GW-BASIC

2022 Summer School on Numerical Relativity and Gravitational Waves

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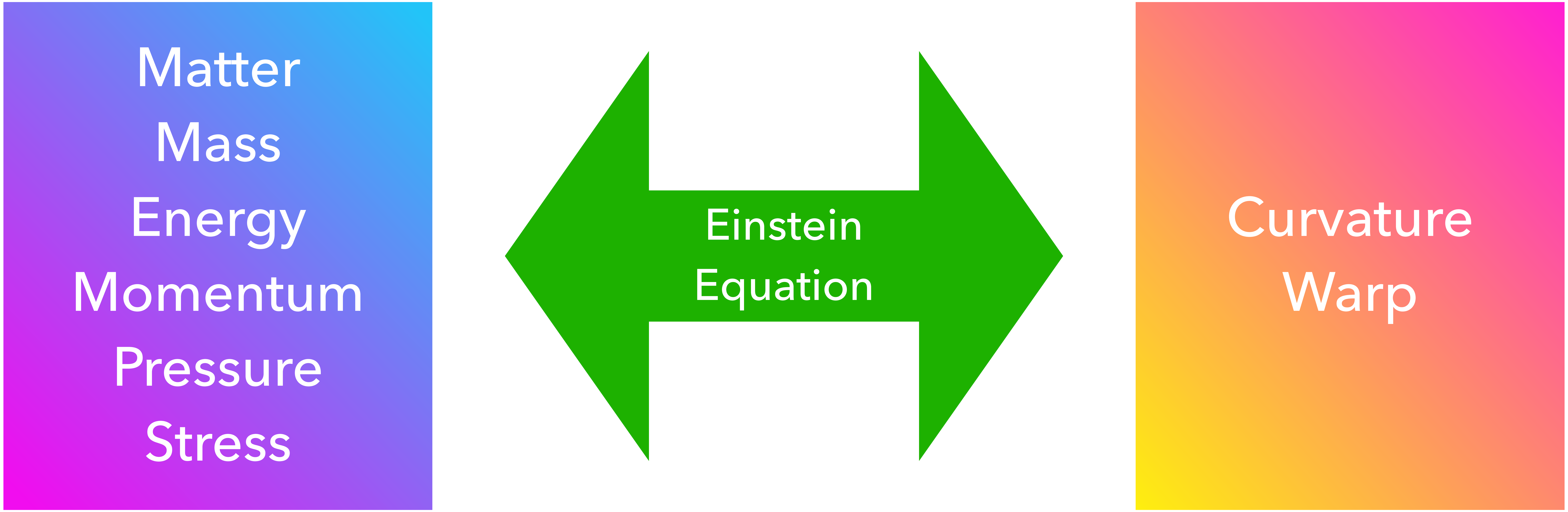
2022.07.26 @ La Valse Hotel Busan

Overview

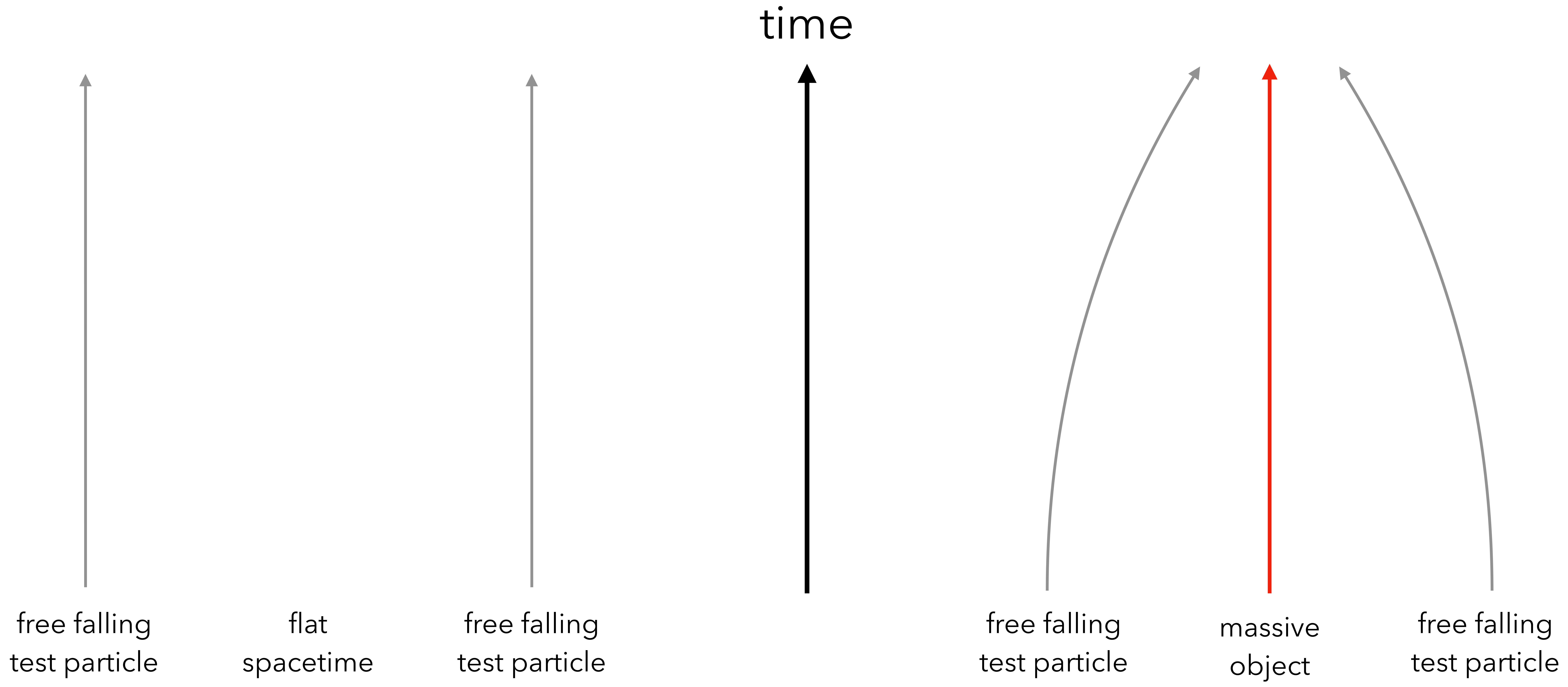
- Public Level
- Undergraduate Level
- Graduate Level
- Expert Level

