Gravitational Waves

Theoretical Basics

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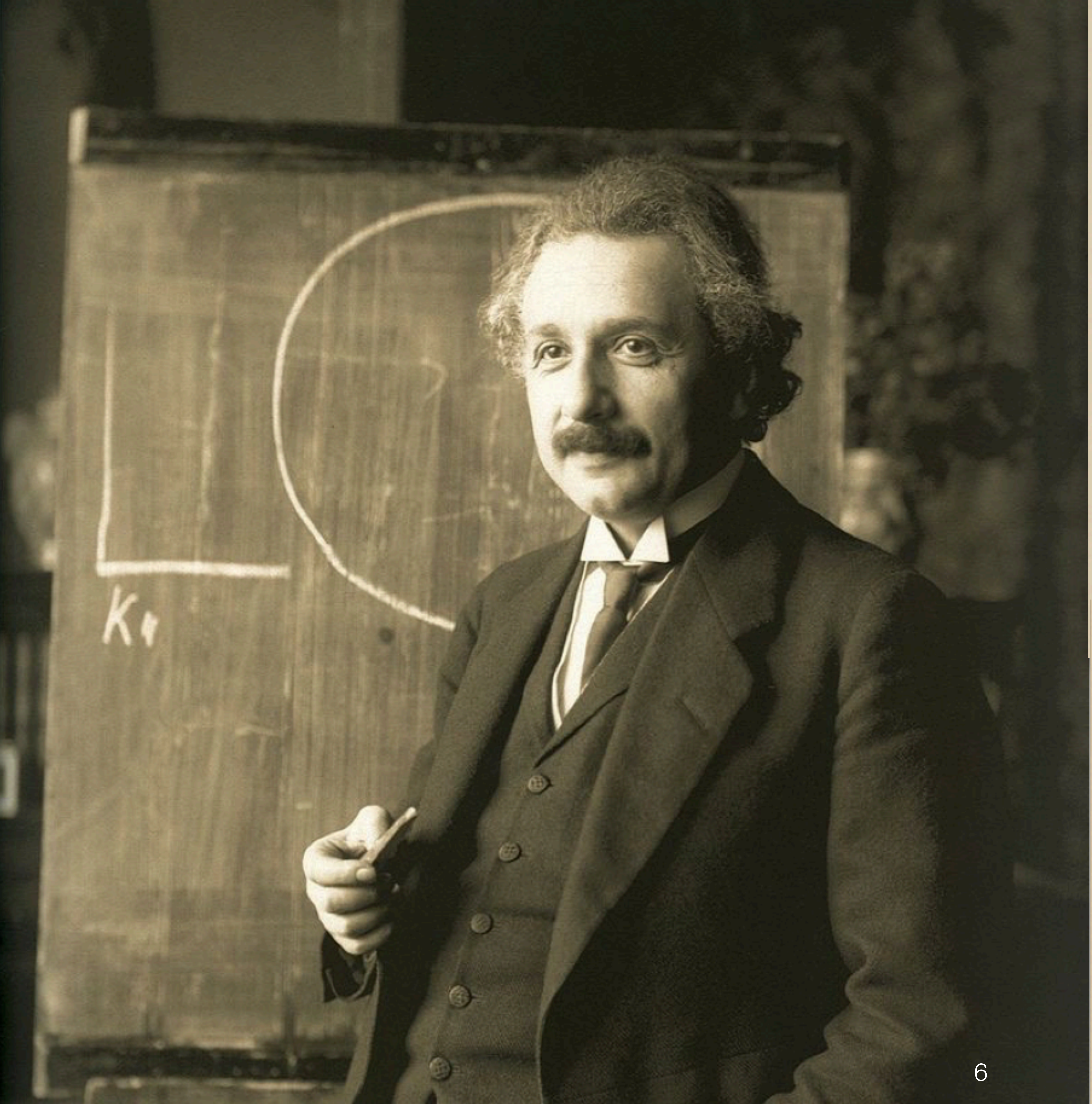
Calculation makes concepts fruitful!

- Juan Muchi

1915

1916



Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Approximate Integration of the Gravitational Field Equation

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$\gamma_{\mu\nu}$ und $\gamma'_{\mu\nu}$ nicht beliebigen, sondern nur linearen, orthogonalen Substitutionen gegenüber Tensorcharakter besitzen.

Plane Gravitational Wave

§ 2. Ebene Gravitationswellen.

Aus den Gleichungen (6) und (9) folgt, daß sich Gravitationsfelder stets mit der Geschwindigkeit 1, d. h. mit Lichtgeschwindigkeit, fortpflanzen. Ebene, nach der positiven x-Achse fortschreitende Gravitationswellen sind daher durch den Ansatz zu finden

$$\gamma'_{\mu\nu} = \alpha_{\mu\nu} f(x_1 + i x_4) = \alpha_{\mu\nu} f(x - t). \qquad (15)$$

Dabei sind die $\alpha_{\mu\nu}$ Konstante; f ist eine Funktion des Arguments $x - t$. Ist der betrachtete Raum frei von Materie, d. h. verschwinden die $T_{\mu\nu}$, so sind die Gleichungen (6) durch diesen Ansatz erfüllt. Die Gleichungen (4) liefern zwischen den $\alpha_{\mu\nu}$ die Beziehungen

$$\left.\begin{aligned} \alpha_{11} + i\alpha_{14} &= 0 \\ \alpha_{12} + i\alpha_{24} &= 0 \\ \alpha_{13} + i\alpha_{34} &= 0 \end{aligned}\right\} . \qquad (16)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \longrightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

1. Propagation of Gravitational Waves

1.1. Linearized Gravity Solution

Assuming: $g_{ab} = \eta_{ab} + h_{ab}$, **where** $||h_{ab}|| << 1$,

then we compute the Christoffel symbol in the order of $\mathcal{O}(h)$**:**

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_c g_{bd} + \partial_b g_{dc} - \partial_d g_{bc})$$

$$= \frac{1}{2}(\eta^{ad} - h^{ad})\left[\partial_c(\eta_{bd} + h_{bd}) + \partial_b(\eta_{dc} + h_{dc}) - \partial_d(\eta_{bc} + h_{bc})\right]$$

$$= \frac{1}{2}\eta^{ad}(\partial_c h_{bd} + \partial_b h_{dc} - \partial_d h_{bc}) + \mathcal{O}(h^2)$$

$$\simeq \frac{1}{2}\left(\partial_c h^a{}_b + \partial_b h^a{}_c - \partial^a h_{bc}\right)$$

Next, the Riemann tensor can be computed:

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \mathcal{O}(h^2)$$

$$\simeq \frac{1}{2}\partial_c(\partial_d h^a{}_b + \partial_b h^a{}_d - \partial^a h_{bd}) - \frac{1}{2}\partial_d(\partial_c h^a{}_b + \partial_b h^a{}_c - \partial^a h_{bc})$$

$$= \frac{1}{2}\left(\partial_c \partial_b h^a{}_d + \partial_d \partial^a h_{bc} - \partial_c \partial^a h_{bd} - \partial_d \partial_b h^a{}_c\right)$$

The Ricci tensor and the Ricci scalar are:

$$R_{bd} = R^a{}_{bad} = \frac{1}{2}\left[\partial_a\partial_b h^a{}_d + \partial_d\partial^a h_{ba} - \Box h_{bd} - \partial_d\partial_b h^a{}_a\right]$$

$$R = g^{ab}R_{ab} = \partial_c\partial^a h^c{}_a - \Box h$$

Ref) d'Alembertian operator

$$\Box = g^{ab}\partial_a\partial_b = \frac{1}{c^2}\partial_t^2 - \nabla^2$$

The Einstein tensor:

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$$

$$= \frac{1}{2}\left[\partial_c\partial_b h^c{}_a + \partial_a\partial^c h_{bc} - \Box h_{ab} - \partial_a\partial_b h\right] - \frac{1}{2}\eta_{ab}(\partial_c\partial^d h^c{}_d - \Box h) + \mathcal{O}(h^2)$$

$$\simeq \frac{1}{2}\left(\partial_c\partial_b h^c{}_a + \partial_a\partial^c h_{bc} - \Box h_{ab} - \partial_a\partial_b h - \eta_{ab}\partial_c\partial^d h^c{}_d + \eta_{ab}\Box h\right)$$

1.2. Trace-reversed Perturbation

We set: $\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h$, then we get: $h_{ab} = \bar{h}_{ab} + \frac{1}{2}\eta_{ab}h$, and the Einstein tensor becomes:

$$G_{ab} = \frac{1}{2}\left[\partial_c\partial_b\left(\bar{h}^c{}_a + \frac{1}{2}\eta^c{}_a h\right) + \partial_a\partial^c\left(\bar{h}_{bc} + \frac{1}{2}\eta_{bc}h\right) - \Box\left(\bar{h}_{ab} + \frac{1}{2}\eta_{ab}h\right) + \partial_a\partial_b\bar{h}\right.$$
$$\left. -\eta_{ab}\partial_c\partial_d\left(\bar{h}^{cd} + \frac{1}{2}\eta^{cd}h\right) + \eta_{ab}\Box h\right]$$
$$= \frac{1}{2}\left(\partial_c\partial_b\bar{h}^c{}_a + \partial_a\partial^c\bar{h}_{bc} - \Box\bar{h}_{ab} - \eta_{ab}\partial_c\partial^d\bar{h}^c{}_d\right)$$

1.3. Gauge Fixing

Consider a general infinitesimal transformation as $x'^a = x^a + \xi^a$, then

$$g'_{ab} = \eta'_{ab} + h'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}g_{cd} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}(\eta_{cd} + h_{cd})$$

This yields:

$$\eta_{cd} + h_{cd} = (\delta^a{}_c + \partial_c \xi^a)(\delta^b{}_d + \partial_d \xi^d)(\eta'_{ab} + h'_{ab})$$

$$= \eta_{cd} + \partial_c \xi_d + \partial_d \xi_c + h'_{cd} + \mathcal{O}(\xi \cdot h)$$

Then we have: $h'_{ab} = h_{ab} - 2\partial_{(a}\xi_{b)}$

Therefore, the trace-reversed metric is transformed as:

$$\begin{aligned}\bar{h}'_{ab} &= h'_{ab} - \frac{1}{2}\eta_{ab}h' \\ &= h_{ab} - 2\partial_{(a}\xi_{b)} - \frac{1}{2}\eta_{ab}(h - 2\partial^c\xi_c) \\ &= h_{ab} - \frac{1}{2}\eta_{ab}h - 2\partial_{(a}\xi_{b)} + \eta_{ab}\partial^c\xi_c \\ &= \bar{h}_{ab} - 2\partial_{(a}\xi_{b)} + \eta_{ab}\partial^c\xi_c\end{aligned}$$

Taking derivatives in both sides yields: $\partial^a \bar{h}'_{ab} = \partial^a \bar{h}_{ab} - \Box \xi_b$: whenever we choose $\partial^a \bar{h}_{ab} = \Box\, \xi_b$, we always put into a Lorentz gauge as $\partial^a \bar{h}_{ab} = 0$.

In this gauge fixing, the Einstein tensor can be expressed as a simple form:

$$G_{ab} = -\frac{1}{2}\Box \bar{h}_{ab}$$

1.4. Linearized Einstein Equation and Plane GWs

The vacuum Einstein equation ($T_{ab} = 0$) in a Lorentz gauge is

$$\Box \bar{h}_{ab} = 0$$

(wave equation)

The equation has a plane wave solution of :

$$\bar{h}_{ab}(\vec{x}, t) = \mathfrak{Re}\int d^3k A_{ab}(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

Plugging the solution into the equation yields $k^a k_a = 0$, meaning that k^a is a null vector propagating with the speed of light.

Ref) $k^a k_a = k^0 k_0 - \vec{k}\cdot\vec{k} = \omega^2/c^2 - k^2 = 0, \quad \therefore v_{gw} = w/k = c$

The Lorentz gauge condition,

$$\partial^a \bar{h}_{ab} = \partial^a \Re \int d^3k A_{ab}(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$
$$= \Re \int d^3k A_{ab}(\vec{k}) \partial^a e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$
$$= \Re \int d^3k \boxed{A_{ab}(\vec{k}) k^a} e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$
$$= 0$$

$\Rightarrow$ $A_{ab}k^a = 0$

: implying that A_{ab} is orthogonal to k^a

* $A_{ab} = A_{ba}$ for the symmetry of the metric

1.5. Transverse-Traceless (TT) gauge

Consider a wave propagating in the x^3-direction, then we have $k^a = (k,0,0,k)$, $k_a = (k,0,0,-k)$

The Lorentz gauge condition yields $kA^{a0} - kA^{a3} = 0$: $A^{a0} = A^{a3}$

Therefore, the amplitude matrix becomes:

$$A^{a0} = A^{a3}$$

4-constraints!

$$\begin{pmatrix} A^{00} & A^{01} & A^{02} & A^{00} \\ A^{01} & A^{11} & A^{12} & A^{01} \\ A^{02} & A^{12} & A^{22} & A^{02} \\ A^{00} & A^{01} & A^{02} & A^{00} \end{pmatrix}$$

Only 6-independent components!

The symmetric tensor has 10 components!
16-6 (symmetric) = 10

The Lorentz gauge has additional four degrees of freedom by choosing ξ^a : $\partial^a \bar{h}_{ab} = \Box \xi_b$

Recall the trace-reversed transformation:

$$\bar{h}'_{ab} = \bar{h}_{ab} - \partial_a \xi_b - \partial_b \xi_a + \eta_{ab} \partial^c \xi_c \quad \Longleftrightarrow \quad A'_{ab} = A_{ab} - k_a \epsilon_b - k_b \epsilon_a + \eta_{ab} k^c \epsilon_c$$

where assuming: $\xi^a = -\Re e[i\epsilon^a e^{ik_b x^b}]$

$$\begin{cases} A'_{00} = A_{00} - k_0\epsilon_0 - k_0\epsilon_0 + \eta_{00}k^c\epsilon_c = A_{00} - k(\epsilon_0 + \epsilon_3) \\ A'_{01} = A_{01} - k_0\epsilon_1 - k_1\epsilon_0 + \eta_{01}k^c\epsilon_c = A_{01} - k\epsilon_1 \\ A'_{02} = A_{02} - k_0\epsilon_2 - k_2\epsilon_0 + \eta_{02}k^c\epsilon_c = A_{02} - k\epsilon_2 \\ A'_{11} = A_{11} - k_1\epsilon_1 - k_1\epsilon_1 + \eta_{11}k^c\epsilon_c = A_{11} - k(\epsilon_0 - \epsilon_3) \\ A'_{12} = A_{12} - k_1\epsilon_2 - k_2\epsilon_1 + \eta_{12}k^c\epsilon_c = A_{12} \\ A'_{22} = A_{22} - k_2\epsilon_2 - k_2\epsilon_2 + \eta_{22}k^c\epsilon_c = A_{22} - k(\epsilon_0 - \epsilon_3) \end{cases} \Rightarrow \begin{cases} \epsilon_0 = (2A_{00} + A_{11} + A_{22})/4k \\ \epsilon_1 = A_{01}/k \\ \epsilon_2 = A_{02}/k \\ \epsilon_3 = (2A_{00} - A_{11} - A_{22})/4k \end{cases}$$

This is nothing but : $h_{a0} = 0$ (3) and $h^a{}_a = 0$ (1), which is called transverse-traceless gauge (TT-gauge)

So remaining degrees of freedom is only two! Then the matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & 0 \\ 0 & A^{12} & -A^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A^{11}\epsilon_+^{ab} + A^{12}\epsilon_\times^{ab} \equiv A_+\epsilon_+^{ab} + A_\times\epsilon_\times^{ab}$$

where $\epsilon_+^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\epsilon_\times^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Two independent polarizations!

