GW-BASIC

2022 Summer School on Numerical Relativity and Gravitational Waves

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1 Perturbations

1.1 Definition

We can understand perturbations in general relativity intuitively by introducing 5-dimensional manifold $\mathcal{F}$ which is foliated by a one-parameter family of perturbed spacetime $(\mathcal{M}_\epsilon, g(\epsilon))$ where ϵ is a perturbation parameter and $g(\epsilon)$ is metric of $\mathcal{M}_\epsilon$. To discuss difference between two quantities live in background and perturbed spacetime, we need a one-parameter group of diffeomorphism $\phi_\epsilon : \mathcal{M}_0 \to \mathcal{M}_\epsilon$ which maps points in $\mathcal{M}_0$ to points in $\mathcal{M}_\epsilon$. Then, the perturbed quantity of a geometrical quantity Q with left superscript ϵ is defined by

$$ {}^\epsilon Q \equiv \phi^*_{-\epsilon} Q(\epsilon), \tag{1}$$

where $\phi^*_{-\epsilon}$ is the push-forward (or pull-back) through $\phi_{-\epsilon}$ (or ϕ_ϵ). Its Taylor expansion is given by

$$ {}^\epsilon Q = Q + \epsilon \dot{Q} + \frac{1}{2}\epsilon^2 \ddot{Q} + O\left(\epsilon^3\right), \tag{2}$$

where $\dot{Q}$ and $\ddot{Q}$ is the first and second order derivative with respect to ϵ, respectively. From eq. (1), we identify that

$$ \dot{Q} = \mathcal{L}_V Q, \tag{3}$$

$$ \ddot{Q} = \mathcal{L}_V \mathcal{L}_V Q, \tag{4}$$

where V is the generator of ϕ_ϵ as

$$ V(f) \equiv \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(f \circ \phi_\epsilon - f\right), \tag{5}$$

for an arbitrary scalar field f.

1.2 Gauges

Notice that there are infinite ways to choose diffeomorphism ϕ_ϵ. It corresponds to the gauge freedom of perturbations in general relativity. The gauge transformation of $\dot{Q}$ between two different gauges, ϕ_ϵ and ϕ'_ϵ, becomes

$$ \mathcal{L}_{V'} Q - \mathcal{L}_V Q = \mathcal{L}_\xi Q, \tag{6}$$

where V and V' are generators of ϕ_ϵ and ϕ'_ϵ, respectively, and $\xi \equiv V' - V$. Moreover, ξ is tangent to $\mathcal{M}_0$ because

$$ \xi(\epsilon) = V'(\epsilon) - V(\epsilon) = 1 - 1 = 0, \tag{7}$$

where ϵ is understood scalar field on $\mathcal{F}$. Hence, we can evaluate $\mathcal{L}_\xi Q$ on $\mathcal{M}_0$ which menas

$$ \left[\mathcal{L}_\xi Q\right](0) = \mathcal{L}_{\xi(0)}\left[Q(0)\right]. \tag{8}$$

$\dot{Q}$ is gauge invariant if the above vanishes for all ξ. This is possible only when $Q(0)$ is zero, a constant scalar, or constructed by Kronecker delta with constant coefficients. This approach was first introduced in [Stewart and Walker(1974)] and reviewed in [Stewart(1990)].

1.3 Metric and Levi-Civita Tensor

The perturbed metric is expanded into

$$ {}^\epsilon g_{ab} = g_{ab} + \epsilon h_{ab} + O\left(\epsilon^2\right), \tag{9}$$

where $h \equiv \mathcal{L}_V g$. The perturbation of identity endormorphism δ vanishes because

$$\dot{v}^a = \mathcal{L}_V \left(\delta^a{}_b v^b\right) = \dot{\delta}^a{}_b v^b + \delta^a{}_b \dot{v}^b, \tag{10}$$
$$0 = \dot{\delta}^a{}_b v^b \tag{11}$$

for any vector v. Then, the perturbation of inverse metric becomes $\mathcal{L}_V g^{ab} = -h^{ab}$ because

$$0 = \dot{\delta}^a{}_b = g_{bc}\mathcal{L}_V g^{ac} + g^{ac}h_{bc}, \tag{12}$$
$$\mathcal{L}_V g^{ab} = -g^{ac}g^{bd}h_{cd} = -h^{ab}. \tag{13}$$

1.4 Levi-Civita Tensor

The normalization condition of Levi-Civita tensor,

$$-4! = \epsilon_{abcd}\epsilon^{abcd} \tag{14}$$

is perturbed by

$$0 = \mathcal{L}_V \left(g^{ae}g^{bf}g^{cg}g^{dh}\epsilon_{abcd}\epsilon_{efgh}\right) \tag{15}$$
$$= 2\epsilon^{abcd}\dot{\epsilon}_{abcd} - 4h^{ab}\epsilon_{acde}\epsilon_b{}^{cde} \tag{16}$$
$$= 2\epsilon^{abcd}\dot{\epsilon}_{abcd} + 4!h^{ab}g_{ab} \tag{17}$$