Public Level

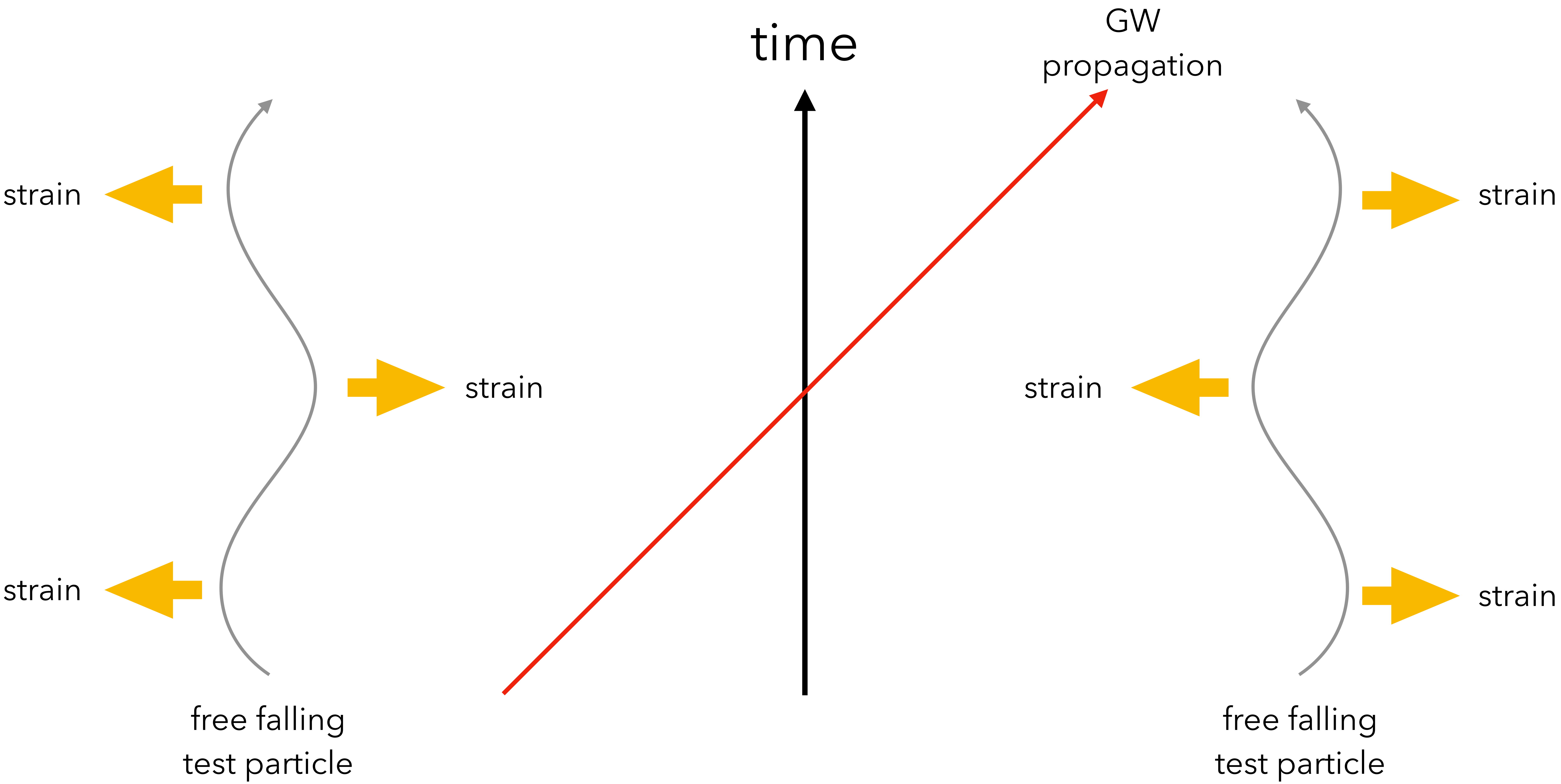
Einstein Equation



Spacetime Curvature

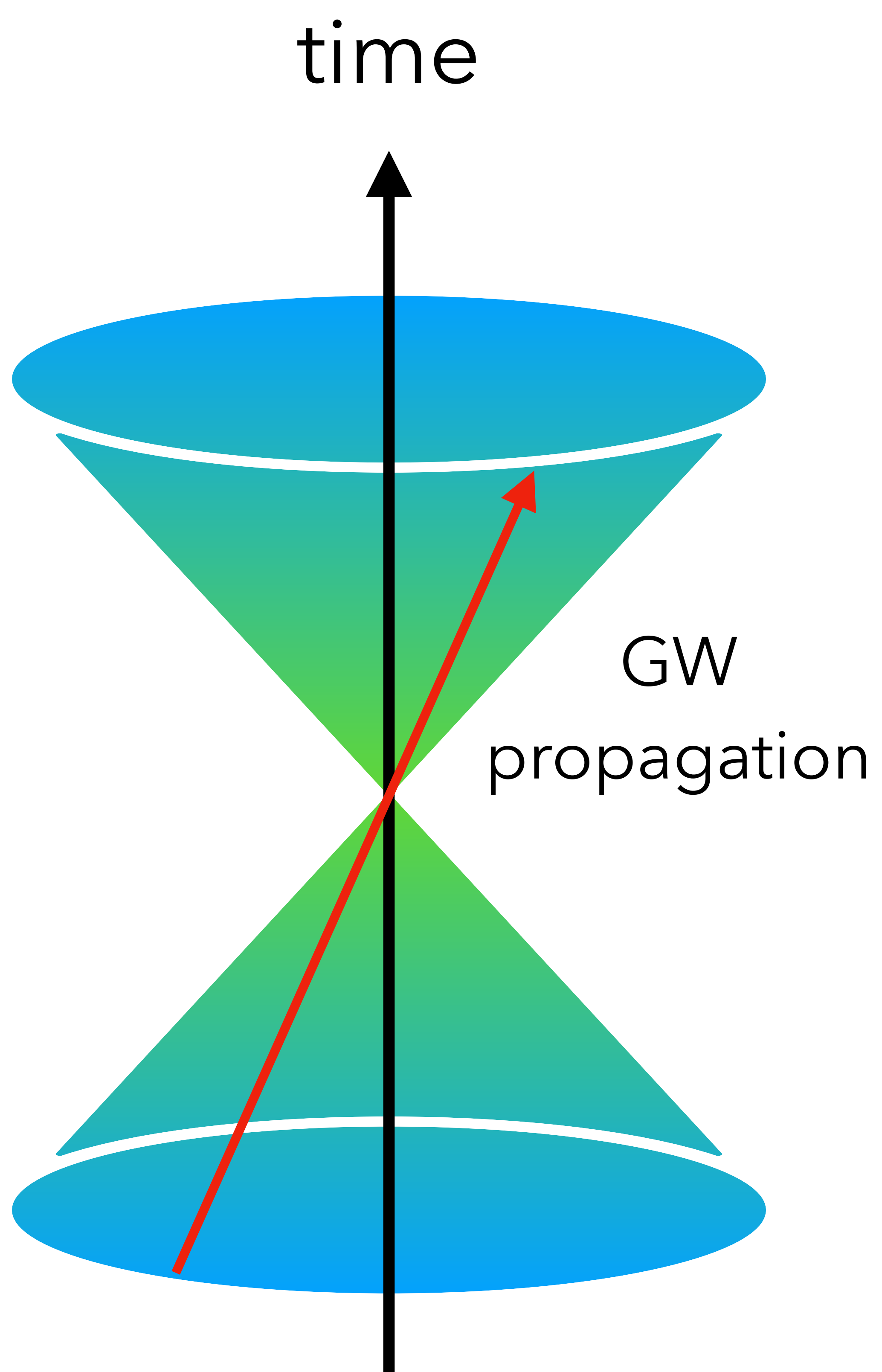


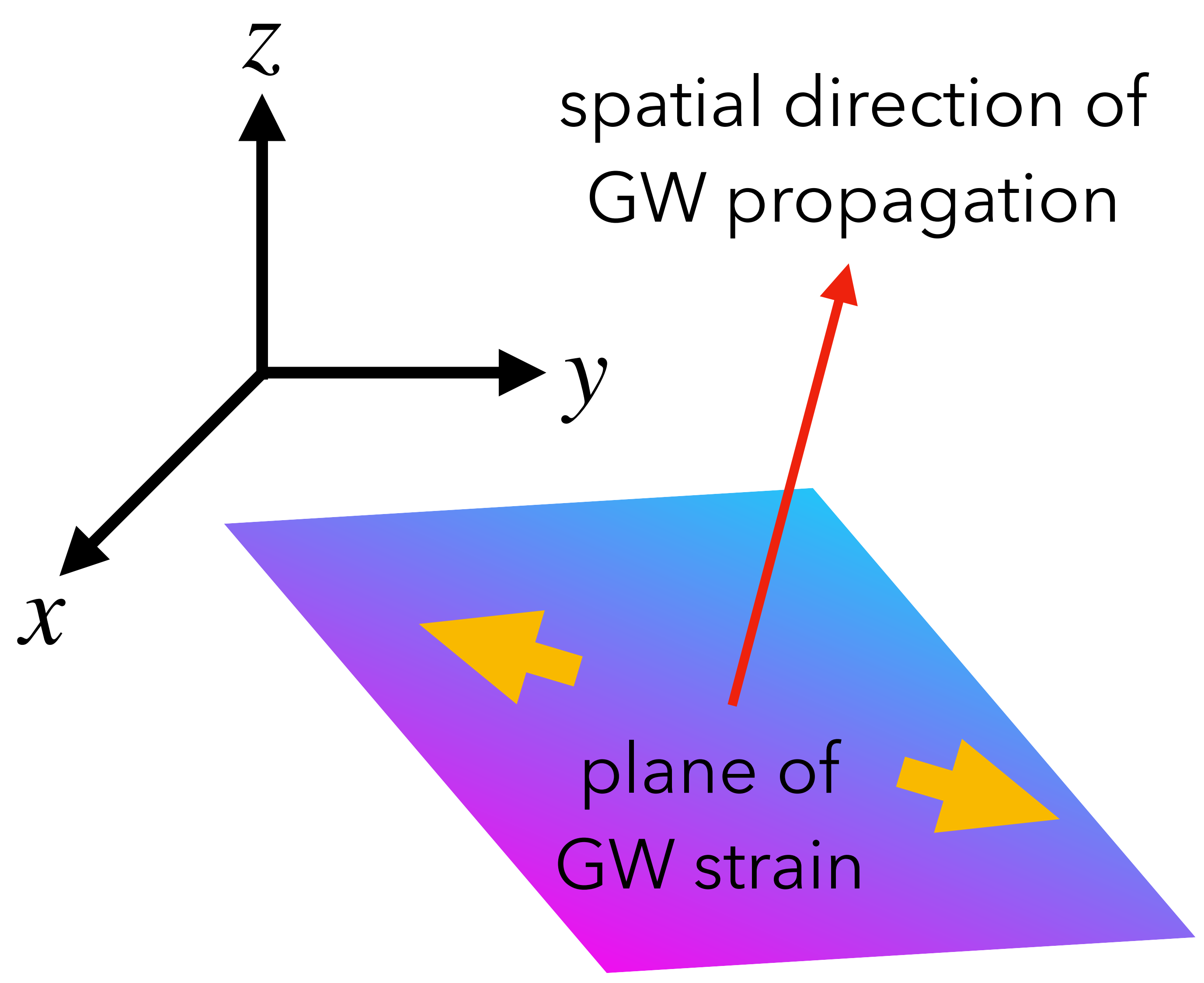
Gravitational Waves



Properties of Gravitational Waves

- Propagation speed: speed of light
- Transverse wave: propagation direction ⊥ strain direction
- No expansion of strain plane: GW does not change the area, but the shape.

