A radically brief introduction to

General relativistic hydrodynamics

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SIMULATING EXTREME SPACETIMES

Caltech

Cornell University — Founded A.D. 1865

University of New Hampshire

California State University Fullerton

Max Planck Institute for Gravitational Physics
ALBERT EINSTEIN INSTITUTE

The University of Mississippi

Washington State University

https://www.black-holes.org/

https://github.com/sxs-collaboration/WelcomeToSXS

Contents

- What is numerical relativity?
- Numerical relativity with matter
  - What equations do we solve?
  - What kinds of physics are involved?
- What are issues & challenges?

Numerical relativity (NR)

Solving Einstein's equations on computers

- modeling GW sources
- astrophysical phenomena with strong gravity & dynamic spacetime

Einstein's GR (1915)

ADM / 3+1 formulation (circa 1960)

Breakthroughs in 2005 (annus mirabilis)

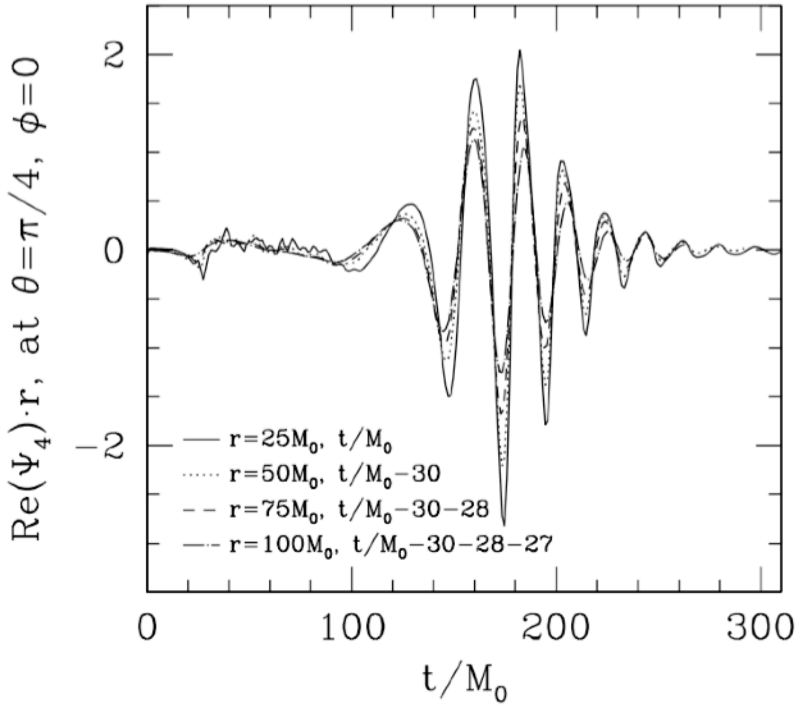
Pretorius 2005

Campanelli+ 2006

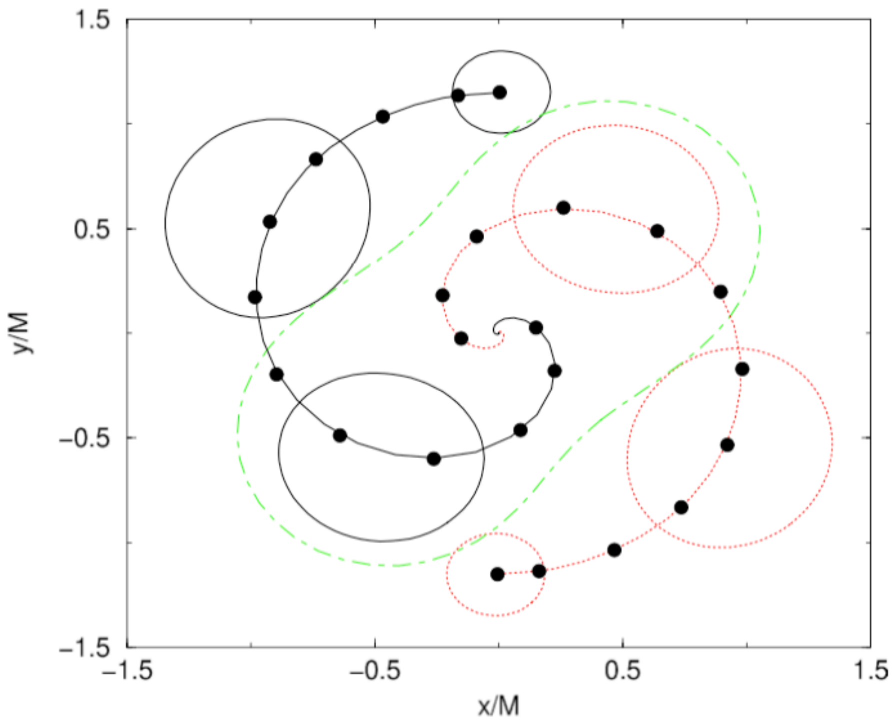
Baker+ 2006

LIGO's first detection of GW (2015)

Kilonova event (2017)