Summary: GWs

1. Propagating with speed of light

2. Two independent polarizations

1.6. Geodesic equation of a moving particle

Let us consider two particles with a separation vector s^μ in the presence of GWs in order to see the effect of GWs. The metric is $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$ and the proper time τ is given by $(d\tau)^2 = -\frac{1}{c^2}(ds)^2$.

For a fixed two points A and B, the proper time is

$$\tau_{AB} = \int_A^B \sqrt{-(ds)^2} = \int_A^B \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} \equiv \int_A^B d\sigma \mathscr{L}\left[\frac{dx^\mu}{d\sigma}, x^\mu\right].$$

Then the Euler-Lagrange equation determines the geodesic trajectory of a moving particle as

$$\frac{d}{d\sigma}\left(\frac{\partial \mathscr{L}}{\partial(dx^\alpha/d\sigma)}\right) - \frac{\partial \mathscr{L}}{\partial x^\alpha} = 0,$$

where the Lagrangian is

$$\mathscr{L} = \left(-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right)^{1/2}.$$

Precisely,

$$\Rightarrow \frac{\partial \mathscr{L}}{\partial x^\lambda} = -\frac{1}{2\mathscr{L}}\partial_\lambda g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} = -\frac{1}{2\mathscr{L}}\partial_\lambda g_{\alpha\beta}\underbrace{\left(\frac{d\tau}{d\sigma}\right)^2}_{=\mathscr{L}^2}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

$$= -\frac{\mathscr{L}}{2}\partial_\lambda g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

and

$$\Rightarrow \frac{\partial \mathscr{L}}{\partial (dx^\gamma/d\sigma)} = -\frac{2\mathscr{L}}{g_{\alpha\beta}}\left(\frac{dx^\beta}{d\sigma}\delta^\alpha_\gamma + \frac{dx^\alpha}{d\sigma}\delta^\beta_\gamma\right) = -\frac{1}{2\mathscr{L}}\left[g_{\gamma\beta}\frac{dx^\beta}{d\sigma} + g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right] = -\frac{1}{\mathscr{L}}g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}$$

In addition,

$$\Rightarrow -\frac{d}{d\sigma}\left(\frac{\partial \mathscr{L}}{\partial (dx^\gamma/d\sigma)}\right) = \frac{d}{d\sigma}\left(\frac{1}{\mathscr{L}}g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right) = \underbrace{\frac{d\tau}{d\sigma}}_{=\mathscr{L}}\frac{d}{d\tau}\left(\frac{1}{\mathscr{L}}g_{\alpha\gamma}\mathscr{L}\frac{dx^\alpha}{d\tau}\right) = \mathscr{L}\frac{d}{d\tau}\left(g_{\alpha\gamma}\frac{dx^\alpha}{d\tau}\right)$$

$$= \mathscr{L}\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \partial_\beta g_{\alpha\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right] = \mathscr{L}\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}(\partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\beta\gamma})\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right]$$

Together with all these, the Euler-Lagrange equation is

$$0 = -\frac{d}{d\sigma}\left(\frac{\partial \mathscr{L}}{\partial(dx^\gamma/d\sigma)}\right) + \frac{\partial \mathscr{L}}{\partial x^\gamma}$$

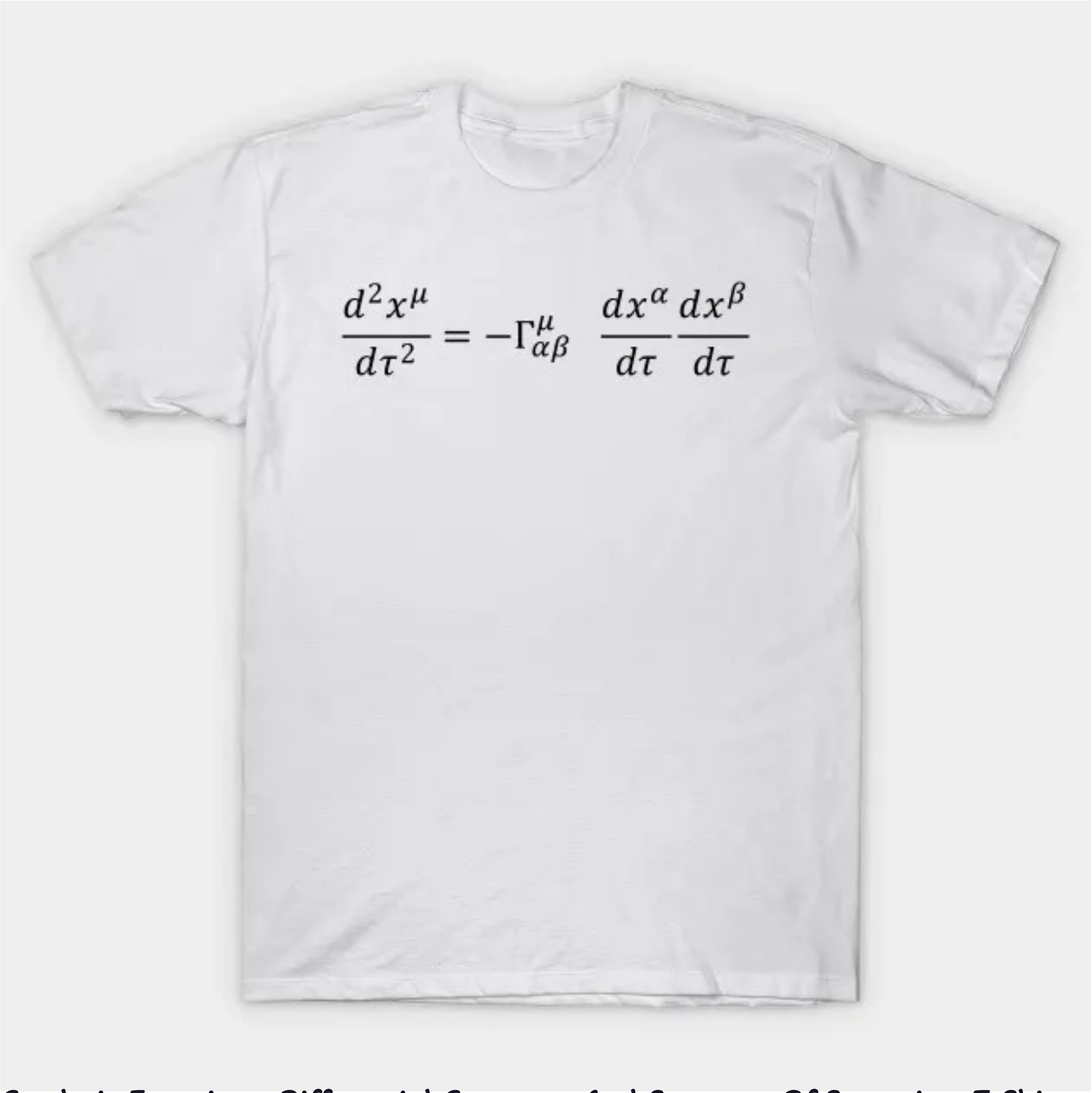
$$= \mathscr{L}\underbrace{\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}(\partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\beta\gamma} - \partial_\gamma g_{\alpha\beta})\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right]}_{=0}$$

Then,

$$g^{\gamma\sigma} \times \| \, g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}(\partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\beta\gamma} - \partial_\gamma g_{\alpha\beta})\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} = 0$$

$$\Rightarrow \frac{d^2x^\sigma}{d\tau^2} + \underbrace{\frac{1}{2}g^{\gamma\sigma}(\partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\beta\gamma} - \partial_\gamma g_{\alpha\beta})}_{=\Gamma^\sigma_{\alpha\beta}}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} = 0$$

Therefore, the geodesic equation is $\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} = 0$



Geodesic Equation - Differential Geometry And Structure Of Spacetime T-Shirt

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1.7. Geodesic deviation

Consider the small deviation from the geodesic motion as $x^\mu \to x^\mu + \xi^\mu$, then the geodesic equation becomes:

$$\frac{d^2}{d\tau^2}(x^\mu + \xi^\mu) + \Gamma^\mu_{\nu\rho}(x+\xi)\frac{d}{d\tau}(x^\nu + \xi^\nu)\frac{d}{d\tau}(x^\rho + \xi^\rho)$$

$$= \frac{d^2 x^\mu}{d\tau^2} + \frac{d^2\xi^\mu}{d\tau^2} + \left[\Gamma^\mu_{\nu\rho}(x) + \xi^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho}(x)\right]\left(\frac{dx^\nu}{d\tau} + \frac{d\xi^\nu}{d\tau}\right)\left(\frac{dx^\rho}{d\tau} + \frac{d\xi^\rho}{d\tau}\right)$$

$$= \underbrace{\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}}_{=0} + \frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{d\xi^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

$$= \underbrace{\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau}}_{(*)} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

Now defining the covariant derivative as:

$$\frac{D\xi^\mu}{D\tau} = \frac{d\xi^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau} \equiv V^\mu$$

We can compute:

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{DV^\mu}{D\tau} = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} V^\nu \frac{dx^\rho}{d\tau}$$

$$= \frac{d}{d\tau}\left(\frac{d\xi^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}\xi^\nu \frac{dx^\rho}{d\tau}\right) + \Gamma^\mu_{\nu\rho}\left(\frac{d\xi^\nu}{d\tau} + \Gamma^\nu_{\alpha\beta}\xi^\alpha \frac{dx^\beta}{d\tau}\right)\frac{dx^\rho}{d\tau}$$

$$= \frac{d^2\xi^\mu}{d\tau^2} + \frac{d}{d\tau}\left(\Gamma^\mu_{\nu\rho}\xi^\nu \frac{dx^\rho}{d\tau}\right) + \Gamma^\mu_{\nu\rho}\frac{d\xi^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\Gamma^\nu_{\alpha\beta}\xi^\alpha \frac{dx^\beta}{d\tau}\frac{dx^\rho}{d\tau}$$

$$= \frac{d^2\xi^\mu}{d\tau^2} + \frac{d\Gamma^\mu_{\nu\rho}}{d\tau}\xi^\nu \frac{dx^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\frac{d\xi^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\xi^\nu \frac{d^2x^\rho}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{d\xi^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\Gamma^\nu_{\alpha\beta}\xi^\alpha \frac{dx^\beta}{d\tau}\frac{dx^\rho}{d\tau}$$

$$= \frac{d^2\xi^\mu}{d\tau^2} + \partial_\alpha \Gamma^\mu_{\nu\rho}\xi^\nu \frac{dx^\alpha}{d\tau}\frac{dx^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\frac{d\xi^\nu}{d\tau}\frac{dx^\rho}{d\tau} - \Gamma^\mu_{\nu\rho}\xi^\nu \Gamma^\rho_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \Gamma^\mu_{\nu\rho}\frac{d\xi^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\Gamma^\nu_{\alpha\beta}\xi^\alpha \frac{dx^\beta}{d\tau}\frac{dx^\rho}{d\tau}$$

$$= \boxed{\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}\frac{d\xi^\nu}{d\tau}\frac{dx^\rho}{d\tau}} + \partial_\alpha \Gamma^\mu_{\nu\rho}\xi^\nu \frac{dx^\alpha}{d\tau}\frac{dx^\rho}{d\tau} - \Gamma^\mu_{\nu\rho}\Gamma^\rho_{\alpha\beta}\xi^\nu \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \Gamma^\mu_{\nu\rho}\Gamma^\nu_{\alpha\beta}\xi^\alpha \frac{dx^\beta}{d\tau}\frac{dx^\rho}{d\tau}$$

Plugging this into (*)

Continued from p.18:

$$= \frac{D^2\xi^\mu}{D\tau^2} + \xi^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} - \partial_\alpha \Gamma^\mu_{\nu\rho}\xi^\nu \frac{dx^\alpha}{d\tau}\frac{dx^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\Gamma^\rho_{\alpha\beta}\xi^\nu \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} - \Gamma^\mu_{\nu\rho}\Gamma^\nu_{\alpha\beta}\xi^\alpha \frac{dx^\beta}{d\tau}\frac{dx^\rho}{d\tau}$$

$$= \frac{D^2\xi^\mu}{D\tau^2} + \xi^\sigma \underbrace{\left[\partial_\sigma \Gamma^\mu_{\nu\rho} - \partial_\nu \Gamma^\mu \sigma\rho + \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\sigma\nu}\right]}_{=R^\mu_{\rho\sigma\nu}} \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

$$= \frac{D^2\xi^\mu}{D\tau^2} + \xi^\sigma R^\mu_{\rho\sigma\nu} \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0.$$

Finally, we get the geodesic deviation equation:

$$\boxed{\frac{D^2\xi^\mu}{D\tau^2} + \xi^\sigma R^\mu_{\rho\sigma\nu} \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0}$$

1.8. Gravitational Wave Polarization

Now we set the velocity $v^a = \frac{dx^a}{d\tau} = (1,0,0,0)$ and the separation vector s^a between two points (a=1,2):

And assuming the slowly moving particles yields: $\frac{D^2 s^a}{D\tau^2} \simeq \frac{\partial^2 s^a}{\partial t^2}$, then we have:

$$\frac{\partial^2 s^a}{\partial t^2} = R^a{}_{00d} s^d$$

In the TT-gauge, the Riemann tensor:

$$R^a{}_{00d} = \frac{1}{2}\left(\partial_0 \partial_0 h^a{}_d{}^{TT} + \cancel{\partial_d \partial^a h_{00}^{TT}} - \cancel{\partial_0 \partial^a h_{0d}^{TT}} - \cancel{\partial_d \partial_0 h^a{}_0{}^{TT}}\right)$$

$$= \frac{1}{2}\partial_0^2 h^a{}_d{}^{TT}$$

Therefore, we have:

