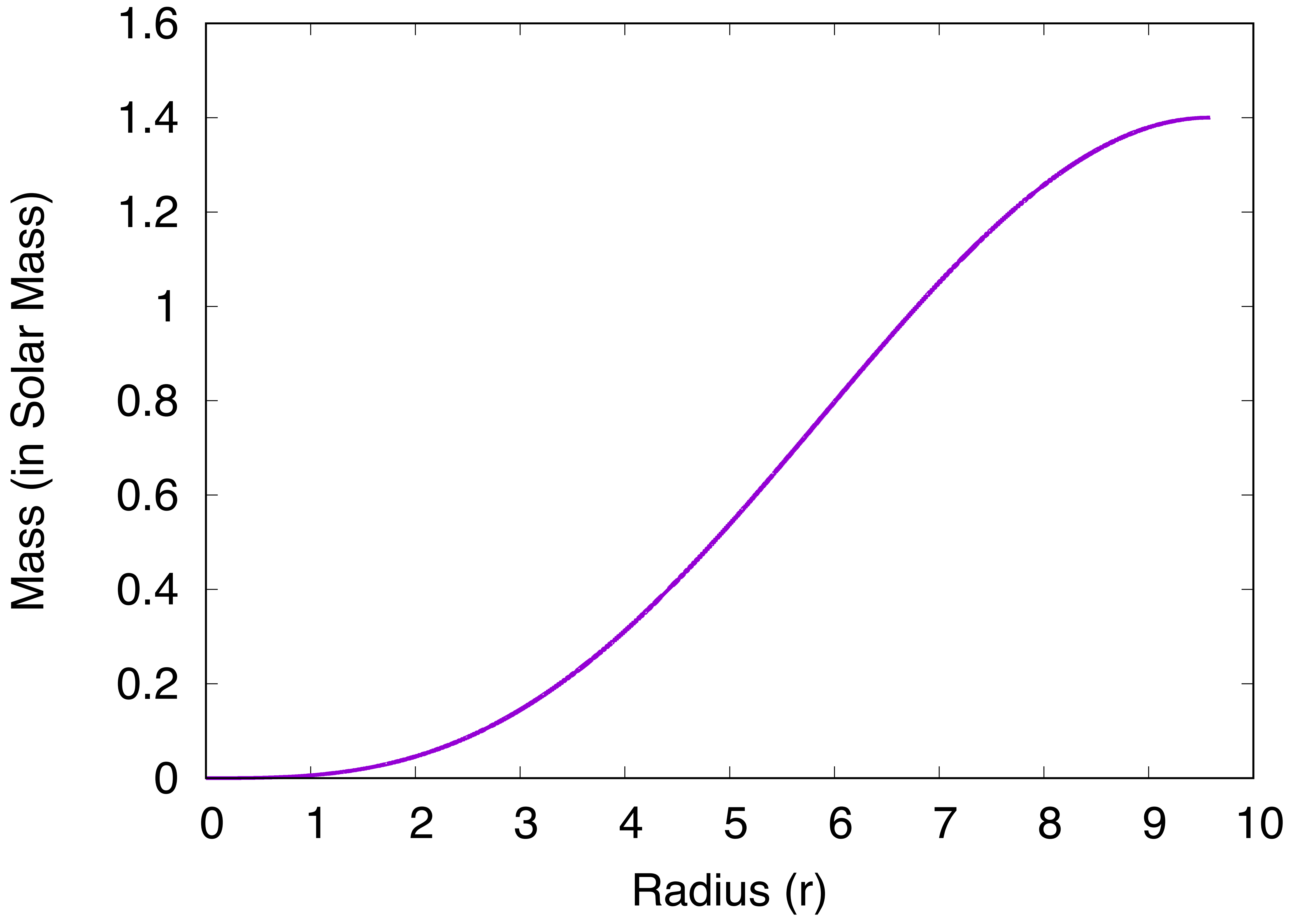
$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3 P}{mc^2}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

Solution of TOV Equation

Solution of TOV equation using polytropic equation of state with N=1, K=100.





Realistic EoS

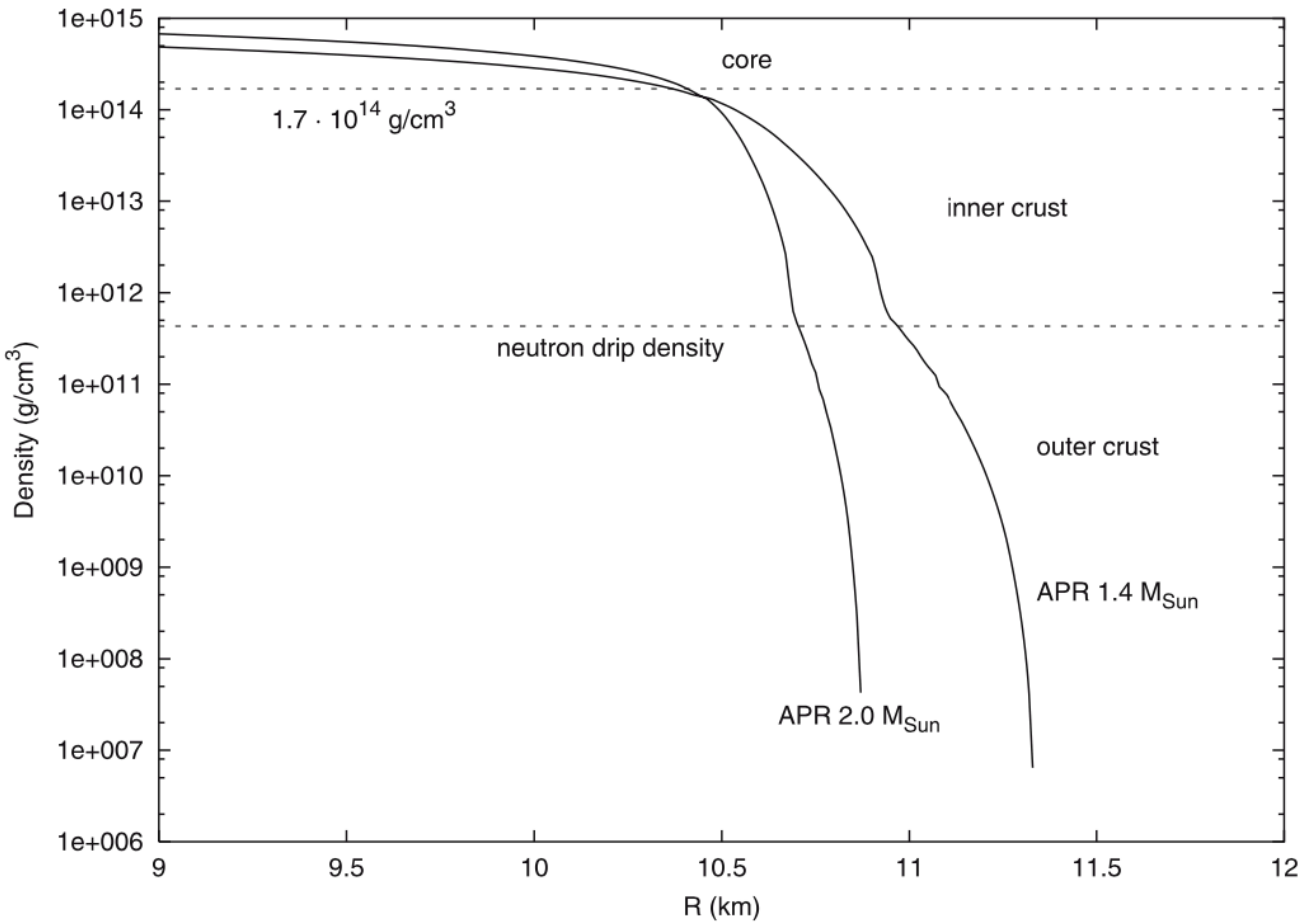
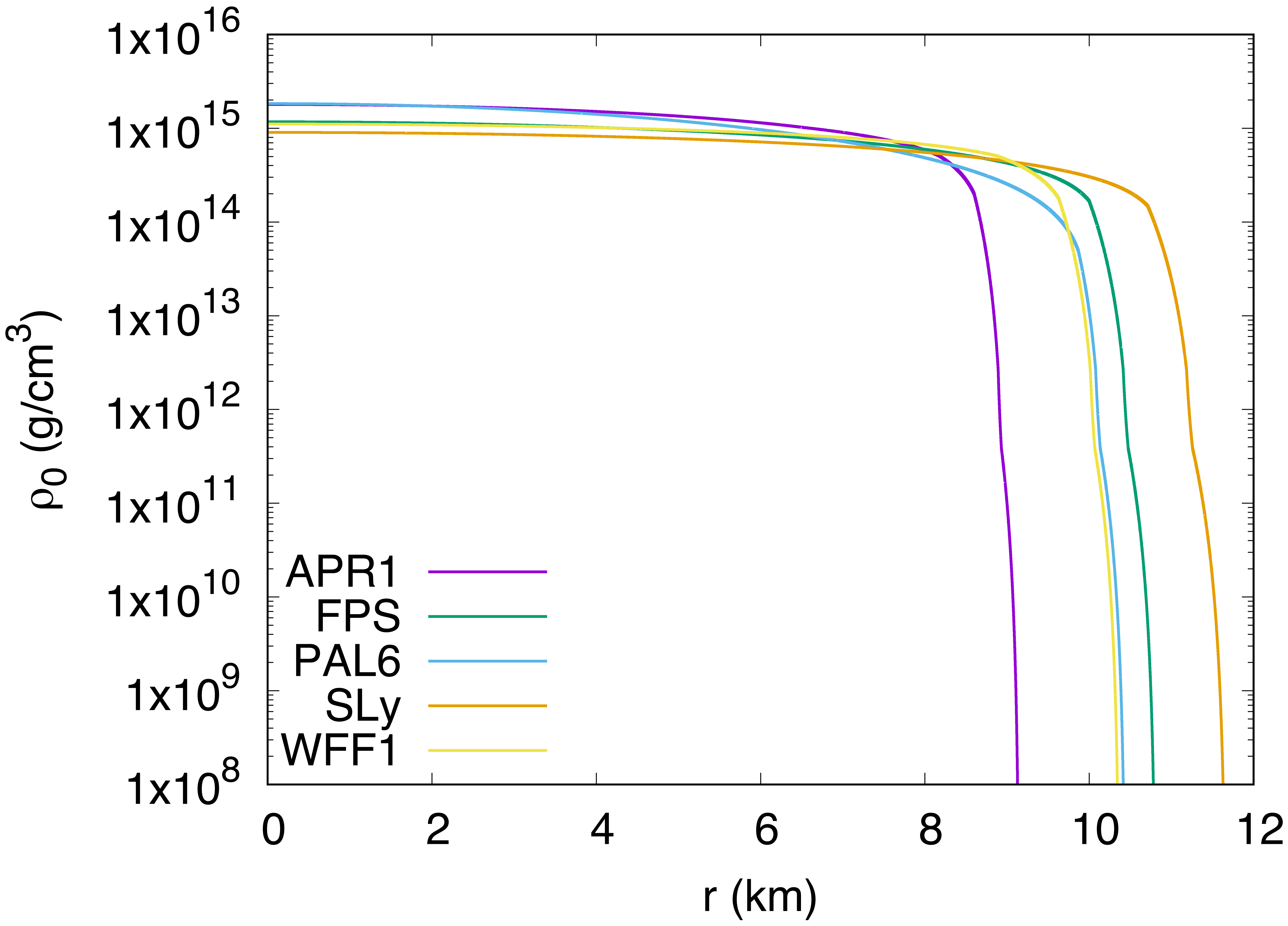
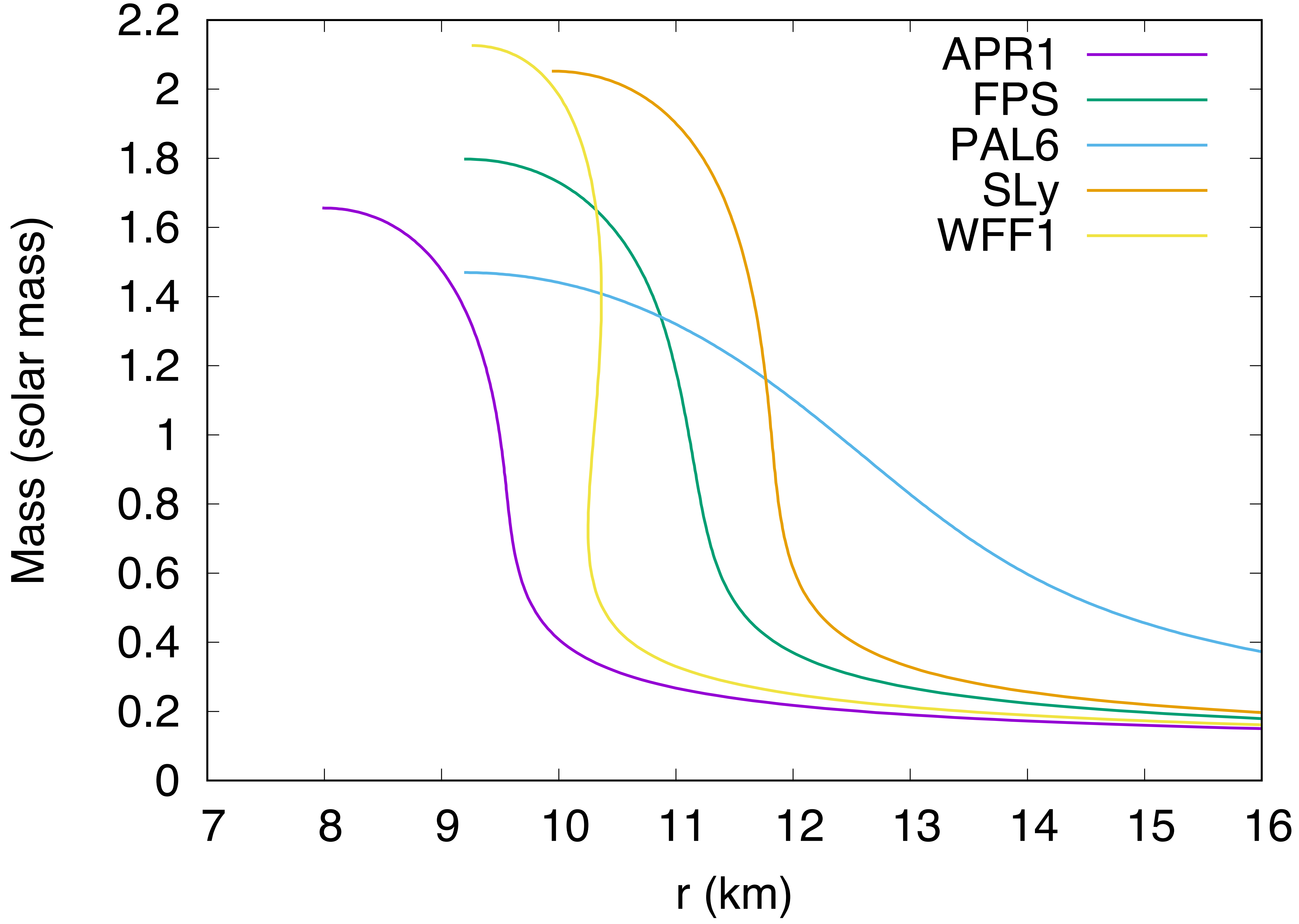


Image from Bauswein (2006)