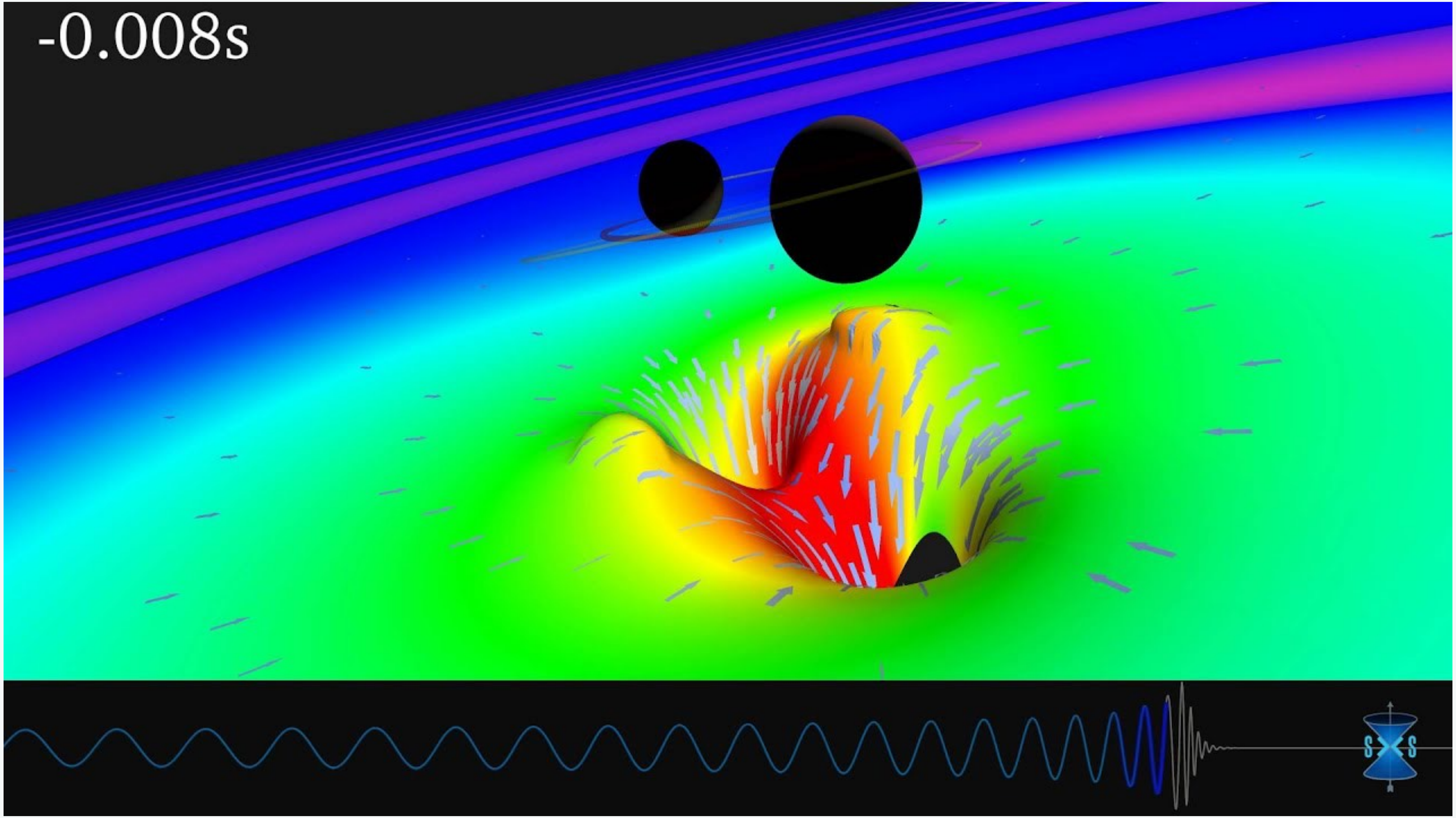
Pretorius 2005



Campanelli+ 2006

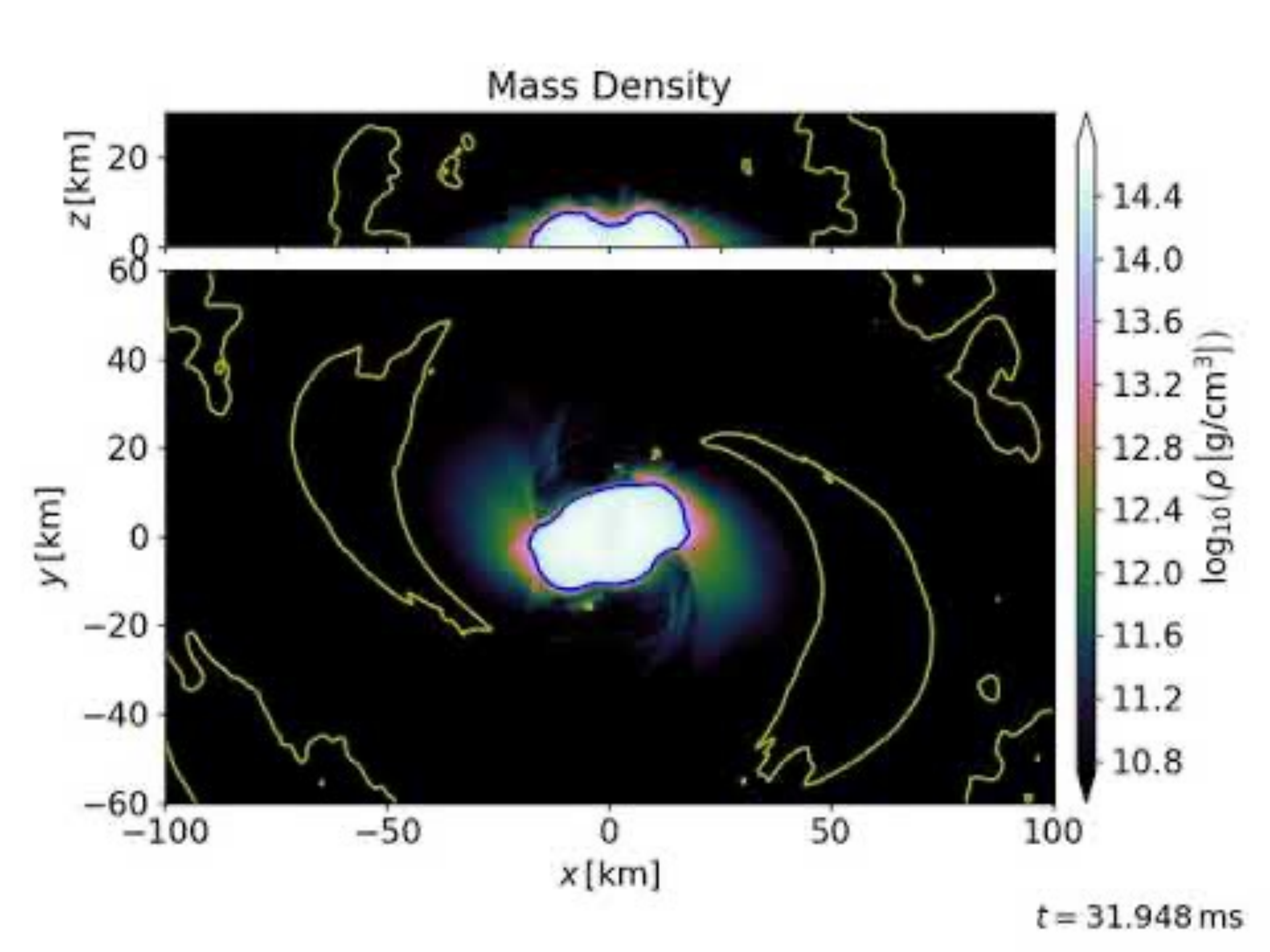
Image credit : SXS collaboration (black-holes.org)

Binary black hole merger



Video credit : SXS collaboration (https://www.youtube.com/watch?v=c-2XIuNFgD0)

Binary neutron star merger



Video credit : Max Planck Institute for Gravitational Physics
(https://www.youtube.com/watch?v=7jMw_EpLuSs)

Modern NR codes

Code	Open Source	Catalog	Formulation	Hydro	Beyond GR
AMSS-NCKU [43–46]	Yes	No	BSSN/Z4c	No	Yes
BAM [47–49]	No	[18]	BSSN/Z4c	Yes	No
BAMPS [50, 51]	No	No	GHG	Yes	No
COFFEE[52, 53]	Yes	No	GCFE	No	Yes
Dendro-GR [54–56]	Yes	No	BSSN/CCZ4	No	Yes
Einstein Toolkit [57, 58]	Yes	No	BSSN/Z4c	Yes	Yes
*Canuda [59–62]	Yes	No	BSSN	No	Yes
*IllinoisGRMHD [63]	Yes	No	BSSN	Yes	No
*LazEv [37, 64]	No	[65–68]	BSSN+CCZ4	No	No
*Lean [69, 70]	Partially	No	BSSN	No	Yes
*MAYA [71]	No	[71]	BSSN	No	Yes
*NRPy+ [72]	Yes	No	BSSN	Yes	No
*SphericalNR [73, 74]	No	No	spherical BSSN	Yes	No
*THC [75–77]	Yes	[18]	BSSN/Z4c	Yes	No
ExaHyPE [78]	Yes	No	CCZ4	Yes	No
FIL[79]	No	No	BSSN/Z4c/CCZ4	Yes	No
FUKA [80, 81]	Yes	No	XCTS	Yes	No
GR-Athena++ [82]	Yes	No	Z4c	Yes	No
GRChombo [83–85]	Yes	No	BSSN+CCZ4	No	Yes
HAD [86–88]	No	No	CCZ4	Yes	Yes
Illinois GRMHD [89, 90]	No	Yes	BSSN	Yes	No
MANGA/NRPy+ [91]	Partially	No	BSSN	Yes	No
MHDuet [92, 93]	No	No	CCZ4	Yes	Yes
SACRA-MPI [94]	No		BSSN+Z4c	Yes	No
SpEC [95, 96]	No	[96, 97]	GHG	Yes	Yes
SpECTRE [98, 99]	Yes	No	GHG	Yes	No
SPHINCS_BSSN [100]		No	BSSN	SPH	No

Foucart+ 2203.08139

Numerical relativity with matter

- *Spacetime tells matter how to move :* $\nabla_\mu T^{\mu\nu} = 0$
- *Matter tells spacetime how to curve :* $G_{\mu\nu} = 8\pi T_{\mu\nu}$