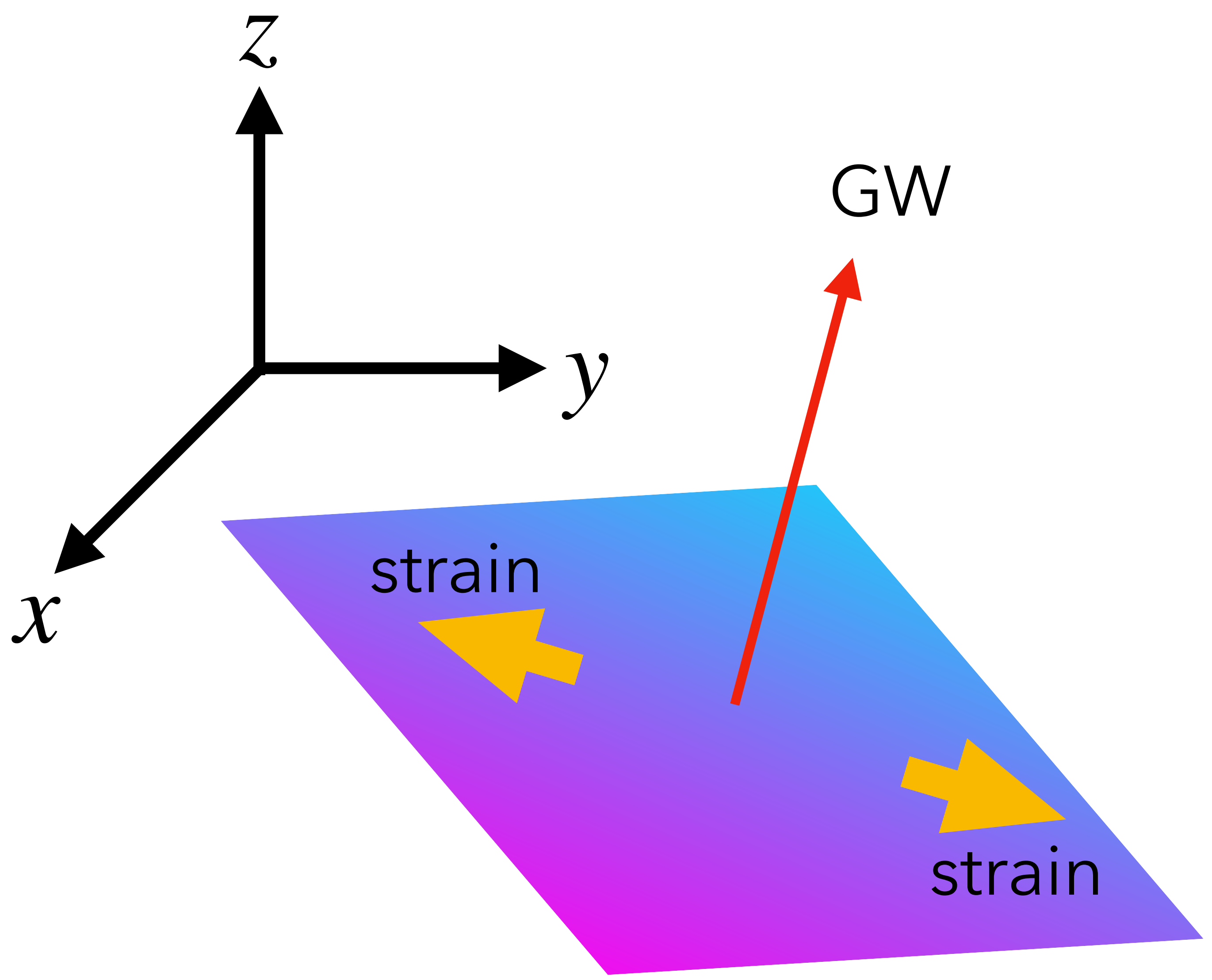


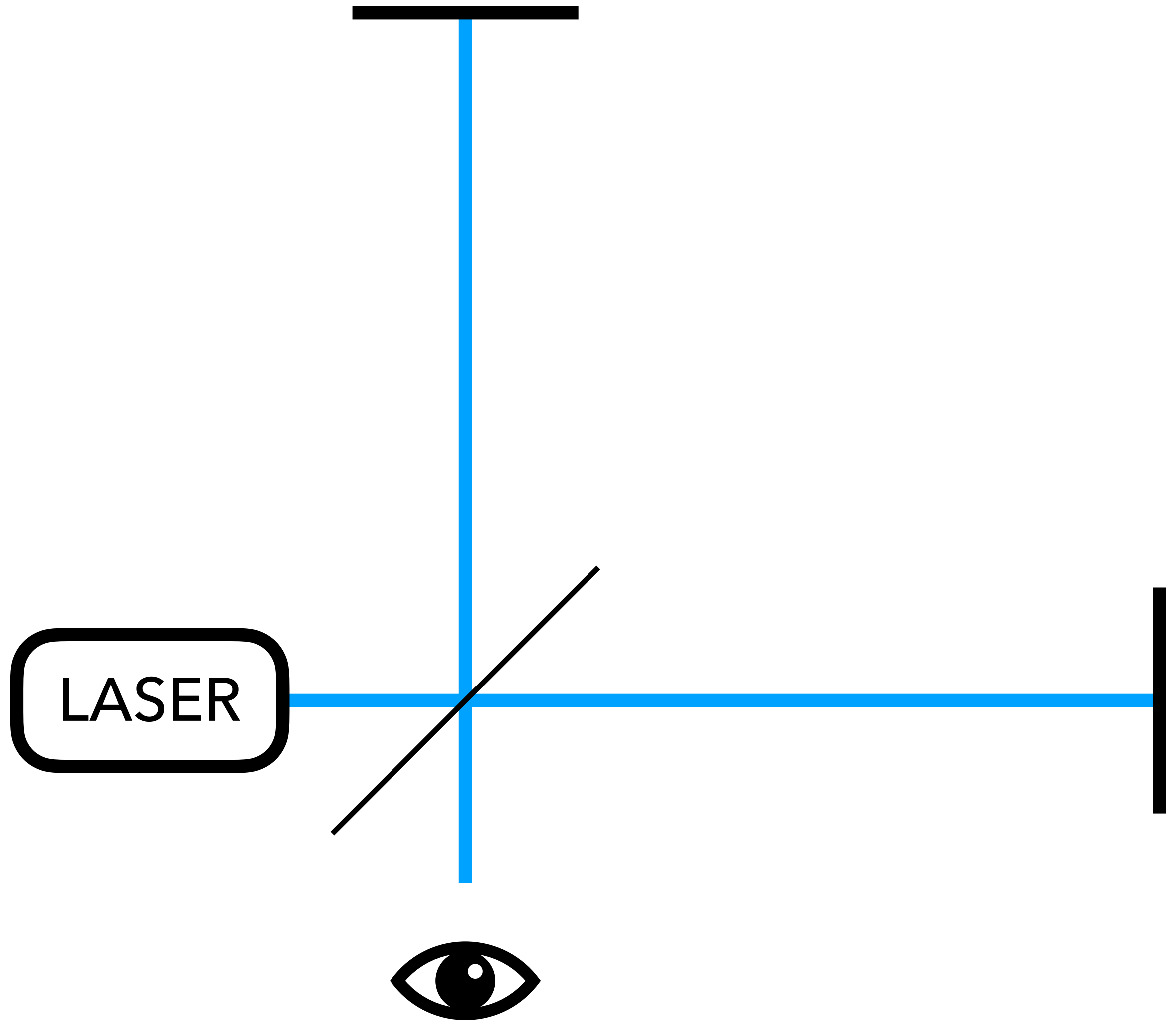


Detection of Gravitational Waves

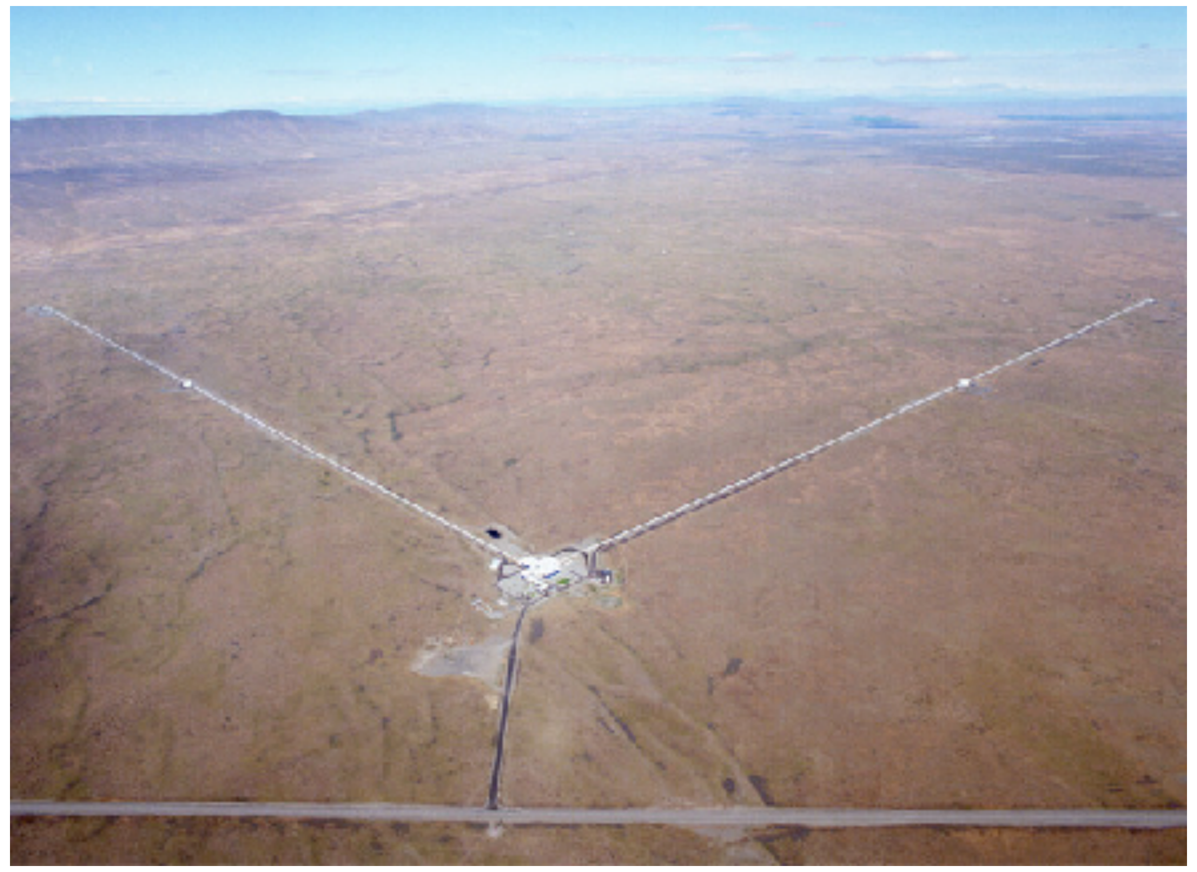
Bar Detector

Joseph Weber and his bar detector





Interferometer

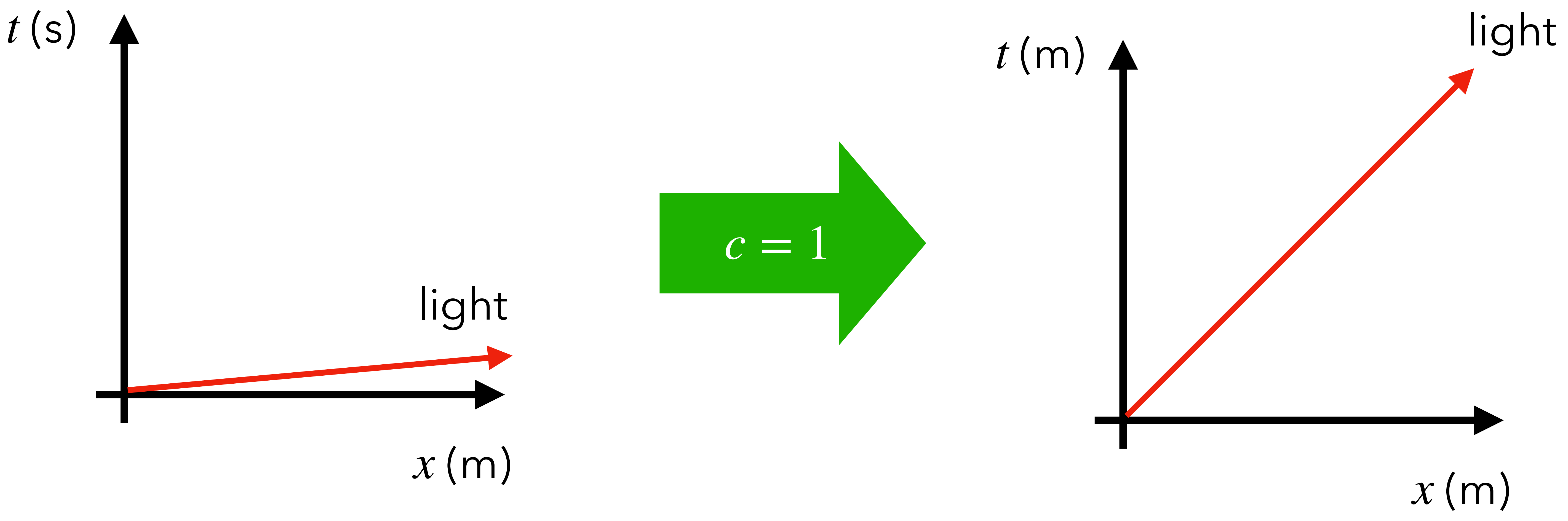


LIGO Hanford

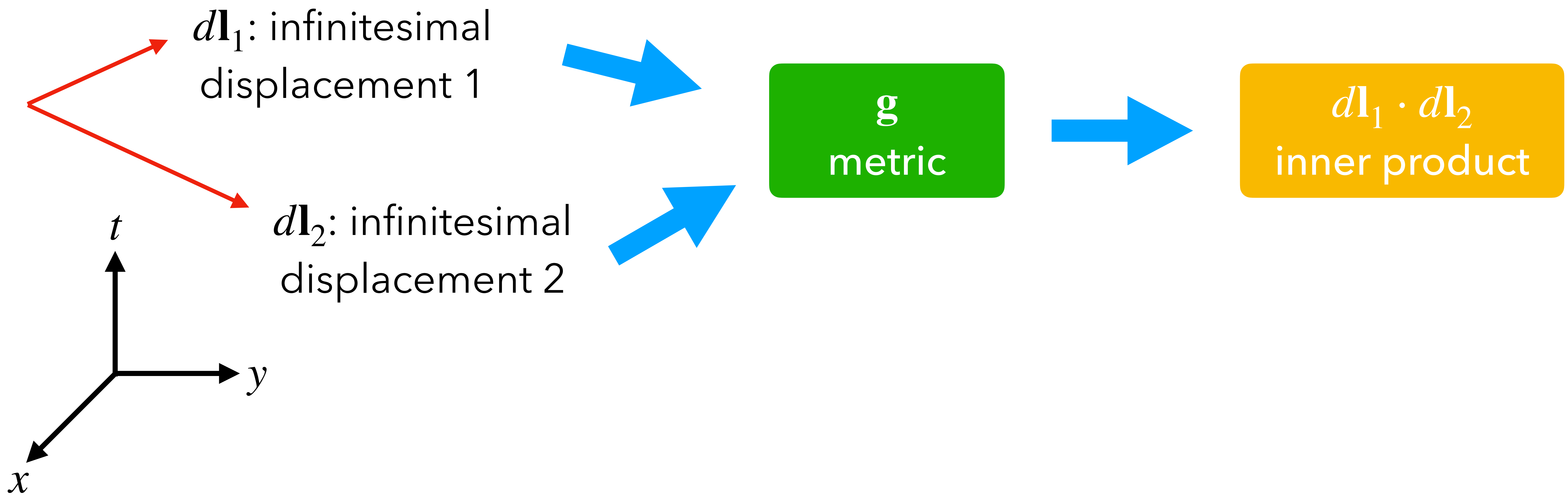
Undergraduate Level

Geometrized Unit

- $c = 299792458\,\mathrm{m/s} = 1$
  - $1\,\mathrm{s} = 299792458\,\mathrm{m}$
- $G = 6.67430 \times 10^{-11}\,\mathrm{m}^3 \cdot \mathrm{s}^{-2} \cdot \mathrm{kg}^{-1} = 1$
  - $1\,\mathrm{kg} = \dfrac{6.67430 \times 10^{-11}}{(299792458)^2}\,\mathrm{m}$