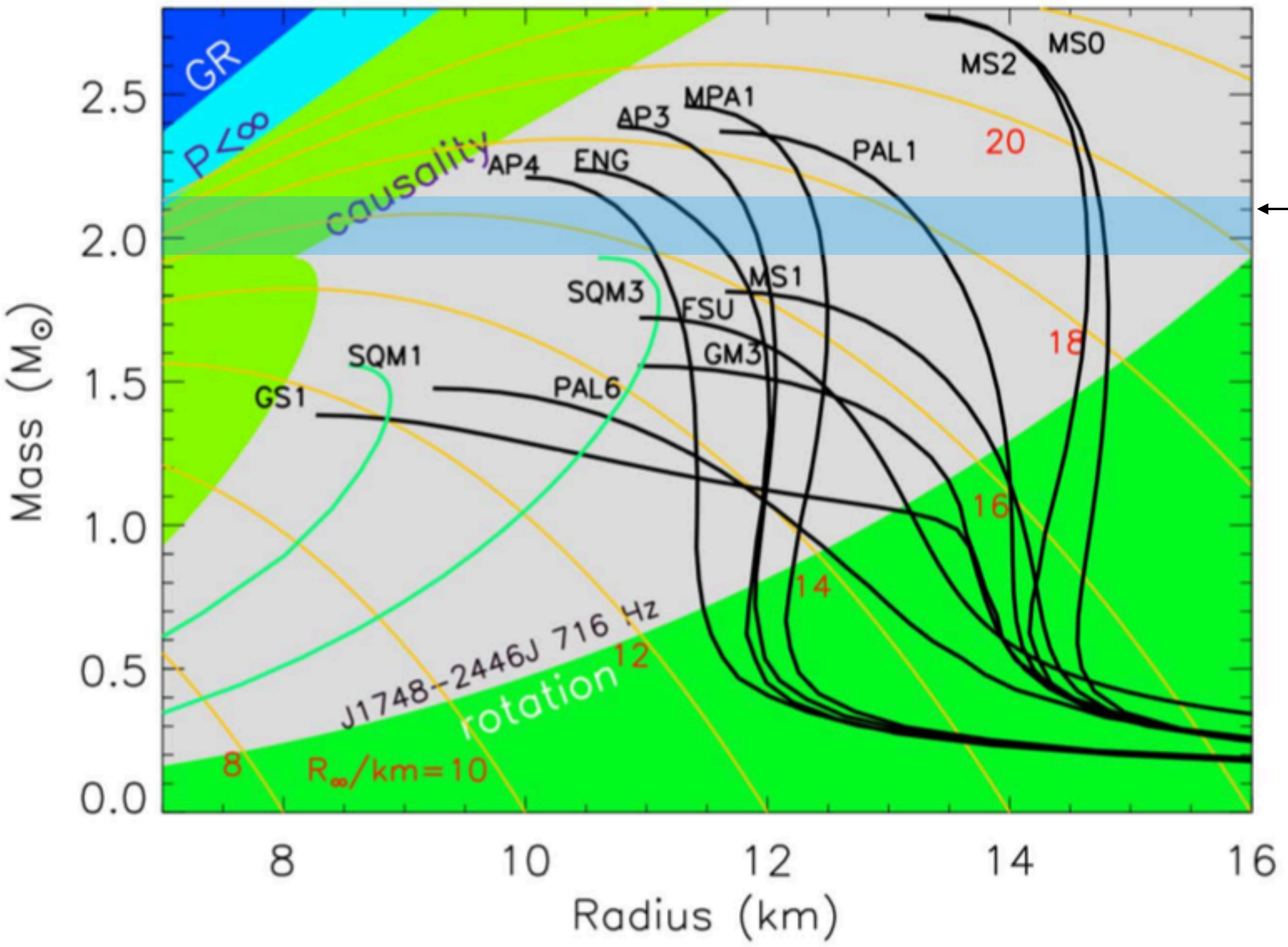
Different EoS

1.4 solar mass neutron star with various EoSs





M-R Relation in Various EoSs



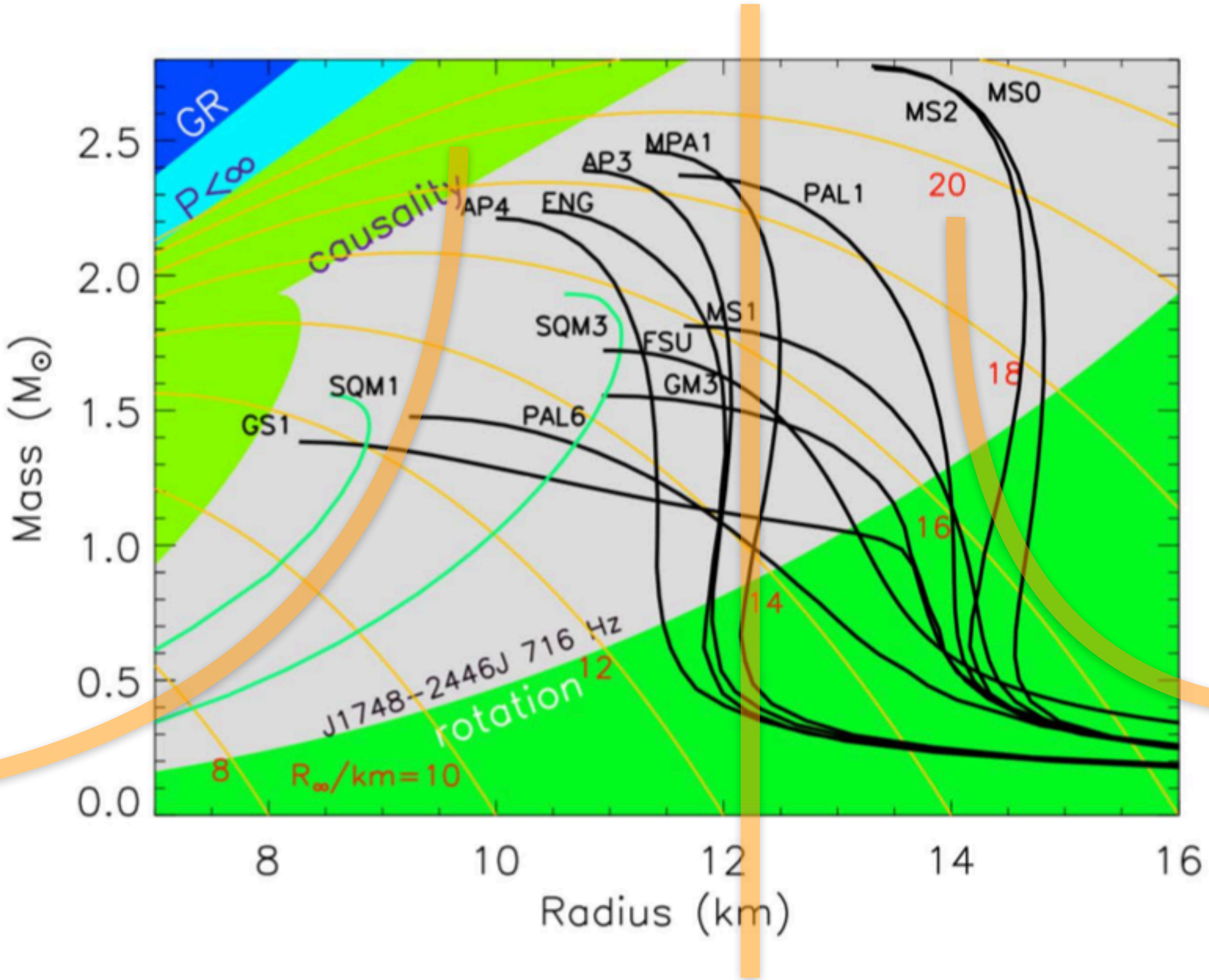
Maximum Mass of NS observed up to now

M-R Relation in Various EoSs

Recall that

$$M(R) = 4\pi R^{\frac{3-N}{1-N}} \left[\frac{(N+1)K}{4\pi G}\right]^{\frac{N}{N-1}} \xi_s^{\frac{5-3N}{1-N}} \left|\theta'(\xi_s)\right|$$

Vertical line follows N=1 polytope in Lane-Emden Solution



$N < 1$

$\Gamma > 2$

$1 < N < 3$

$4/3 < \Gamma < 2$

Models of Rapidly Rotating NS

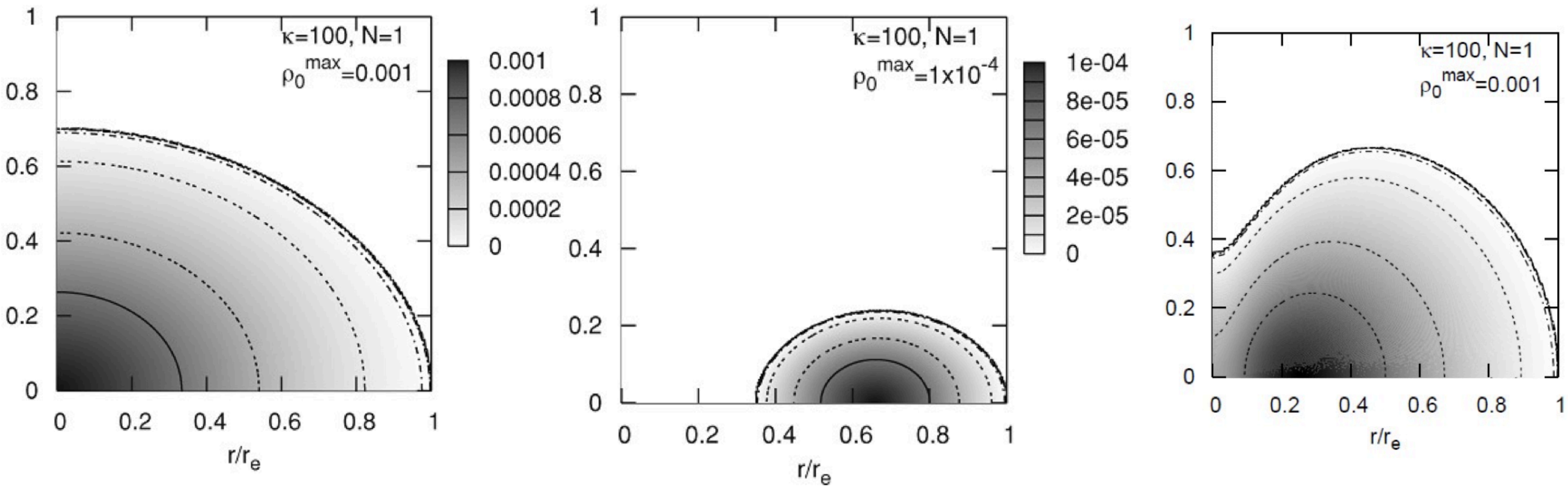
- Rotating star (2D structure) depends on r, z (or theta) coordinates.
  - Integral representation of equilibrium equation (see previous lecture and homework, it is also known as Self-Consistent Field Method).
  - Hachisu (1986): Newtonian

$$H + \Phi - \int \Omega^2 R dR = C.$$

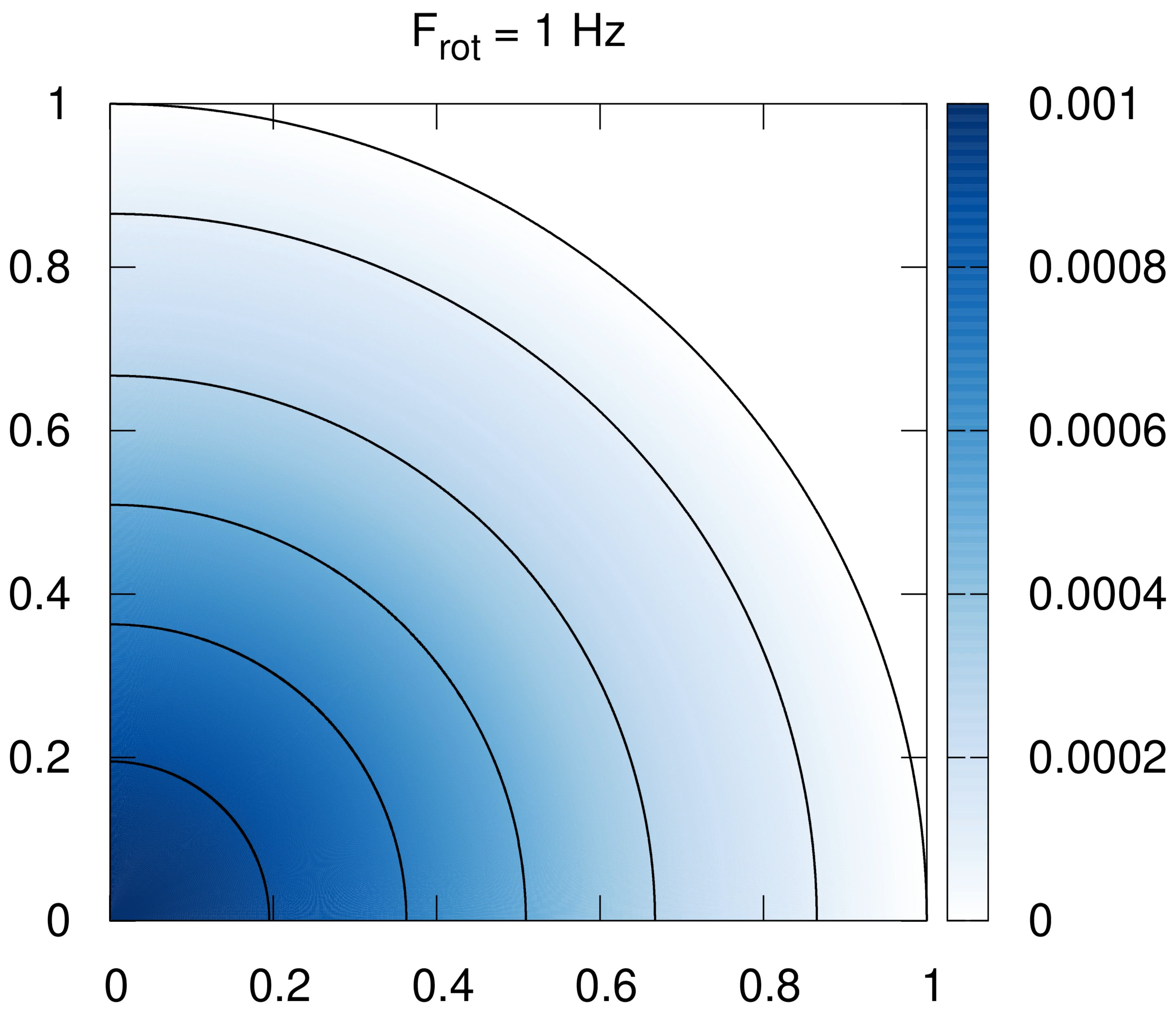
  - Komatsu, Eriguchi & Hachisu (KEH, 1989), Cook, Shapiro & Teukolsky (CST, 1992): GR

Metric: $ds^2 = -e^{2\nu}dt^2 + e^{2\alpha}\left(dr^2 + r^2 d\theta^2\right) + e^{2\beta}r^2\sin^2\theta\left(d\phi - \omega dt\right)^2,$

Integral equation: $\ln H + \nu + \ln\left(1 - v^2\right) + \int v^2 \dfrac{d\Omega}{\Omega - \omega}.$



Limits in Rotation Speed



- In SCF method, rotation speed is to be controlled by axis ratio (ratio between the radius at equator and that at rotation axis).

- One can obtain more rapidly rotating stars by decreasing the axis ratio. No equilibrium body can be found when

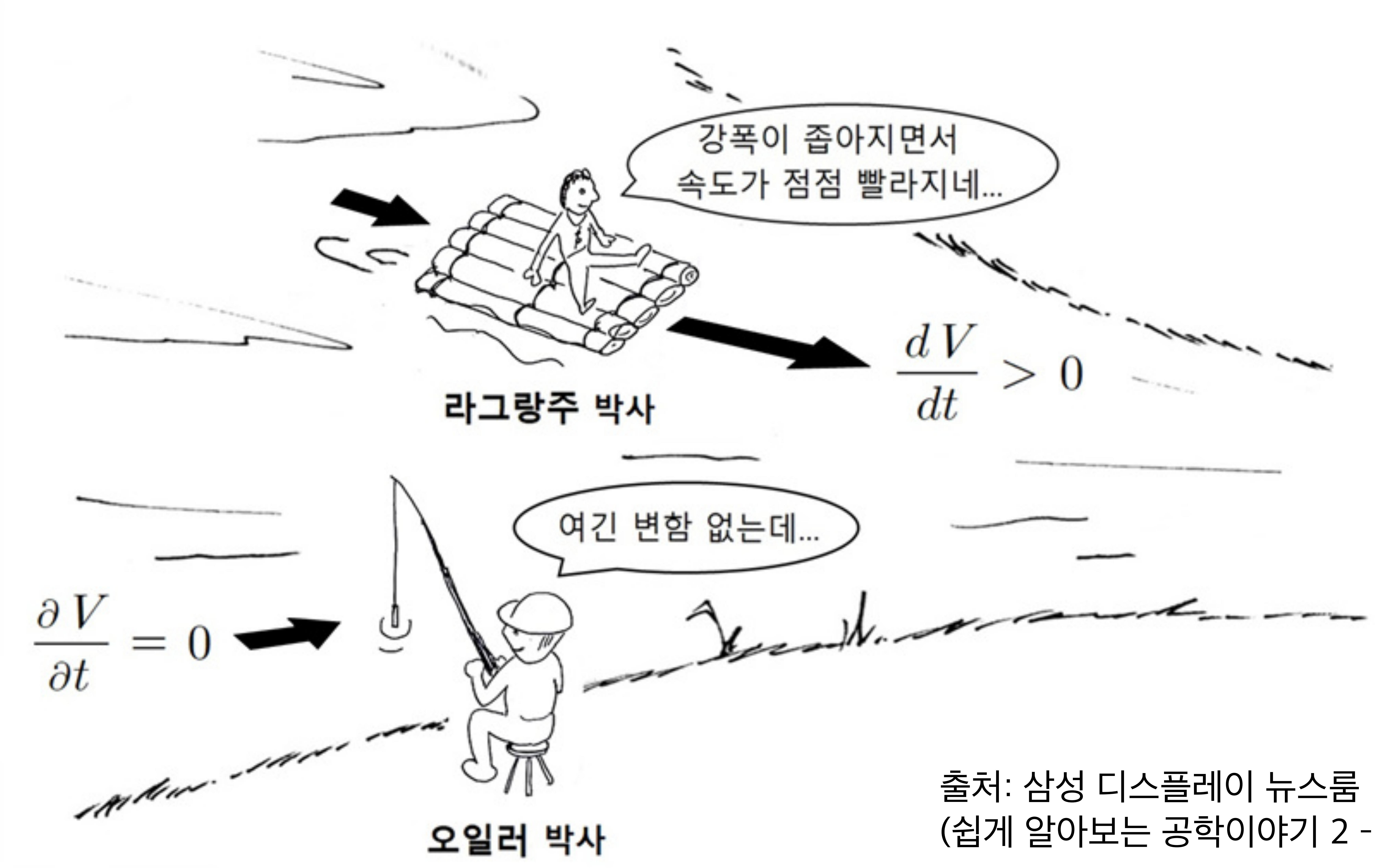
$$\Omega > \Omega_K \,,$$

where Ω_K is the Keplerian angular speed. It is also known as mass shedding limit.