Then, we get

$$\dot{\epsilon}_{abcd} = \frac{1}{2}h^e{}_e\epsilon_{abcd}. \tag{18}$$

1.5 Covariant Derivatives

Let us consider a operation ${}^\epsilon\nabla$ defined by

$${}^\epsilon\nabla \equiv \phi^*_{-\epsilon}\nabla\phi^*_\epsilon, \tag{19}$$

where ∇ is the Levi-Civita connection associated with the spacetime metric g. In fact, this operation is also the Levi-Civita connection associated with the perturbed metric ${}^\epsilon g = \phi^*_{-\epsilon}g$ as shown by

$${}^\epsilon\nabla\,{}^\epsilon g = \phi^*_{-\epsilon}\nabla\phi^*_\epsilon\phi^*_{-\epsilon}g = 0 \tag{20}$$
$${}^\epsilon\nabla_{[a}{}^\epsilon\nabla_{b]}f = \phi^*_{-\epsilon}\nabla_{[a}\nabla_{b]}\phi^*_\epsilon f = 0 \tag{21}$$

where f is an arbitrary function. Then, the difference between covariant derivatives of a tensor T with respect to ${}^\epsilon\nabla$ and ∇ is written by

$$\left({}^\epsilon\nabla_c - \nabla_c\right)T^{a_1\cdots a_k}{}_{b_1\cdots b_l} = \sum_{i=1}^{k} T^{a_1\cdots d\cdots a_k}{}_{b_1\cdots b_l}\,{}^\epsilon C^{a_i}{}_{dc} - \sum_{i=1}^{l} T^{a_1\cdots a_k}{}_{b_1\cdots d\cdots b_l}\,{}^\epsilon C^d{}_{b_i c}, \tag{22}$$

as in [Wald(1984)] where

$${}^\epsilon C^a{}_{bc} = \frac{1}{2}{}^\epsilon g^{ad}\left(\nabla_c{}^\epsilon g_{bd} + \nabla_b{}^\epsilon g_{cd} - \nabla_d{}^\epsilon g_{bc}\right), \tag{23}$$

where ${}^\epsilon g^{ab}$ is the inverse of ${}^\epsilon g_{ab}$.

Defining $\dot{C}$ and $\dot{\nabla}$ as

$$\dot{C}^a{}_{bc} \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left({}^\epsilon C^a{}_{bc} - 0\right), \tag{24}$$
$$\dot{\nabla} \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left({}^\epsilon\nabla - \nabla\right), \tag{25}$$

we get

$$\dot{C}^a{}_{bc} = \frac{1}{2}g^{ad}\left(\nabla_c h_{bd} + \nabla_b h_{cd} - \nabla_d h_{bc}\right) \tag{26}$$
$$\dot{\nabla}_c T^{a_1\cdots a_k}{}_{b_1\cdots b_l} = \sum_{i=1}^{k} T^{a_1\cdots d\cdots a_k}{}_{b_1\cdots b_l}\,\dot{C}^{a_i}{}_{dc} - \sum_{i=1}^{l} T^{a_1\cdots a_k}{}_{b_1\cdots d\cdots b_l}\,\dot{C}^d{}_{b_i c}. \tag{27}$$

As a result, the dot of ∇T, where T is a tensor of any type, becomes

$$\mathcal{L}_V\nabla T = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\phi^*_{-\epsilon}\nabla T - \nabla T\right] \tag{28}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[{}^\epsilon\nabla\left({}^\epsilon T - T\right) + \left({}^\epsilon\nabla - \nabla\right)T\right] \tag{29}$$
$$= \nabla\dot{T} + \dot{\nabla}T. \tag{30}$$

1.6 Riemann Curvatures

Riemann curvature tensor of the Levi-Civita connection ∇ associated with the spacetime metric g is given by

$$R^a{}_{bcd} v^b = 2\nabla_{[c}\nabla_{d]} v^a \tag{31}$$

where v is an arbitrary vector. Let us consider dot of the above along a gauge V which is

$$\mathcal{L}_V \left(R^a{}_{bcd} v^b \right) = 2\nabla_{[c}\nabla_{d]} \dot{v}^a + 2\nabla_{[c}\dot{\nabla}_{d]} v^a + 2\dot{\nabla}_{[c}\nabla_{d]} v^a \tag{32}$$

$$= R^a{}_{bcd} \dot{v}^b + 2v^b \nabla_{[c} \dot{C}^a{}_{d]b}. \tag{33}$$

By the Leibniz' rule of Lie derivatives, we get

$$\dot{R}^a{}_{bcd} = 2\nabla_{[c} \dot{C}^a{}_{d]b} \tag{34}$$

$$= \nabla_{[c}\nabla_{d]} h^a{}_b + \nabla_{[c}\nabla_{|b|} h_{d]}{}^a - \nabla_{[c}\nabla^a h_{d]b} \tag{35}$$

$$= \frac{1}{2}\left(h^e{}_b R^a{}_{ecd} - h^a{}_e R^e{}_{bcd} \right) + \nabla_{[c}\nabla_{|b|} h_{d]}{}^a - \nabla_{[c}\nabla^a h_{d]b} \tag{36}$$