Metric



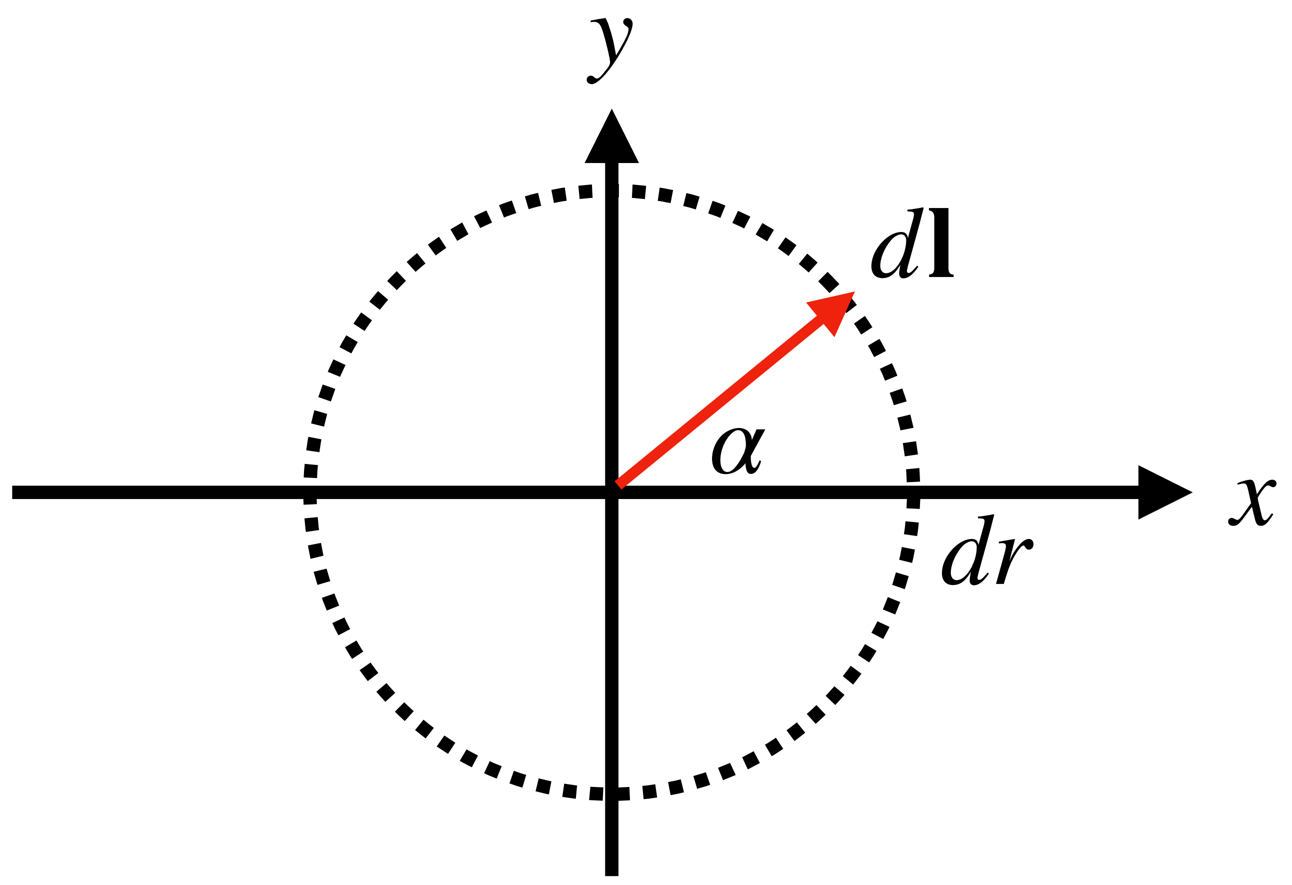
$d\mathbf{l}_1$: infinitesimal displacement 1
$d\mathbf{l}_2$: infinitesimal displacement 2
t
y
x
$\mathbf{g}$ metric
$d\mathbf{l}_1 \cdot d\mathbf{l}_2$ inner product

Metric in Components

- Infinitesimal Displacement
  - $d\mathbf{l}_1 = \left(dt_1, dx_1, dy_1, dz_1\right)$
  - $d\mathbf{l}_2 = \left(dt_2, dx_2, dy_2, dz_2\right)$
- Inner product
  - $$d\mathbf{l}_1 \cdot d\mathbf{l}_2 = g_{\mu\nu} dx_1^\mu dx_2^\nu = \begin{bmatrix} dt_1 & dx_1 & dy_1 & dz_1 \end{bmatrix} \begin{bmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{bmatrix} \begin{bmatrix} dt_2 \\ dx_2 \\ dy_2 \\ dz_2 \end{bmatrix}$$
  - where $g_{\mu\nu}$ is symmetric such that $g_{\mu\nu} = g_{\nu\mu}$.

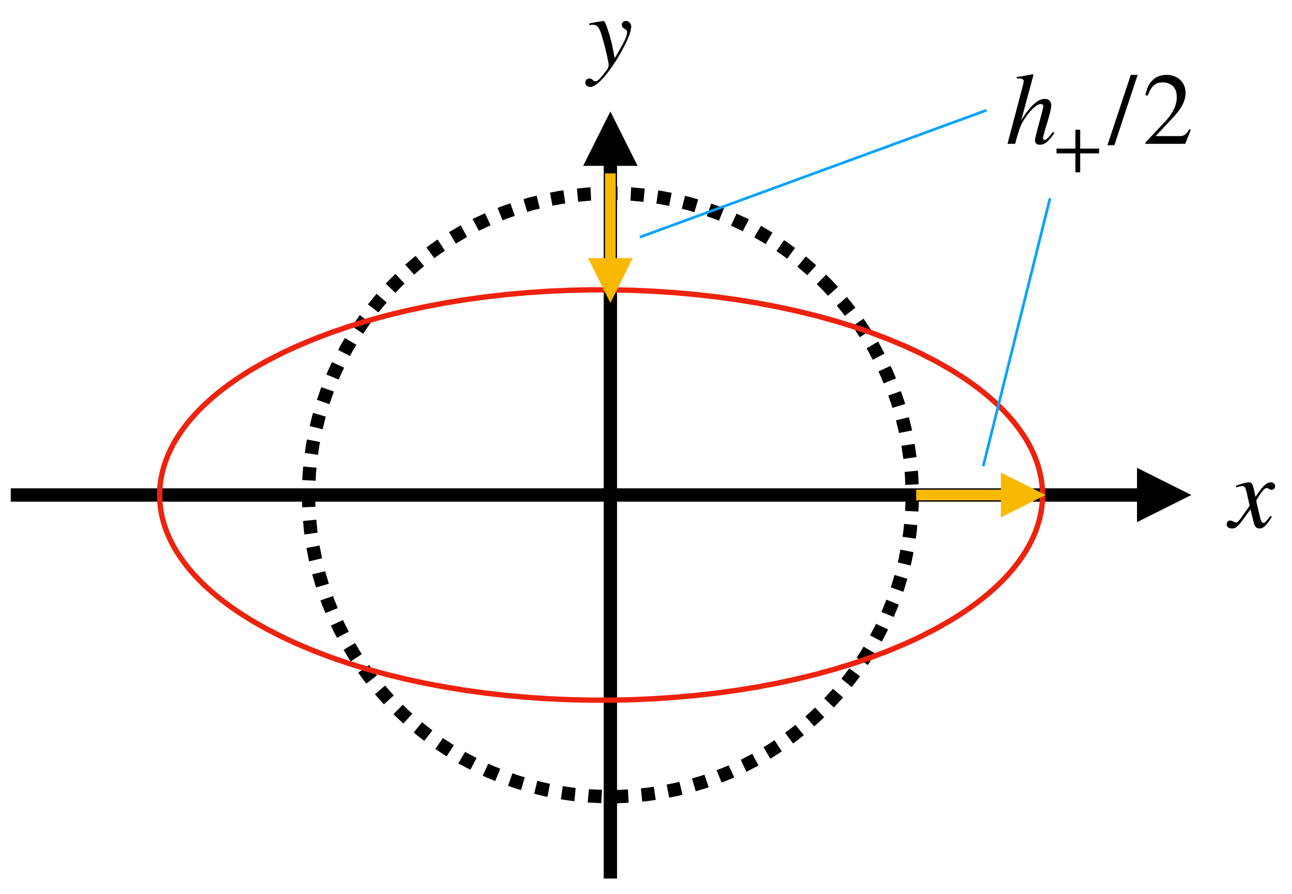
Metric Perturbation of GWs

- Let us consider a metric perturbation $h_{\mu\nu} \ll 1$ preserving area of strain plane
- $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- Let us consider the coordinate displacement $d\mathbf{l} = \left(0, dr\cos\alpha, dr\sin\alpha, 0\right)$