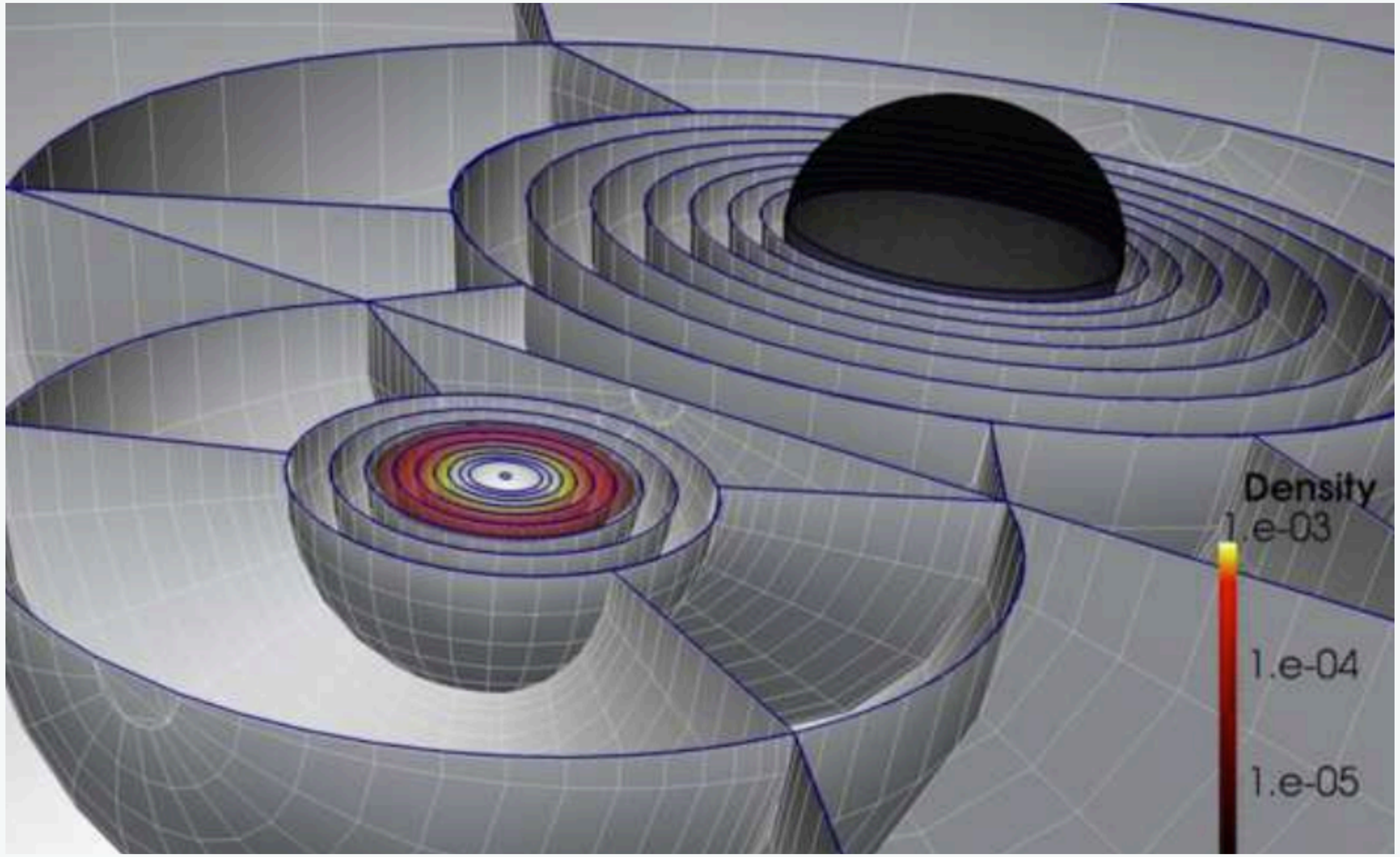
+ hydrodynamics (GRHD)

+ magnetic fields (GRMHD)

+ radiation (GRRMHD)

Applications

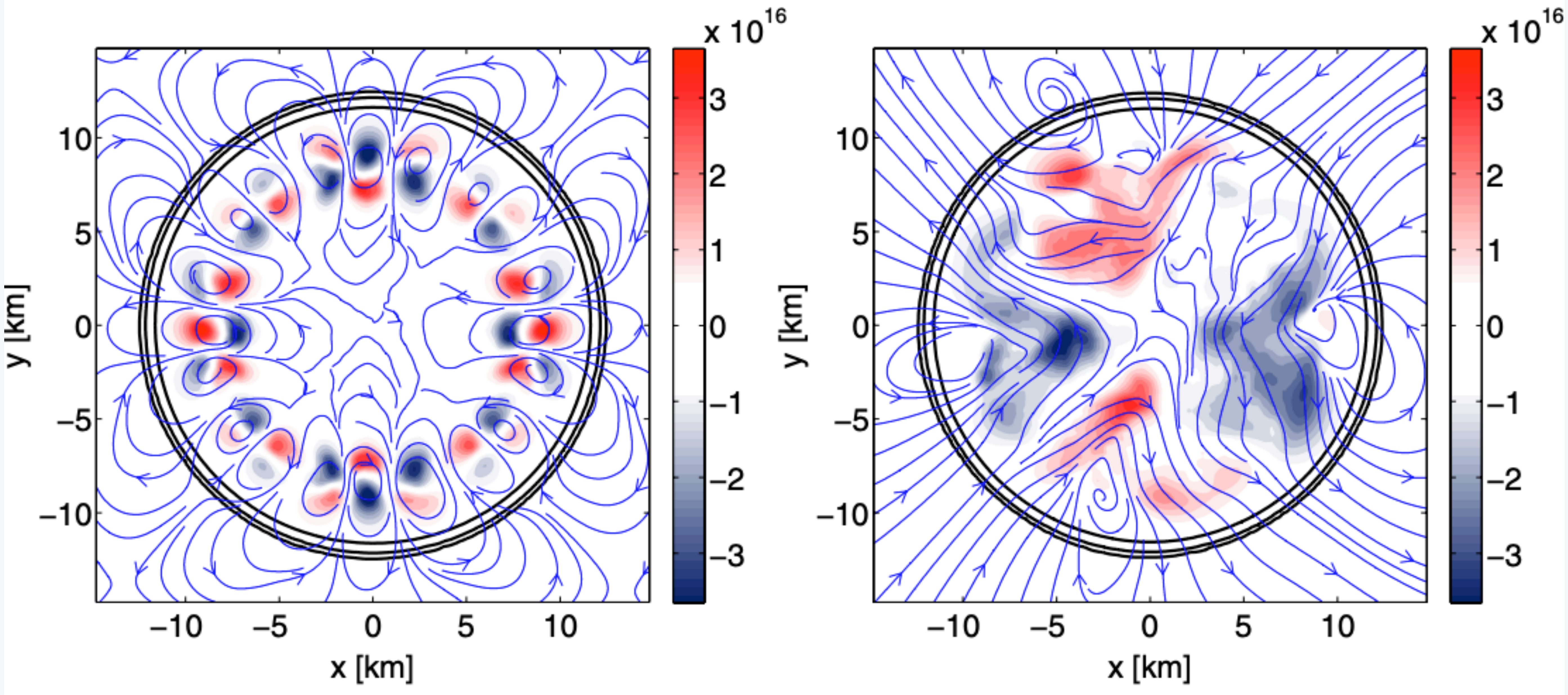
- non-vacuum compact binaries (NS-BH & NS-NS)