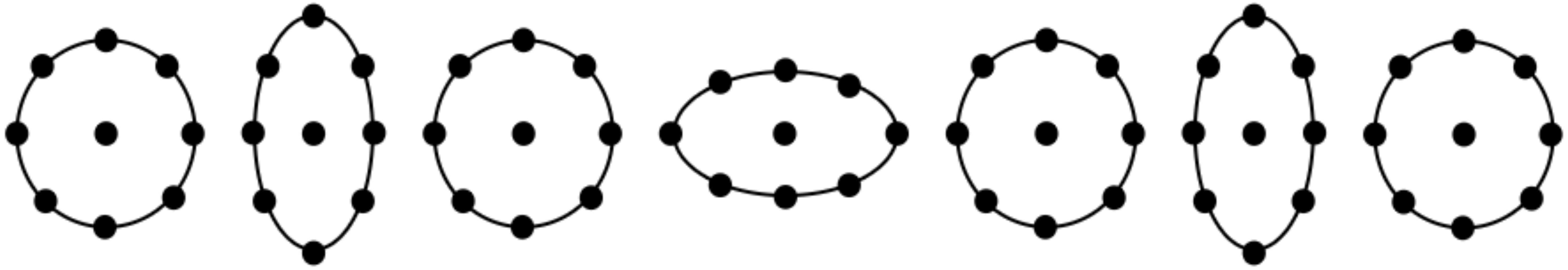
$$\frac{\partial^2 s^a}{\partial t^2} = \frac{1}{2} s^d \frac{\partial^2}{\partial t^2} h^a{}_d{}^{TT}$$

1.8.1. Plus Polarization: $A_\times = 0$

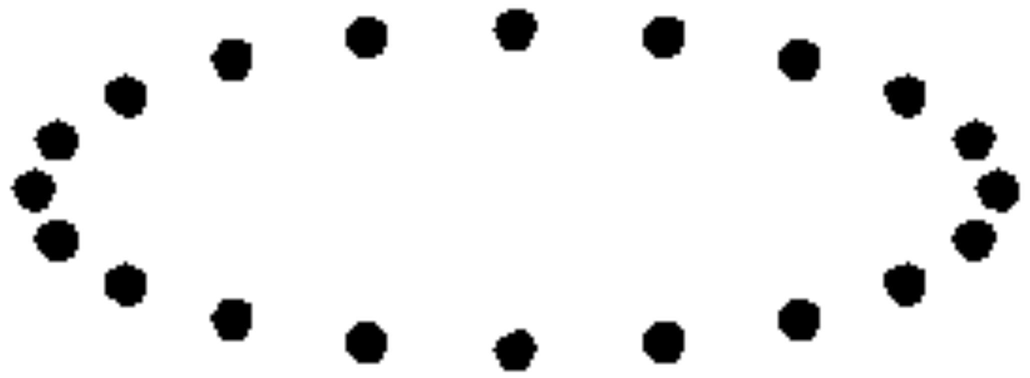
assuming $s^1(t) = s^1(0) + \delta s^1(t)$, **where** $||s^1(0)|| >> ||\delta s^1(t)||$

$$\frac{\partial^2}{\partial t^2} s^1 = \frac{1}{2} s^1 \frac{\partial^2}{\partial t^2}\left(A_+ e^{ik_\sigma x^\sigma}\right) \Rightarrow \frac{\partial^2}{\partial t^2}\delta s^1(t) \simeq \frac{1}{2} s^1(0) \frac{\partial^2}{\partial t^2}\left(A_+ e^{ik_\sigma x^\sigma}\right) \Rightarrow s^1(t) = s^1(0)\left(1 + \frac{1}{2} A_+ e^{ik_\sigma x^\sigma}\right)$$

$$\frac{\partial^2}{\partial t^2} s^2 = -\frac{1}{2} s^2 \frac{\partial^2}{\partial t^2}\left(A_+ e^{ik_\sigma x^\sigma}\right) \Rightarrow \frac{\partial^2}{\partial t^2}\delta s^2(t) \simeq -\frac{1}{2} s^2(0) \frac{\partial^2}{\partial t^2}\left(A_+ e^{ik_\sigma x^\sigma}\right) \Rightarrow s^2(t) = s^2(0)\left(1 - \frac{1}{2} A_+ e^{ik_\sigma x^\sigma}\right)$$



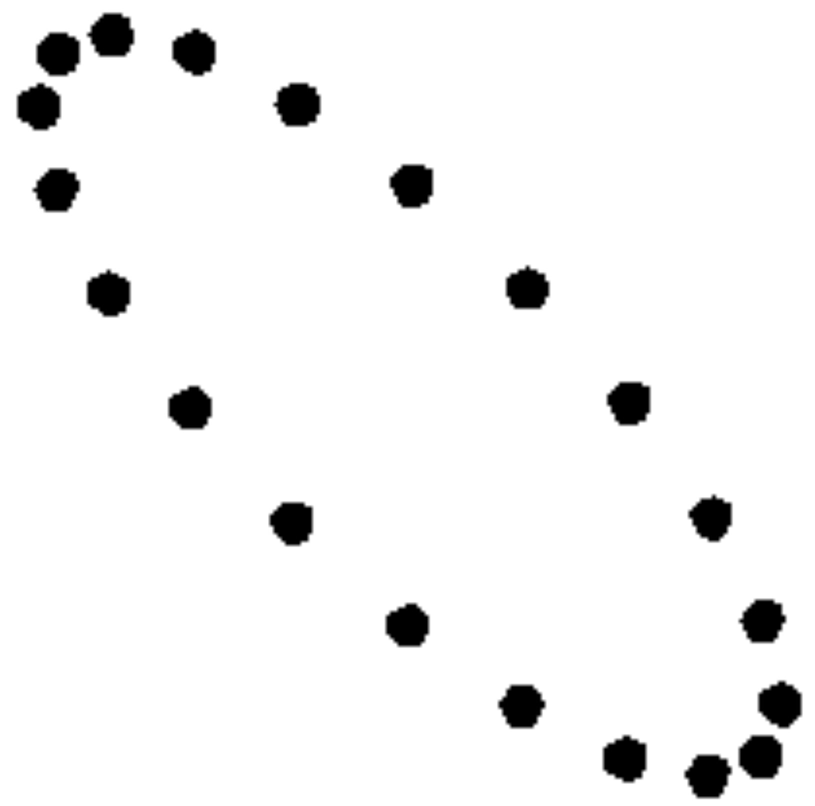
Plus Polarization of Gravitational-Waves

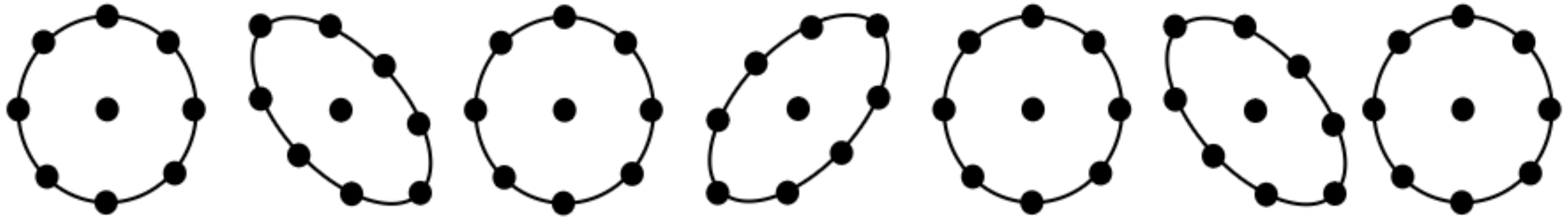


1.8.2. Cross Polarization: $A_+ = 0$

In the similar way,

$$s^1(t) = s^1(0) + \frac{1}{2} s^2(0) A_\times e^{ik_\sigma x^\sigma}$$
$$s^2(t) = s^2(0) + \frac{1}{2} s^1(0) A_\times e^{ik_\sigma x^\sigma}$$



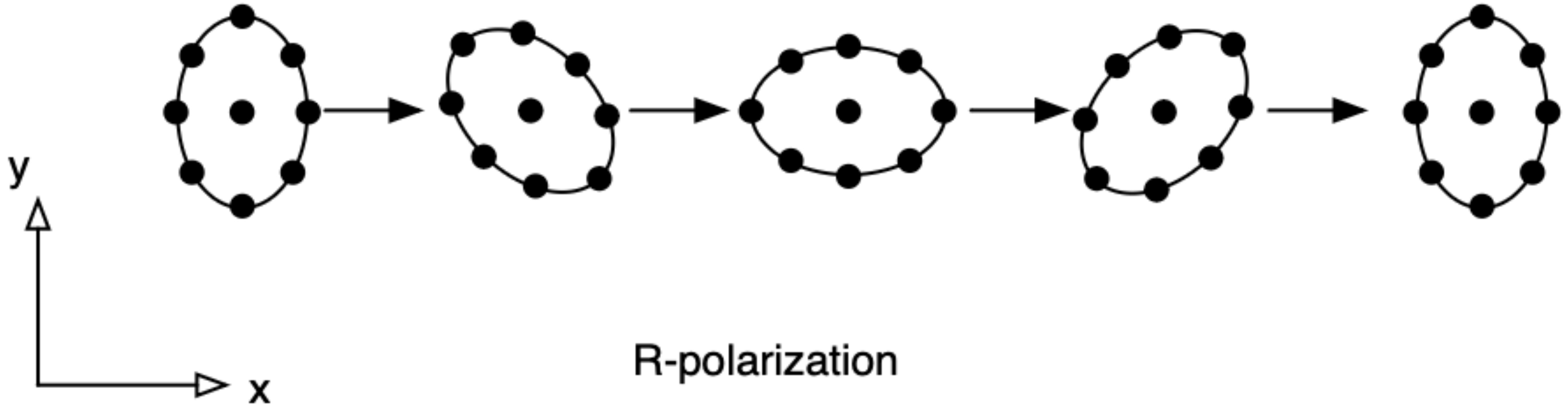


Cross Polarization of Gravitational-Waves

1.8.3. R-Polarization

Circularly polarized mode defined by

$$h_R \equiv \frac{1}{\sqrt{2}}(A_+ + iA_\times), \quad h_L \equiv \frac{1}{\sqrt{2}}(A_+ - iA_\times).$$



R-polarization

2. Generation of Gravitational Waves

2.1. Einstein Equation with Gravitational Wave Sources

Consider the linearized Einstein equation with matter source as: $\Box \bar{h}_{ab} = -\frac{16\pi G}{c^4} T_{ab}$

, which has a solution of

$$\bar{h}_{ab}(x^c) = -\frac{16\pi G}{c^4} \int d^4y \, G(x^c - y^c) T_{ab}(y^c)$$

Green's function satisfying:

$$\Box_x G(x^c - y^c) = \delta^{(4)}(x^c - y^c)$$

We define the retarded Green's function for considering the causality as:

$$G(x^c - y^c) = -\frac{1}{4\pi |\vec{x} - \vec{y}|} \delta\left(\frac{1}{c}|\vec{x} - \vec{y}| - (x^0 - y^0)\right) \Theta(x^0 - y^0)$$

where the Heaviside function is

$$\Theta(x^0 - y^0) = \begin{cases} 1 & (\text{if } x^0 > y^0) \\ 0 & (\text{if } x^0 < y^0) \end{cases}$$

and the retarded time defined as:

$$t_r \equiv t - \frac{1}{c}|\vec{x} - \vec{y}| = x^0 - \frac{1}{c}|\vec{x} - \vec{y}|$$

Assumptions:

1) The GW source is isolated and composed of non-relativistic matter

2) The GW source is fairly far-way from the detector (Earth)

Then we have a Fourier transformed GW solution as:

$$\tilde{\bar{h}}_{ab} = \frac{1}{\sqrt{2\pi}} \int dt e^{i\omega t} \bar{h}_{ab}(t, \vec{x})$$

$$= \frac{4G}{c^4\sqrt{2\pi}} \int dt \int d^3y e^{i\omega t} \frac{1}{|\vec{x}-\vec{y}|} T_{ab}\left(t - \frac{1}{c}|\vec{x}-\vec{y}|, \vec{y}\right)$$

$$= \frac{4G}{c^4\sqrt{2\pi}} \int dt_r \int d^3y e^{i\omega t_r} e^{i\omega\frac{|\vec{x}-\vec{y}|}{c}} \frac{1}{|\vec{x}-\vec{y}|} T_{ab}(t_r, \vec{y})$$

$$= \frac{4G}{c^4} \int d^3y \frac{e^{i\omega\frac{|\vec{x}-\vec{y}|}{c}}}{|\vec{x}-\vec{y}|} \underbrace{\frac{1}{\sqrt{2\pi}} \int dt_r e^{i\omega t_r} T_{ab}(t_r, \vec{y})}_{=\tilde{T}_{ab}(\omega,\vec{y})} = \frac{4G}{c^4} \int d^3y \tilde{T}_{ab}(\omega, \vec{y}) \frac{e^{i\omega\frac{|\vec{x}-\vec{y}|}{c}}}{|\vec{x}-\vec{y}|} \sim \frac{e^{i\omega r/c}}{r}$$

by above assumptions

Recall: Fourier Transformation:

$$\tilde{\phi}(\omega, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int dt e^{i\omega t} \phi(t, \vec{x})$$

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int d\omega e^{-i\omega t} \tilde{\phi}(\omega, \vec{x})$$

We, therefore, have:

$$\tilde{\bar{h}}_{ab}(\omega, \vec{x}) = \frac{4G}{c^4}\frac{e^{i\omega\frac{r}{c}}}{r}\boxed{\int d^3y \tilde{T}_{ab}(\omega, \vec{y})}$$

Applying the Lorentz gauge, then one finds:

$$\partial_a \bar{h}^{ab}(t, \vec{x}) = 0 = \partial_a \left[\frac{1}{\sqrt{2\pi}}\int d^3y e^{-i\omega t}\tilde{\bar{h}}^{ab}(\omega, \vec{y})\right] = \frac{1}{\sqrt{2\pi}}\int d^3y \partial_a \left[e^{-i\omega t}\tilde{\bar{h}}^{ab}(\omega, \vec{y})\right]$$

Then we get: $\partial_t(e^{-i\omega t}\tilde{\bar{h}}^{0b}) + \partial_i(e^{-i\omega t}\tilde{\bar{h}}^{ib}) = 0$

Equivalently, $\tilde{\bar{h}}^{0b} = \frac{1}{i\omega}\partial_i \tilde{\bar{h}}^{ib}$ ⇐ the recursive relation: $\tilde{\bar{h}}^{00} \leftarrow \tilde{\bar{h}}^{i0} \leftarrow \tilde{\bar{h}}^{ij}$