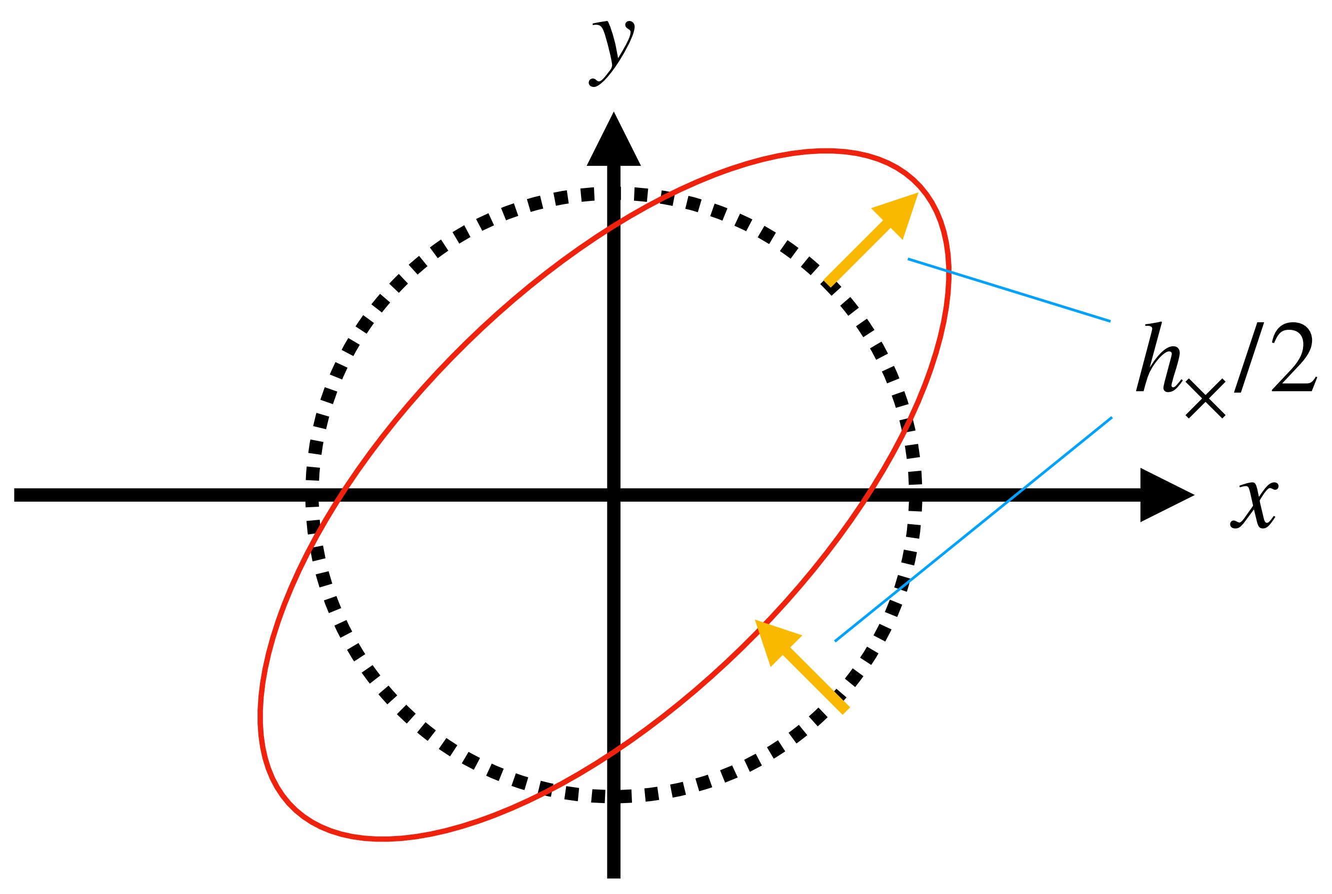


Metric Perturbation of GWs

- The physical length of $d\mathbf{l}$ is given by
- $$d\mathbf{l}\cdot d\mathbf{l} = [0 \quad dr\cos\alpha \quad dr\sin\alpha \quad 0]\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+h_+ & h_\times & 0 \\ 0 & h_\times & 1-h_+ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ dr\cos\alpha \\ dr\sin\alpha \\ 0 \end{bmatrix}$$
- $$\sqrt{d\mathbf{l}\cdot d\mathbf{l}} = dr\left\{1+\frac{1}{2}h_+\cos(2\alpha)+\frac{1}{2}h_\times\sin(2\alpha)\right\}$$



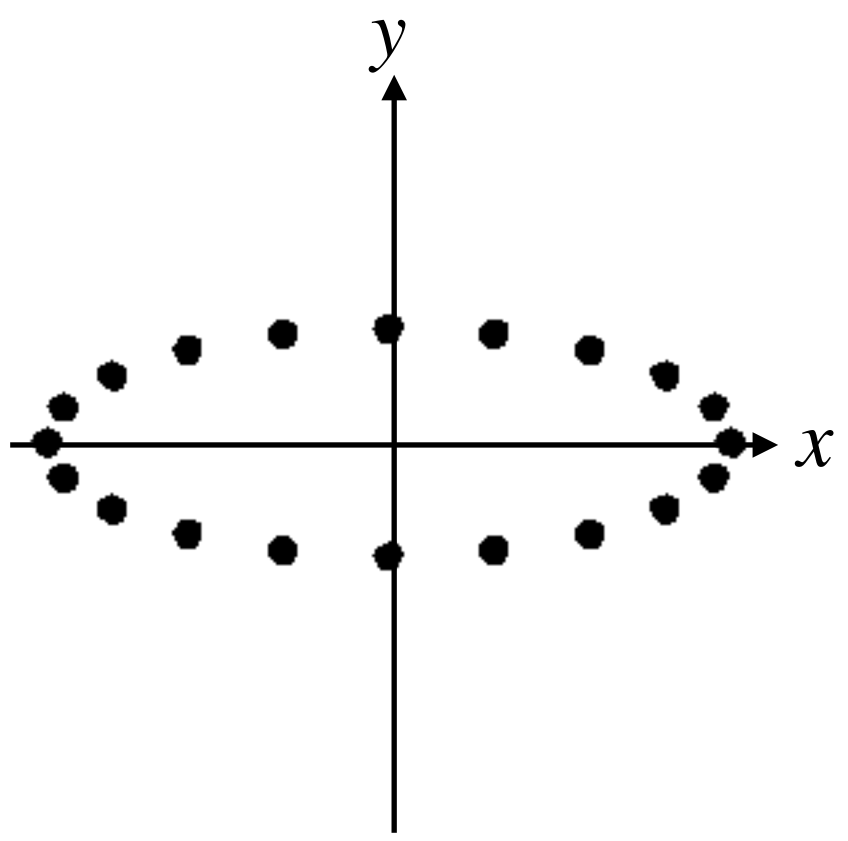
+ polarization



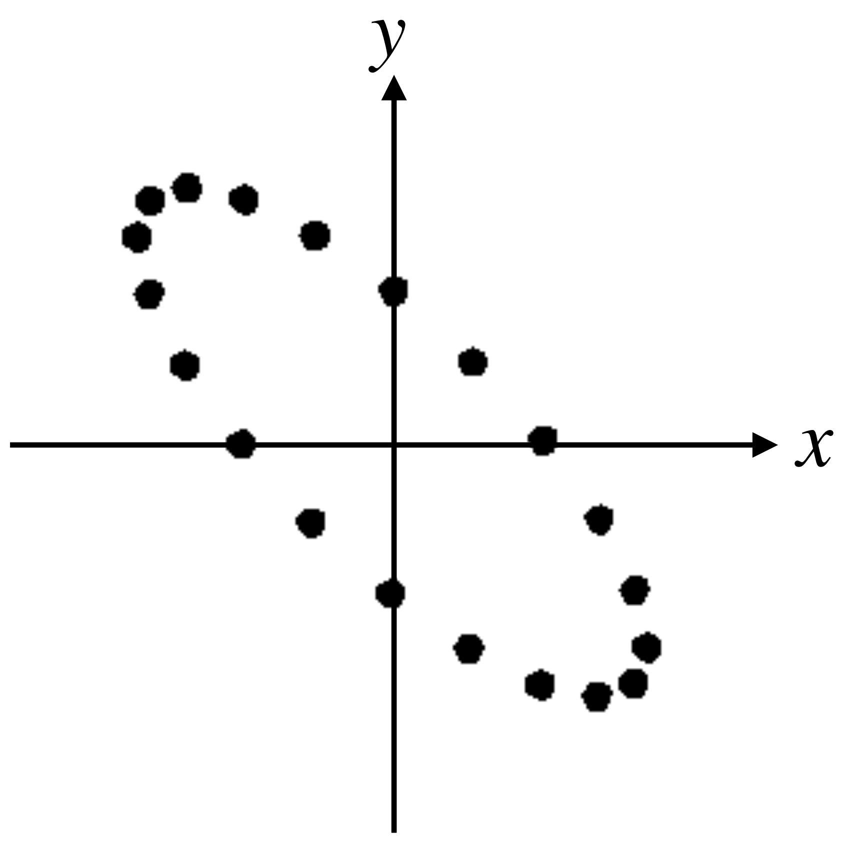
x polarization

Gravitational Waves

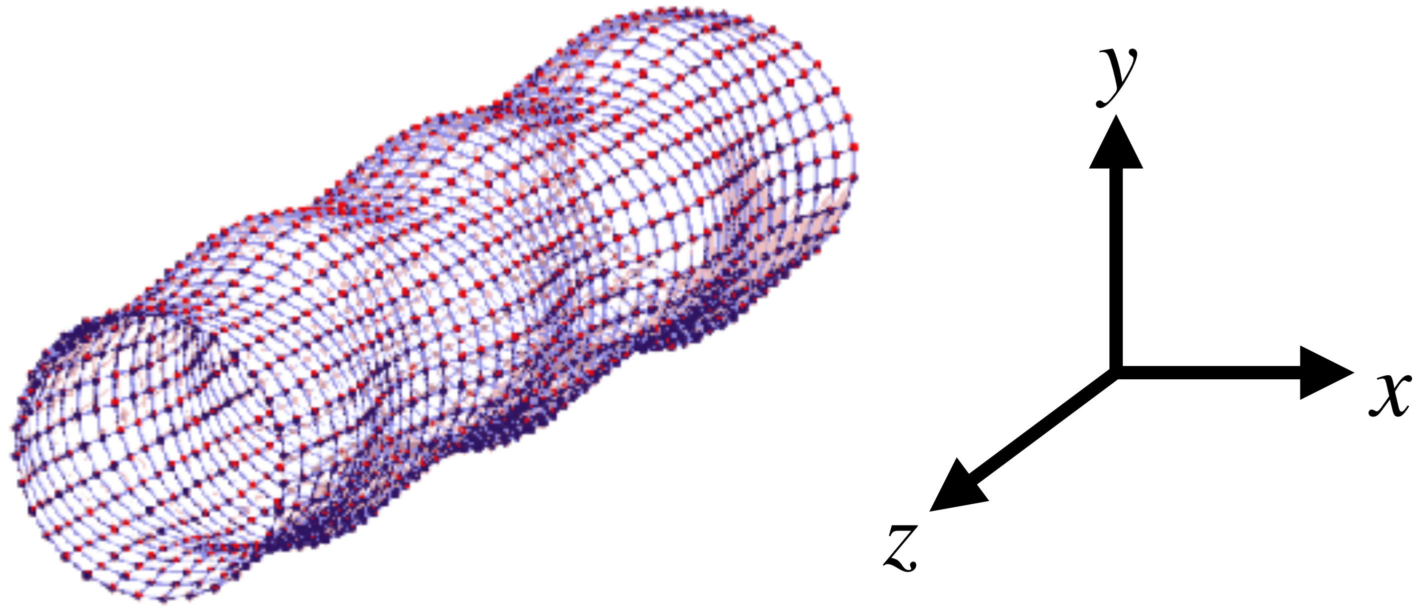
- Monochromatic Plane GWs propagating to $+z$ axis
- $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cos\left\{\omega_{\rm g}(-t+z)+\phi\right\}$$