Foucart+2013

Applications

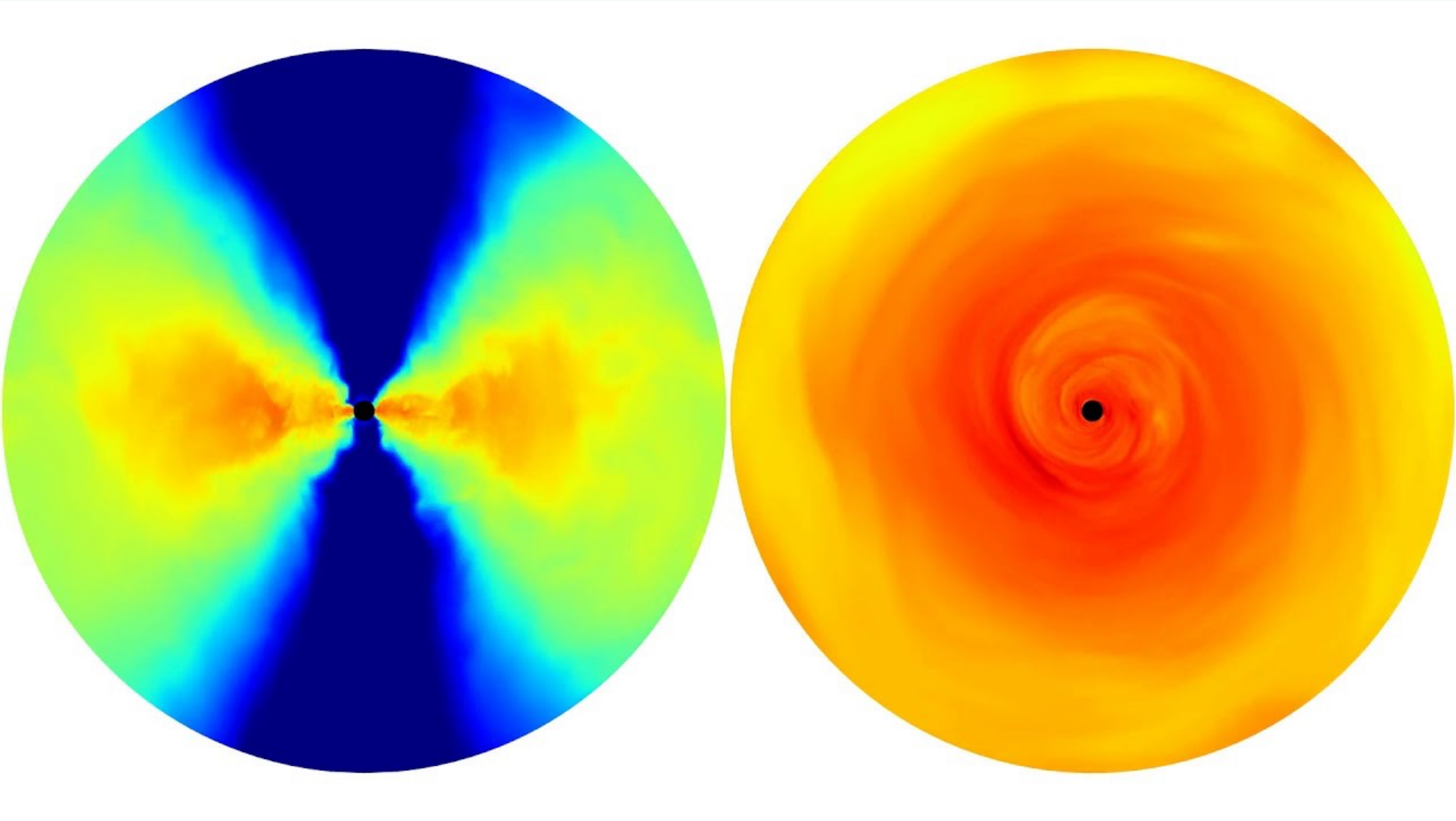
- relativistic stars



Ciolfi+2011

Applications

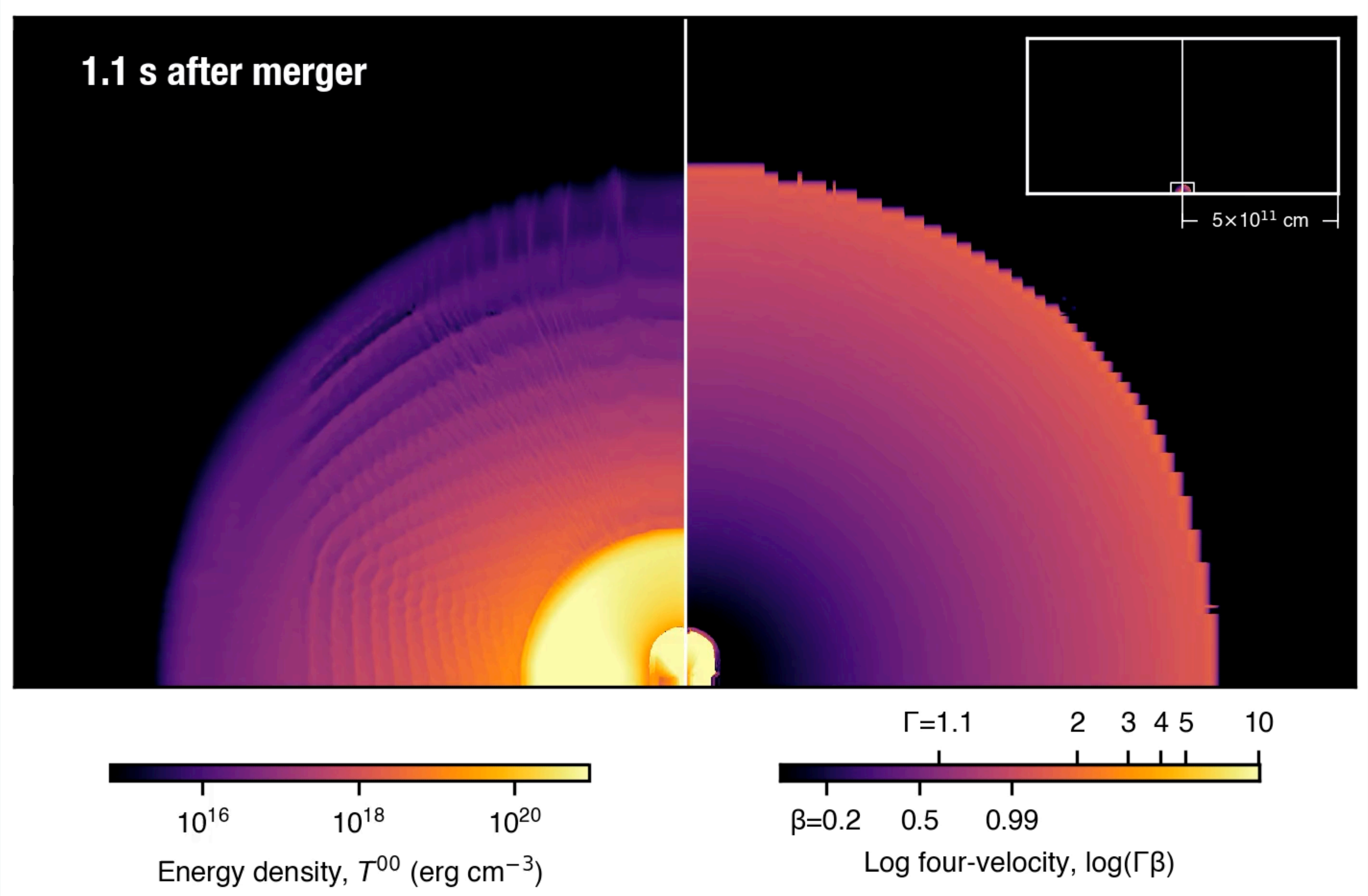
- accretion disk



Video credit : Illinois Physics
(https://www.youtube.com/watch?v=pjJIA4AjHiQ)

Applications

- jets



Video credit : Ore Gottlieb
(https://www.oregottlieb.com/videos.html)

Evolution of matter

begin with Newtonian hydrodynamics :