1.7 Einstein Equation

The dot of Ricci tensor $R_{ab} \equiv R^c{}_{acb}$ and scalar $R \equiv R^a{}_a$ are given by

$$\dot{R}_{ab} = \delta^d{}_c \dot{R}^c{}_{adb} + \dot{\delta}^d{}_c R^c{}_{adb} \tag{37}$$

$$= \nabla^c \nabla_{(a} h_{b)c} - \frac{1}{2}\nabla^c\nabla_c h_{ab} - \frac{1}{2}\nabla_b\nabla_a h^c{}_c \tag{38}$$

$$= \nabla_{(a}\nabla^c h_{b)c} - \frac{1}{2}\nabla^c\nabla_c h_{ab} - \frac{1}{2}\nabla_b\nabla_a h^c{}_c - R_a{}^c{}_b{}^d h_{cd} + R^c{}_{(a} h_{b)c} \tag{39}$$

$$\dot{R} = g^{ab}\dot{R}_{ab} - h^{ab} R_{ab} \tag{40}$$

$$= \nabla_b\nabla_a h^{ab} - \nabla^b\nabla_b h^a{}_a - h^{ab} R_{ab} \tag{41}$$

For the Einstein tensor, we get

$$\dot{G}_{ab} = \dot{R}_{ab} - \frac{1}{2} h_{ab} R - \frac{1}{2} g_{ab} \dot{R} \tag{42}$$

$$= -\frac{1}{2}\nabla^c\nabla_c h_{ab} + \nabla_c\nabla_{(a} h_{b)}{}^c - \frac{1}{2} g_{ab}\nabla_d\nabla_c h^{cd} - \frac{1}{2}\nabla_b\nabla_a h^c{}_c + \frac{1}{2} g_{ab}\nabla^d\nabla_d h^c{}_c + \frac{1}{2} g_{ab} h^{cd} R_{cd} - \frac{1}{2} h_{ab} R. \tag{43}$$

Spacetime is governed by the Einstein equation given by

$$G_{ab} = 8\pi T_{ab}, \tag{44}$$

where T is the stress-energy. Its linear perturbation is given by

$$\dot{G}_{ab} = 8\pi \dot{T}_{ab}. \tag{45}$$

1.8 Geodesics

Let us consider a dust given by

$$T_{ab} = \rho u_a u_b \tag{46}$$

where ρ is energy-density and u is 4-velocity satisfying

$$-1 = u \cdot u. \tag{47}$$

The stress-energy satisfies the perturbation of contracted Bianchi identity,

$$\nabla^b T_{ab} = 0, \tag{48}$$

by the Einstein equation. Their perturbation is given by

$$\dot{T}_{ab} = \dot{\rho} u_a u_b - \rho h_{ac} u^c u_b - \rho u_a h_{bc} u^c + \rho \dot{u}_a u_b + \rho u_a \dot{u}_b \tag{49}$$

$$0 = h_{ab} u^a u^b + 2 u_a \dot{u}^a \tag{50}$$

$$0 = \nabla^b \dot{T}_{ab} - \dot{C}^c{}_a{}^b T_{cb} - \dot{C}^c{}_b{}^b T_{ac} - h^{bc}\nabla_c T_{ab} \tag{51}$$

$$0 = \nabla^b \ddot{T}_{ab} - 2\dot{C}^c{}_a{}^b \dot{T}_{cb} - 2\dot{C}^c{}_b{}^b \dot{T}_{ac} - \ddot{C}^c{}_a{}^b T_{cb} - \ddot{C}^c{}_b{}^b T_{ac} + h^{bd}\dot{C}^c{}_{ad} T_{cb} + h^{bd}\dot{C}^c{}_{bd} T_{ac} - \left(j^{bc} - 2h^{ac} h^b{}_c \right) \nabla_c T_{ab} - h^{bc}\left(\nabla_c \dot{T}_{ab} - \dot{C}^c{}_a{}^b T_{cb} - \dot{C}^c{}_b{}^b T_{ac} \right) \tag{52}$$

At background we assume that

$$
\begin{aligned}
{}^{\epsilon}\rho &= \epsilon\rho + \epsilon^2\sigma + O\left(\epsilon^3\right), && (53)\\
{}^{\epsilon}u^a &= u^a + \epsilon v^a + O\left(\epsilon^2\right), && (54)\\
{}^{\epsilon}T_{ab} &= \epsilon\left(\rho u_a u_b\right) + \epsilon^2\left\{\sigma u_a u_b + \rho\left(v_a u_b + u_a v_b\right)\right\} + O\left(\epsilon^3\right) && (55)
\end{aligned}
$$

Then, the perturbation of contracted Bianchi identity in the leading-order becomes

$$
\begin{aligned}
0 &= \nabla^b\left(\rho u_a u_b\right) && (56)\\
&= \rho u^b\nabla_b u_a + u_a\nabla^b\left(\rho u_b\right). && (57)
\end{aligned}
$$