+ polarization



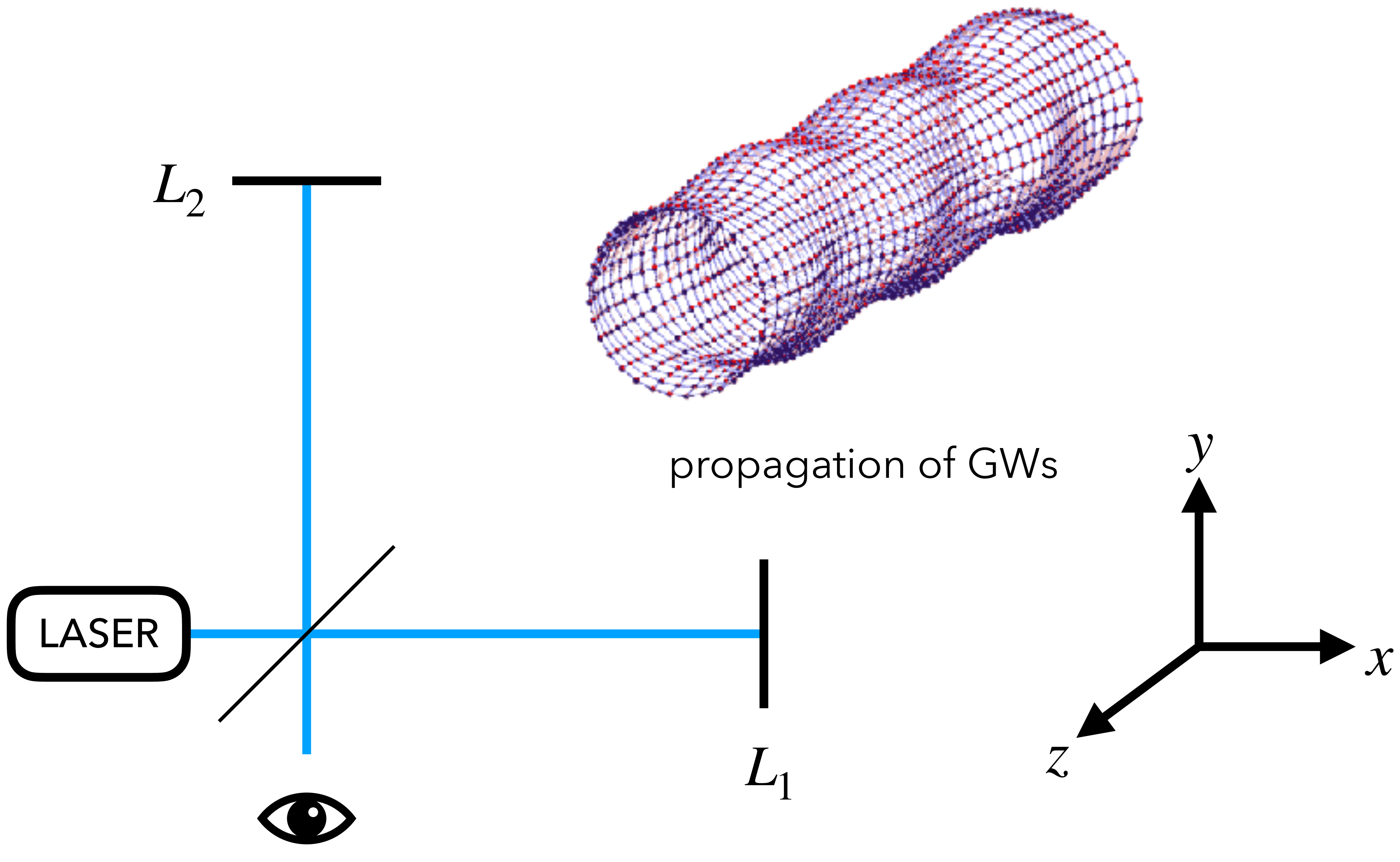
× polarization



propagation of GWs

Interferometric GW detector

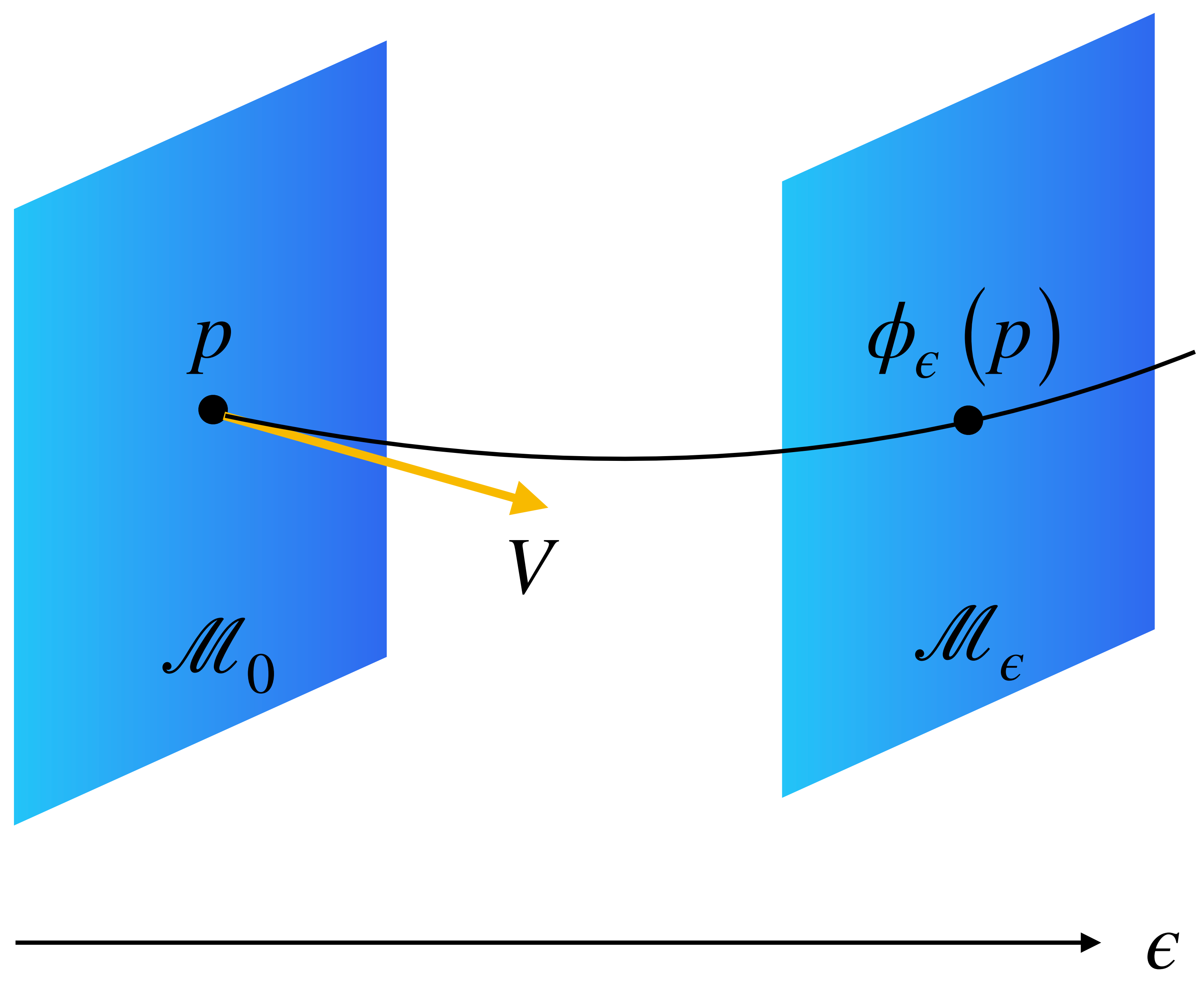
- GWs at interferometer ($z = 0$)
  - $h_{\mu\nu} = \begin{bmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{bmatrix} \cos\left(-\omega_g t + \phi\right)$
- Phase difference without GWs
  - $\Delta = 2\omega_e \left(L_2 - L_1\right)$
- Phase difference with GWs
  - $L_1'(t) = L_1 \left\{1 + \frac{1}{2} h_+ \cos\left(-\omega_g t + \phi\right)\right\}$
  - $L_2'(t) = L_2 \left\{1 - \frac{1}{2} h_+ \cos\left(-\omega_g t + \phi\right)\right\}$
  - $\Delta'(t) = \Delta + \omega_e \left(L_1 + L_2\right) h_+ \cos\left(-\omega_g t + \phi\right)$