This means that we only have to know about the spatial component of h_{ij}, then from the recursive relation, we can get all informations of the metric. So, we only focus on the spatial component of the source, T_{ij}

2.2. Quadrupole Formula of GWs

Consider:

$$\int d^3y \tilde{T}^{ij}(\omega,\vec{y}) = \underbrace{\int \partial_k(y^i \tilde{T}^{kj}) d^3y}_{\text{vanishes:surface term}} - \int y^i(\partial_k \tilde{T}^{ij}) d^3y$$

$$= i\omega \int y^i \tilde{T}^{0j} d^3y$$

$$= \frac{i\omega}{2}\int \left(y^i \tilde{T}^{0j} + y^j \tilde{T}^{0i}\right) d^3y$$

$$= \frac{i\omega}{2}\int d^3y \left[\underbrace{\partial_k(y^i y^j \tilde{T}^{0k})}_{=0 \text{ (surface term)}} - \underbrace{y^i y^j \partial_k \tilde{T}^{0k}}_{-i\omega y^i y^j \tilde{T}^{00}} \right]$$

$$= -\frac{\omega^2}{2}\tilde{\mathcal{I}}_{ij}(\omega)$$

Energy-momentum conservation:

$$\partial_\mu T^{\mu\nu}(t,\vec{x}) = 0 = \frac{1}{\sqrt{2\pi}}\int d\omega \partial_\mu \left(e^{i\omega t}\tilde{T}^{\mu\nu}(\omega,\vec{y})\right)$$

$$\Rightarrow \quad \partial_i \tilde{T}^{i\nu} = -i\omega \tilde{T}^{0\nu}$$

Quadrupole momentum tensor:

$$\mathcal{I}_{ij}(t) \equiv \int y^i y^j T^{00}(t,\vec{y}) d^3y$$

$$\tilde{\mathcal{I}}^{FT}_{ij}(\omega) = \int y^i y^j \tilde{T}^{00}(\omega,\vec{y}) d^3y$$

We, therefore, have:

$$\tilde{\bar{h}}_{ij}(\omega,\vec{x}) = \frac{4G}{c^4}\frac{e^{i\omega r/c}}{r}\int d^3y\tilde{T}_{ij}(\omega,\vec{y}) = -\frac{2G}{c^4}\omega^2\frac{e^{i\omega r/c}}{r}\tilde{\mathcal{I}}_{ij}(\omega)$$

The strain tensor in the time-domain can be obtained by the inverse Fourier Transform as:

$$\begin{aligned}\bar{h}_{ij}(t,\vec{x}) &= \frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega t}\tilde{\bar{h}}_{ij}(\omega,\vec{x})\\ &= -\frac{2G}{c^4 r}\frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega t}e^{i\omega r/c}\omega^2\tilde{\mathcal{I}}_{ij}(\omega)\\ &= \frac{2G}{c^4 r}\frac{1}{\sqrt{2\pi}}\int d\omega\frac{d^2}{dt^2}e^{-i\omega t}e^{i\omega r/c}\tilde{\mathcal{I}}_{ij}(\omega)\\ &= \frac{2G}{c^4 r}\frac{d^2}{dt^2}\left[\frac{1}{\sqrt{2\pi}}\int d\omega\,\underbrace{e^{-i\omega(t-r/c)}}_{\equiv e^{-i\omega t_r}}\tilde{\mathcal{I}}_{ij}(\omega)\right]\\ &= \frac{2G}{c^4 r}\frac{d^2}{dt^2}\mathcal{I}_{ij}(t_r)\end{aligned}$$

The quadrupole formula of the GW strain tensor

$$\bar{h}_{ij}(t,\vec{x}) = \frac{2G}{c^4 r}\frac{d^2}{dt^2}\mathcal{I}_{ij}(t_r)$$

2.3. Energy Loss from Gravitational Radiation

2.3.1. Weak Field Approximation

We assume the weak field approximation of the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, keeping $\mathcal{O}(h^2)$ and work in the TT-gauge frame. Then the Christoffel symbols are:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}) - \frac{1}{2}h^{\alpha\beta}(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu})$$

The Ricci tensor is:

$$\begin{aligned}
R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} &= \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}) - \frac{1}{2}\partial_\alpha h^{\alpha\beta}(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}) \\
&- \frac{1}{2}h^{\alpha\beta}\partial_\alpha(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}) - \frac{1}{2}\eta^{\alpha\beta}\partial_\nu(\partial_\alpha h_{\mu\beta} + \partial_\mu h_{\beta\alpha} - \partial_\beta h_{\mu\alpha}) \\
&+ \frac{1}{2}\partial_\nu h^{\alpha\beta}(\partial_\alpha h_{\mu\beta} + \partial_\mu h_{\beta\alpha} - \partial_\beta h_{\mu\alpha}) + \frac{1}{2}h^{\alpha\beta}\partial_\nu(\partial_\alpha h_{\mu\beta} + \partial_\mu h_{\beta\alpha} - \partial_\beta h_{\mu\alpha}) \\
&+ \frac{1}{4}\left[\eta^{\beta\sigma}(\partial_\nu h_{\mu\sigma} + \partial_\mu h_{\sigma\nu} - \partial_\sigma h_{\mu\nu}) - h^{\beta\sigma}(\partial_\nu h_{\mu\sigma} + \partial_\mu h_{\sigma\nu} - \partial_\sigma h_{\mu\nu})\right] \\
&\qquad \times \left[\eta^{\alpha\beta}(\partial_\alpha h_{\beta\rho} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\beta\alpha}) - h^{\alpha\beta}(\partial_\alpha h_{\beta\rho} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\beta\alpha})\right] \\
&- \frac{1}{4}\left[\eta^{\beta\sigma}(\partial_\alpha h_{\mu\sigma} + \partial_\mu h_{\sigma\alpha} - \partial_\sigma h_{\mu\alpha}) - h^{\rho\sigma}(\partial_\alpha h_{\mu\sigma} + \partial_\mu h_{\sigma\alpha} - \partial_\sigma h_{\mu\alpha})\right] \\
&\qquad \times \left[\eta^{\alpha\rho}(\partial_\nu h_{\rho\beta} + \partial_\beta h_{\rho\nu} - \partial_\rho h_{\beta\nu}) - h^{\alpha\rho}(\partial_\nu h_{\rho\beta} + \partial_\beta h_{\rho\nu} - \partial_\rho h_{\beta\nu})\right]
\end{aligned}$$

Consider the vacuum Einstein equation, $R_{\mu\nu} = 0$, to the 2nd order perturbation, then

$$0 = R_{\mu\nu} = R^{(\eta)}_{\mu\nu} + R^{(h)}_{\mu\nu} + R^{(h^2)}_{\mu\nu}$$

So we have:

$$R^{(h)}_{\mu\nu} = -R^{(h^2)}_{\mu\nu} \equiv \frac{8\pi G}{c^4} t_{\mu\nu}$$

This term can be interpreted as the first-order perturbation term of h in flat space being affected by the second-order perturbation term. In other words, by interpreting the second-order perturbation term of h as energy, we can describe the energy term where the influence of the first-order perturbation term of h is stored.

Now we consider:

$$R^{(2)}_{\mu\nu} = -\frac{1}{2}h^{\alpha\beta}\partial_\alpha(\partial_\nu h_{\mu\beta}+\partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}) + \frac{1}{2}\partial_\nu h^{\alpha\beta}(\partial_\alpha h_{\mu\beta} + \partial_\mu h_{\beta\alpha} - \partial_\beta h_{\mu\alpha}) + \frac{1}{2}h^{\alpha\beta}\partial_\nu(\partial_\alpha h_{\mu\beta}+\partial_\mu h_{\beta\alpha} - \partial_\beta h_{\mu\alpha})$$

$$+\frac{1}{4}\eta^{\beta\sigma}\eta^{\alpha\rho}(\partial_\nu h_{\mu\sigma} + \partial_\mu h_{\sigma\nu} - \partial_\sigma h_{\mu\nu})\underbrace{(\partial_\alpha h_{\beta\rho} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\beta\alpha})}_{=0,\ \text{resp.: } h^\alpha_{\ \alpha}=0,\ \text{or } \partial_\alpha h^{\alpha\beta}=0} - \frac{1}{4}\eta^{\beta\sigma}\eta^{\alpha\rho}(\partial_\alpha h_{\mu\sigma} + \partial_\mu h_{\sigma\alpha} - \partial_\sigma h_{\mu\alpha})(\partial_\nu h_{\rho\beta} + \partial_\beta h_{\rho\nu} - \partial_\rho h_{\beta\nu})$$

$$= -\frac{1}{2}h^{\alpha\beta}\partial_\alpha\partial_\mu h_{\beta\nu} + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \frac{1}{2}h^{\alpha\beta}\partial_\nu\partial_\mu h_{\alpha\beta} - \frac{1}{2}h^{\alpha\beta}\partial_\nu\partial_\beta h_{\mu\alpha} + \frac{1}{2}\partial_\nu h^{\alpha\beta}\partial_\alpha h_{\mu\beta} + \frac{1}{2}\partial_\nu h^{\alpha\beta}\partial_\mu h_{\beta\alpha} - \frac{1}{2}\partial_\nu h^{\alpha\beta}\partial_\beta h_{\mu\alpha}$$

$$-\frac{1}{4}\eta^{\beta\sigma}\eta^{\alpha\rho}\left(\partial_\alpha h_{\mu\sigma}\partial_\nu h_{\rho\beta} + \partial_\alpha h_{\mu\sigma}\partial_\beta h_{\rho\nu} - \partial_\alpha h_{\mu\sigma}\partial_\rho h_{\beta\nu} + \partial_\mu h_{\sigma\alpha}\partial_\nu h_{\rho\beta} + \partial_\mu h_{\sigma\alpha}\partial_\beta h_{\rho\nu} - \partial_\mu h_{\sigma\alpha}\partial_\rho h_{\beta\nu} - \partial_\sigma h_{\mu\alpha}\partial_\nu h_{\rho\beta} - \partial_\sigma h_{\mu\alpha}\partial_\beta h_{\mu\nu} + \partial_\sigma h_{\mu\alpha}\partial_\rho h_{\beta\nu}\right)$$

$$= -h^{\alpha\beta}\partial_\alpha\partial_{(\mu}h_{\nu)\beta} + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \frac{1}{2}h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + \frac{1}{2}\partial_\nu h^{\alpha\beta}\partial_\alpha h_{\mu\beta} + \frac{1}{2}\partial_\nu h^{\alpha\beta}\partial_\mu h_{\beta\alpha} - \frac{1}{2}\partial_\nu h^{\alpha\beta}\partial_\beta h_{\mu\alpha}$$

$$-\frac{1}{4}\left[\partial^\rho h_\mu^{\ \beta}\partial_\nu h_{\rho\beta}+\partial^\rho h_{\mu\sigma}\partial^\sigma h^\alpha_{\ \nu} - \partial^\rho h_\mu^{\ \beta}\partial_\rho h_{\beta\nu} + \partial_\mu h^\beta_{\ \alpha}\partial_\nu h^\alpha_{\ \beta}+\partial_\mu h^\beta_{\ \alpha}\partial_\beta h^\alpha_{\ \nu}-\overbrace{\partial_\mu h^\beta_{\ \alpha}\partial^\alpha h_{\beta\nu}}^{=\partial_\mu h^\beta_{\ \alpha}\partial_\beta h^\alpha_{\ \nu}} - \underbrace{\partial^\beta h_{\mu\alpha}\partial_\nu h^\alpha_{\ \beta}}_{=\partial^\rho h_\mu^{\ \beta}\partial_\nu h_{\beta\rho}} - \underbrace{\partial^\beta h_{\mu\alpha}\partial_\beta h^\alpha_{\ \nu}}_{=\partial^\rho h_\mu^{\ \beta}\partial_\rho h_{\rho\nu}} + \underbrace{\partial^\beta h_{\mu\alpha}\partial^\alpha h_{\beta\nu}}_{=\partial^\rho h_{\mu\sigma}\partial^\sigma h_{\rho\nu}}\right]$$

$$= - h^{\alpha\beta}\partial_\alpha\partial_{(\mu}h_{\nu)\beta} + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \frac{1}{2}h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + \frac{1}{2}\partial_\nu h^{\alpha\beta}\partial_\alpha h_{\mu\beta}$$

$$+\frac{1}{2}\partial_\nu h^{\alpha\beta}\partial_\mu h_{\beta\alpha} - \frac{1}{2}\underbrace{\partial_\nu h^{\alpha\beta}\partial_\beta h_{\mu\alpha}}_{=\partial_\nu h^{\alpha\beta}\partial_\alpha h_{\mu\beta}} - \frac{1}{2}\partial^\rho h_{\mu\sigma}\partial^\sigma h_{\rho\nu} + \frac{1}{2}\partial^\rho h_\mu{}^\beta\partial_\rho h_{\beta\nu} - \frac{1}{4}\partial_\mu h^\beta{}_\alpha\partial_\nu h^\alpha{}_\beta$$

$$= - h^{\alpha\beta}\partial_\alpha\partial_{(\mu}h_{\nu)\beta} + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \frac{1}{2}h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} - \frac{1}{2}\partial^\alpha h_{\mu\beta}\partial^\beta h_{\alpha\nu} + \frac{1}{2}\eta^{\beta\lambda}\partial^\alpha h_{\mu\lambda}\partial_\alpha h_{\beta\nu}$$

2.3.2. Averaged Bracket

We wish to interpret $t_{\mu\nu}$ as an energy-momentum tensor but there are some limitation unfortunately since it is not a tensor in the full theory and not invariant under gauge transformations (no diffeomorphism symmetry). Thus one way of circumventing this difficulty is to average the energy-momentum tensor over wavelengths. Since it has no diffeomorphism, it is difficult to choose an appropriate Riemann normal coordinates to measure an energy-momentum tensor that is purely local. However, we might choose enough physical curvature in a small region to have an gauge-invariant measurement by averaging over serveral wavelengths. In this sense, we introduce an averaged bracket such that

$$< A > \equiv \int_{all} dx^\mu A \qquad \text{and} \qquad < \partial_\mu A > = \int_{all} dx^\mu \partial_\mu A = 0$$

Now we finally obtain:

$$< R^{(2)}_{\mu\nu} > = -\frac{1}{4} < \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} > - \frac{1}{2} < \eta^{\beta\lambda} h_{\mu\lambda} \underbrace{\Box\, h_{\beta\nu}}_{=0} >$$

$$= -\frac{1}{4} < \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} >$$

Therefore, the averaged energy-momentum tensor:

$$\mathrm{t}_{\mu\nu} \equiv < t_{\mu\nu} > = -\frac{c^4}{8\pi G} < R^{(h^2)}_{\mu\nu} > = \frac{c^4}{32\pi G} < \partial_\mu h^{\alpha\beta}_{TT} \partial_\nu h^{TT}_{\alpha\beta} >$$

2.4. Energy Loss from Gravitational Waves

From the energy-momentum conservation, $\partial_\mu \mathfrak{t}^{\mu\nu} = 0$, then we have for a certain volume V:

$$\int_V dx^3 (\partial_0 \mathfrak{t}^{00} + \partial_i \mathfrak{t}^{i0}) = 0,$$

which gives $\partial_0 \mathfrak{t}^{00} = -\partial_i \mathfrak{t}^{i0}$.