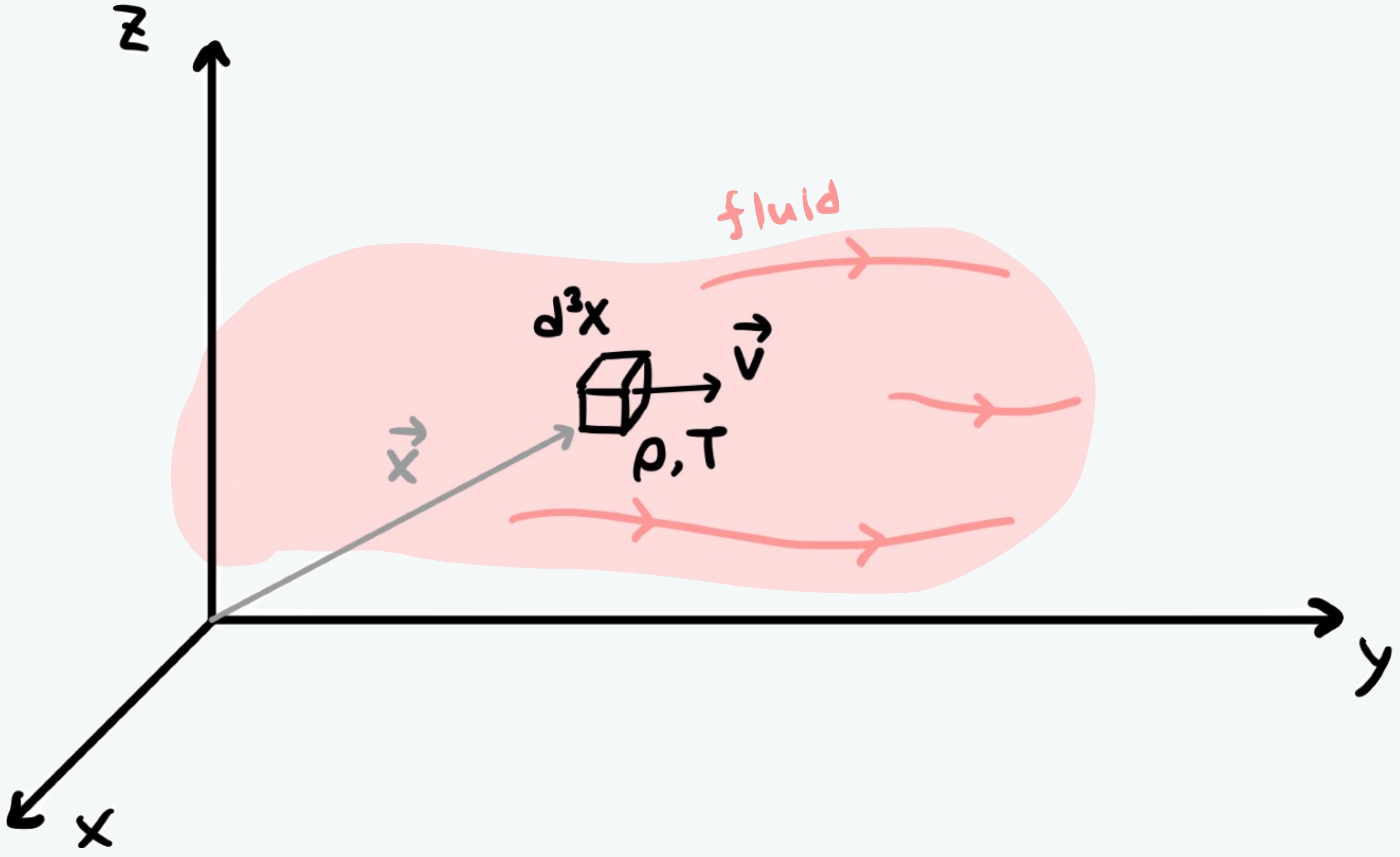
Physical quantities describing the fluid at $(\mathbf{x}, t)$

- mass density ρ
- velocity $\mathbf{v}$
- energy density (~temperature) ϵ

*pressure from $P = P(\rho, \epsilon)$: **equations of state** (EOS)

e.g. Classical ideal gas $$P = \frac{2}{3}\epsilon \quad (= nk_BT)$$

e.g. Electron degeneracy $$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m_e^{8/3}}\rho^{5/3}$$



Evolution of matter

Which physical laws do we have?

- mass conservation
- $\mathbf{F} = m\mathbf{a}$ *(~momentum conservation)*
- energy conservation

Basic equations in Newtonian hydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{\rho}$$

$$\frac{\partial \epsilon}{\partial t} + (\mathbf{v} \cdot \nabla)\epsilon = -\frac{P(\nabla \cdot \mathbf{v})}{\rho}$$

note) convective derivative $\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$

makes equations *nonlinear*

Evolution of matter

<u>Conservative form</u> of PDEs

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F} = \mathbf{S}$$

- $\mathbf{U}$: evolved variables (conserved variables)
- $\mathbf{F}$: flux
- $\mathbf{S}$: source

e.g. charge continuity $\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{J} = 0$

- Important for correct shock speeds
- Advantageous when solving numerically (Lax & Wendroff 1960, Hou & LeFloch 1994)

this is a good form !

Can we write Newtonian hydro equations into this form? Yes

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho v^j \\ \rho\epsilon + \frac{1}{2}\rho v^2 \end{bmatrix}, \qquad \mathbf{F}^i = \begin{bmatrix} \rho v^i \\ \rho v^i v^j + P\delta^{ij} \\ (\rho\epsilon + \frac{1}{2}\rho v^2 + P)v^i \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

conserved variables $\mathbf{U}$ vs primitive variables $\{\rho, v^i, \epsilon\}$

Evolution of matter

GR hydrodynamics

Physical quantities describing the fluid at $(\mathbf{x}, t)$: $T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu}$ stress-energy tensor (ideal fluid)

- conservation of mass $\nabla_\mu(\rho u^\mu) = 0$
- conservation of energy and momentum $\nabla_\mu T^{\mu\nu} = 0$

$\}$ → $\partial_t \mathbf{U} + \nabla \cdot \mathbf{F} = \mathbf{S}$

cast to the conservative form

- **Valencia formulation** (Marti+1991; Banyuls+1997)

$$\mathbf{U} = \begin{bmatrix} \tilde{D} \\ \tilde{S}_j \\ \tilde{\tau} \end{bmatrix} = \sqrt{\gamma} \begin{bmatrix} \rho W \\ \rho h W^2 v_j \\ \rho h W^2 - P - \rho W \end{bmatrix}, \quad \mathbf{F}^i = \alpha \begin{bmatrix} \tilde{D}\tilde{v}^i \\ \tilde{S}_j \tilde{v}^i + \sqrt{\gamma} P \delta^i_j \\ \tilde{\tau}\tilde{v}^i + \sqrt{\gamma} P v^i \end{bmatrix}, \quad \mathbf{S} = \alpha\sqrt{\gamma} \begin{bmatrix} 0 \\ T^{\mu\nu} g_{\nu\sigma} \Gamma^\sigma_{\mu j} \\ T^{\mu 0}\partial_\mu \alpha - \alpha T^{\mu\nu}\Gamma^0_{\mu\nu} \end{bmatrix}$$

$W = \alpha u^0$
$= (1 - \gamma_{ij} v^i v^j)^{-1/2}$

$\tilde{v}^i = v^i - \beta^i/\alpha$

used by current NR codes

See analogies:

Newtonian hydro

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho v^j \\ \rho\epsilon + \frac{1}{2}\rho v^2 \end{bmatrix}$$

$$\mathbf{F}^i = \begin{bmatrix} \rho v^i \\ \rho v^i v^j + P\delta^{ij} \\ (\rho\epsilon + \frac{1}{2}\rho v^2 + P)v^i \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

GR hydro

$$\mathbf{U} = \sqrt{\gamma}\begin{bmatrix} \rho W \\ \rho h W^2 v_j \\ \rho h W^2 - P - \rho W \end{bmatrix}$$

$$\mathbf{F}^i = \alpha \begin{bmatrix} \tilde{D}\tilde{v}^i \\ \tilde{S}_j\tilde{v}^i + \sqrt{\gamma}P\delta^i_j \\ \tilde{\tau}\tilde{v}^i + \sqrt{\gamma}Pv^i \end{bmatrix}$$

$$\mathbf{S} = \alpha\sqrt{\gamma}\begin{bmatrix} 0 \\ T^{\mu\nu}g_{\nu\sigma}\Gamma^\sigma_{\mu j} \\ T^{\mu 0}\partial_\mu\alpha - \alpha T^{\mu\nu}\Gamma^0_{\mu\nu} \end{bmatrix}$$

variables
- mass (1)
- momentum (3)
- energy (1)