Graduate Level

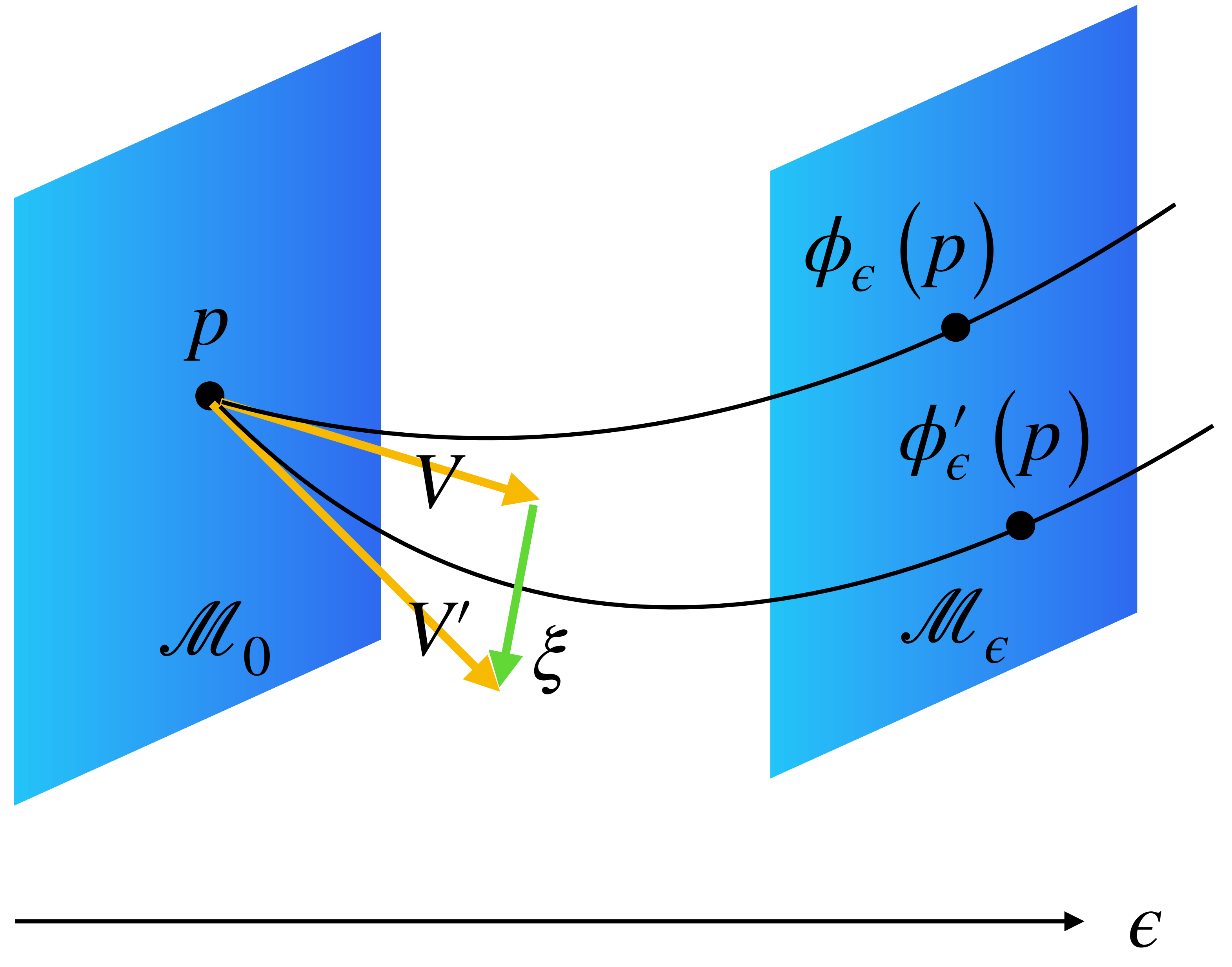
Perturbations

- $\mathscr{F}$: 5-dimensional manifold foliated by a one-parameter family of perturbed spacetime $\left(\mathscr{M}_\epsilon, g(\epsilon)\right)$
- ϵ: perturbation parameter
- ϕ_ϵ: a one-parameter group of diffeomorphism from $\mathscr{M}_0$ to $\mathscr{M}_\epsilon$.
- ${}^\epsilon Q$: perturbed Q in $\mathscr{M}_0$
    - ${}^\epsilon Q = \phi^*_{-\epsilon} Q(\epsilon) = Q + \epsilon \dot{Q} + \frac{1}{2}\epsilon^2 \ddot{Q} + O\left(\epsilon^3\right)$
- $\dot{Q}$: first order perturbation at $\mathscr{M}_0$
    - $\dot{Q} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left({}^\epsilon Q - Q\right) = \mathscr{L}_V Q$



Gauges in Perturbations

- We have gauges because the choice of ϕ_ϵ is not unique.
- Gauge Transformation
  - $\mathscr{L}_{V'}Q - \mathscr{L}_{V}Q = \mathscr{L}_{\xi}Q$
  - ξ is tangent to $\mathscr{M}_0$ because $\xi(\epsilon) = V'(\epsilon) - V(\epsilon) = 1 - 1 = 0$
- Stewart-Walker Lemma
  - $\dot{Q}$ is gauge-invariant if and only if $\mathscr{L}_{\xi}Q = 0$ for all ξ.



Metric and Levi-Civita Tensor

- Perturbed metric
  - ${}^{\epsilon}g_{ab} = g_{ab} + \epsilon h_{ab} + O\left(\epsilon^2\right)$
- Endomorphism
  - $\dot{\delta}^a{}_b = 0$
- Inverse metric
  - $\mathscr{L}_V g^{ab} = -h^{ab}$
- Levi-Civita tensor
  - $\dot{\epsilon}_{abcd} = \frac{1}{2} h^e{}_e \epsilon_{abcd}$

Covariant Derivatives

- Perturbation of the covariant derivative for a rank (k, l) tensor T
  - $\mathscr{L}_V \nabla T = \nabla \dot{T} + \dot{\nabla} T$
- where
  - $\dot{\nabla}_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = \sum_{i=1}^{k} T^{a_1 \cdots d \cdots a_k}{}_{b_1 \cdots b_l} \dot{C}^{a_i}{}_{dc} - \sum_{i=1}^{l} T^{a_1 \cdots a_k}{}_{b_1 \cdots d \cdots b_l} \dot{C}^{d}{}_{b_i c}$
  - $\dot{C}^a{}_{bc} = \frac{1}{2} g^{ad} \left(\nabla_c h_{bd} + \nabla_b h_{cd} - \nabla_d h_{bc} \right)$

GWs in Minkowski Background

- Perturbed metric
  - ${}^{\epsilon}g_{ab} = g_{ab} + \epsilon h_{ab} + O\left(\epsilon^2\right)$
- Ansatz
  - Minkowski background
    - ${}^{\epsilon}R^{a}{}_{bcd} = \epsilon \dot{R}^{a}{}_{bcd} + O\left(\epsilon^2\right)$
  - Stress-energy
    - ${}^{\epsilon}T_{ab} = O\left(\epsilon^2\right)$
- Perturbed Einstein equation
  - $\nabla^c \nabla_c h_{ab} = 0$
  - $\nabla^b h_{ab} = 0$
  - $h^{a}{}_{a} = 0$
- Wave solution
  - $h_{ab} = \int_{\mathcal{N}} d^3\mathcal{N}\left(k\right)\, \tilde{h}_{ab}\left(k\right) e^{\mathrm{i}P(;k)}$
  - $\mathcal{N} = \{k : k \cdot k = 0\} - \{0\}$
  - $\tilde{h}_{ab}\, d^3\mathcal{N}$: infinitesimal amplitude
  - $k^a = \nabla^a P$
- Gauge conditions
  - $k^b \tilde{h}_{ab}\left(k\right) = 0$
  - $\tilde{h}^{a}{}_{a}\left(k\right) = 0$

GWs in Transverse-Traceless Gauge

- We introduce the Eulerian observer with 4-velocity n for a globally inertial coordinate system $\{t, \vec{x}\}$.
- Wavevector k is decomposed into
  - $k^a = \omega \left(n^a + \kappa^a \right)$
- Imposing an additional gauge condition
  - $h_{ab} n^b = 0$
- the gauge conditions are summarized in
  - $\tilde{h}_{ab} \left(k \right) n^b = 0$
  - $\tilde{h}_{ab} \left(k \right) \kappa^b = 0$
  - $\tilde{h}^a{}_a \left(k \right) = 0$

Expert Level

Please Refer PDF File

- Perturbations
  - Definition
  - Gauges
  - Metric
  - Levi-Civita Tensor
  - Covariant Derivatives
  - Riemann Curvature
  - Einstein Equation
  - Geodesics
- Gravitational Waves
  - Minkowski Background
  - Lorenz Gauge
  - Wave Solution
  - Traceless Gauge
  - Riemann Tensor
  - Introducing Observer
  - Polarization of Amplitude
  - Changing Observer
  - Perturbation of Observer
- Detection of GWs
  - Geometrical Optics
  - Perturbation of Rays
  - Beyond Geometrical Optics

Summary

- Metric perturbation induces a strain for matter.
- GWs are a propagation of metric perturbation over spacetime.
- GWs waves propagates in the light speed and gives a strain perpendicular to spatial propagation direction without expanding the area of strain plane.
- GWs have + and x polarizations.
- Interferometric and bar detectors can observe GWs by measuring detector response with respect to GWs.