Noticing $u^b\nabla_b u_a$ is spatial with respect to u, we obtain

$$
\begin{aligned}
0 &= u^b\nabla_b u^a, && (58)\\
0 &= \nabla^a\left(\rho u_a\right), && (59)
\end{aligned}
$$

where the first is the equation of geodesics and the second is the law of conservation. In the next-to-leading-order we get

$$
v\cdot u = -\frac{1}{2}h_{ab}u^a u^b \tag{60}
$$

$$
\begin{aligned}
0 &= \nabla^b\left\{\sigma u_a u_b + \rho\left(v_a u_b + u_a v_b\right)\right\} - \frac{1}{2}\left(\nabla_a h^{bc} + \nabla^b h_a{}^c - \nabla^c h_a{}^b\right)\left(\rho u_c u_b\right) - \frac{1}{2}\left(\nabla_b h^{bc} + \nabla^b h_b{}^c - \nabla^c h^b{}_b\right)\left(\rho u_a u_c\right)\\
&\quad - \frac{1}{2}h^{bc}\nabla_c\left(\rho u_a u_b\right) && (61)\\
&= u_a\nabla^b\left(\sigma u_b\right) + \rho u_b\nabla^b v_a + \nabla^b\left(\rho u_a v_b\right) - \frac{1}{2}\rho u_b u_c\nabla_a h^{bc} - \frac{1}{2}\left(\nabla_b h^{bc} + \nabla^b h_b{}^c - \nabla^c h^b{}_b\right)\left(\rho u_a u_c\right)\\
&\quad - \frac{1}{2}h^{bc}\nabla_c\left(\rho u_a u_b\right) && (62)
\end{aligned}
$$

2 Gravitational Waves

In this lecture, we only consider perturbations with Minkowski background.

2.1 Minkowski Background

The Riemann tensor of Minkowski spacetime vanishes:

$$
R^a{}_{bcd} = 0. \tag{63}
$$

Thus, the Levi-Civita connection for any tensors is commute:

$$
\nabla_{[a}\nabla_{b]} = 0. \tag{64}
$$

Assuming ${}^{\epsilon}T = O\left(\epsilon^2\right)$, we get the perturbed Einstein equation in the next-to-leading-order as

$$
0 = -\frac{1}{2}\nabla^c\nabla_c h_{ab} + \nabla_c\nabla_{(a}h_{b)}{}^c - \frac{1}{2}g_{ab}\nabla_d\nabla_c h^{cd} - \frac{1}{2}\nabla_b\nabla_a h^c{}_c + \frac{1}{2}g_{ab}\nabla^d\nabla_d h^c{}_c \tag{65}
$$

Defining $\bar{h}$ as

$$
\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}g_{ab}h^c{}_c, \tag{66}
$$

we get

$$
0 = -\frac{1}{2}\nabla^c\nabla_c\bar{h}_{ab} + \nabla_{(a}\nabla^c\bar{h}_{b)c} - \frac{1}{2}g_{ab}\nabla^c\nabla^d\bar{h}_{cd}. \tag{67}
$$

2.2 Lorenz Gauge

We impose a gauge condition, so-called Lorenz gauge, given by

$$
\nabla^b\bar{h}_{ab} = 0. \tag{68}
$$

Then, the perturbed Einstein equation becomes the wave equation:

$$
0 = \nabla^c\nabla_c\bar{h}_{ab}. \tag{69}
$$

Note that eq. (68) does not determine gauge uniquely. We can generate a set of gauges satisfying the Lorenz gauge condition by the gauge transformation given in eq. (6):

$$h'_{ab} = h_{ab} + \mathcal{L}_\xi g_{ab}, \tag{70}$$

where h and h' are metric perturbations in given gauge and new gauge, respectively, and ξ is a vector. Their relation in $\bar{h}$ becomes

$$\bar{h}'_{ab} = \bar{h}_{ab} + \nabla_a \xi_b + \nabla_b \xi_a - g_{ab} \nabla^c \xi_c. \tag{71}$$

If both gauges satisfy the Lorenz gauge condition, we obtain

$$\nabla^b \nabla_b \xi^a = 0. \tag{72}$$

2.3 Wave Solution

Let us consider a wave solution of the metric perturbation with the wave equation in eq. (69) given by

$$h_{ab} = \int_{\mathcal{N}} d^3\mathcal{N}\left(k\right)\, \tilde{h}_{ab}\left(k\right) e^{\mathrm{i}P(;k)}, \tag{73}$$

where $\mathcal{N} \equiv \{k : k \cdot k = 0\} - \{0\}$ is the set of null vectors, $\tilde{h}\, d^3\mathcal{N}$ is the infinitesimal amplitude, and P is the phase such that $k^a = \nabla^a P$. $\mathcal{N}$ is decomposed into the future-directed subset $\mathcal{N}^+$ and the past-directed subset $\mathcal{N}^-$, respectively. Because h is real, $\mathcal{N}^+$ and $\mathcal{N}^-$ have one-to-one correspondence given by

$$\tilde{h}_{ab}\left(-k\right) = \tilde{h}^*_{ab}\left(k\right). \tag{74}$$

2.4 Traceless Gauge

Let us investigate possibilities of the further gauge restriction given by

$$h^a{}_a = 0, \tag{75}$$

which is the traceless gauge. For the above, we have to transform gauge with ξ satisfying

$$0 = h^a{}_a + 2\nabla_a \xi^a, \tag{76}$$

where h is a metric perturbation in given gauge. Because we only consider gauges satisfying the Lorenz gauge condition, ξ is the solution of eq. (72). Thus, it has form of

$$\xi^a = \int_{\mathcal{N}} d^3\mathcal{N}\left(k\right)\, \tilde{\xi}^a\left(k\right) e^{\mathrm{i}P(;k)}. \tag{77}$$