~ variables x velocity

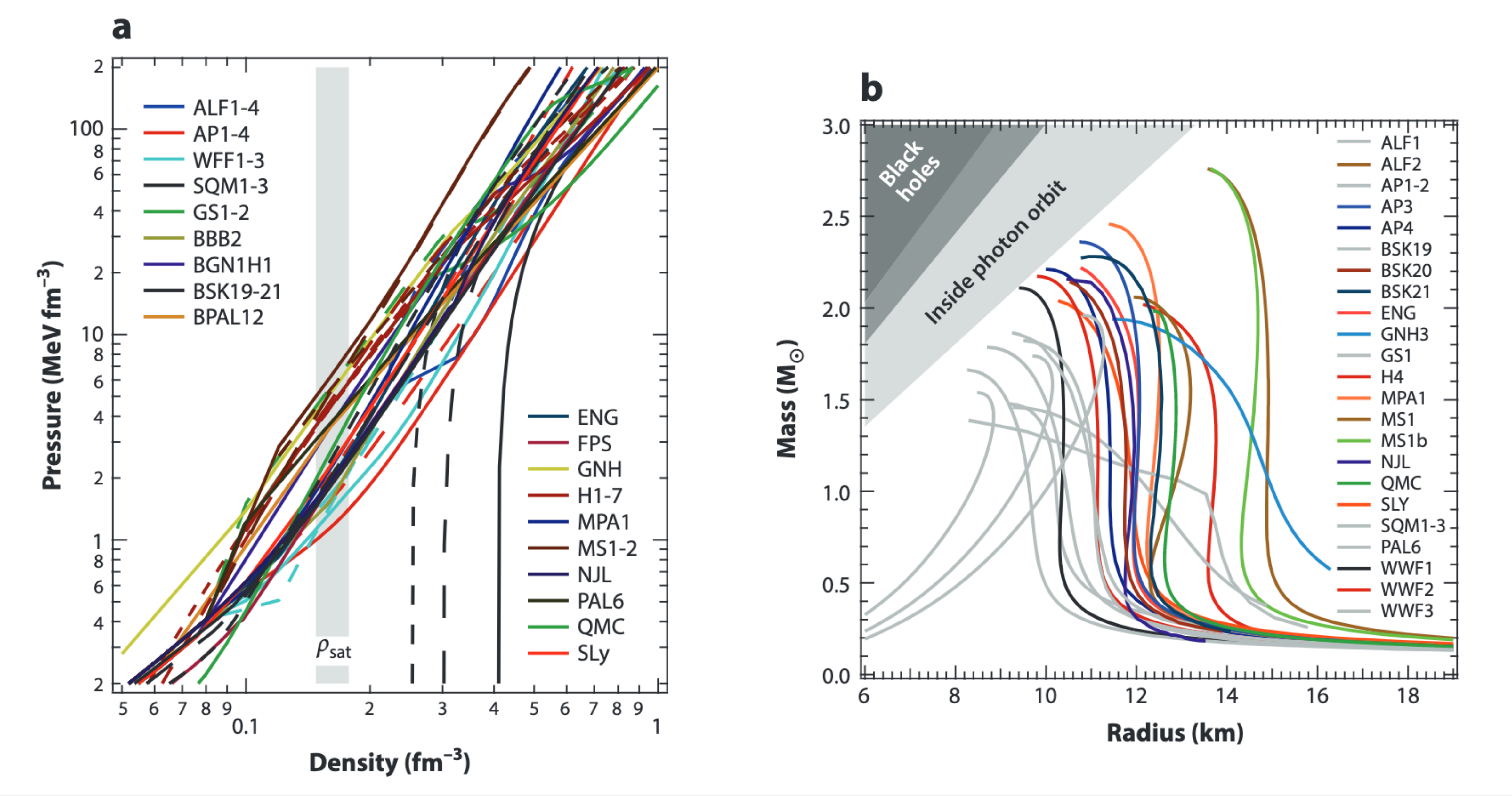
relativistic effects

* flat space : $\alpha = 1, \quad \beta^i = 0, \quad \sqrt{\gamma} = 1, \quad \Gamma^\sigma_{\mu\nu} = 0$
* non-relativistic : $v^j \ll 1, \quad \rho \gg \epsilon, P$

More physics !