Then the gravitational wave energy inside the volume V is given by

$$E_V = \int_V dx^3 \mathfrak{t}^{00}$$

Now the power (the energy change by time) can be defined by

$$P = -\frac{dE_V}{dt} = -\int_V dx^3 \partial_0 \mathfrak{t}^{00} = \int_V dx^3 \partial_i \mathfrak{t}^{i0} = \int_S dA n_i \mathfrak{t}^{i0} = \int_S dA n_r \mathfrak{t}^{r0} = \int_S r^2 d\Omega n_r \mathfrak{t}^{r0},$$

where $n^\mu = (0,1,0,0)$

Defining the spatial projection tensor as: $\mathscr{P}_{ij} = \delta_{ij} - n_i n_j$ with properties of

$$\begin{cases} \mathscr{P}^i_i = \delta^i_i - n^i n_i = 3 - 1 = 2, \\ \mathscr{P}_{ij}\mathscr{P}^{ij} = 2, \\ n^i \mathscr{P}_{ij} = n^i(\delta_{ij} - n_i n_j) = n_j - n_j = 0 \end{cases}$$

then we can construct the TT-version of arbitrary spatial symmetric tensor as

$$X^{TT}_{ij} = \left(\mathscr{P}^k_i \mathscr{P}^l_j - \frac{1}{2}\mathscr{P}_{ij}\mathscr{P}^{kl} \right) X_{kl}$$

Now we consider the TT-version of the strain tensor:

$$\begin{aligned} h^{TT}_{ij} = \bar{h}^{TT}_{ij} &= \left(\mathscr{P}^k_i \mathscr{P}^l_j - \frac{1}{2}\mathscr{P}_{ij}\mathscr{P}^{kl} \right) \bar{h}_{kl} \\ &= \frac{2G}{c^4 r}\frac{d^2}{dt^2} \underbrace{\left(\mathscr{P}^k_i \mathscr{P}^l_j - \frac{1}{2}\mathscr{P}_{ij}\mathscr{P}^{kl} \right) \mathscr{I}_{kl}(t_r)}_{\equiv \mathscr{I}^{TT}_{kl}} \quad = \frac{2G}{c^4 r}\frac{d^2}{dt^2}\mathscr{I}^{TT}_{ij}(t - r/c) \end{aligned}$$

Defining the reduced quadrupole moment as

$$\mathfrak{I}_{ij} \equiv \mathscr{I}_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}\mathscr{I}_{kl}$$

then we have

$$\mathfrak{I}_{ij}^{TT} = \left(\mathscr{P}_i^k \mathscr{P}_j^l - \frac{1}{2}\mathscr{P}_{ij}\mathscr{P}^{kl}\right)\mathscr{I}_{kl}$$

$$= \left(\mathscr{P}_i^k \mathscr{P}_j^l - \frac{1}{2}\mathscr{P}_{ij}\mathscr{P}^{kl}\right)\mathscr{I}_{kl} - \frac{1}{3}\left(\mathscr{P}_i^k \mathscr{P}_j^l - \frac{1}{2}\mathscr{P}_{ij}\mathscr{P}^{kl}\right)\delta_{kl}\delta^{mn}\mathscr{I}_{mn}$$

$$= \mathscr{I}_{ij}^{TT} - \underbrace{\left(\mathscr{P}_i^k \mathscr{P}_{jk} - \frac{1}{2}\mathscr{P}_{ij}\mathscr{P}_k^k\right)}_{=\mathscr{P}_i^k(\delta_{jk}-n_jn_k)-\mathscr{P}_{ij}=0}\frac{1}{3}\delta^{mn}\mathscr{I}_{mn}$$

$$= \mathscr{I}_{ij}^{TT}$$

Properties:

$$\mathfrak{I}_i^i = \delta_{ij}\left(\mathscr{I}^{ij} - \frac{1}{3}\delta^{ij}\delta^{kl}\mathscr{I}_{kl}\right) = 0$$

$$\mathscr{P}_{ij}\mathscr{I}^{ij} = (\delta_{ij} - n_in_j)\mathscr{I}^{ij} = \underbrace{\delta_{ij}\mathscr{I}^{ij}}_{=0} - n_in_j\mathscr{I}^{ij} = -n_in_j\mathscr{I}^{ij}$$

Finally, we get:

$$h_{ij}^{TT} = \frac{2G}{c^4 r}\frac{d^2}{dt^2}\mathfrak{I}_{ij}^{TT}(t - r/c)$$

2.4.1. Computing Power

We compute:

$$P = \int \mathfrak{t}_{0\mu} n^{\mu} r^2 d\Omega = \int \mathfrak{t}_{0r} r^2 d\Omega$$

where

$$\mathfrak{t}_{0r} = \frac{c^4}{32\pi G} < (\partial_0 h^{TT}_{\alpha\beta})(\partial_r h^{\alpha\beta}_{TT}) >$$

$$= -\frac{G}{8\pi r^2 c^5} < \dddot{\mathfrak{J}}^{TT}_{\alpha\beta} \dddot{\mathfrak{J}}^{\alpha\beta}_{TT} >$$

$$\begin{cases} \partial_0 h^{TT}_{\alpha\beta} = \dfrac{2G}{c^4 r} \dfrac{d^3}{dt^3} \mathfrak{J}^{TT}_{\alpha\beta}(t - r/c) \\ \partial_r h^{TT}_{\alpha\beta} = \dfrac{2G}{c^4 r} \dfrac{d^2}{dt^2} \partial_r \mathfrak{J}^{TT}_{\alpha\beta}(t - r/c) - \underbrace{\dfrac{2G}{c^4 r^2} \dfrac{d^2}{dt^2} \mathfrak{J}^{TT}_{\alpha\beta}(t - r/c)}_{\text{vanish as } r \to \infty} \\ \qquad = -\dfrac{2G}{c^4 r} \dfrac{d^3}{dt^3} \mathfrak{J}^{TT}_{\alpha\beta}(t - r/c) \end{cases}$$

Then the power is

$$P = -\frac{G}{8\pi c^5} \int < \dddot{\mathfrak{J}}^{TT}_{ij} \dddot{\mathfrak{J}}^{ij}_{TT} > d\Omega$$

Converting back to $\mathfrak{I}_{ij}$ from $\mathfrak{I}_{ij}^{TT}$ using the projection tensor:

$$\mathfrak{I}_{ij}^{TT} = \left(\mathscr{P}_i^k \mathscr{P}_j^l - \frac{1}{2}\mathscr{P}_{ij}\mathscr{P}^{kl}\right)\mathfrak{I}_{kl} = \left[(\delta_i^k - n_i n^k)(\delta_j^l - n_j n^l) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta^{kl} - n^k n^l)\right]\mathfrak{I}_{kl}$$

$$= \left[\delta_i^k \delta_j^l - n_i n^k \delta_j^l - n_j n^l \delta_i^k + n_i n^k n_j n^l - \frac{1}{2}(\delta_{ij}\delta^{kl} - n_i n_j \delta^{kl} - n^k n^l \delta_{ij} + n_i n_j n^k n^l)\right]\mathfrak{I}_{kl}$$

$$= \left[\mathfrak{I}_{ij} - n_i n^k \mathfrak{I}_{kj} - n_j n^l \mathfrak{I}_{il} - \frac{1}{2}\delta_{ij}\underbrace{\mathfrak{I}_k^k}_{=0} + \frac{1}{2}n_i n_j \underbrace{\mathfrak{I}_k^k}_{=0} + n_i n_j n^k n^l \mathfrak{I}_{kl} + \frac{1}{2}n^k n^l \delta_{ij}\mathfrak{I}_{kl} - \frac{1}{2}n^k n^l n_i n_j \mathfrak{I}_{kl}\right]$$

$$= \mathfrak{I}_{ij} - n_i n^k \mathfrak{I}_{kj} - n_j n^l \mathfrak{I}_{il} + n_i n_j n^k n^l \mathfrak{I}_{kl} + \frac{1}{2}\left(\delta_{ij} - n_i n_j\right) n^k n^l \mathfrak{I}_{kl}$$

$$= \mathfrak{I}_{ij} - n_i n^k \mathfrak{I}_{kj} - n_j n^l \mathfrak{I}_{il} + n_i n_j n^k n^l \mathfrak{I}_{kl} + \frac{1}{2}n^k n^l \mathscr{P}_{ij}\mathfrak{I}_{kl}$$

So now we can compute:

$$\ddddot{\mathfrak{I}}^{TT}_{ij}\ddddot{\mathfrak{I}}^{ij}_{TT} = \left[\ddddot{\mathfrak{I}}_{ij} - n_i n^k \ddddot{\mathfrak{I}}_{kj} - n_j n^l \ddddot{\mathfrak{I}}_{il} + n_i n_j n^k n^l \ddddot{\mathfrak{I}}_{kl} + \frac{1}{2} n^k n^l \mathscr{P}_{ij} \ddddot{\mathfrak{I}}_{kl}\right]\left[\ddddot{\mathfrak{I}}^{ij} - n^i n_a \ddddot{\mathfrak{I}}^{aj} - n^j n_a \ddddot{\mathfrak{I}}^{ia} + n^i n^j n_a n_b \ddddot{\mathfrak{I}}^{ab} + \frac{1}{2} n_a n_b \mathscr{P}^{ij}\ddddot{\mathfrak{I}}^{ab}\right]$$

$$= \ddddot{\mathfrak{I}}_{ij}\ddddot{\mathfrak{I}}^{ij} - n_i n^k \ddddot{\mathfrak{I}}^{ij}\ddddot{\mathfrak{I}}_{kj} - n_j n^l \ddddot{\mathfrak{I}}^{ij}\ddddot{\mathfrak{I}}_{il} + n_i n_j n^k n^l \ddddot{\mathfrak{I}}^{ij}\ddddot{\mathfrak{I}}_{kl} + \frac{1}{2} n^k n^l \underbrace{P_{ij}\ddddot{\mathfrak{I}}^{ij}}_{=-n_i n_j \ddddot{\mathfrak{I}}^{ij}} \ddddot{\mathfrak{I}}_{kl}$$

$$- n^i n_a \ddddot{\mathfrak{I}}_{ij}\ddddot{\mathfrak{I}}^{ja} + \underbrace{n_i n^i}_{=1} n_a n^k \ddddot{\mathfrak{I}}^{ja}\ddddot{\mathfrak{I}}_{kj} + n^i n_j n^l n_a \ddddot{\mathfrak{I}}^{ja}\ddddot{\mathfrak{I}}_{il} - \underbrace{n_i n^i}_{=1} n_j n_a n^k n^l \ddddot{\mathfrak{I}}^{ja}\ddddot{\mathfrak{I}}_{kl}$$

$$- \frac{1}{2} n^k n^l n_a \ddddot{\mathfrak{I}}^{ja}\ddddot{\mathfrak{I}}_{kl} \underbrace{n^i \mathscr{P}_{ij}}_{=0} - n^j n_a \ddddot{\mathfrak{I}}_{ij}\ddddot{\mathfrak{I}}^{ia} + n_i n^k n^j n_a \ddddot{\mathfrak{I}}_{kj}\ddddot{\mathfrak{I}}^{ia} + \underbrace{n^j n_j}_{=1} n^l n_a \ddddot{\mathfrak{I}}_{kl}\ddddot{\mathfrak{I}}^{ia}$$

$$- \underbrace{n^j n_j}_{=1} n_i n_a n^k n^l \ddddot{\mathfrak{I}}^{ia}\ddddot{\mathfrak{I}}_{kl} - \frac{1}{2} n^k n^l n_a \underbrace{n^j \mathscr{P}_{ij}}_{=0} \ddddot{\mathfrak{I}}^{ia}\ddddot{\mathfrak{I}}_{kl} + n^i n^j n^a n^b \ddddot{\mathfrak{I}}_{ij}\ddddot{\mathfrak{I}}^{ab} - \underbrace{n^i n_i}_{=1} n^k n^j n_a n_b \ddddot{\mathfrak{I}}^{ab}\ddddot{\mathfrak{I}}_{kj} - \underbrace{n^j n_j}_{=1} n^l n^i n_a n_b \ddddot{\mathfrak{I}}^{ab}\ddddot{\mathfrak{I}}_{il} + \underbrace{n_i n^i n_j n^j}_{=1} n_a n_b n^k n^l \ddddot{\mathfrak{I}}_{kl}\ddddot{\mathfrak{I}}^{ab}$$

$$+ \frac{1}{2} n_a n_b n^k n^l n^i \underbrace{n^j \mathscr{P}_{ij}}_{=0} \ddddot{\mathfrak{I}}_{kl}\ddddot{\mathfrak{I}}^{ab} + \frac{1}{2} n_a n_b \underbrace{\mathscr{P}^{ij}\ddddot{\mathfrak{I}}_{ij}}_{=-n_i n_j \ddddot{\mathfrak{I}}^{ij}} \ddddot{\mathfrak{I}}^{ab} - \frac{1}{2} n_a n_b n k \underbrace{n_i \mathscr{P}^{ij}}_{=0} \ddddot{\mathfrak{I}}_{kj}\ddddot{\mathfrak{I}}^{ab}$$

$$- \frac{1}{2} n^l n_a n_b \underbrace{n_j \mathscr{P}^{ij}}_{=0} \ddddot{\mathfrak{I}}_{il}\ddddot{\mathfrak{I}}^{ab} + \frac{1}{2} n^k n^l n_a n_b n_i \underbrace{n_j \mathscr{P}^{ij}}_{=0} \ddddot{\mathfrak{I}}_{kl}\ddddot{\mathfrak{I}}^{ab} + \frac{1}{4} n_a n_b n^k n^l \underbrace{\mathscr{P}_{ij}\mathscr{P}^{ij}}_{=2} \ddddot{\mathfrak{I}}_{kl}\ddddot{\mathfrak{I}}^{ab}$$