Then, eq. (76) becomes

$$0 = \int_{\mathcal{N}} d^3\mathcal{N} \left(\tilde{h}^a{}_a + 2\mathrm{i}k_a \tilde{\xi}^a\right) e^{\mathrm{i}P}. \tag{78}$$

Choices of $\tilde{\xi}$ to vanish the integrand of the above for all $k \in \mathcal{N}$ realize the traceless gauge.

In summary, our gauge choice have been

$$0 = \nabla^b h_{ab}, \tag{79}$$
$$0 = h^a{}_a, \tag{80}$$

which implies

$$0 = k^b \tilde{h}_{ab}\left(k\right), \tag{81}$$
$$0 = \tilde{h}^a{}_a\left(k\right), \tag{82}$$

for all $k \in \mathcal{N}$. Among gauges satisfying the above condition, the gauge transformations are given by ξ satisfying

$$0 = \nabla_a \xi^a, \tag{83}$$

which implies

$$k_a \tilde{\xi}^a\left(k\right) = 0, \tag{84}$$

for all $k \in \mathcal{N}$.

2.5 Riemann Tensor

From eq. (36), we obtain

$$\dot{R}^{ab}{}_{cd} = -2\nabla^{[a}\nabla_{[c}h^{b]}{}_{d]} \tag{85}$$

$$= \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\, 4k^{[a}k_{[c}\tilde{h}^{b]}{}_{d]}e^{\mathrm{i}P}. \tag{86}$$

Note that the perturbation of Reimann tensor is gauge-invariant because its background value vanishes.

2.6 Introducing Observer

Let us consider the Eulerian observer with 4-velocity n of a globally inertial coordinate system $\{t, \vec{x}\}$ in Minkowski background spacetime. The 3+1 decomposition of a wave vector $k \in \mathcal{N}$ is given by

$$k^a = \omega\left(n + \kappa\right), \tag{87}$$

where $\omega = -n \cdot k$ is the frequency and $\kappa = k/\omega - n$ is the spatial unit vector of propagation. We define a projection operator P onto a vector subspace orthogonal to n and κ as

$$P^a{}_b = \delta^a{}_b + n^a n_b - \kappa^a \kappa_b. \tag{88}$$

Then, it is idempotent,

$$P^a{}_b = P^a{}_c P^c{}_b, \tag{89}$$

and its trace is

$$P^a{}_a = 2. \tag{90}$$

Let us give an additional gauge condition as

$$h_{ab}n^b = 0. \tag{91}$$

The the gauge conditions we have chosen including the above is so-called the transverse-traceless (TT) gauge condition. All gauge conditions we impose are summarized in

$$0 = n^b\tilde{h}_{ab}\left(k\right), \tag{92}$$

$$0 = \kappa^b\tilde{h}_{ab}\left(k\right), \tag{93}$$

$$0 = \tilde{h}^a{}_a\left(k\right), \tag{94}$$

for all $k \in \mathcal{N}$.

Let us show that the TT gauge condition determines a gauge uniquely. We orthogonally decompose $\tilde{\xi}$ for traceless gauges into

$$\tilde{\xi}^a = \tilde{\alpha}n^a + \tilde{\beta}\kappa^a + \tilde{X}^a, \tag{95}$$

where $\tilde{X}^a = P^a{}_b\tilde{\xi}^b$. Because eq. (84) implies $\tilde{\alpha} = \tilde{\beta}$, we get form of

$$\tilde{\xi}^a = \tilde{\gamma}k^a + \tilde{X}^a. \tag{96}$$

For two gauges satisfying the TT gauge condition have a transformation with $\tilde{\xi}$ satisfying

$$0 = \mathrm{i}\left(k_a\tilde{\xi}_b + \tilde{\xi}_a k_b\right)n^b \tag{97}$$

$$= \mathrm{i}\left\{k_a\tilde{\gamma}\left(k \cdot n\right) + \left(\tilde{\gamma}k_a + \tilde{X}_a\right)\left(k \cdot n\right)\right\} \tag{98}$$

$$= \mathrm{i}\left(2\tilde{\gamma}k_a + \tilde{X}_a\right)\left(k \cdot n\right). \tag{99}$$

It implies that $\tilde{\xi}$ vanishes and both gauges are indentical.