Properties of dense matter

- Uncertainties in $P(\rho, T)$ inside neutron stars

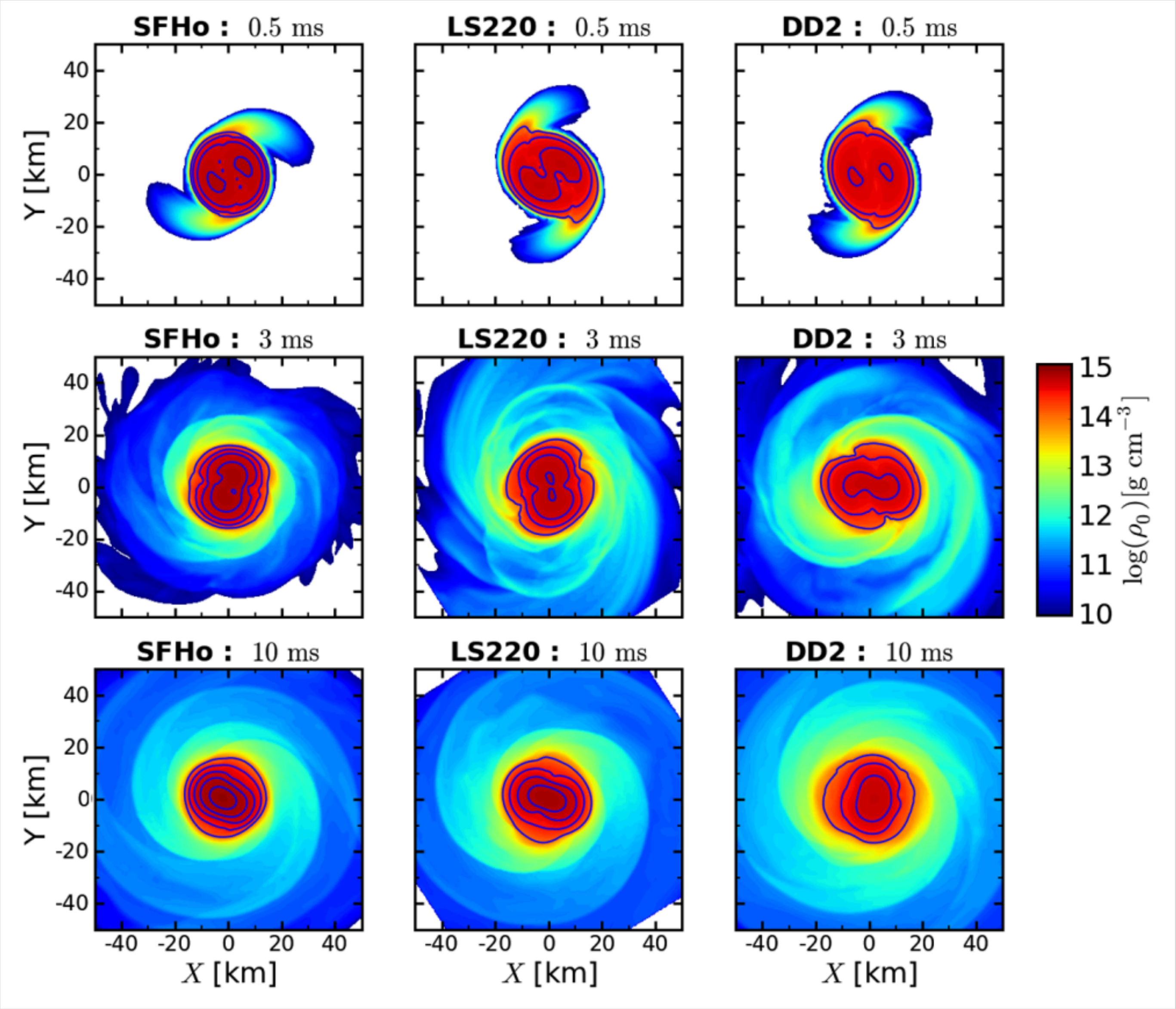


Özel & Freire 2016

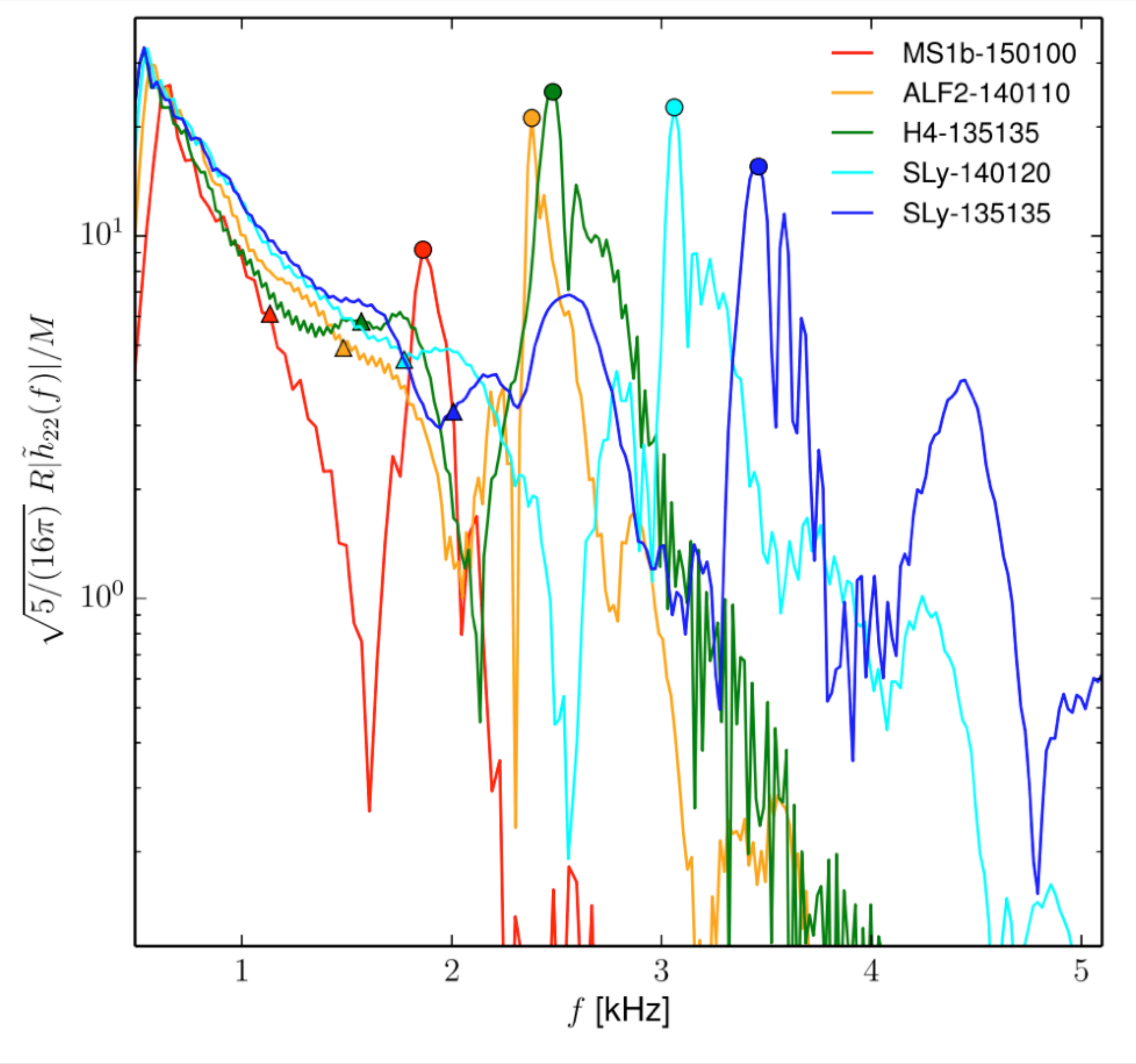
More physics !

Properties of dense matter

- BH-NS / NS-NS merger as a probe of nuclear physics
- Constrain EOSs from merger detections



Foucart+2016



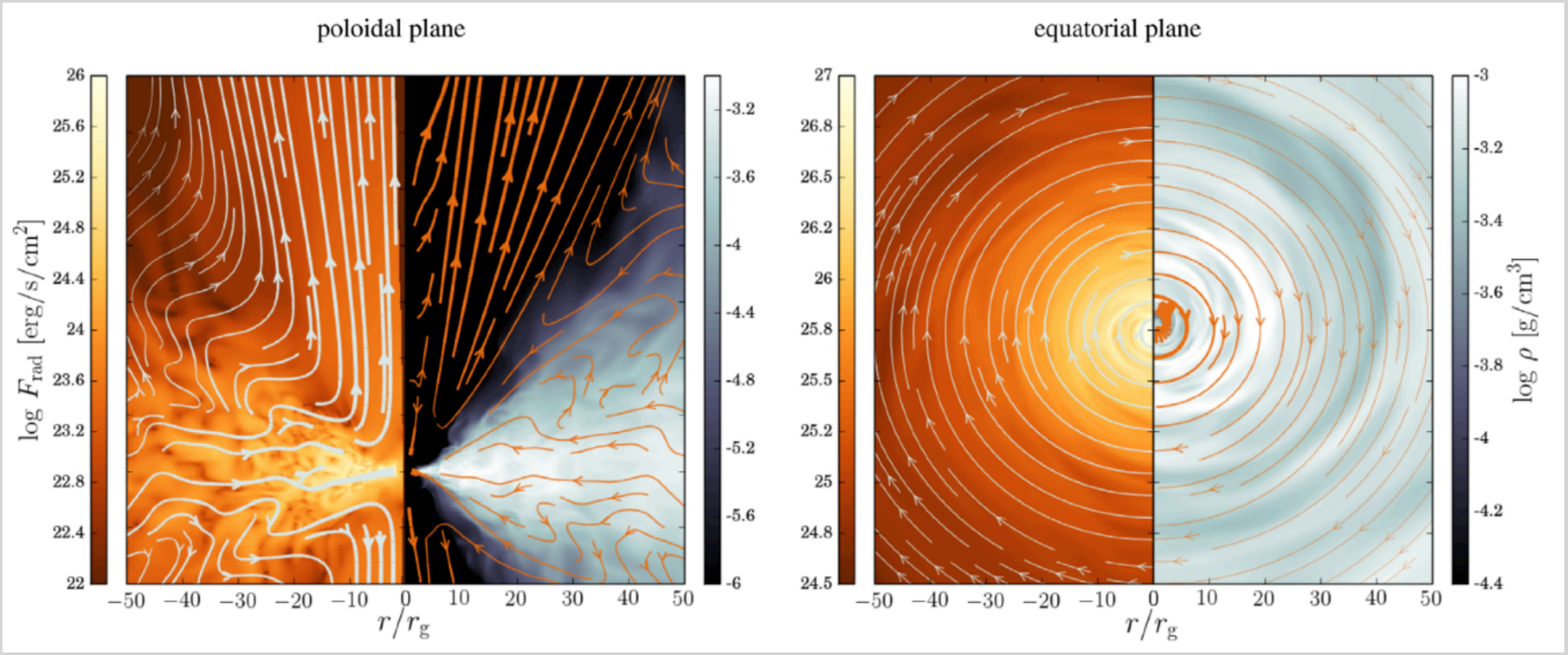
Bernuzzi+2015

More physics !