$$= \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} - n_i n^k \dddot{\mathfrak{I}}^{ij}\dddot{\mathfrak{I}}_{kj} - n_j n^l \dddot{\mathfrak{I}}^{ij}\dddot{\mathfrak{I}}_{il} + n_i n_j n^k n^l \dddot{\mathfrak{I}}^{ij}\dddot{\mathfrak{I}}_{kl} - \frac{1}{2} n^k n^l n_i n_j \dddot{\mathfrak{I}}^{ij}\dddot{\mathfrak{I}}_{kl}$$

$$-n^i n_a \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ja} + n_a n^k \dddot{\mathfrak{I}}^{ja}\dddot{\mathfrak{I}}_{kj} + n^i n_j n^l n_a \dddot{\mathfrak{I}}^{ja}\dddot{\mathfrak{I}}_{il} - n_j n_a n^k n^l \dddot{\mathfrak{I}}^{ja}\dddot{\mathfrak{I}}_{kl}$$

$$-n^j n_a \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ia} + n_i n^k n^j n_a \dddot{\mathfrak{I}}_{kj}\dddot{\mathfrak{I}}^{ia} + n^l n_a \dddot{\mathfrak{I}}_{kl}\dddot{\mathfrak{I}}^{ia} - n_i n_a n^k n^l \dddot{\mathfrak{I}}^{ia}\dddot{\mathfrak{I}}_{kl} + n^i n^j n^a n^b \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ab}$$

$$-n^k n^j n_a n_b \dddot{\mathfrak{I}}^{ab}\dddot{\mathfrak{I}}_{kj} - n^l n^i n_a n_b \dddot{\mathfrak{I}}^{ab}\dddot{\mathfrak{I}}_{il} + n_a n_b n^k n^l \dddot{\mathfrak{I}}_{kl}\dddot{\mathfrak{I}}^{ab} - \frac{1}{2} n_a n_b n_i n_j \dddot{\mathfrak{I}}^{ij}\dddot{\mathfrak{I}}^{ab} + \frac{1}{2} n_a n_b n^k n^l \dddot{\mathfrak{I}}_{kl}\dddot{\mathfrak{I}}^{ab}$$

$$= \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} - (4-2) n_i n^k \dddot{\mathfrak{I}}^{ij}\dddot{\mathfrak{I}}_{kj} + \left[\frac{11}{2} - 5\right] n_i n_j n^k n^l \dddot{\mathfrak{I}}_{kl}\dddot{\mathfrak{I}}^{ij}$$

$$= \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} - 2 n_i n^k \dddot{\mathfrak{I}}_{kj}\dddot{\mathfrak{I}}^{ij} + \frac{1}{2} n_i n_j n^k n^l \dddot{\mathfrak{I}}_{kl}\dddot{\mathfrak{I}}^{ij}$$

$$\therefore \ \dddot{\mathfrak{I}}^{TT}_{ij}\dddot{\mathfrak{I}}^{ij}_{TT} = \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} - 2 n_i n^k \dddot{\mathfrak{I}}_{kj}\dddot{\mathfrak{I}}^{ij} + \frac{1}{2} n_i n_j n^k n^l \dddot{\mathfrak{I}}_{kl}\dddot{\mathfrak{I}}^{ij}$$

Now we compute the power:

$$P = -\frac{G}{8\pi c^5}\int < \dddot{\mathfrak{I}}_{ij}^{TT}\dddot{\mathfrak{I}}^{ij}_{TT} > d\Omega$$

$$= -\frac{G}{8\pi c^5}\left[< \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} > \int d\Omega - 2 < \dddot{\mathfrak{I}}^{k}{}_{j}\dddot{\mathfrak{I}}^{ij} > \int n_i n_k d\Omega + \frac{1}{2} < \dddot{\mathfrak{I}}^{kl}\dddot{\mathfrak{I}}^{ij} > \int n_i n_j n_k n_l d\Omega\right]$$

$$= -\frac{G}{8\pi c^5}\left[4\pi < \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} > - \frac{8\pi}{3} < \dddot{\mathfrak{I}}^{k}{}_{j}\dddot{\mathfrak{I}}^{ij} > \delta_{ik} + \frac{4\pi}{30} < \dddot{\mathfrak{I}}^{kl}\dddot{\mathfrak{I}}^{ij} > (\underbrace{\delta_{ij}\delta_{kl}}_{\mathfrak{I}=0} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]$$

$$= -\frac{G}{8\pi c^5}4\pi < \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} > \underbrace{\left[1 - \frac{2}{3} + \frac{1}{15}\right]}_{=\frac{2}{5}}$$

$$= -\frac{G}{5c^5} < \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} >$$

$$\therefore \; P = -\frac{G}{5c^5} < \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} >$$

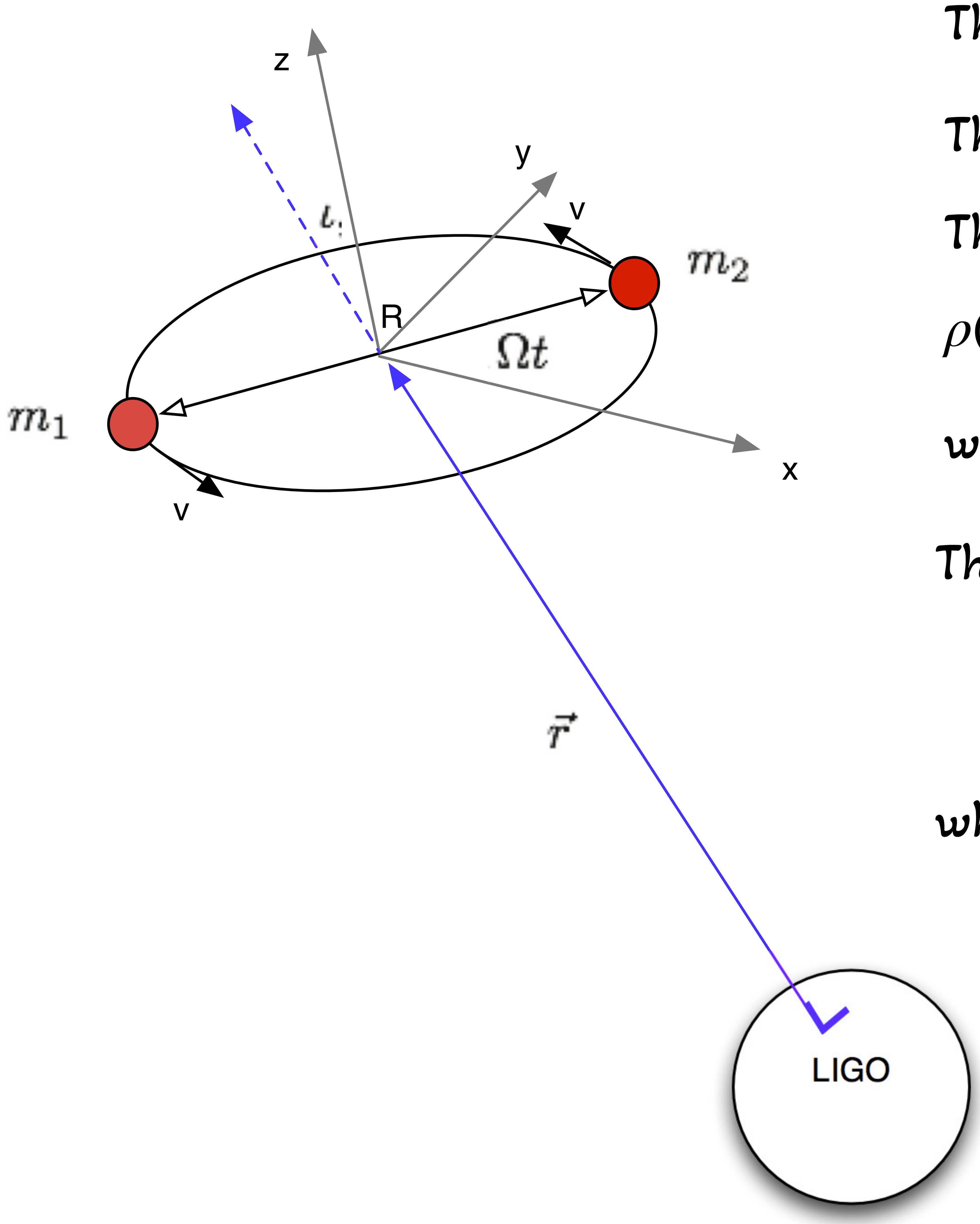
$$\int d\Omega = 4\pi$$

$$\int n_i n_j d\Omega = \frac{4\pi}{3}\delta_{ij}$$

$$\int n_i n_j n_k n_l d\Omega = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

3. Example: Gravitational Waves from the Binary Inspiral Source

3.1. Setup



The total mass: $M = m_1 + m_2$

The reduced mass: $\mu = \dfrac{m_1 m_2}{M}$

The mass distribution:

$$\rho(\vec{x}) = m_1 \delta(x - R_1 \cos \Omega t)\delta(y - R_1 \sin \Omega t) + m_2 \delta(x + R_2 \cos \Omega t)\delta(y + R_2 \sin \Omega t)\delta(z)$$

where $R_1 = \dfrac{m_2}{M} R$ and $R_2 = \dfrac{m_1}{M} R$

The orbital energy is computed as:

$$E = \frac{1}{2}\mu v^2 - \frac{M\mu}{R} = -\frac{GM\mu}{2R}$$

where $v = \sqrt{\dfrac{GM\mu}{R}}$ since we have $GM = \Omega^2 R^3 = 4\pi^2 \dfrac{R^3}{T^2}$ by Kepler's law and $v = \dfrac{2\pi R}{T}$

The quadrupole moment:

$$\mathcal{I}_{ij} = \int \rho(\mathbf{x}) x_i x_j d^3 x$$

$$\mathscr{I}_{xx} = \int \left\{m_1\delta(x - R_1\cos\Omega t)\delta(y - R_1\sin\Omega t)\delta(z) + m_2\delta(x + R_2\cos\Omega t)\delta(y + R_2\sin\Omega t)\delta(z)\right\} x^2 d^3x$$

$$= (m_1R_1^2 + m_2R_2^2)\cos^2\Omega t$$

$$= \left[m_1\left(\frac{m_2}{M}\right)^2 + m_2\left(\frac{m_1}{M}\right)^2\right] R^2\cos^2\Omega t = \mu R^2\cos^2\Omega t = \frac{1}{2}\mu R^2(1 + \cos 2\Omega t)$$

$$\Rightarrow \quad \ddot{\mathscr{I}}_{xx} = -2\mu\Omega^2R^2\cos 2\Omega t$$

$$\mathscr{I}_{yy} = \int \left\{m_1\delta(x - R_1\cos\Omega t)\delta(y - R_1\sin\Omega t)\delta(z) + m_2\delta(x + R_2\cos\Omega t)\delta(y + R_2\sin\Omega t)\delta(z)\right\} y^2 d^3x$$

$$= (m_1R_1^2 + m_2R_2^2)\sin^2\Omega t$$

$$= \left[m_1\left(\frac{m_2}{M}\right)^2 + m_2\left(\frac{m_1}{M}\right)^2\right] R^2\sin^2\Omega t = \mu R^2\sin^2\Omega t = \frac{1}{2}\mu R^2(1 - \cos 2\Omega t)$$

$$\Rightarrow \quad \ddot{\mathscr{I}}_{yy} = 2\mu\Omega^2R^2\cos 2\Omega t$$

$$\mathscr{I}_{xy} = \int \left\{m_1\delta(x - R_1\cos\Omega t)\delta(y - R_1\sin\Omega t)\delta(z) + m_2\delta(x + R_2\cos\Omega t)\delta(y + R_2\sin\Omega t)\delta(z)\right\} xy d^3x$$

$$= (m_1R_1^2 + m_2R_2^2)\cos\Omega t\sin\Omega t$$

$$= \left[m_1\left(\frac{m_2}{M}\right)^2 + m_2\left(\frac{m_1}{M}\right)^2\right] R^2\cos\Omega t\sin\Omega t = \mu R^2\cos\Omega t\sin\Omega t = \frac{1}{2}\mu R^2\sin 2\Omega t = \mathscr{I}_{yx}$$

$$\Rightarrow \quad \ddot{\mathscr{I}}_{xy} = -2\mu\Omega^2R^2\sin 2\Omega t = \ddot{\mathscr{I}}_{yx}$$

3.2. Detector's Frame

Coordinate transformation in terms of observer in (r, ι, ϕ) from (x, y, z) using tensor transformation:

$$A'_{ij} = \frac{\partial x'_i}{\partial x^k}\frac{\partial x'_j}{\partial x^l}A_{kl}, \quad \hat{\mathbf{e}}'_i = \frac{\partial x'_i}{\partial x^j}\hat{\mathbf{e}}_j$$

then we have

$$\hat{\mathbf{e}}_\iota = \cos\iota\cos\phi\hat{\mathbf{e}}_x + \cos\iota\sin\phi\hat{\mathbf{e}}_y - \sin\iota\hat{\mathbf{e}}_z$$

$$\hat{\mathbf{e}}_\phi = -\sin\phi\hat{\mathbf{e}}_x + \cos\phi\hat{\mathbf{e}}_y$$