2.7 Polarization of Amplitude

We consider a set of rank $(0,2)$ tensors $\mathcal{T}$ satisfying eqs. (92) to (94). It has projection operator Λ given by

$$\Lambda^{ab}{}_{cd} \equiv P^{a}{}_{(c}P^{b}{}_{d)} - \frac{1}{2}P^{ab}P_{cd}. \tag{100}$$

A right-handed orthonormal basis $\left\{e^{A} : A = +, \times\right\}$ for $\mathcal{T}$ has properties,

$$e^{A} \cdot e^{B} = \delta^{AB}, \tag{101}$$

$$e^{A}_{ab}e^{B}_{cd}g^{ac}\epsilon^{bd}{}_{ef}n^{e}\kappa^{f} = \varepsilon^{AB} \tag{102}$$

where δ is the Kronecker delta, ε is the Levi-Civita symbol for two dimension, ϵ is the spacetime Levi-Civita tensor, and $\cdot$ is the inner product for $\mathcal{T}$ defined by

$$x \cdot y = g^{ac}g^{bd}x_{ab}y_{cd}, \tag{103}$$

for $x, y \in \mathcal{T}$. The collection of right-handed orthonormal bases are parametrized by ϑ with the relation,

$$e^{+}_{ab}(\vartheta) = \cos(2\vartheta)\, e^{+}_{ab}(0) + \sin(2\vartheta)\, e^{\times}_{ab}(0), \tag{104}$$

$$e^{\times}_{ab}(\vartheta) = -\sin(2\vartheta)\, e^{+}_{ab}(0) + \cos(2\vartheta)\, e^{\times}_{ab}(0), \tag{105}$$

where $\left\{e^{A}(0)\right\}$ is a fiducial basis.

Let us consider a phase adjustment of $\tilde{h}$ by α as

$$\tilde{h}_{ab} = \left\{\Re\left(\tilde{h}_{ab}e^{-\mathrm{i}\alpha}\right) + \mathrm{i}\Im\left(\tilde{h}_{ab}e^{-\mathrm{i}\alpha}\right)\right\} e^{\mathrm{i}\alpha}, \tag{106}$$

such that the real part and imaginary part are orthogonal to each other. Note that α for the orthogonalization is not unique. So, we introduce a "standard" process for the orthogonalization given by

$$-\pi/2 \leq \alpha = \frac{1}{2}\,\mathrm{Arg}\left(\tilde{h} \cdot \tilde{h}\right) \leq \pi/2, \tag{107}$$

$$\tilde{h}_{+} = \sqrt{\Re\left(\tilde{h}e^{-\mathrm{i}\alpha}\right) \cdot \Re\left(\tilde{h}e^{-\mathrm{i}\alpha}\right)} > 0, \tag{108}$$

$$e^{+} = \frac{1}{\tilde{h}_{+}}\Re\left(\tilde{h}e^{-\mathrm{i}\alpha}\right) \in \mathcal{T}. \tag{109}$$

Then, $e^{\times}$ is uniquely determined by the right-handedness of the basis $\left\{e^{A} : A = +, \times\right\}$ for $\mathcal{T}$ and

$$\tilde{h}_{\times} = \mathrm{i}\Im\left(\tilde{h}e^{-\mathrm{i}\alpha}\right) \cdot e^{\times}. \tag{110}$$

Finally, we get the form

$$\tilde{h}_{ab} = \tilde{h}_{A}e^{A}_{ab}e^{\mathrm{i}\alpha}. \tag{111}$$

This is an analogy to the form for elliptically polarized electromagnetic waves given in [Landau and Lifshitz(1975)]. Introducing unit eigenvectors x and y for e^{+} with positive and negative eiginvalues, respectively, we get

$$e^{+}_{ab} = \frac{1}{\sqrt{2}}\left(x_a x_b - y_a y_b\right), \tag{112}$$

$$e^{\times}_{ab} = \frac{1}{\sqrt{2}}\left(x_a y_b + y_a x_b\right). \tag{113}$$

Exercise: Show that $\left|\tilde{h}_{+}\right| > \left|\tilde{h}_{\times}\right|$ in the above process.

2.8 Changing Observer

A metric perturbation in the traceless gauge has decomposition as

$$h_{ab} = \mathfrak{A}\hat{k}_a\hat{k}_b + \mathfrak{B}_a\hat{k}_b + \hat{k}_a\mathfrak{B}_b + \mathfrak{C}_{ab}, \tag{114}$$

where

$$\hat{k}^{a} = n^{a} + \kappa^{a} \tag{115}$$

$$\mathfrak{A} = h_{ab}n^{a}n^{b}, \tag{116}$$

$$\mathfrak{B}_a = -h_{ac}n^{a}P^{c}{}_{b}, \tag{117}$$

$$\mathfrak{C}_{ab} = h_{cd}P^{c}{}_{a}P^{d}{}_{b}. \tag{118}$$

We find that the metric perturbation in the TT gauge is only taking $\mathfrak{C}$.

In another observer with 4-velocity n', we have

$$h_{ab} = \mathfrak{A}'\hat{k}'_a\hat{k}'_b + \mathfrak{B}'_a\hat{k}'_b + \hat{k}'_a\mathfrak{B}'_b + \mathfrak{C}'_{ab}. \tag{119}$$

Equating the above and eq. (114), we get

$$\mathfrak{C}'_{ab} = \mathfrak{C}_{cd}P'^c{}_a P'^d{}_b, \tag{120}$$

where P' is the projection operator orthogonal to n' and κ' as defined in eq. (88). It is the transformation rule for metric perturbations in the TT gauge.