- Maximum rotation limit of the model also gives a constraint of EoS.

Dynamics

Two Approaches in Hydrodynamics



출처: 삼성 디스플레이 뉴스룸
(쉽게 알아보는 공학이야기 2 - 유체역학 편)

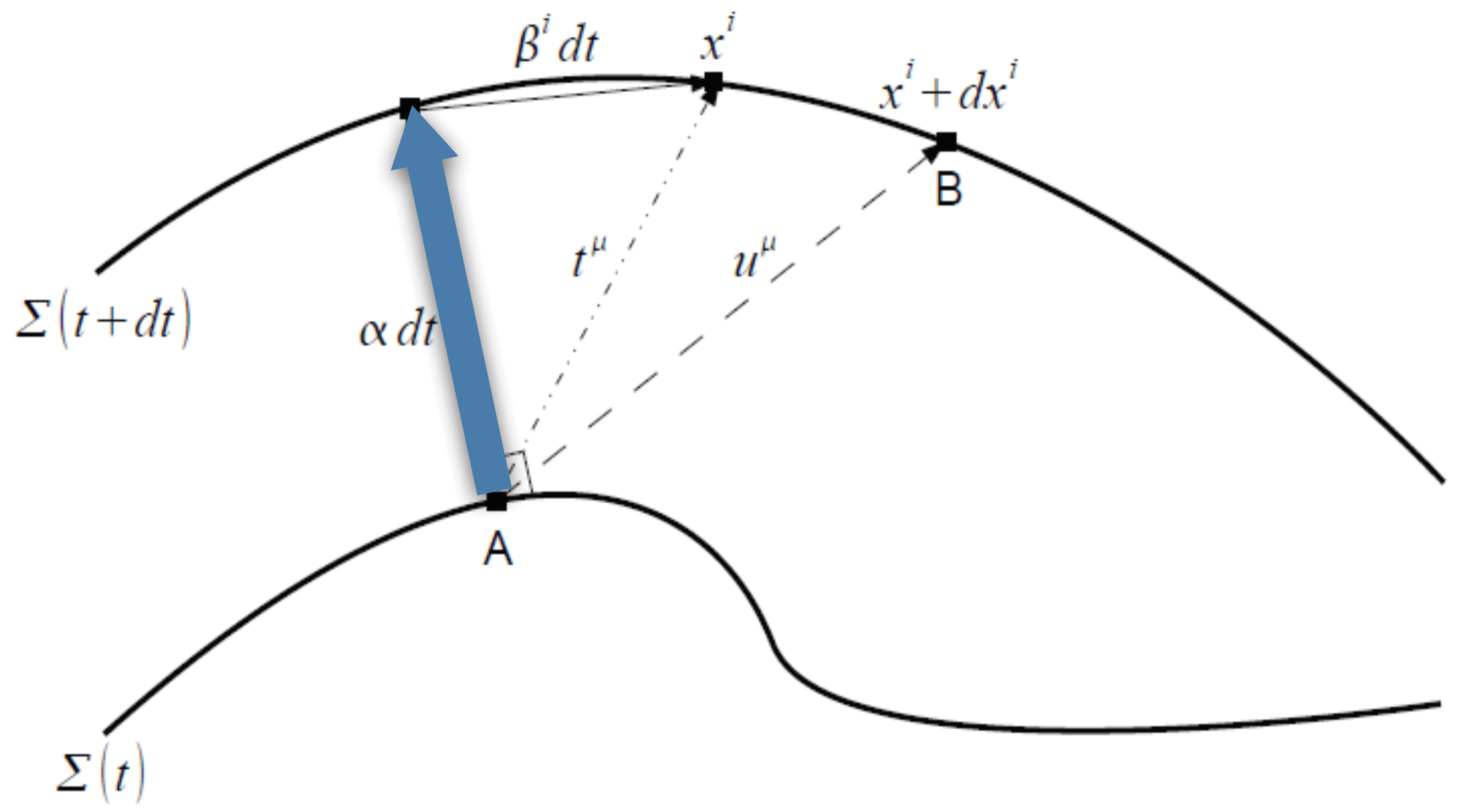
Eulerian Observer

- Eulerian observer stays at rest in Newtonian hydrodynamics.

- In general relativity (3+1 formalism), Eulerian observer is an observer who moves normal to the hypersurface.
  - Zero Angular Momentum Observer (ZAMO) is an example of the Eulerian observer in Kerr geometry or rotating star.

- The 3-velocity of the fluid measured by the Eulerian observer can be expressed as a ratio between spatial projection to normal projection of the 4-velocity.

$$v^i = \frac{\gamma^i_\mu u^\mu}{-n_\mu u^\mu} = \frac{u^i}{\alpha u^t} + \frac{\beta^i}{\alpha}$$



현영환 박사님 강의 참고

Hydrodynamics in General Relativity

The general relativistic (magneto-)hydrodynamics equations consist of the local conservation laws of the the matter current density and the stress energy tensor (Bianchi identity, energy & momentum conservation).

$$\nabla_\mu J^\mu = 0$$

$$\nabla_\mu T^{\mu\nu} = 0$$

Baryon number conservation or total mass conservation

$$\nabla_\mu \left(\rho u^\mu\right) = 0$$

Spatial projection gives momentum conservation equation

$$\gamma_i^\nu \nabla_\mu T_\nu^\mu = 0$$

Normal projection gives energy conservation equation

$$n^\nu \nabla_\mu T_\nu^\mu = 0$$

Flux Conservative Form of Governing Equation

Hydrodynamic equation in 3+1 formalism.
It is also known as Valencia formulation (Banyuls et al. 1997).

$$\frac{\partial\left(\sqrt{\gamma}U\right)}{\partial t} + \frac{\partial\left(\sqrt{-g}F^i\right)}{\partial x^i} = \sqrt{-g}\Sigma$$

$$U = \begin{pmatrix} D \\ S_j \\ \tau \end{pmatrix} = \begin{pmatrix} \rho W \\ \rho h W^2 v_j \\ \rho h W^2 - P - D \end{pmatrix}$$

$$F^i = \begin{bmatrix} D\left(v^i - \beta^i/\alpha\right) \\ S_j\left(v^i - \beta^i/\alpha\right) + P\delta^i_j \\ \tau\left(v^i - \beta^i/\alpha\right) + Pv^i \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0 \\ T^{\mu\nu}\left(\partial_\mu g_{\nu j} - \Gamma^\lambda_{\mu\nu} g_{\lambda j}\right) \\ \alpha\left(T^{\mu t}\partial_\mu(\ln\alpha) - T^{\mu\nu}\Gamma^t_{\mu\nu}\right) \end{bmatrix}$$

Mass conservation (continuity) equation

Momentum conservation equation

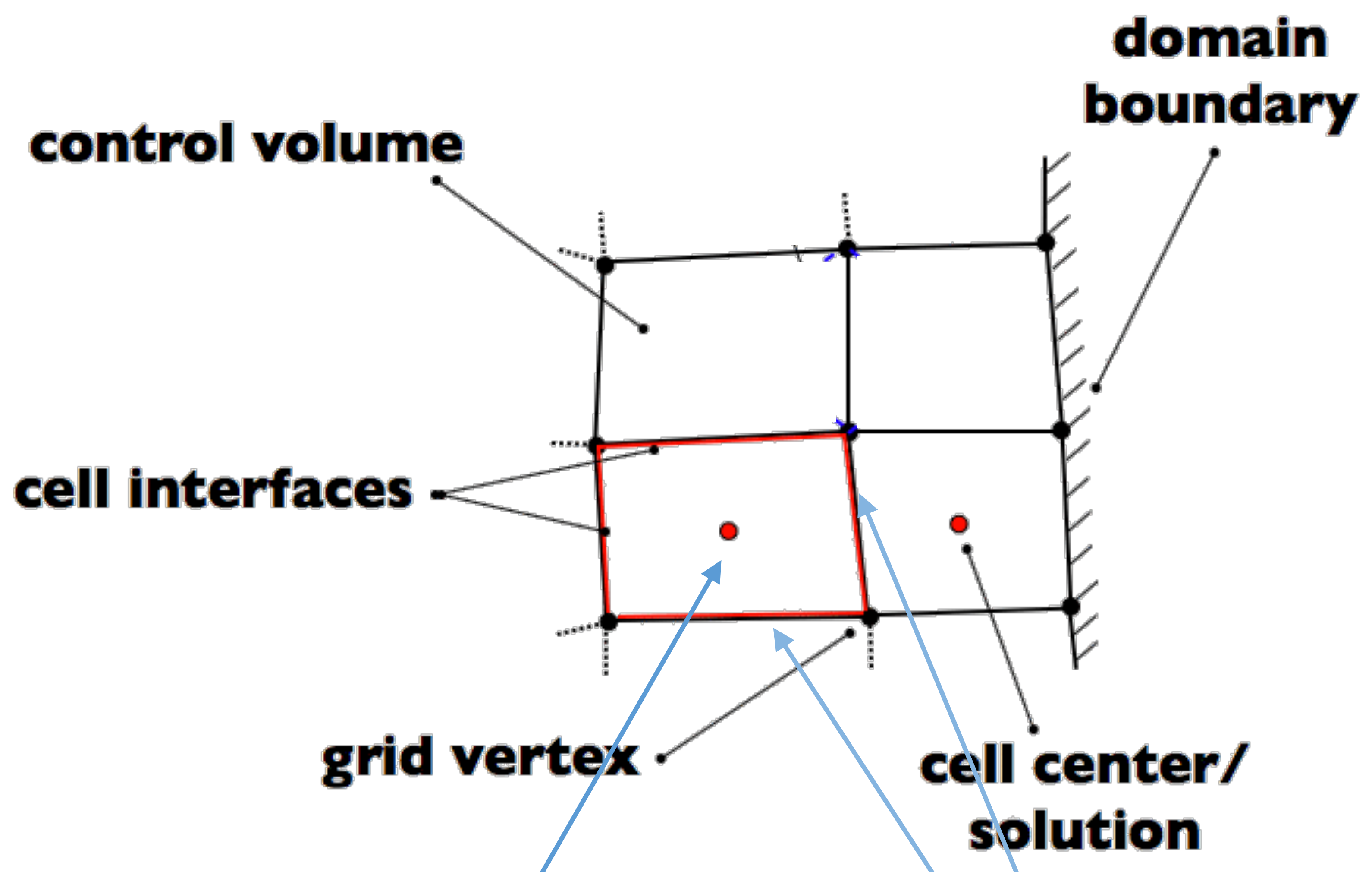
Energy equation

Numerical Solution Strategy: Finite Volume Method

Finite volume method enforces the **local conservation** of the fluid conservative quantities in an control volume.

The conservative quantities on each grid represent the **volume averaged quantities**.

Fluxes are evaluated on the face of the mesh (interface between the control volume).



$$\bar{U} = \frac{\int_{\Delta V} \sqrt{\gamma} U d^3x}{\int_{\Delta V} \sqrt{\gamma} d^3x}$$

$\bar{F}^i$

Solution Strategy

Conservative form of the equation

$$\frac{\partial\left(\sqrt{\gamma}U\right)}{\partial t}+\frac{\partial\left(\sqrt{-g}F^i\right)}{\partial x^i}=\sqrt{-g}\Sigma$$

Volume integration over time and spatial volume gives:

$$\int_{\Delta V^{(4)}}\frac{\partial\left(\sqrt{\gamma}U\right)}{\partial t}dV^{(4)}+\int_{\Delta V^{(4)}}\frac{\partial\left(\sqrt{-g}F^i\right)}{\partial x^i}dV^{(4)}=\int_{\Delta V^{(4)}}\sqrt{-g}\Sigma dV^{(4)}$$

Final form of the numerically adapted equation

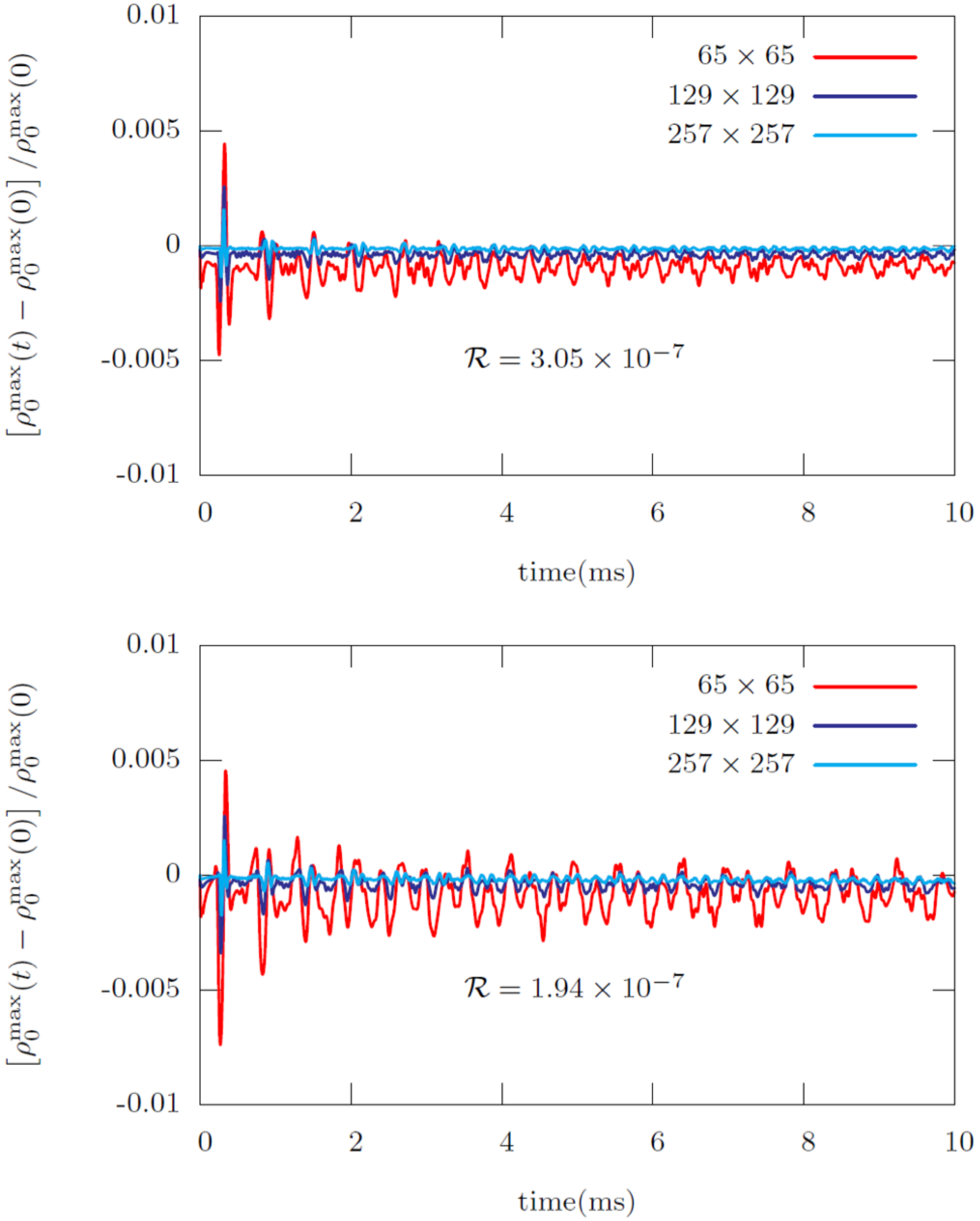
$$\begin{aligned}
&\bar{U}\Delta V^{(3)}\Big|_{x^0+\Delta x^0}-\bar{U}\Delta V^{(3)}\Big|_{x^0}= \\
&-\left(\int_{\Sigma x^1+\frac{\Delta x^1}{2}}\sqrt{-g}F^1dx^0dx^2dx^3-\int_{\Sigma x^1-\frac{\Delta x^1}{2}}\sqrt{-g}F^1dx^0dx^2dx^3\right) \\
&-\left(\int_{\Sigma x^2+\frac{\Delta x^2}{2}}\sqrt{-g}F^1dx^0dx^1dx^3-\int_{\Sigma x^2-\frac{\Delta x^2}{2}}\sqrt{-g}F^1dx^0dx^1dx^3\right) \\
&-\left(\int_{\Sigma x^3+\frac{\Delta x^3}{2}}\sqrt{-g}F^1dx^0dx^1dx^2-\int_{\Sigma x^3-\frac{\Delta x^3}{2}}\sqrt{-g}F^1dx^0dx^1dx^2\right) \\
&+\int_{\Delta V^{(4)}}\sqrt{-g}\Sigma dV^{(4)}
\end{aligned}$$