The transformed quadrupole moment in the detector's frame:

$$\begin{aligned}
\ddot{\mathcal{I}}_{\iota\iota} &= \frac{\partial x'_\iota}{\partial x^k}\frac{\partial x'_\iota}{\partial x^l}\ddot{\mathcal{I}}_{kl} = \frac{\partial x'_\iota}{\partial x^x}\frac{\partial x'_\iota}{\partial x^x}\ddot{\mathcal{I}}_{xx} + 2\frac{\partial x'_\iota}{\partial x^x}\frac{\partial x'_\iota}{\partial x^y}\ddot{\mathcal{I}}_{xy} + \frac{\partial x'_\iota}{\partial x^y}\frac{\partial x'_\iota}{\partial x^y}\ddot{\mathcal{I}}_{yy} \\
&= \cos^2\iota\cos^2\phi\ddot{\mathcal{I}}_{xx} + 2\cos^2\iota\cos\phi\sin\phi\ddot{\mathcal{I}}_{xy} + \cos^2\iota\sin^2\phi\ddot{\mathcal{I}}_{yy} \\
&= -2\mu\Omega^2R^2\cos 2\Omega t\cos^2\iota(\cos^2\phi - \sin^2\phi) - 2\mu\Omega^2R^2\sin 2\Omega t\cos^2\iota\sin 2\phi \\
&= -2\mu\Omega^2R^2\cos 2\Omega t\cos^2\iota\cos 2\phi - 2\mu\Omega^2R^2\sin 2\Omega t\cos^2\iota\sin 2\phi \\
&= -2\mu\Omega^2R^2\cos^2\iota(\cos 2\Omega t\cos 2\phi + \sin 2\Omega t\sin 2\phi) \\
&= -2\mu\Omega^2R^2\cos^2\iota\cos 2(\Omega t - \phi)
\end{aligned}$$

$$\ddot{\mathcal{I}}_{\phi\phi} = \frac{\partial x'_\phi}{\partial x^k}\frac{\partial x'_\phi}{\partial x^l}\ddot{\mathcal{I}}_{kl} = \frac{\partial x'_\phi}{\partial x^x}\frac{\partial x'_\phi}{\partial x^x}\ddot{\mathcal{I}}_{xx} + 2\frac{\partial x'_\phi}{\partial x^x}\frac{\partial x'_\phi}{\partial x^y}\ddot{\mathcal{I}}_{xy} + \frac{\partial x'_\phi}{\partial x^y}\frac{\partial x'_\phi}{\partial x^y}\ddot{\mathcal{I}}_{yy}$$

$$= \sin^2\phi\ddot{\mathcal{I}}_{xx} - 2\sin\phi\cos\phi\ddot{\mathcal{I}}_{xy} + \cos^2\phi\ddot{\mathcal{I}}_{yy}$$

$$= -2\mu\Omega^2R^2\cos 2\Omega t(\sin^2\phi - \cos^2\phi) + 2\mu\Omega^2R^2\sin 2\Omega t\sin 2\phi$$

$$= 2\mu\Omega^2R^2\cos 2\Omega t\cos 2\phi + 2\mu\Omega^2R^2\sin 2\Omega t\sin 2\phi$$

$$= 2\mu\Omega^2R^2(\cos 2\Omega t\cos 2\phi + \sin 2\Omega t\sin 2\phi)$$

$$= 2\mu\Omega^2R^2\cos 2(\Omega t - \phi)$$

$$\ddot{\mathcal{I}}_{\iota\phi} = \frac{\partial x'_\iota}{\partial x^k}\frac{\partial x'_\phi}{\partial x^l}\ddot{\mathcal{I}}_{kl} = \frac{\partial x'_\iota}{\partial x^x}\frac{\partial x'_\phi}{\partial x^x}\ddot{\mathcal{I}}_{xx} + \left(\frac{\partial x'_\iota}{\partial x^x}\frac{\partial x'_\phi}{\partial x^y} + \frac{\partial x'_\iota}{\partial x^y}\frac{\partial x'_\phi}{\partial x^x}\right)\ddot{\mathcal{I}}_{xy} + \frac{\partial x'_\iota}{\partial x^y}\frac{\partial x'_\phi}{\partial x^y}\ddot{\mathcal{I}}_{yy}$$

$$= -\cos\iota\cos\phi\sin\phi\ddot{\mathcal{I}}_{xx} + \cos\iota(\cos^2\phi - \sin^2\phi)\ddot{\mathcal{I}}_{xy} + \cos\iota\cos\phi\sin\phi\ddot{\mathcal{I}}_{yy}$$

$$= 2\mu\Omega^2R^2\cos 2\Omega t\cos\iota\sin 2\phi - 2\mu\Omega^2R^2\sin 2\Omega t\cos\iota\cos 2\phi$$

$$= -2\mu\Omega^2R^2\cos\iota\sin 2(\Omega t - \phi)$$

Now we compute the reduced quadrupole moment by imposing traceless condition:

$$\ddot{\mathfrak{I}}^{TT}_{ij} = \ddot{\mathcal{I}}_{ij} - \frac{1}{2}\delta_{ij}\ddot{\mathcal{I}}^k{}_k$$

$$\ddot{\mathfrak{I}}^{TT}_{\iota\iota} = -\ddot{\mathfrak{I}}^{TT}_{\phi\phi} = \ddot{\mathcal{I}}_{\iota\iota} - \frac{1}{2}\left(\ddot{\mathcal{I}}_{\iota\iota} + \ddot{\mathcal{I}}_{\phi\phi}\right) = \frac{1}{2}\left(\ddot{\mathcal{I}}_{\iota\iota} - \ddot{\mathcal{I}}_{\phi\phi}\right) = -\mu\Omega^2R^2(1+\cos^2\iota)\cos 2(\Omega t - \phi)$$

$$\ddot{\mathfrak{I}}^{TT}_{\iota\phi} = \ddot{\mathfrak{I}}^{TT}_{\phi\iota} = -2\mu\Omega^2R^2\cos\iota\sin 2(\Omega t - \phi)$$

So we finally get the gravitational wave strain tensor:

$$h^{TT}_{ij} = \frac{2G}{c^4 r}\frac{d^2}{dt^2}\mathfrak{I}^{TT}_{ij}(t - r/c)$$

$$h_+(t) \equiv h^{TT}_{\iota\iota} = \frac{2G}{c^4 r}\ddot{\mathfrak{I}}^{TT}_{\iota\iota} = -\frac{2G\mu\Omega^2R^2}{c^4 r}(1+\cos^2\iota)\cos 2(\Omega t - \phi)$$

$$h_\times(t) \equiv h^{TT}_{\iota\phi} = \frac{2G}{c^4 r}\ddot{\mathfrak{I}}^{TT}_{\iota\phi} = -\frac{4G\mu\Omega^2R^2}{c^4 r}\cos\iota\sin 2(\Omega t - \phi)$$

By using the Kepler's law $GM = \Omega^2R^3$ and defining the gravitational wave frequency as $f_{gw} \equiv \Omega/\pi = 2f_r$

$$h_+(t) = -\frac{2G}{c^4 r}\mu(GM\pi f_{gw})^{2/3}(1+\cos^2\iota)\cos 2(\pi f_{gw}t - \phi)$$

$$h_\times(t) = -\frac{4G}{c^4 r}\mu(GM\pi f_{gw})^{2/3}\cos\iota\sin 2(\pi f_{gw}t - \phi)$$

↑ inclination angle (ι)   ↑ orbital phase (ϕ)

When the inclination angle vanishes, the GWs strain becomes maximal:

$$h_0 = \sqrt{h_+^2 + h_\times^2}\,\Big|_{\iota=0} = \frac{4G}{c^4 r}\mu(GM\pi f_{gw})^{2/3} \sim 1.23\times 10^{-22}\left(\frac{Mpc}{r}\right)\left(\frac{\mu}{M_\odot}\right)\left(\frac{M}{M_\odot}\right)^{2/3}\left(\frac{f_{gw}}{Hz}\right)^{2/3}$$

Defining the chirp mass as:

$$M_c = \mu^{3/5}M^{2/5} = \frac{(m_1 m_2)^{3/5}}{M^{1/5}}$$

yields:

$$h_+(t) = -\frac{2G}{c^4 r}M_c^{5/3}(G\pi f_{gw})^{2/3}(1+\cos^2\iota)\cos 2(\pi f_{gw}t - \phi)$$

$$h_\times(t) = -\frac{4G}{c^4 r}M_c^{5/3}(G\pi f_{gw})^{2/3}\cos\iota\,\sin 2(\pi f_{gw}t - \phi)$$

For GW150914: $m_1/M_\odot = 36$, $m_2/M_\odot = 30$, $r = 410Mpc$, and $f_{gw} = 150Hz$,

$$h_0 \sim 2.263\times 10^{-21}$$

3.3. Energy Loss by Gravitational Waves

Recall the TT-reduced quadrupole moment:

$$\mathfrak{I}_{ij}^{TT} \equiv \mathscr{I}_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}\mathscr{I}_{kl}$$

and the results in page 43: $\mathscr{I}_{xx} = \frac{1}{2}\mu R^2(1+\cos 2\Omega t)$ $\quad \mathscr{I}_{yy} = \frac{1}{2}\mu R^2(1-\cos 2\Omega t)$ $\quad \mathscr{I}_{xy} = \frac{1}{2}\mu R^2 \sin 2\Omega t = \mathscr{I}_{yx}$

we have TT-gauged reduced quadrupole moment in the source frame:

$$\begin{aligned}
\mathfrak{I}_{xx}^{TT} &= \mathscr{I}_{xx} - \frac{1}{3}\left(\mathscr{I}_{xx} + \mathscr{I}_{yy} + \mathscr{I}_{zz}\right) \\
&= \frac{1}{2}\mu R^2(1+\cos 2\Omega t) - \frac{1}{3}\left(\frac{1}{2}\mu R^2(1+\cos 2\Omega t) + \frac{1}{2}\mu R^2(1-\cos 2\Omega t)\right) = \mu R^2\left(\cos^2\Omega t - \frac{1}{3}\right) \\
\mathfrak{I}_{yy}^{TT} &= \mathscr{I}_{yy} - \frac{1}{3}\left(\mathscr{I}_{xx} + \mathscr{I}_{yy} + \mathscr{I}_{zz}\right) \\
&= \frac{1}{2}\mu R^2(1-\cos 2\Omega t) - \frac{1}{3}\left(\frac{1}{2}\mu R^2(1+\cos 2\Omega t) + \frac{1}{2}\mu R^2(1-\cos 2\Omega t)\right) = \mu R^2\left(\sin^2\Omega t - \frac{1}{3}\right)
\end{aligned}$$

$$\mathfrak{I}_{zz}^{TT} = \mathscr{I}_{zz} - \frac{1}{3}\left(\mathscr{I}_{xx} + \mathscr{I}_{yy} + \mathscr{I}_{zz}\right)$$

$$= -\frac{1}{3}\left(\frac{1}{2}\mu R^2(1+\cos 2\Omega t) + \frac{1}{2}\mu R^2(1-\cos 2\Omega t)\right) = -\frac{1}{3}\mu R^2$$

$$\mathfrak{I}_{xy}^{TT} = \mathfrak{I}_{yx}^{TT} = \mathscr{I}_{xy} = \frac{1}{2}\mu R^2 \sin 2\Omega t$$

Next, we compute the triple derivatives :

$$\dddot{\mathfrak{I}}_{xx}^{TT} = 4\mu R^2\Omega^3 \sin 2\Omega t = -\dddot{\mathfrak{I}}_{yy}^{TT}$$

$$\dddot{\mathfrak{I}}_{xy}^{TT} = -4\mu R^2\Omega^3 \cos 2\Omega t = \dddot{\mathfrak{I}}_{yx}^{TT}$$

$$\dddot{\mathfrak{I}}_{zz}^{TT} = 0$$

Then we get:

$$\dddot{\mathfrak{I}}_{ij}^{TT\ 2} = \dddot{\mathfrak{I}}_{xx}^{TT\ 2} + \dddot{\mathfrak{I}}_{yy}^{TT\ 2} + \dddot{\mathfrak{I}}_{zz}^{TT\ 2} + 2\dddot{\mathfrak{I}}_{xy}^{TT\ 2} = 2\left(\dddot{\mathfrak{I}}_{xx}^{TT\ 2} + \dddot{\mathfrak{I}}_{xy}^{TT\ 2}\right) = 32\mu^2 R^4\Omega^6(\sin^2 2\Omega t + \cos^2 2\Omega t)$$

$$= 32\mu^2 R^4 \Omega^6$$

Finally, the power, describing the energy change in time:

$$P = \dot{E} = -\frac{G}{5c^5} < \dddot{\mathfrak{I}}_{ij}\dddot{\mathfrak{I}}^{ij} > = -\frac{32G\mu^2R^4\Omega^6}{5c^5} = -\frac{32G\mu^2R^4}{5c^5}\left(\frac{GM}{R^3}\right)^3 = -\frac{32\mu^2G^4M^3}{5c^5R^5}$$

The inspiral rate for circular orbit:

$$\frac{dR}{dt} = \frac{dR}{dE}\frac{dE}{dt} = -\left(\frac{2R^2}{GM\mu}\right)\frac{32G^4\mu^2M^3}{5c^5R^5} = -\frac{64G^3\mu M^2}{5c^5R^3}$$

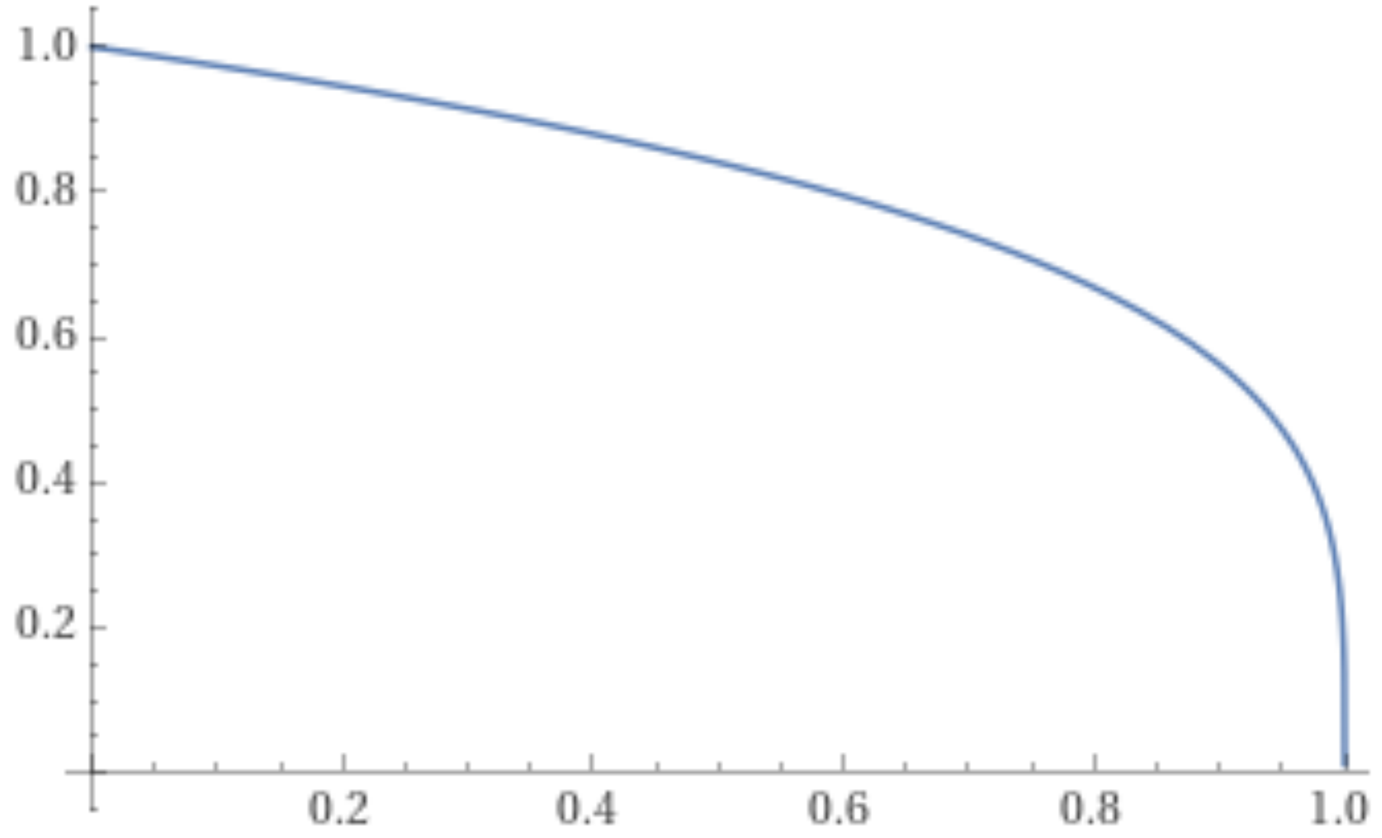
$\longleftarrow$ $E = -\frac{GM\mu}{2R}$

which can be integrated as

$$\frac{1}{4}R^4 = \int_0^R R^3 dR = -\frac{64G^3\mu M^2}{5c^5}\int_{t_c}^{t} dt = \frac{64G^3\mu M^2}{5c^5}(t_c - t)$$

yielding

$$R(t) = \left(\frac{256G^3\mu M^2}{5c^5}\right)^{\frac{1}{4}}(t_c - t)^{\frac{1}{4}}$$