2.9 Perturbation of Observer

We consider observers that follow dusts with the 4-velocity field given by

$${}^{\epsilon}u^a = n^a + \epsilon v^a + O\left(\epsilon^2\right). \tag{121}$$

From eqs. (60) and (62), we obtain

$$0 = v \cdot n, \tag{122}$$

$$0 = n_a\left\{n^b\nabla_b\sigma + \nabla^b\left(\rho v_b\right)\right\} + \rho n^b\nabla_b v_a, \tag{123}$$

in the TT gauge. If $\sigma = 0$ and $v = 0$ before arrival of GWs, $v = 0$ is maintained even though GW is passing. So we set

$$v = 0, \tag{124}$$

over the spacetime and the observerse are fixed on the globally inertial coordinate system $\{t, \vec{x}\}$.

3 Detection of GWs

3.1 Geometrical Optics

Let us consider an electromagnetic field whose 4-potential can be given by

$$A_a = \Re\left[\left\{\tilde{A}_a + \omega^{-1}\tilde{B}_a + O\left(\omega^{-2}\right)\right\}e^{\mathrm{i}\left\{\omega Q + R + O\left(\omega^{-1}\right)\right\}}\right], \tag{125}$$

such that $l^a \equiv \nabla^a Q$ is future-directed, $m^a = \nabla^a R$, and $-n\cdot l \sim 1/\mathcal{R}$ for an observer n and the curvature radius $\mathcal{R}$. Through,

$$\nabla_b A_a = \Re\left[\left\{\mathrm{i}l_b\left(\tilde{A}_a + \omega^{-1}\tilde{B}_a\right) + \nabla_b\left(\tilde{A}_a + \omega^{-1}\tilde{B}_a\right) + O\left(\omega^{-1}\right)\right\}e^{\mathrm{i}\left\{\omega Q + R + O\left(\omega^{-1}\right)\right\}}\right] \tag{126}$$

$$= \Re\left[\left\{\mathrm{i}\omega l_b\tilde{A}_a + \mathrm{i}m_b\tilde{A}_a + \mathrm{i}l_b\tilde{B}_a + \nabla_b\tilde{A}_a + O\left(\omega^{-1}\right)\right\}e^{\mathrm{i}\left\{\omega Q + R + O\left(\omega^{-1}\right)\right\}}\right], \tag{127}$$

$$\nabla^a A_a = \Re\left[\left\{\mathrm{i}\omega\left(l\cdot\tilde{A}\right) + \mathrm{i}\left(m\cdot\tilde{A}\right) + \mathrm{i}\left(l\cdot\tilde{B}\right) + \nabla^a\tilde{A}_a + O\left(\omega^{-1}\right)\right\}e^{\mathrm{i}\left\{\omega Q + R + O\left(\omega^{-1}\right)\right\}}\right], \tag{128}$$

$$\nabla_c\nabla_b A_a = \Re\left[\left\{\mathrm{i}\left(\omega l_c + m_c\right)\left(\mathrm{i}\omega l_b\tilde{A}_a + \mathrm{i}m_b\tilde{A}_a + \mathrm{i}l_b\tilde{B}_a + \nabla_b\tilde{A}_a\right) + \nabla_c\left(\mathrm{i}\omega l_b\tilde{A}_a + \mathrm{i}m_b\tilde{A}_a + \mathrm{i}l_b\tilde{B}_a + \nabla_b\tilde{A}_a\right) + O\left(1\right)\right\}e^{\mathrm{i}\left\{\omega Q + R + O\left(\omega^{-1}\right)\right\}}\right] \tag{129}$$

$$= \Re\left[\left\{-\omega^2 l_c l_b\tilde{A}_a + \omega\left(-m_b l_c\tilde{A}_a - l_b l_c\tilde{B}_a + \mathrm{i}l_c\nabla_b\tilde{A}_a - m_c l_b\tilde{A}_a + \mathrm{i}\nabla_c\left(l_b\tilde{A}_a\right)\right) + O\left(1\right)\right\}e^{\mathrm{i}\left\{\omega Q + R + O\left(\omega^{-1}\right)\right\}}\right], \tag{130}$$

$$\nabla^b\nabla_b A_a = \Re\left[\left\{-\omega^2\left(l\cdot l\right)\tilde{A}_a + \omega\left(-2\left(m\cdot l\right)\tilde{A}_a - \left(l\cdot l\right)\tilde{B}_a + 2\mathrm{i}l^b\nabla_b\tilde{A}_a + \mathrm{i}\tilde{A}_a\nabla_b l^b\right) + O\left(1\right)\right\}e^{\mathrm{i}\left\{\omega Q + R + O\left(\omega^{-1}\right)\right\}}\right], \tag{131}$$

the Maxwell equation without charge,

$$\nabla^b\nabla_b A_a = R^b{}_a A_b, \tag{132}$$

$$\nabla^a A_a = 0, \tag{133}$$

gives

$$0 = l\cdot l \tag{134}$$

in the leading-order of ω and

$$0 = l\cdot\tilde{A}, \tag{135}$$

$$0 = 2l^b\nabla_b\tilde{A}_a + \tilde{A}_a\nabla_b l^b + 2\mathrm{i}\left(m\cdot l\right)\tilde{A}_a \tag{136}$$