여백이 부족하여
더 이상의 자세한 설명은 생략한다…

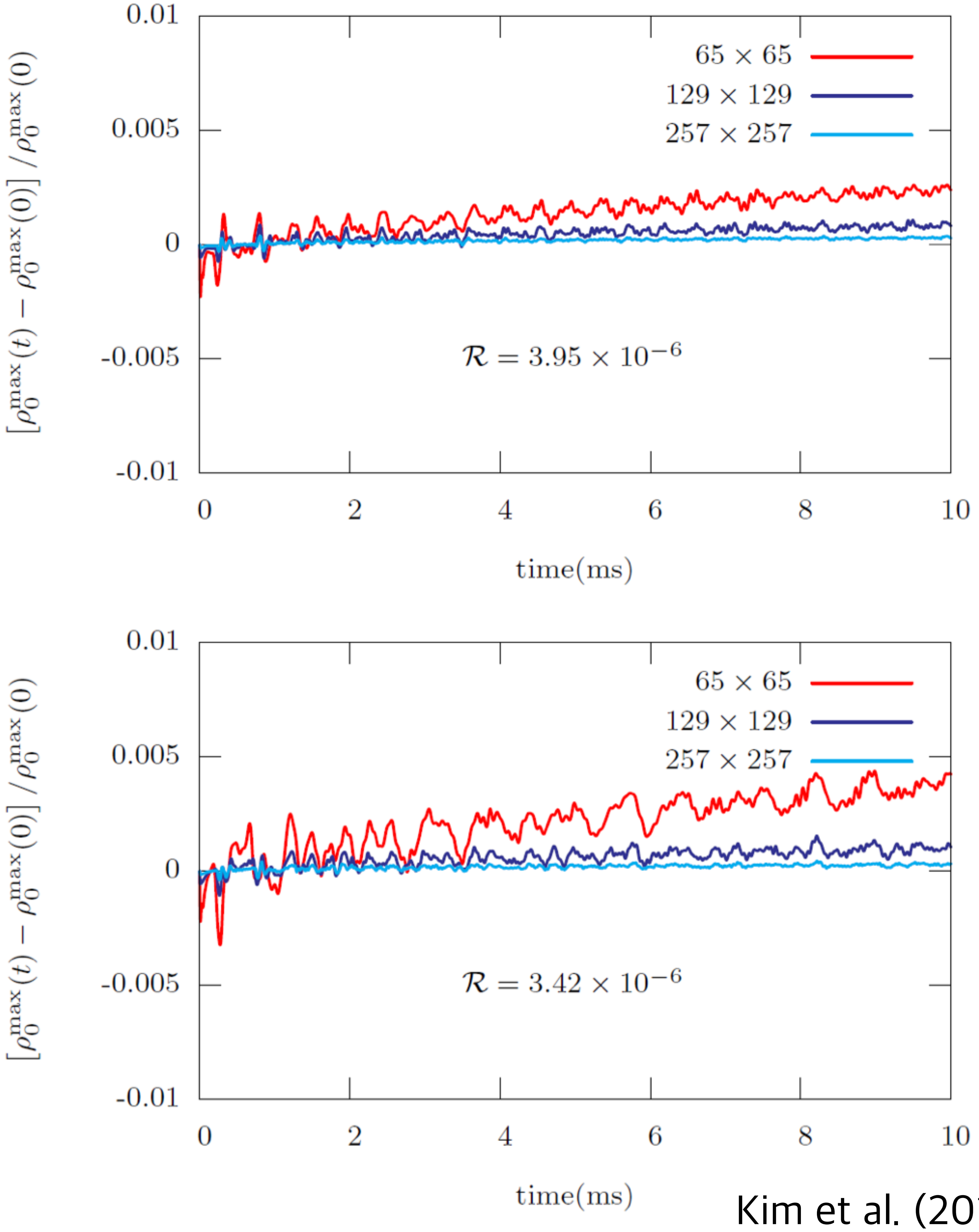
Numerical Simulation

Stationary Star

Non-rotating star



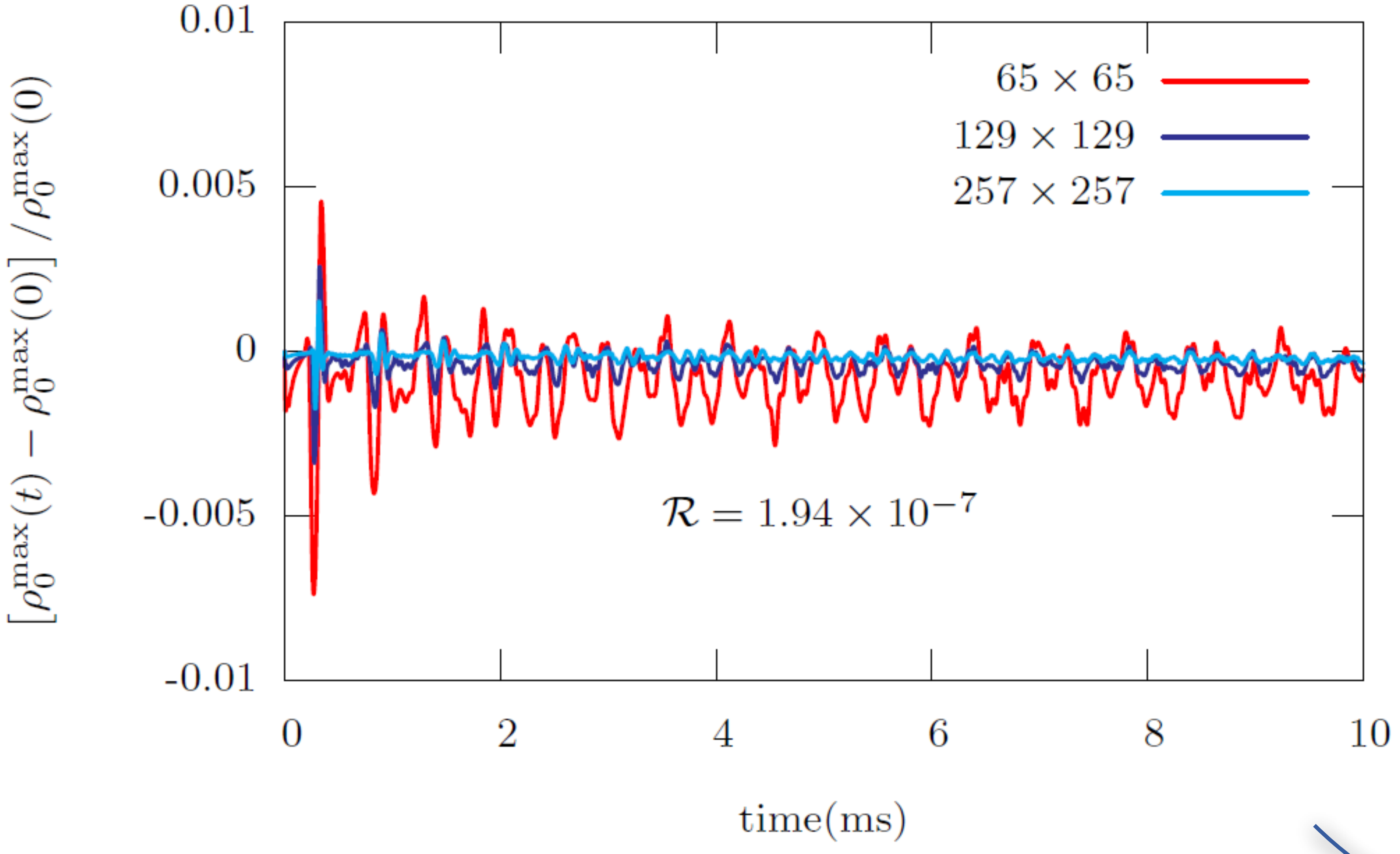
Rotating star

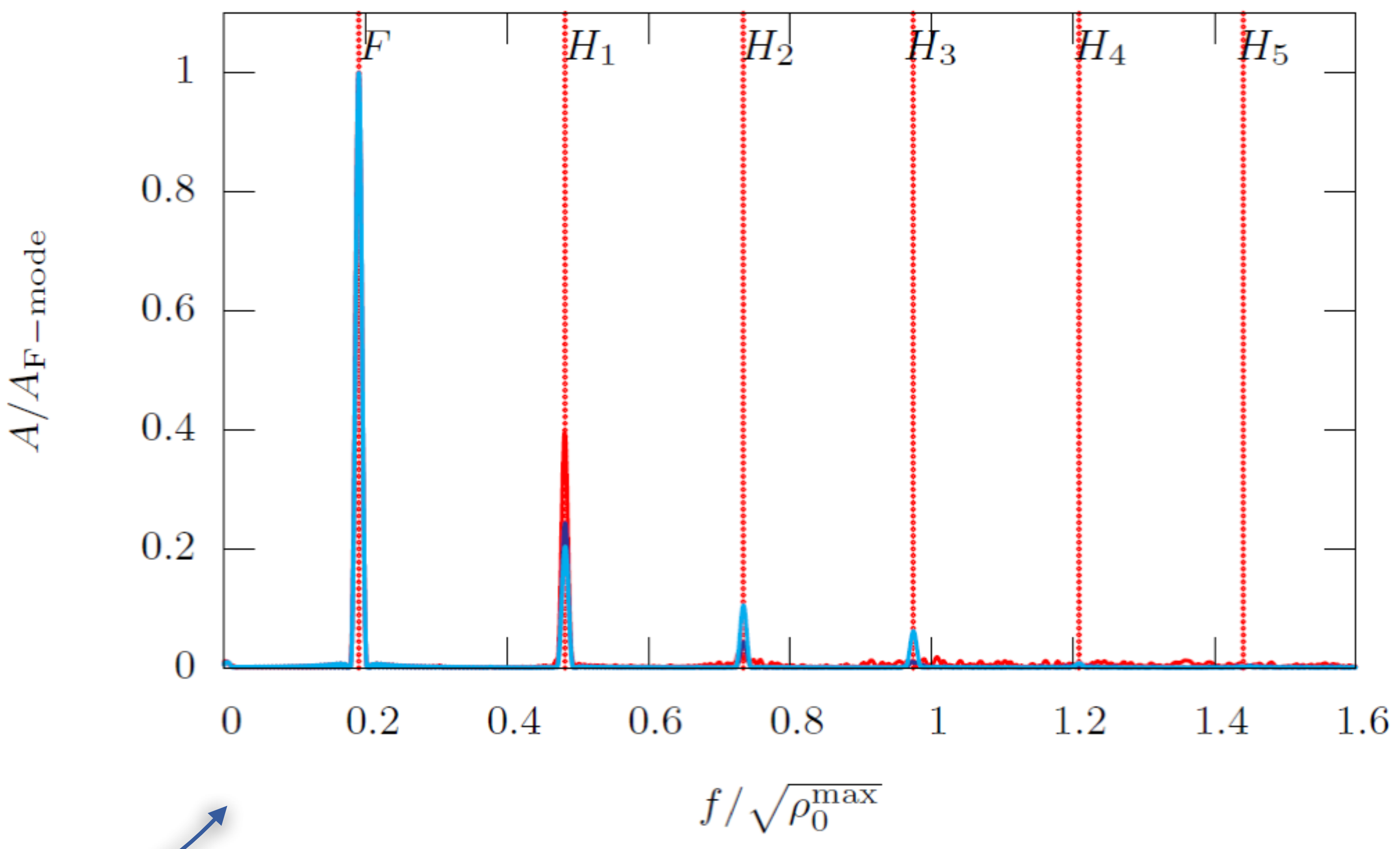


Kim et al. (2012)

Stationary Pulsation Mode Test

Radial oscillation modes of the spherical star

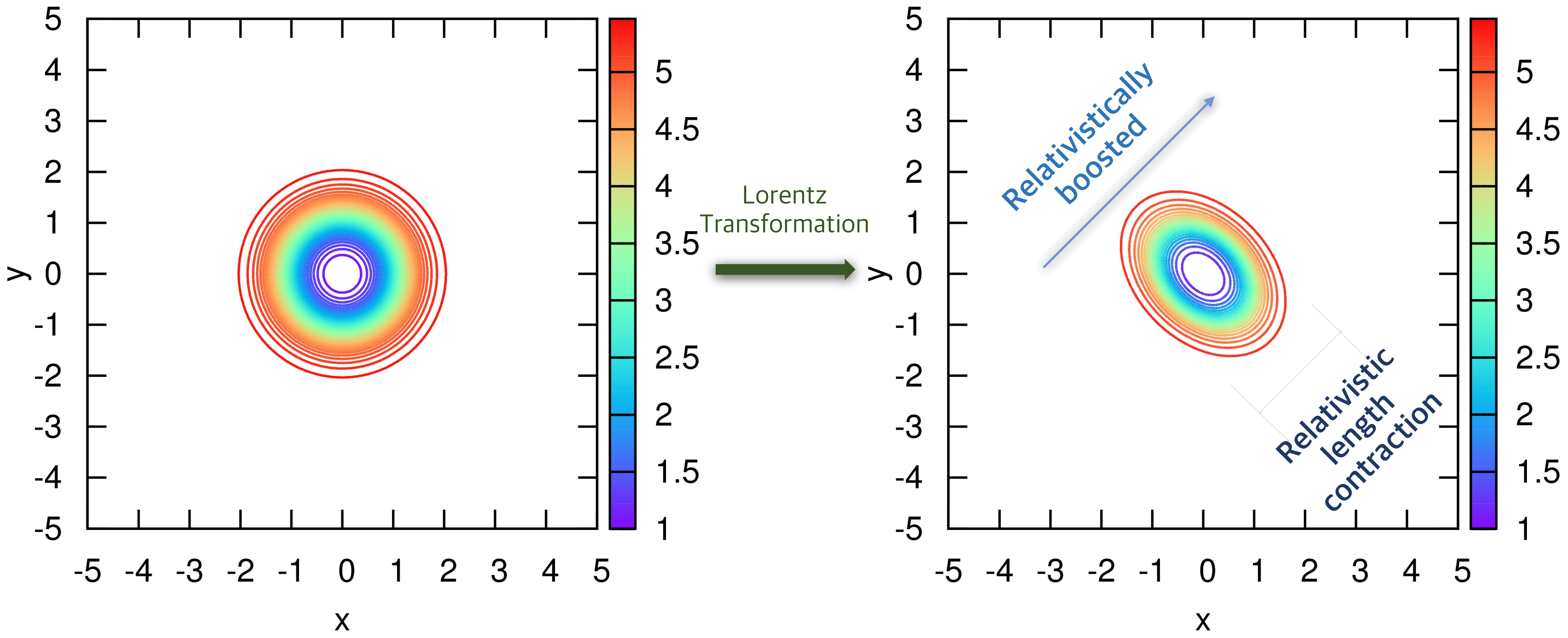




Fourier Transformation

Kim et al. (2012)

Structure of the Relativistic Vortex



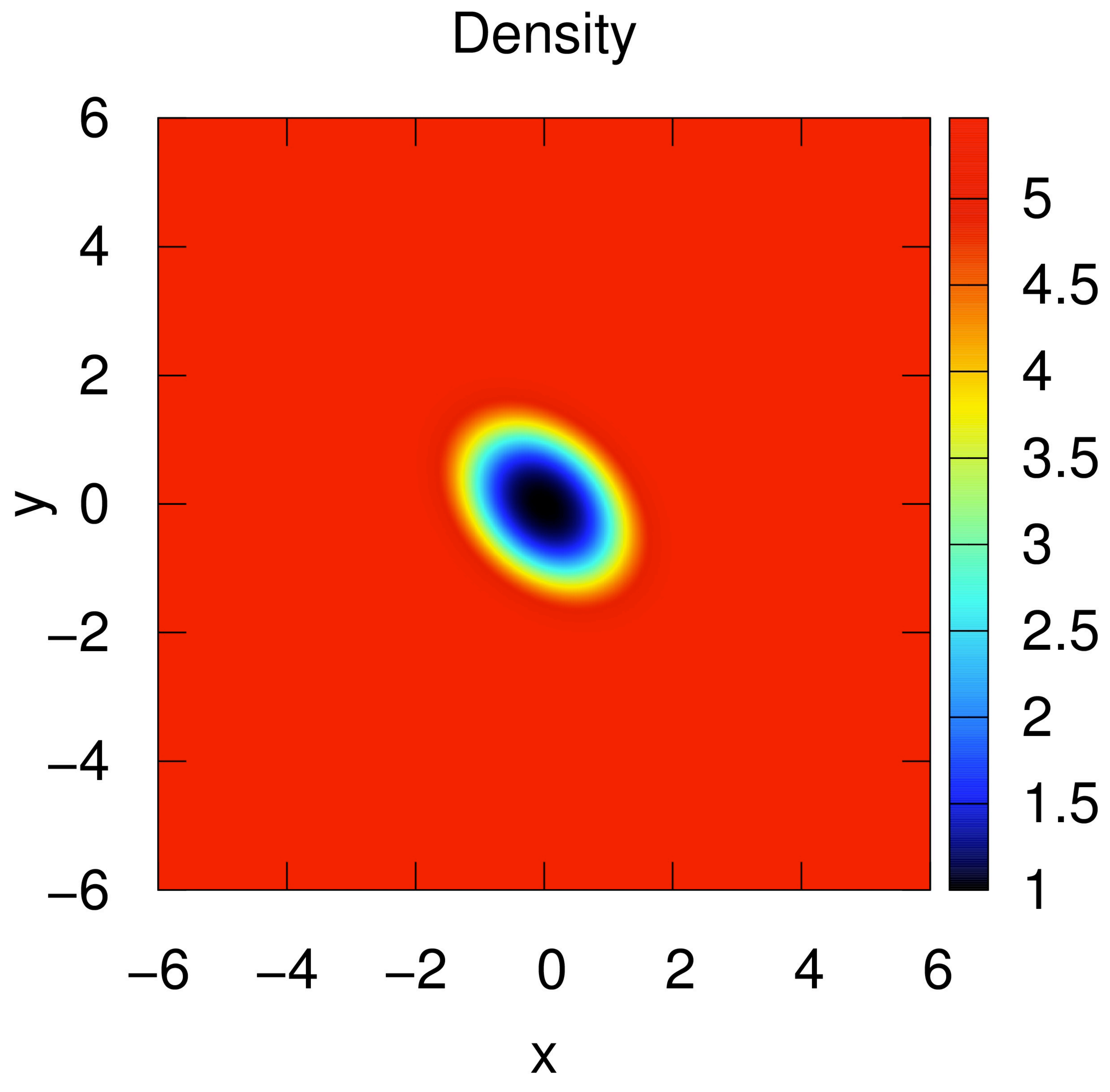
Density profile of the RMHD vortex in the rest frame of the vortex