Inspiral time, $t_{ins} \equiv t_c - t$

$$t_{insp} = \frac{5}{256\pi^{8/3}} \left(\frac{GM}{c^3}\right)^{-2/3} \left(\frac{G\mu}{c^3}\right)^{-1} f_{gw}^{-8/3} = 6.42 \times 10^3 \ sec \left(\frac{M}{M_\odot}\right)^{-2/3} \left(\frac{\mu}{M_\odot}\right)^{-1} \left(\frac{f_{gw}}{Hz}\right)$$

3.4. Gravitational Waveform

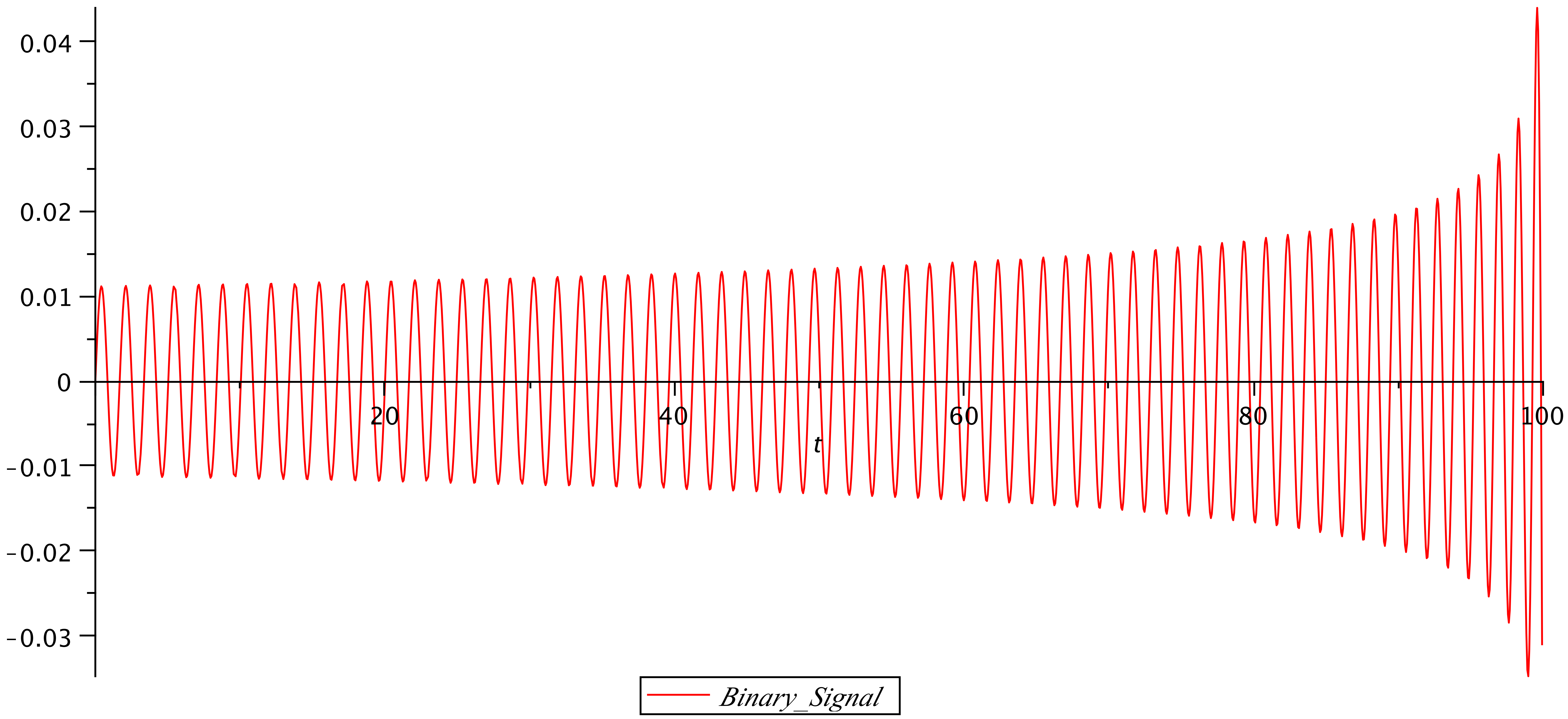
By Kepler's law, $GM = \Omega^2 R^3$

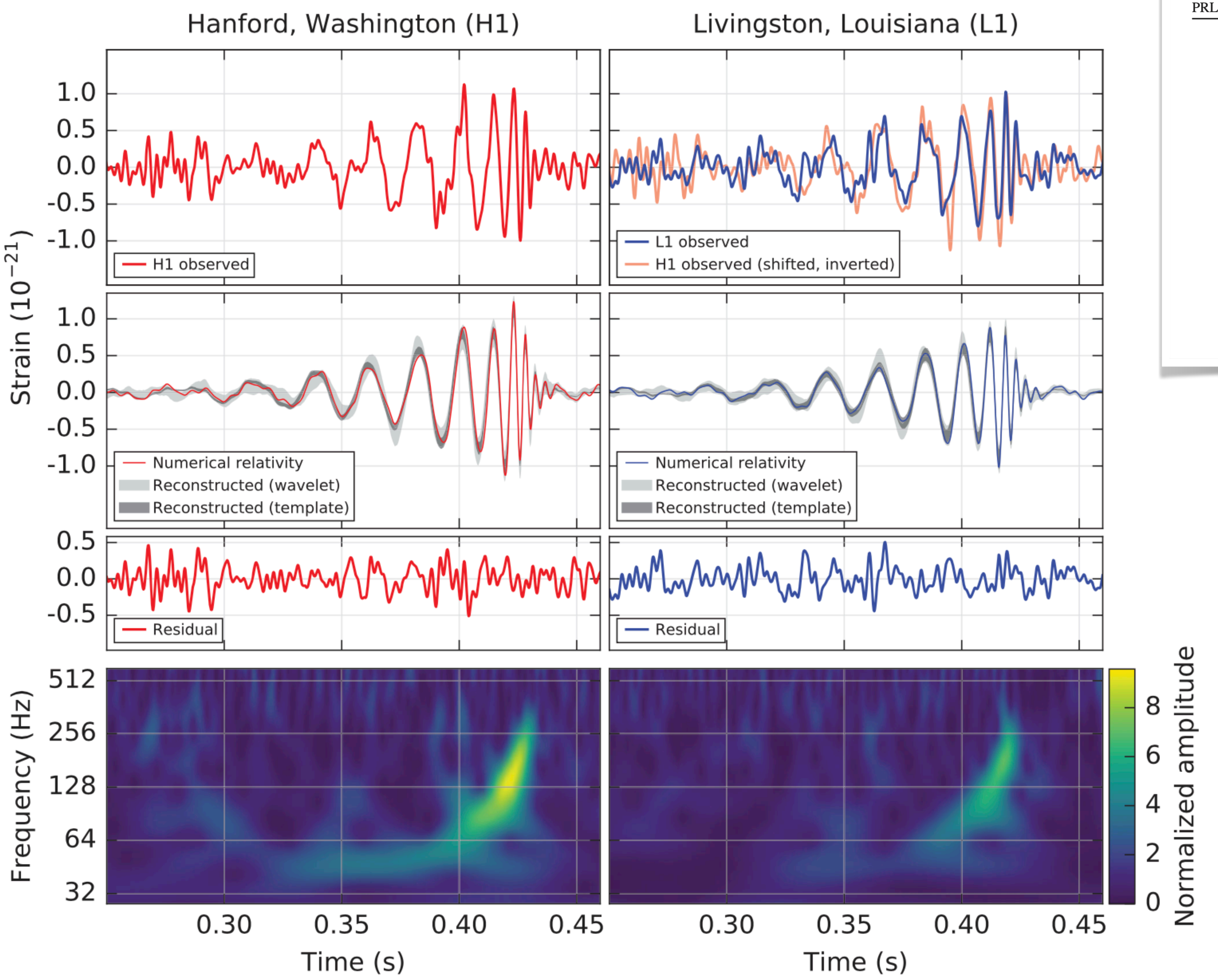
$$\Omega(t) = \frac{\sqrt{GM}}{\sqrt{R(t)}} = \left(\frac{256 G^{5/3}}{5c^5} M_c^{5/3}\right)^{-3/8} (t_c - t)^{-3/8}$$

then GWs strain:

$$h_+ \equiv -h_{xx}^{TT} = h_{yy}^{TT} = \frac{4G\mu}{D} \Omega^2(t) R^2(t) \cos 2\Omega(t) t$$

$$h_\times \equiv -h_{xy}^{TT} = -h_{yx}^{TT} = \underbrace{\frac{4G\mu}{D} \Omega^2(t) R^2(t)}_{\equiv A(t)} \sin 2 \underbrace{\Omega(t)}_{\equiv \Psi(t)} t$$


0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
20
40
60
80
100
t
Binary_Signal



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Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.**

(LIGO Scientific Collaboration and Virgo Collaboration)

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On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with a peak gravitational-wave strain of 1.0×10^{-21}. It matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. The signal was observed with a matched-filter signal-to-noise ratio of 24 and a false alarm rate estimated to be less than 1 event per 203 000 years, equivalent to a significance greater than 5.1σ. The source lies at a luminosity distance of 410^{+160}_{-180} Mpc corresponding to a redshift $z = 0.09^{+0.03}_{-0.04}$. In the source frame, the initial black hole masses are $36^{+5}_{-4} M_\odot$ and $29^{+4}_{-4} M_\odot$, and the final black hole mass is $62^{+4}_{-4} M_\odot$, with $3.0^{+0.5}_{-0.5} M_\odot c^2$ radiated in gravitational waves. All uncertainties define 90% credible intervals. These observations demonstrate the existence of binary stellar-mass black hole systems. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger.

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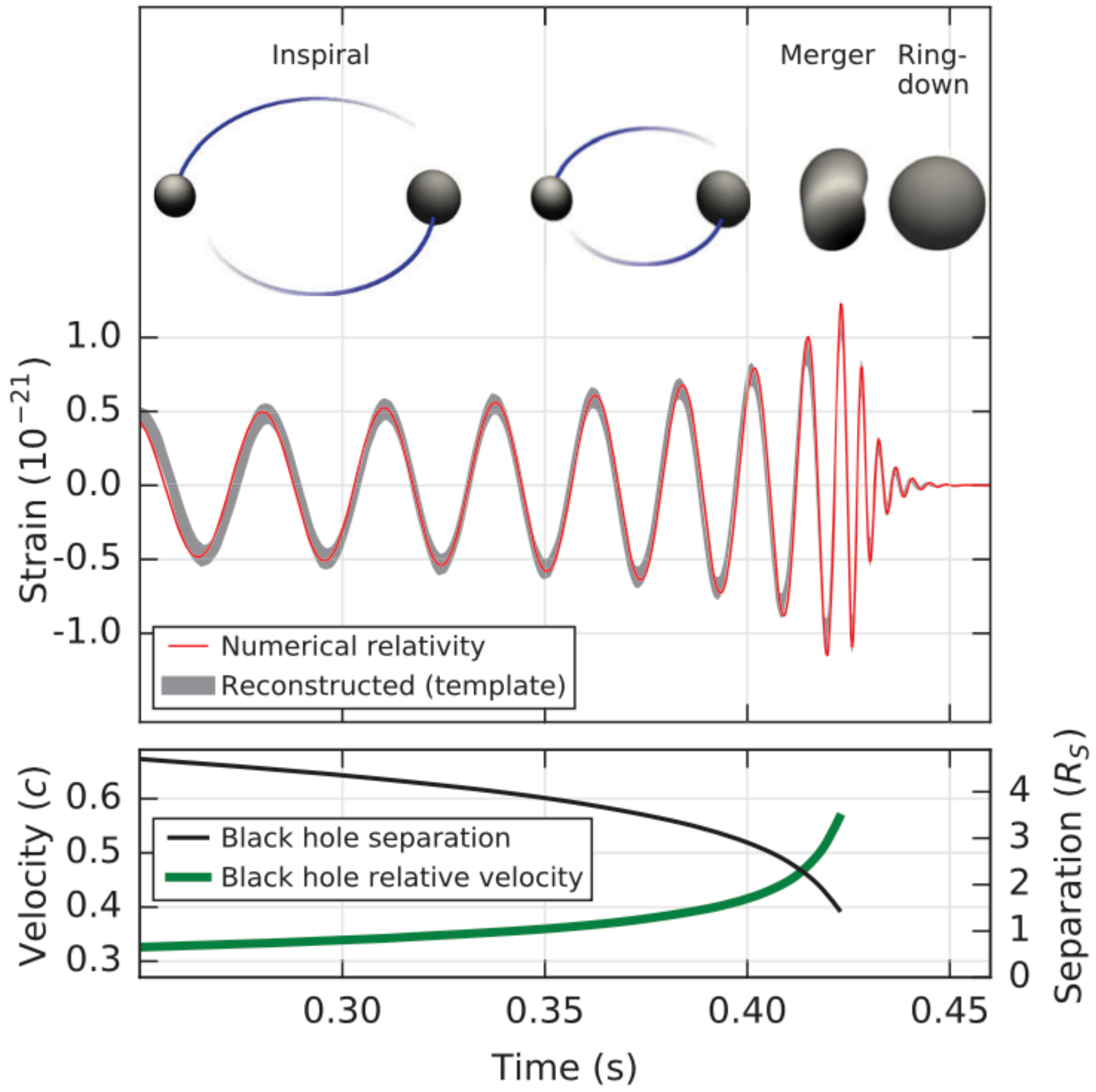

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