in the next-to-leading-order. Let us ignore m. (why?) Rewriting results, we obtain the evolution equations along l as

$$l^a \nabla_a Q = l \cdot l = 0, \tag{137}$$
$$l^b \nabla_b l^a = g^{ac} l^b \nabla_b \nabla_c Q \tag{138}$$
$$= g^{ac} l^b \nabla_c \nabla_b Q \tag{139}$$
$$= \frac{1}{2} g^{ac} \nabla_c (l \cdot l) \tag{140}$$
$$= 0, \tag{141}$$

in the leading-order and

$$0 = \nabla_b \left(\tilde{\mathcal{A}}^2 l^b \right), \tag{142}$$
$$0 = l^b \nabla_b \tilde{f}_a, \tag{143}$$
$$0 = l \cdot \tilde{f}, \tag{144}$$

in the next-to-leading-order where $\tilde{\mathcal{A}} \equiv \sqrt{\tilde{A} \cdot \tilde{A}^*}$, $\tilde{f}_a \equiv \tilde{A}_a / \tilde{\mathcal{A}}$, the first equation is conservation of ray number, the second equation is the parallel transport of polarization, and the third equation is the transverse condition of polarization.

3.2 Perturbation of Rays

Perturbation of $l^a = g^{ab} \nabla_b Q$ is given by

$$\dot{l}^a = -h^{ab} l_b + \nabla^a \dot{Q}. \tag{145}$$

Perturbation of the evolution of Q becomes

$$0 = \dot{l}^a \nabla_a Q + l^a \nabla_a \dot{Q} \tag{146}$$
$$= \left(-h^{ab} l_b + \nabla^a \dot{Q} \right) l_a + l^a \nabla_a \dot{Q}, \tag{147}$$

$$l^a \nabla_a \dot{Q} = \frac{1}{2} h_{ab} l^a l^b. \tag{148}$$

Perturbation of $\alpha \equiv -n^a \nabla_a Q$ is given by

$$\dot{\alpha} = -\dot{n}^a \nabla_a Q - n^a \nabla_a \dot{Q}. \tag{149}$$

At background, we assume that $\alpha = 1$ and $l^a = n^a + \lambda^a$ where λ is a spatial unit vector. Perturbed quantities are given by

$${}^\epsilon Q = Q + \epsilon S + O\left(\epsilon^2\right), \tag{150}$$
$${}^\epsilon \alpha = 1 + \epsilon \beta + O\left(\epsilon^2\right) \tag{151}$$

Then, the equation for S,

$$l^a \nabla_a S = \frac{1}{2} \int_{\mathcal{N}} d^3\mathcal{N}\left(k\right) \tilde{h}_{ab} l^a l^b e^{\mathrm{i}P} \tag{152}$$

solves

$$S = S^{\mathrm{p}} + S^{\mathrm{h}} \tag{153}$$

where

$$S^{\mathrm{p}} = \int_{\mathcal{N}} d^3\mathcal{N}\, \tilde{S}^{\mathrm{p}} e^{\mathrm{i}P}, \tag{154}$$
$$\tilde{S}^{\mathrm{p}} = -\mathrm{i} \frac{1}{2\left(l \cdot k\right)} \tilde{h}_{ab} l^a l^b, \tag{155}$$
$$\tag{156}$$

and S^{h} satisfies

$$\nabla_a S^{\mathrm{h}} = \gamma \left(n_a + \lambda_a\right). \tag{157}$$

Then,

$$\beta = -n^a \nabla_a S \tag{158}$$
$$= -\frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\, \frac{n\cdot k}{l\cdot k}\tilde{h}_{ab} l^a l^b e^{\mathrm{i}P} + \gamma \tag{159}$$

We give boundary condition at the plane $\mathcal{P}$ that is the congruence of emitters as

$$[\beta]_{\mathcal{P}} = 0. \tag{160}$$

Then,

$$\gamma = \frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\, \frac{n\cdot k}{l\cdot k}\tilde{h}_{ab} l^a l^b e^{\mathrm{i}P^{\mathrm{h}}}, \tag{161}$$

where

$$\nabla_a P^{\mathrm{h}} = k_a - (k\cdot l)\,\lambda_a. \tag{162}$$

As a result,

$$\beta = -n^a \nabla_a S \tag{163}$$
$$= -\frac{1}{2}\int_{\mathcal{N}} d^3\mathcal{N}\, \frac{n\cdot k}{l\cdot k}\tilde{h}_{ab} l^a l^b \left(1 - e^{\mathrm{i}\Delta}\right) e^{\mathrm{i}P}, \tag{164}$$

where $\Delta \equiv P^{\mathrm{h}} - P$. Note that

$$P^{\mathrm{h}}(t, \vec{x}) = P\left(t - \vec{x}\cdot\lambda, \vec{x} - (\vec{x}\cdot\lambda)\,\lambda\right) \tag{165}$$

is retarded phase from $\mathcal{P}$.

3.3 Beyond Geometrical Optics

Please refer [Park and Kim(2021), Park(2022a), Park(2022b)].

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