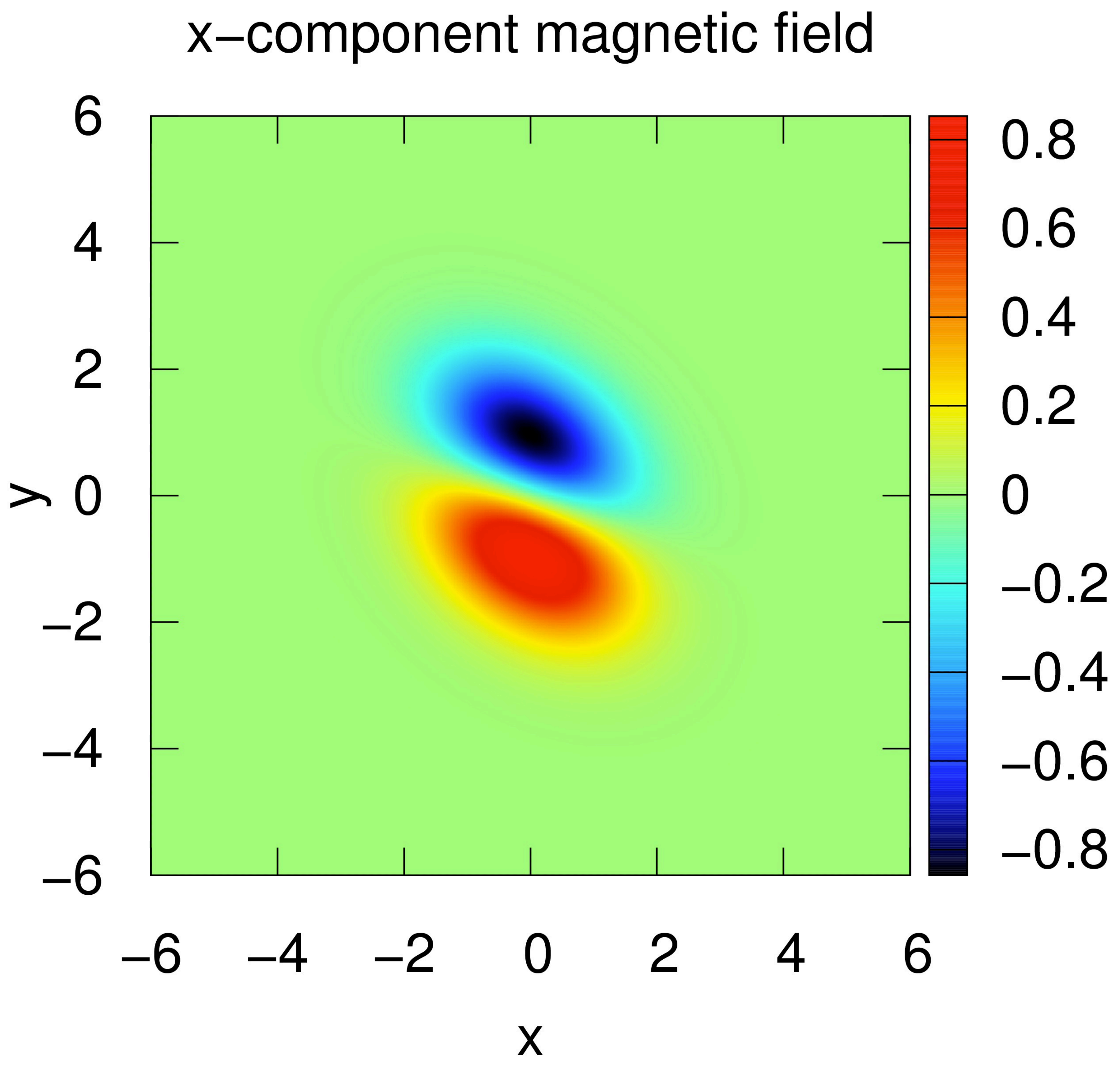
Density profile of the boosted RMHD vortex

Balsara & Kim (2016)

RMHD Vortex Movie

Time = 0.0

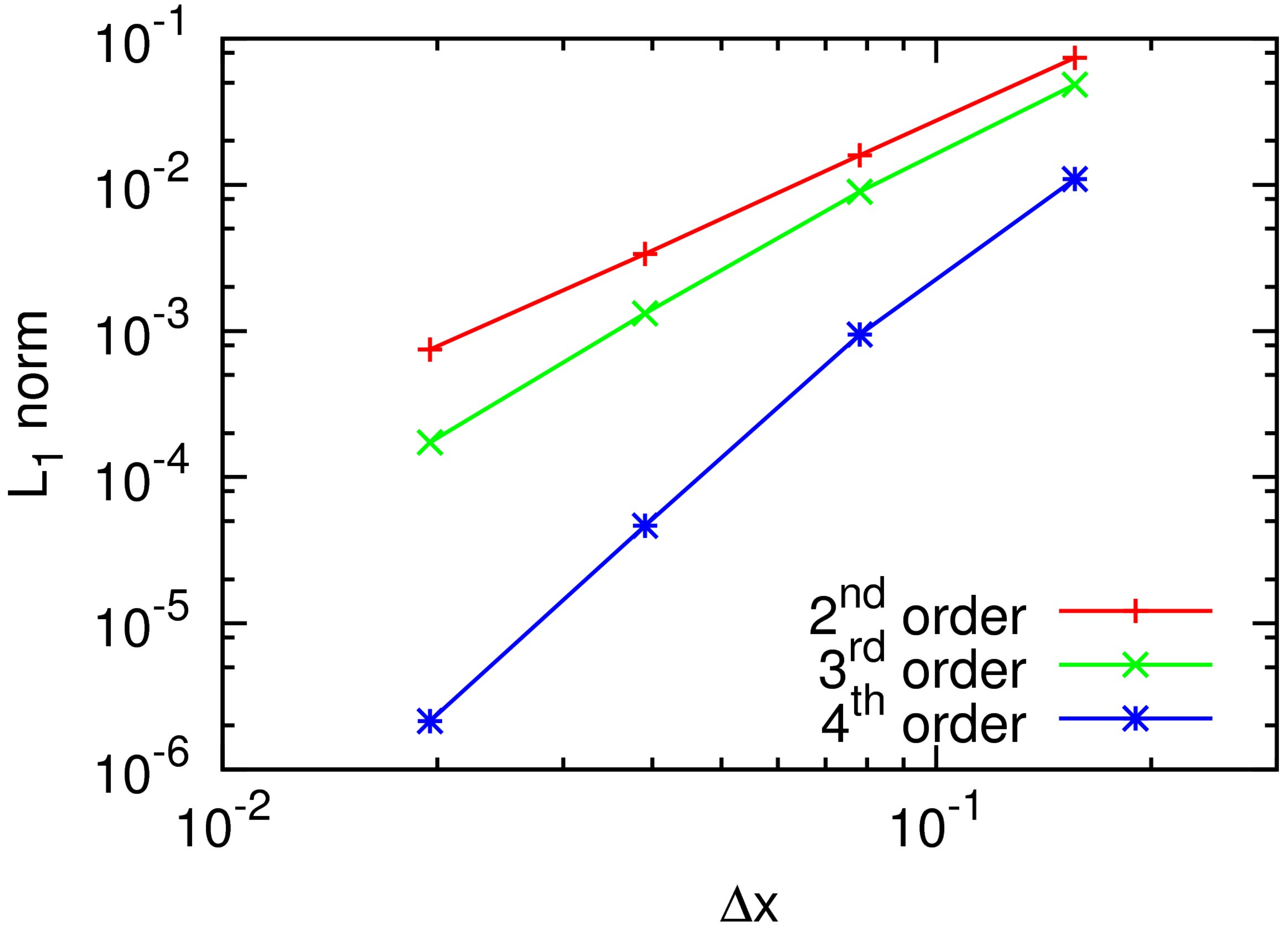




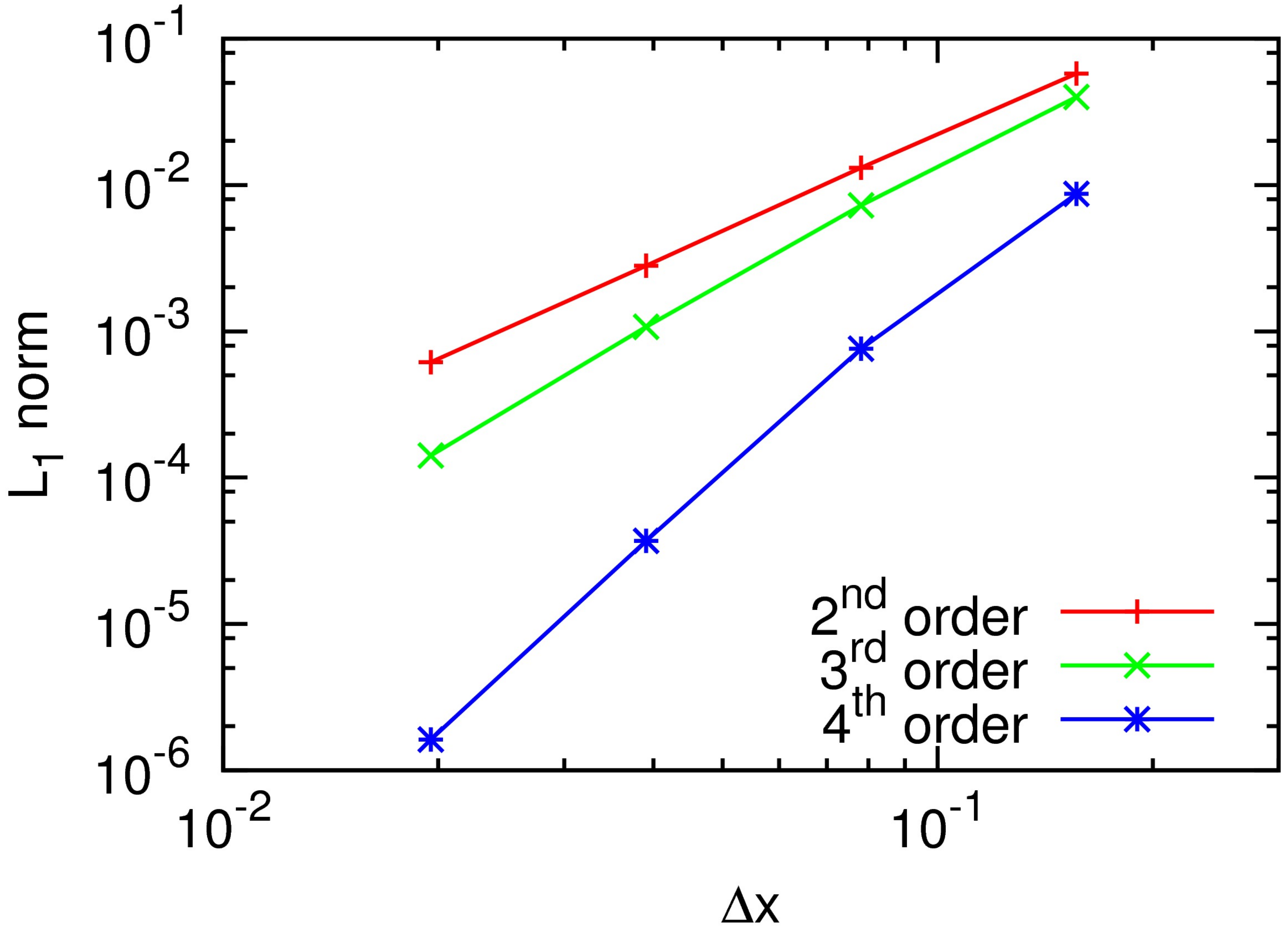
Balsara & Kim (2016)

Accuracy Analysis

Hydrodynamic Vortex



MHD Vortex

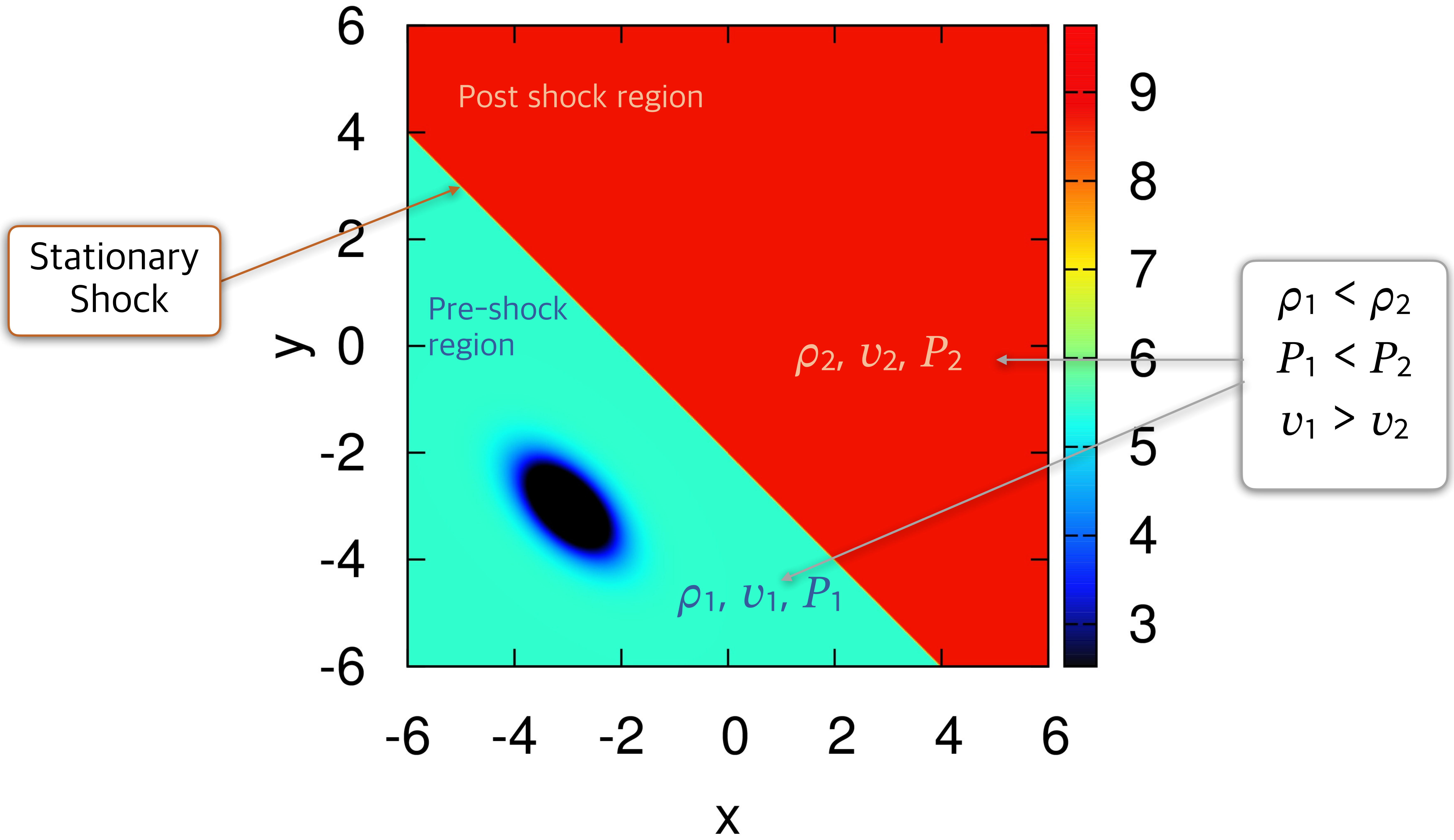


L_1 norm ~ $(\Delta x)^n$, n is the order of accuracy

Our code is matched with the designed order of accuracy.

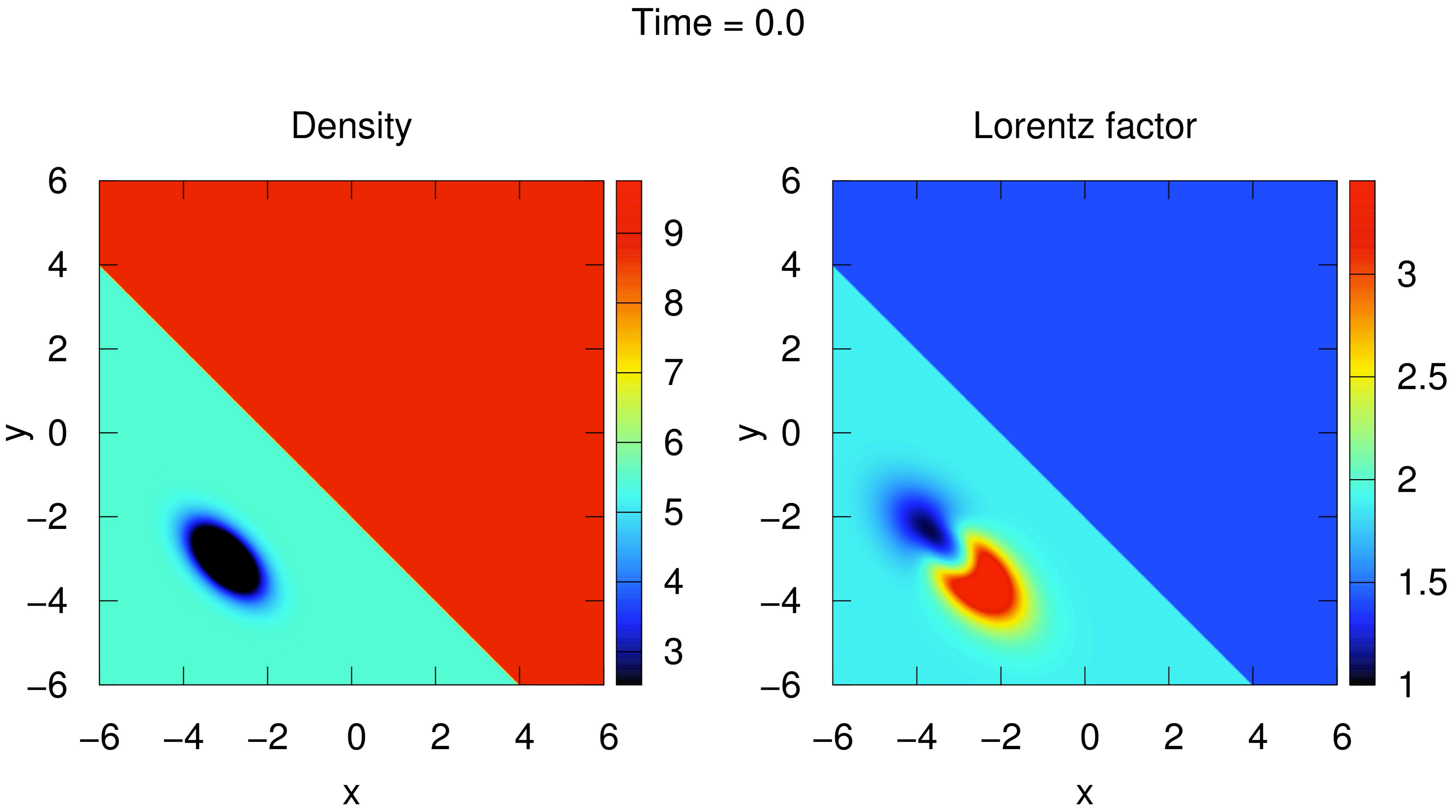
Balsara & Kim (2016)

RMHD Vortex Interacting with a Shock



Balsara & Kim (2016)

RMHD Vortex Interacting with a Shock

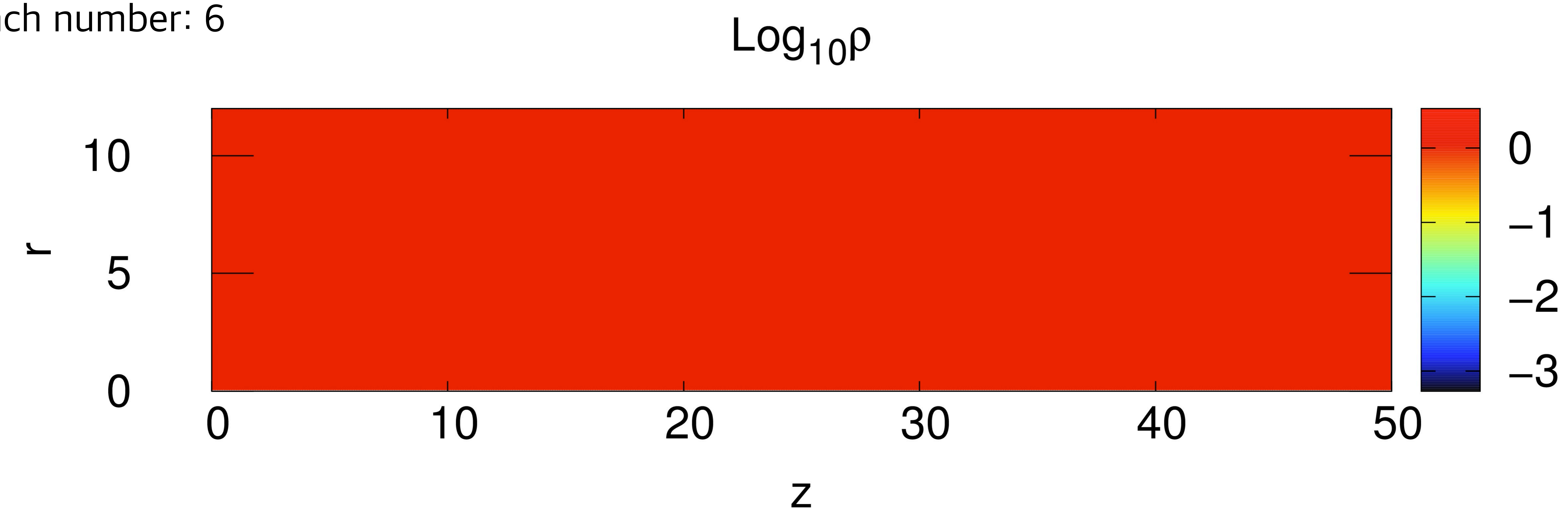


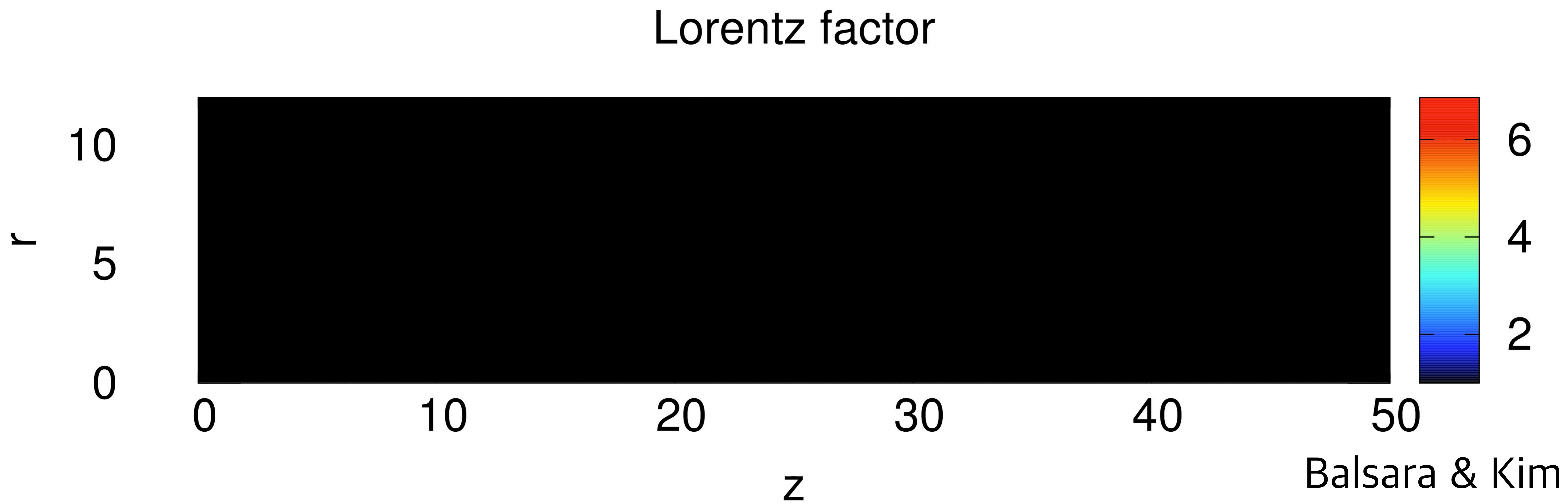
Balsara & Kim (2016)

Relativistic Jet Propagation

Jet to ambient density ratio = 0.01
Mach number: 6

Time = 0.0

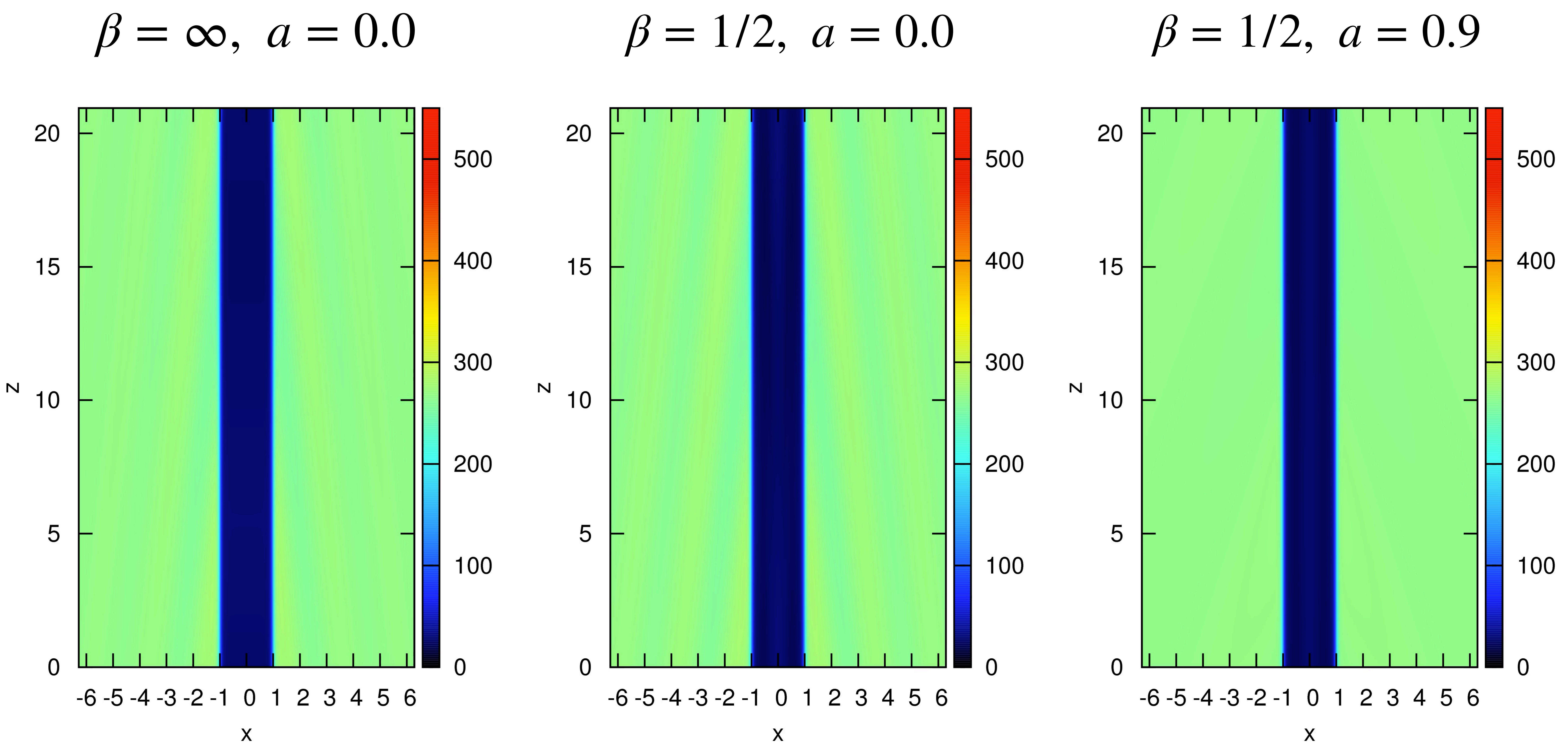




Balsara & Kim (2016)

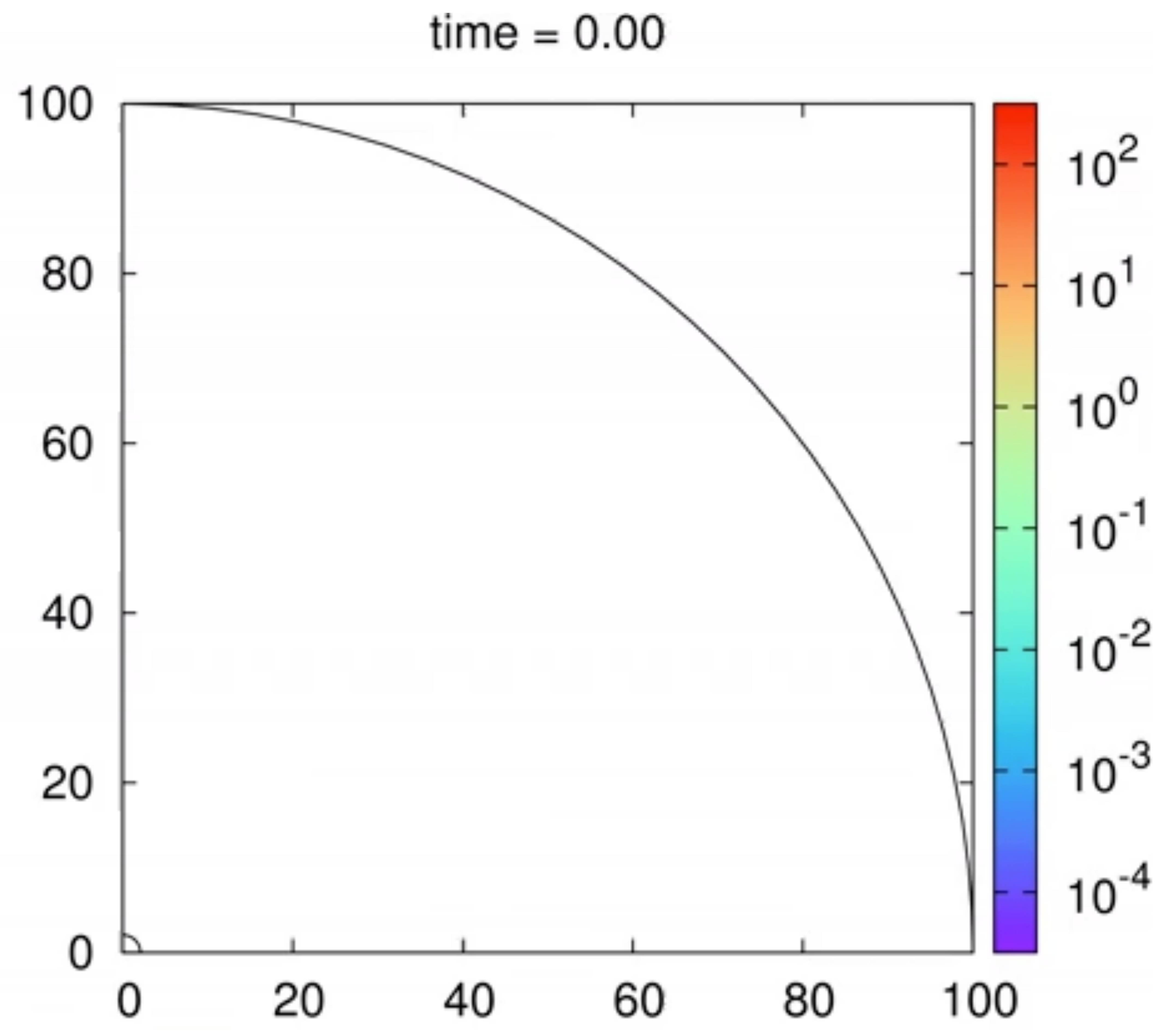
Recent Works

Jet simulation



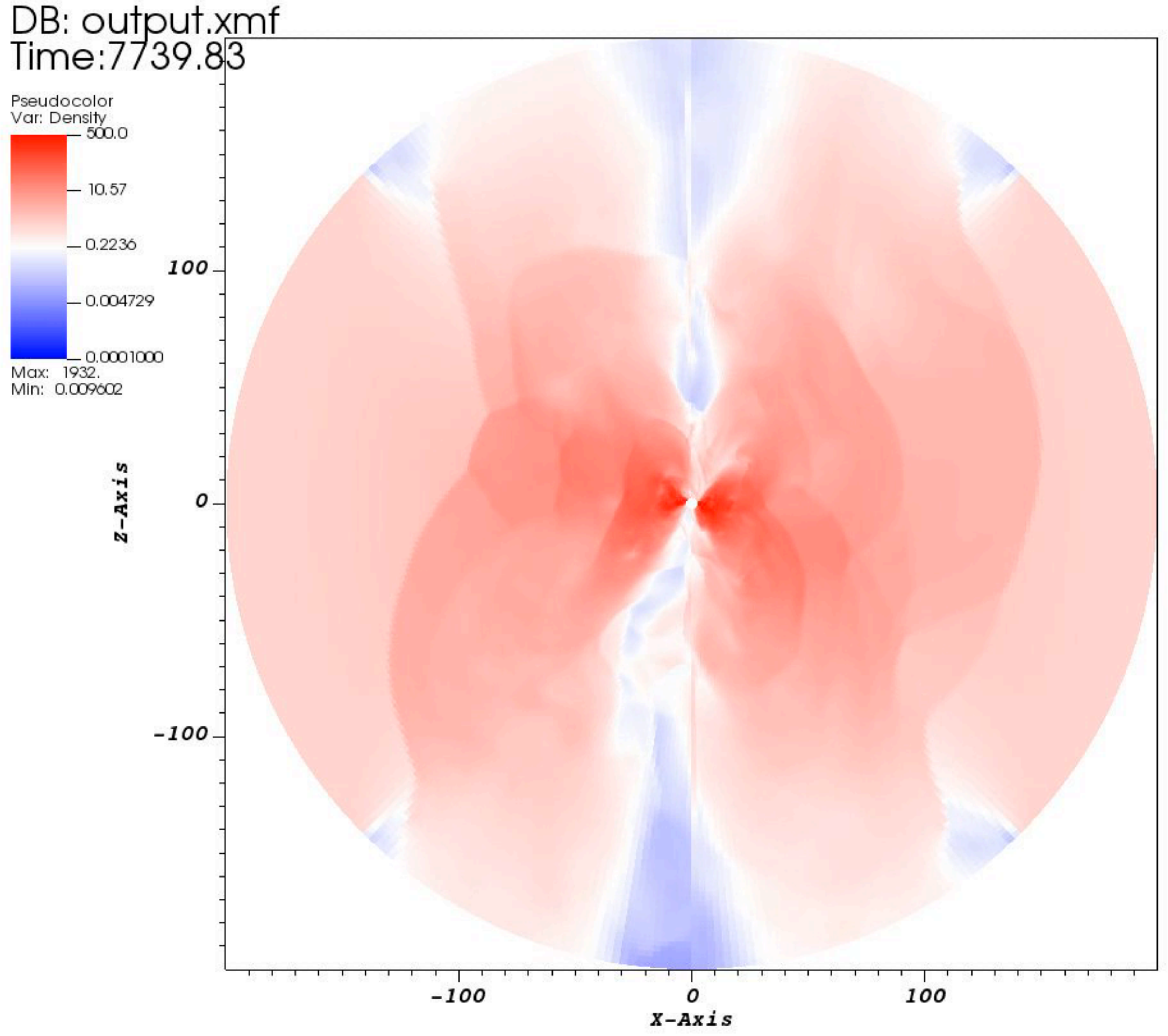
Kim et al. in preparation

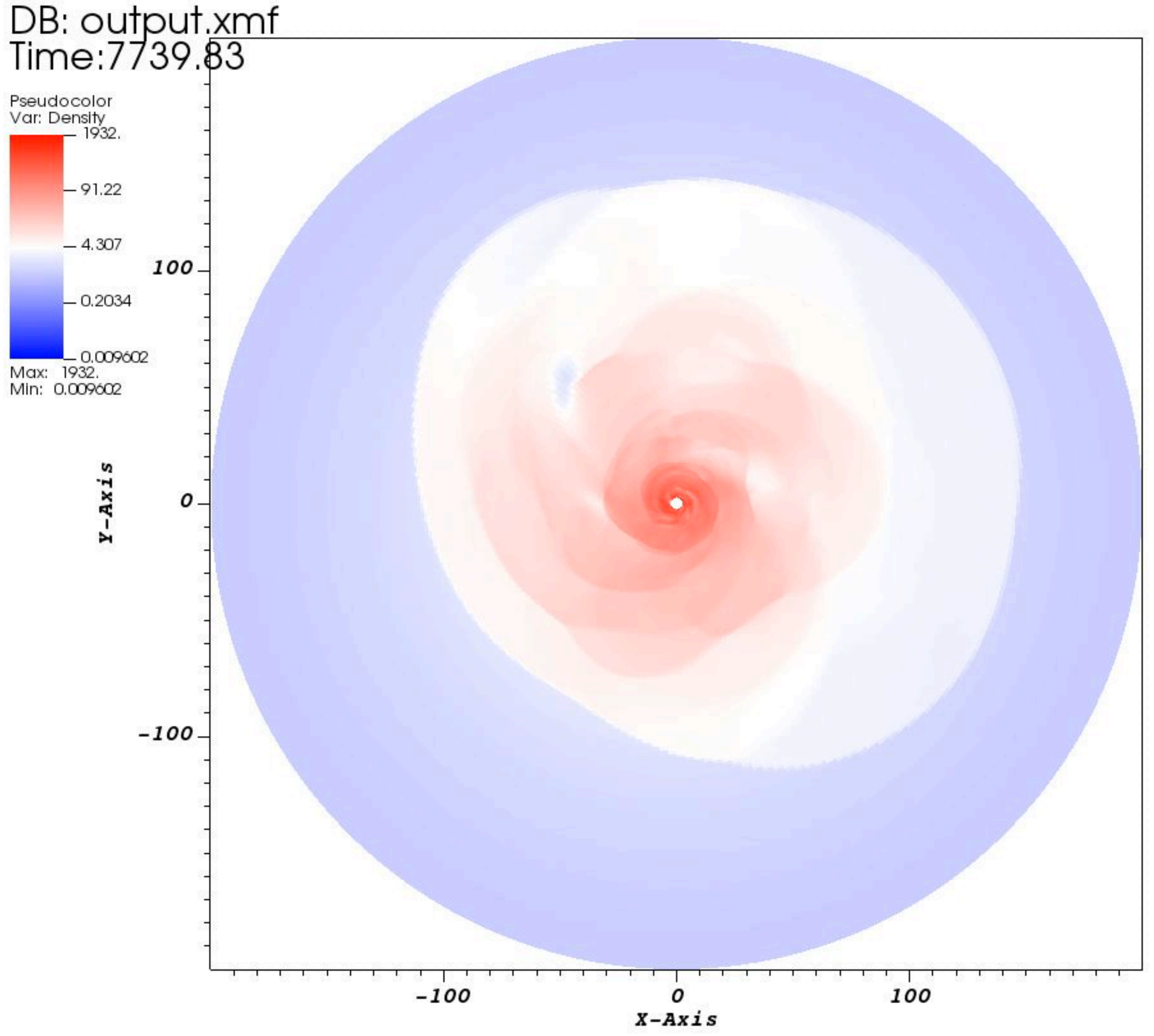
Accretion Disk around BH and Launching of Jets



Kim et al. (2017)

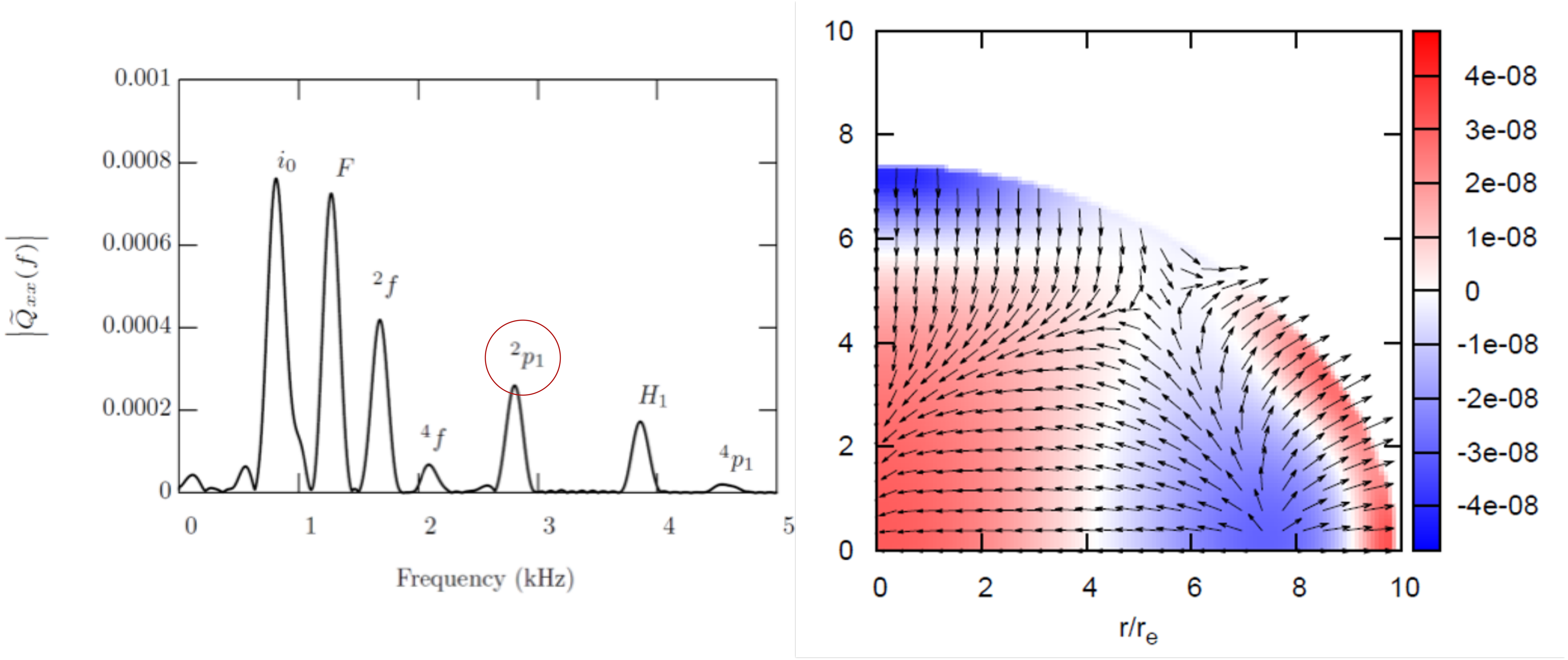
Three dimensional simulation





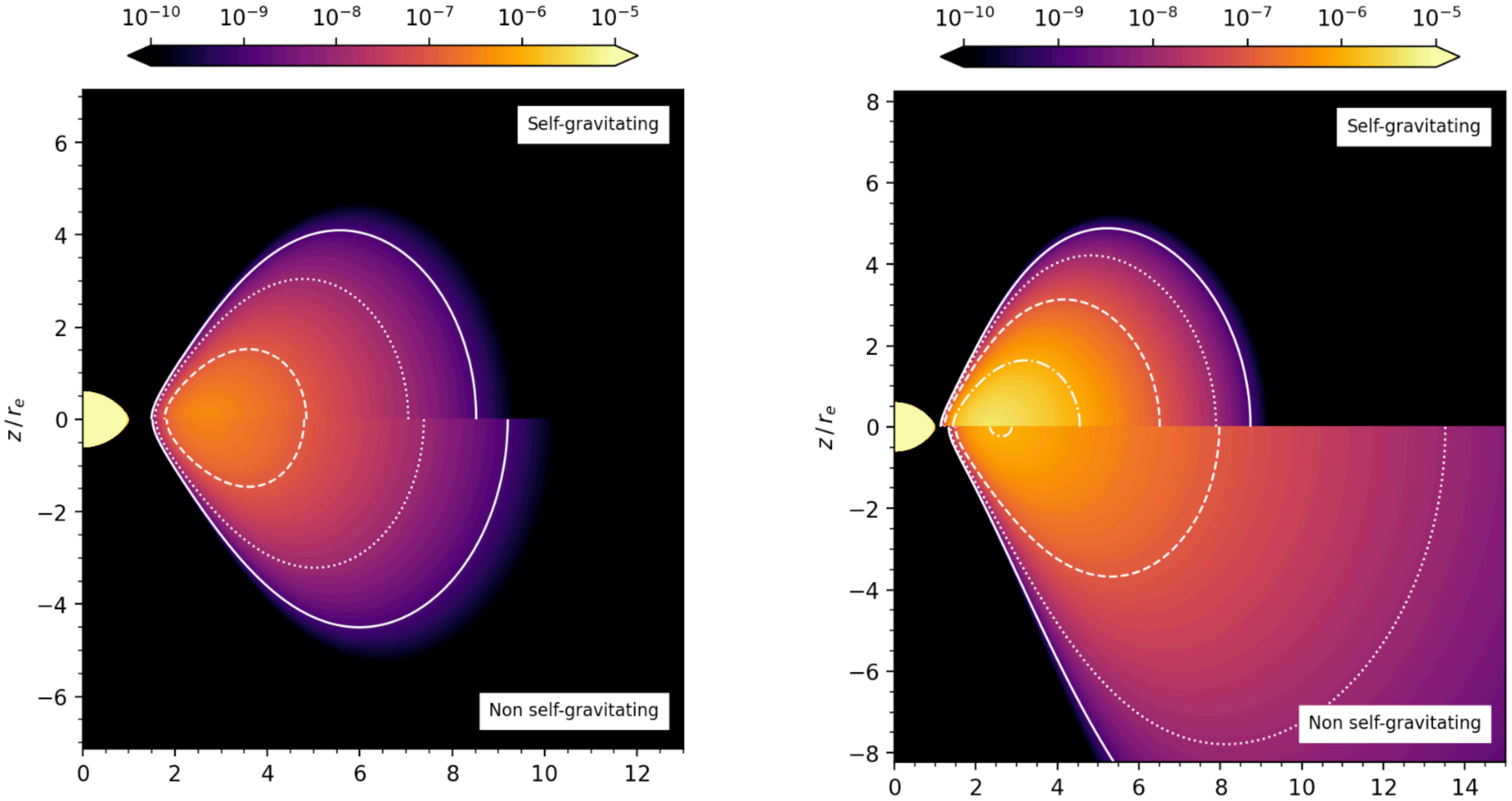
Preliminary result using KISTI Nurion with 28,000 cpus

Pulsation of Rapidly Rotating Neutron Star



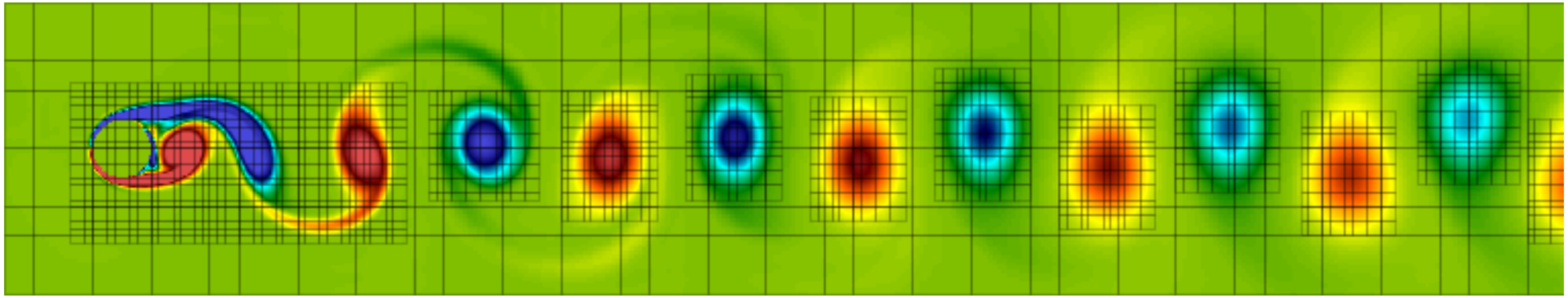
Kim et al. in preparation

Self-Gravitating Disk around NS



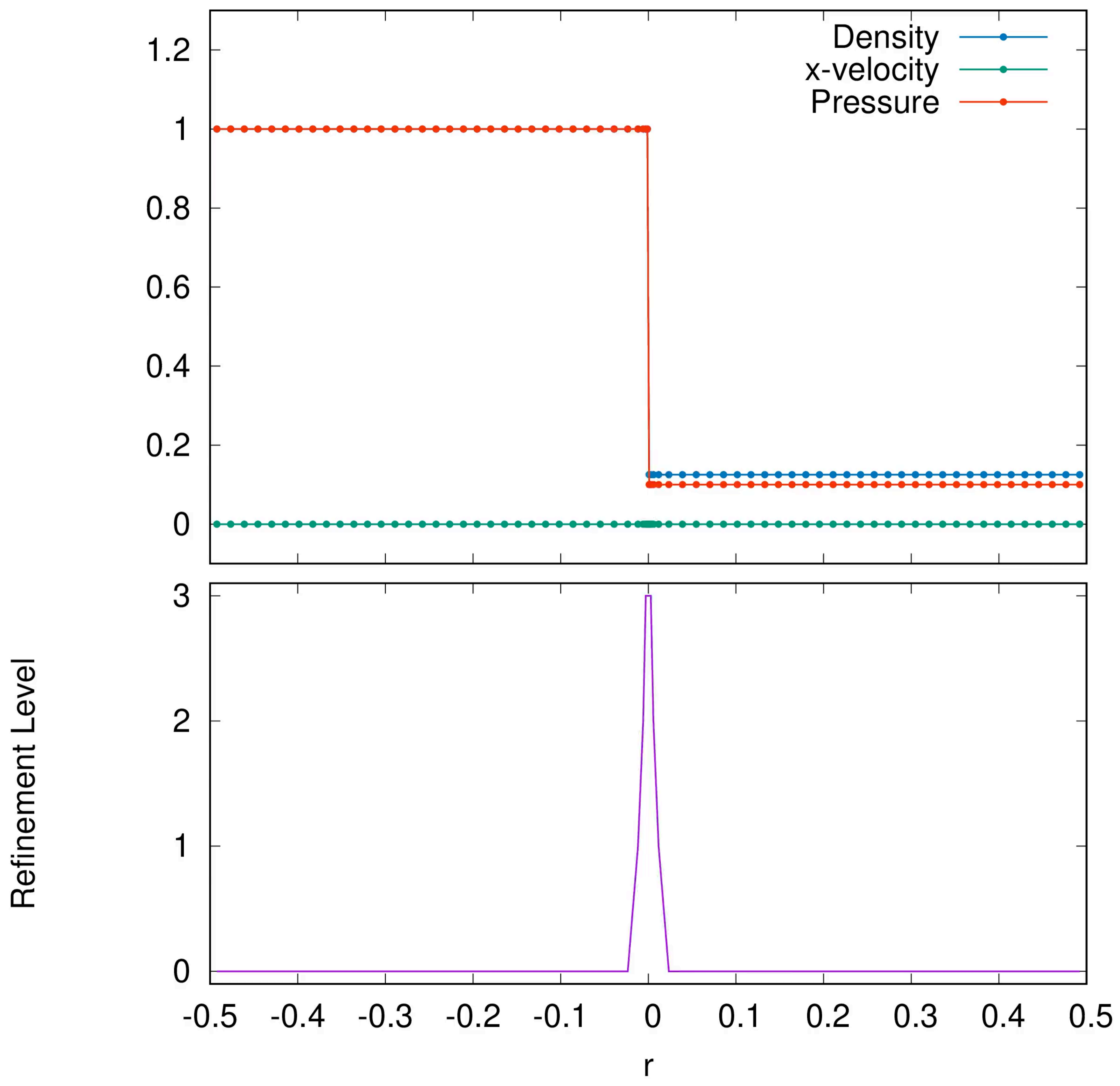
Kim et al. (2024)

Adaptive Mesh Refinement (AMR)

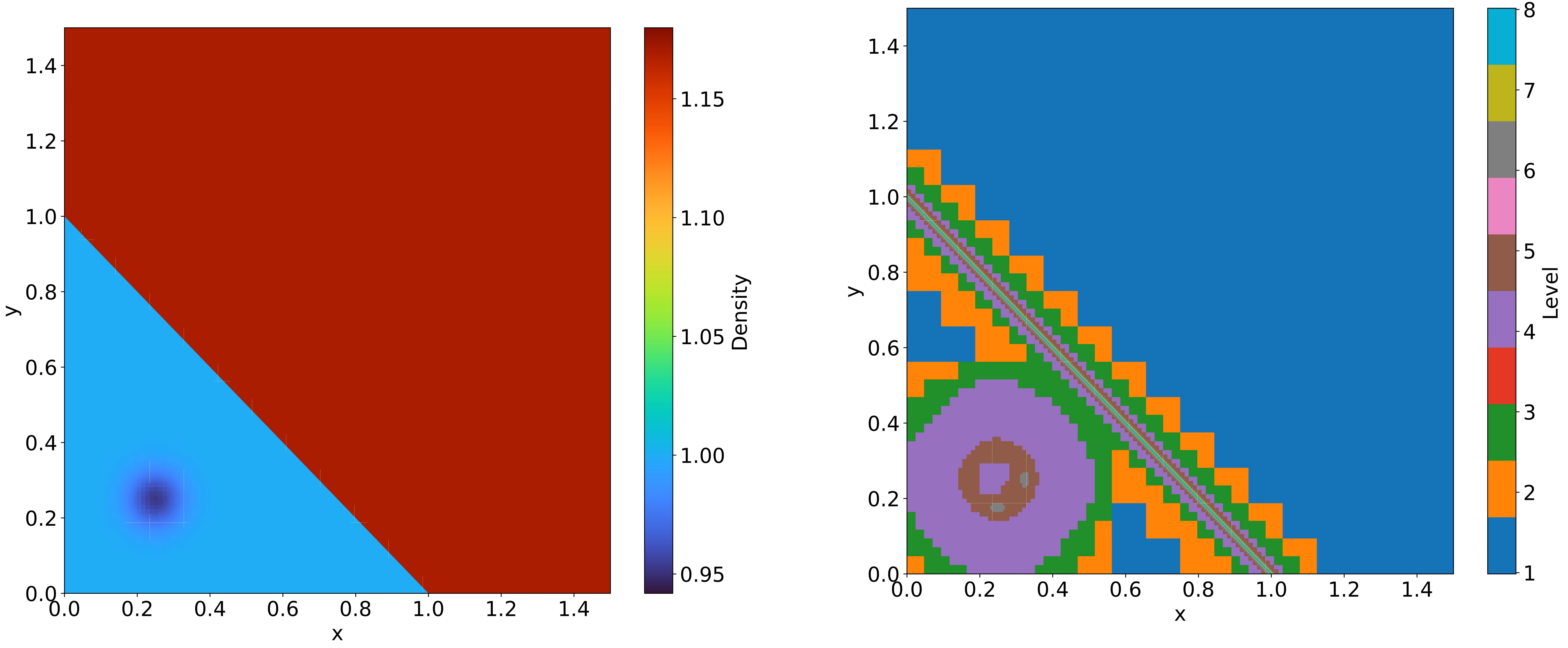


Credit: Chombo webpage

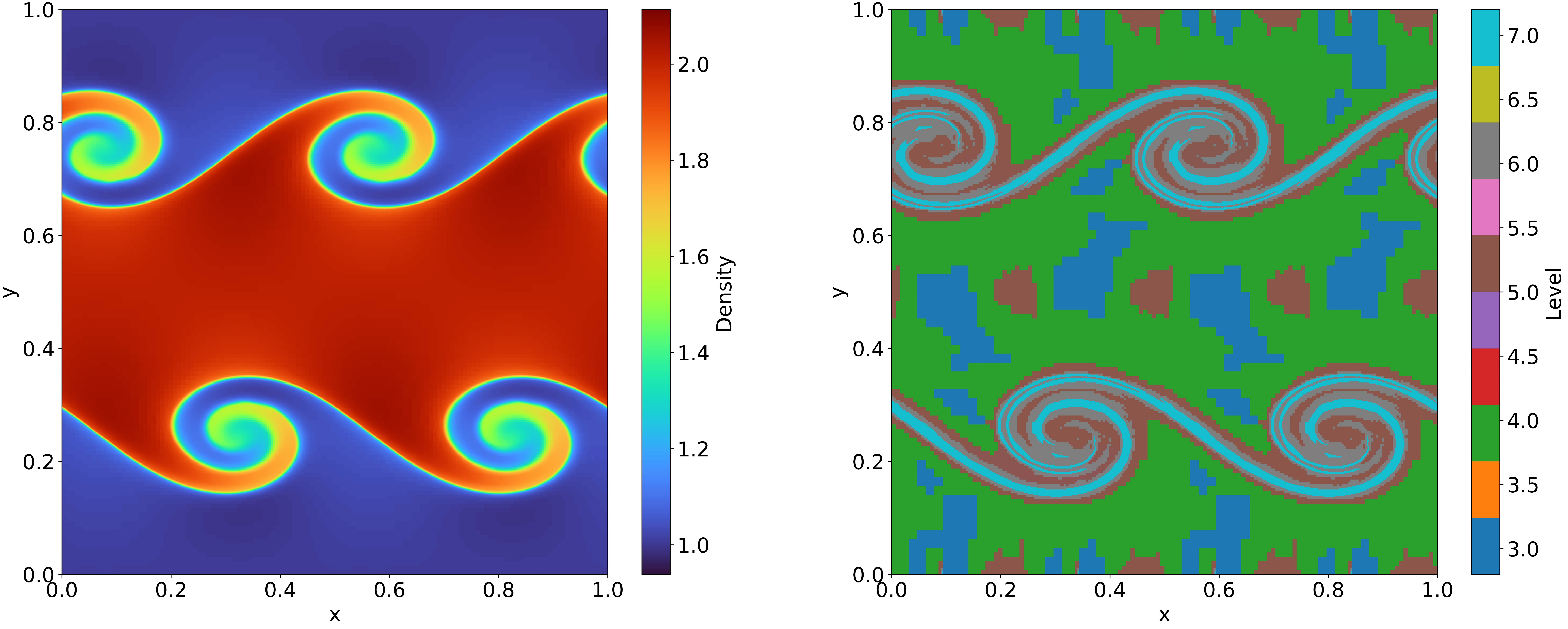
Shocktube with AMR



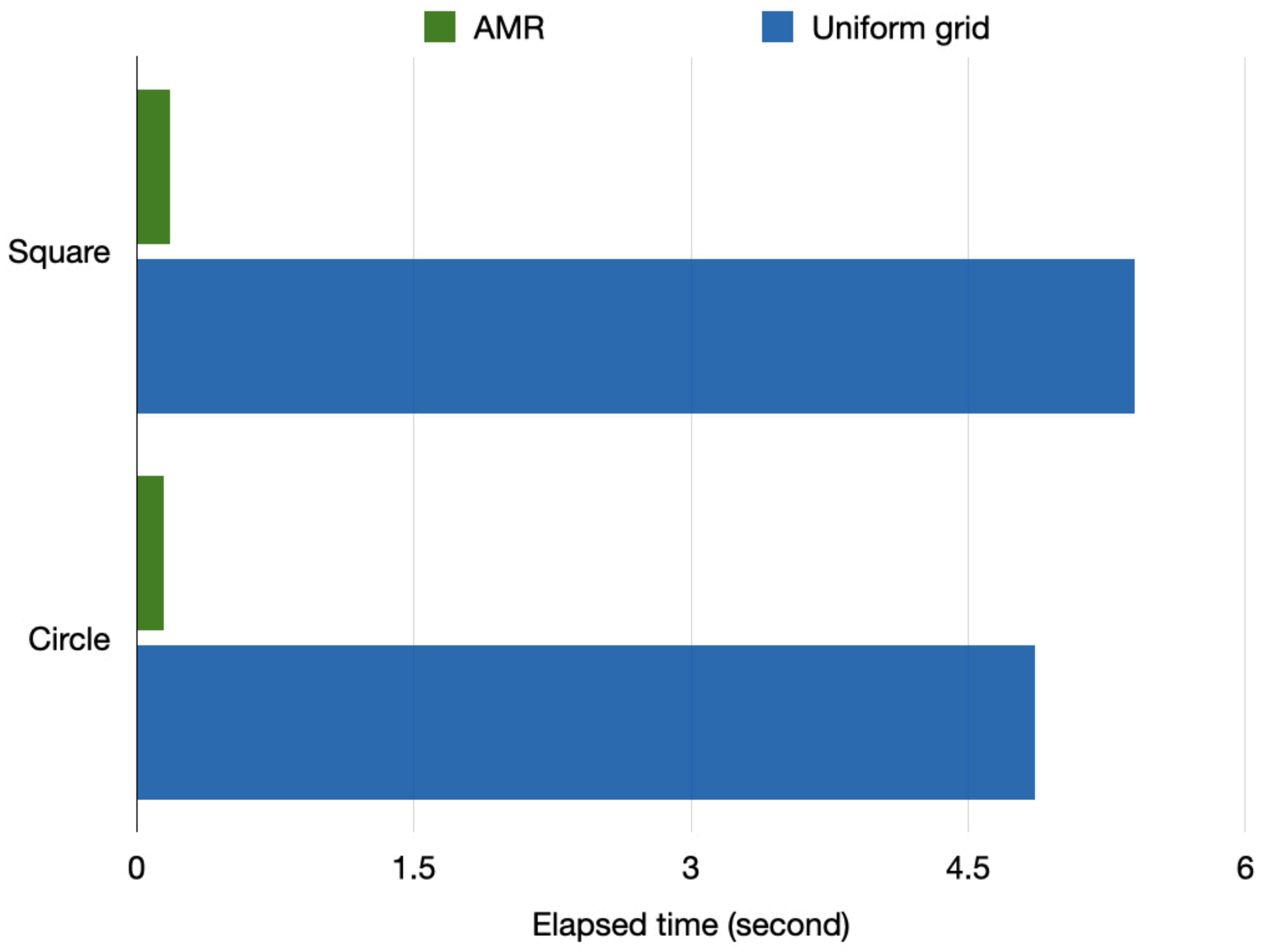
Vortex interacting with shock



Kelvin-Helmholtz Instability



Comparison: Advection



Relativistic Ray-tracing with GPU acceleration

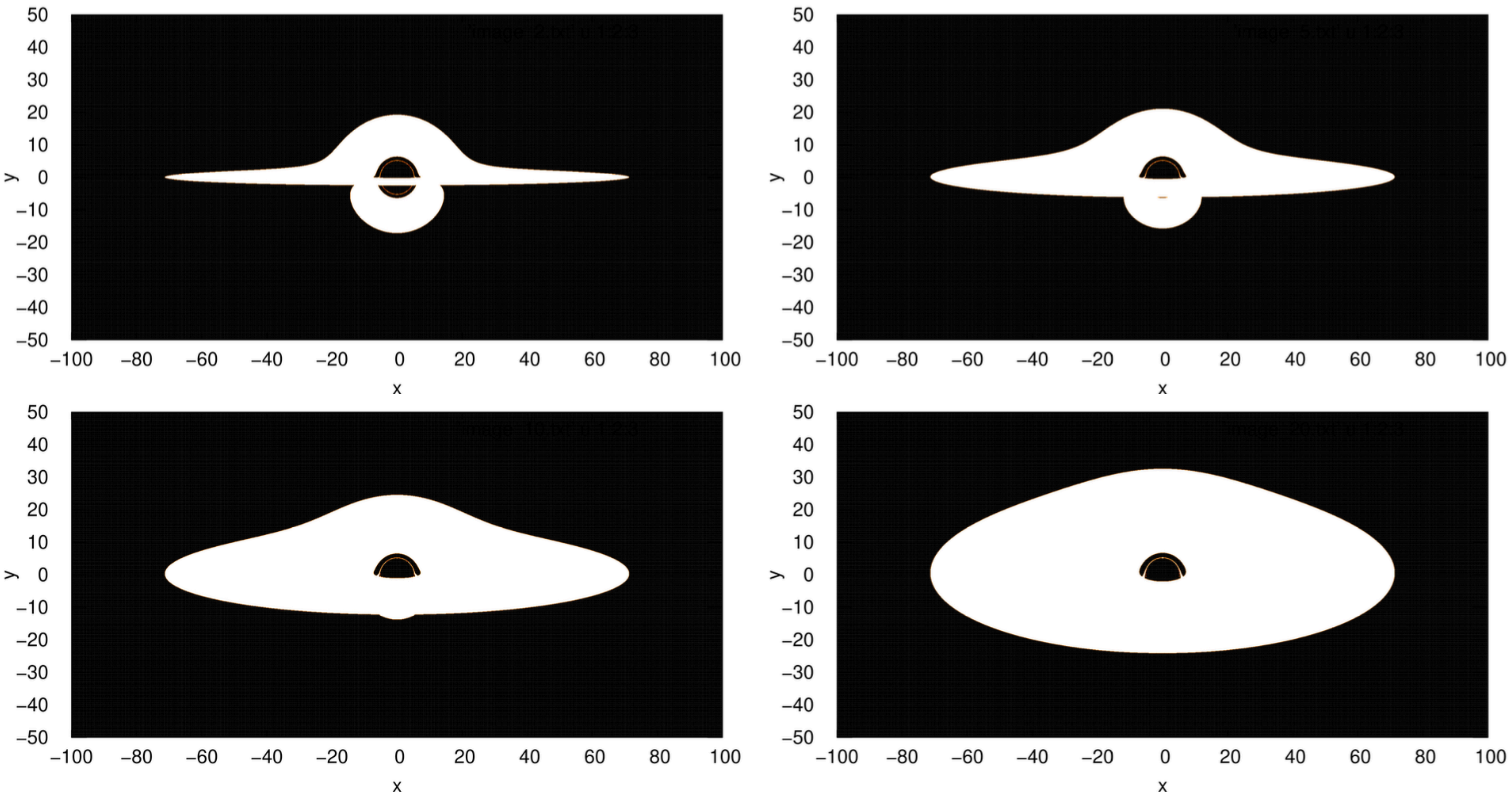


그림 5: 블랙홀 주변 디스크에 대한 이미지. Top Left: $\theta = 2°$, Top Right: $\theta = 5°$, Bottom Left: $\theta = 10°$, Botton Right: $\theta = 20°$