Magnetic field

- GRMHD : $T^{\mu\nu} = T^{\mu\nu}_{\text{matter}} + \boxed{T^{\mu\nu}_{\text{EM}}}$
- introduces (even) higher nonlinearity

✓ capturing instabilities & turbulence

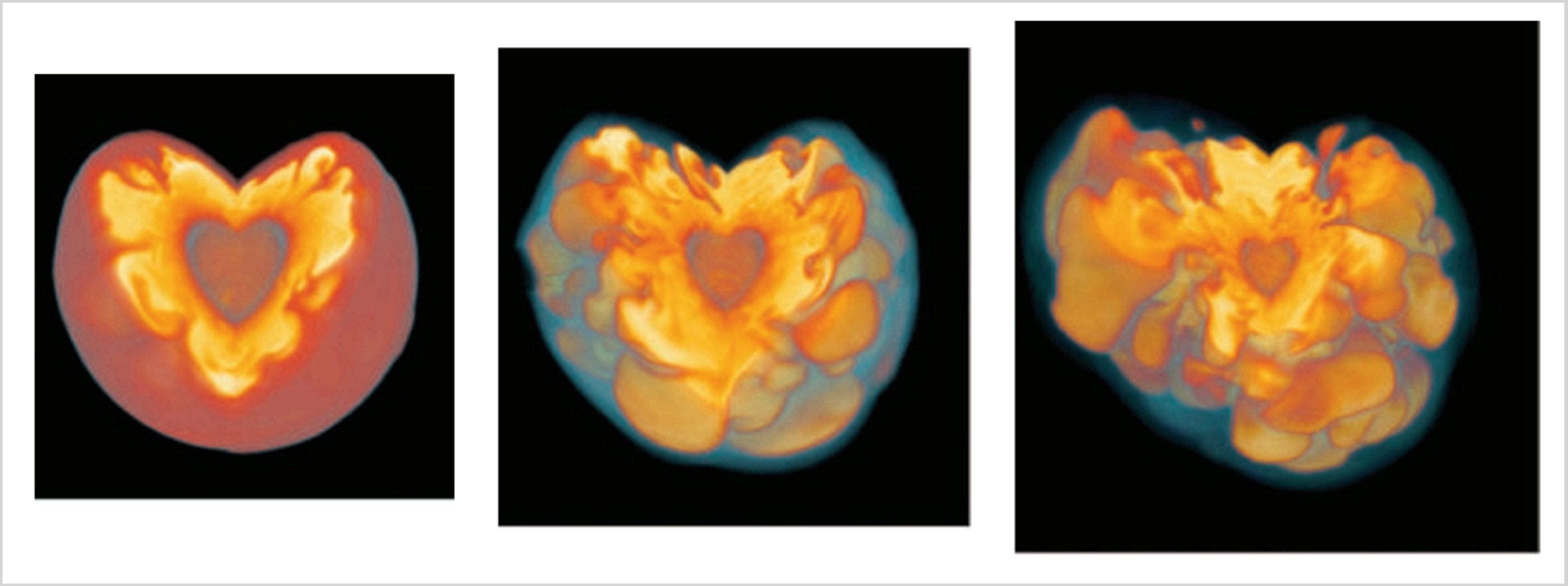


Sądowski & Narayan 2016

Neutrino effects

- cooling and heating of matter
- evolution of matter composition (e.g. nucleosynthesis)

✓ expensive to compute full equations



Woosley & Janka 2005

Challenges

Shocks & Discontinuities

e.g. shockwave, stellar surface

- **Gibbs phenomena** may lead to unphysical states

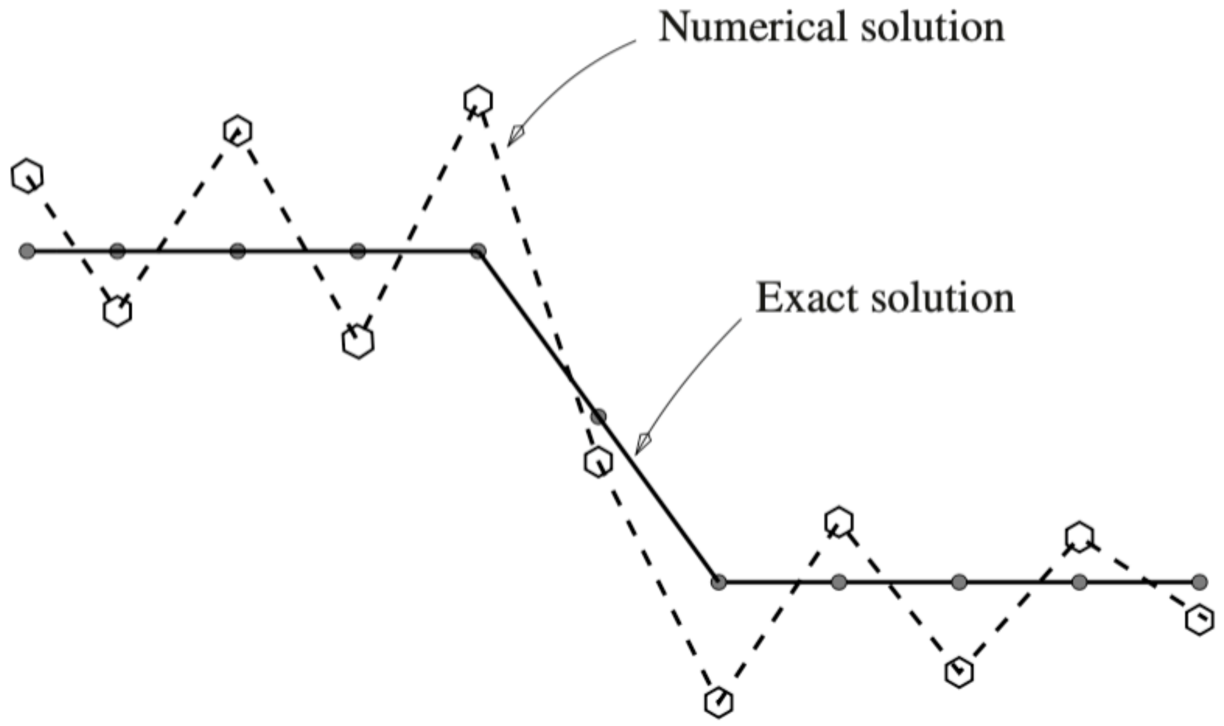
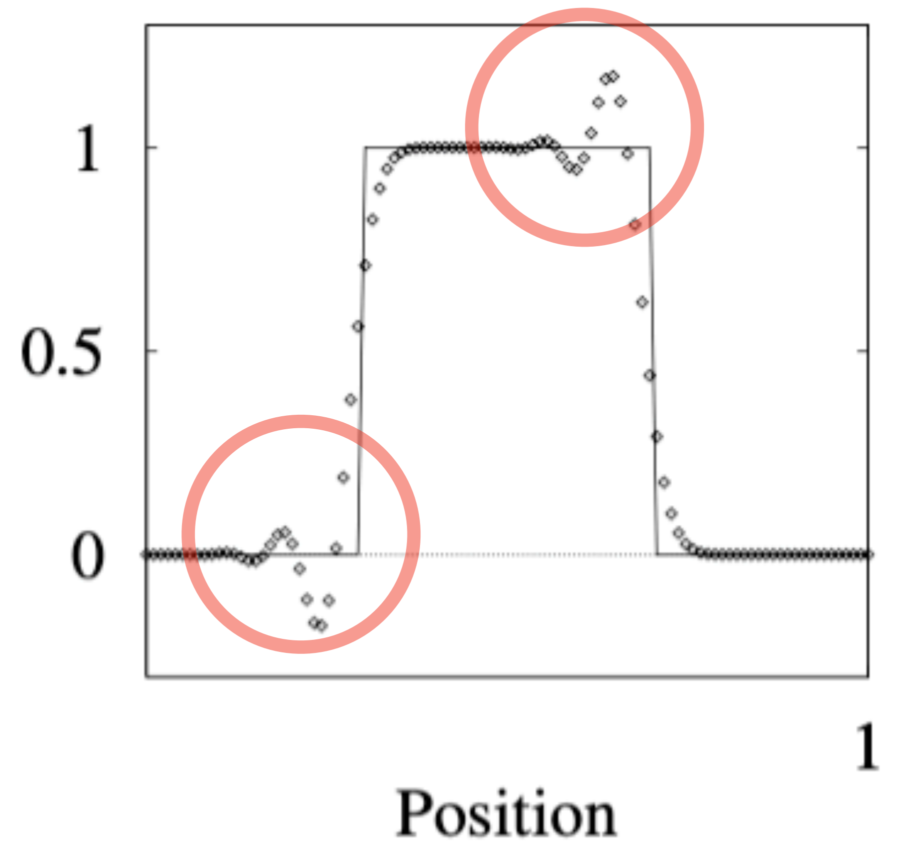


Fig. 13.11. Illustration of the numerical phenomenon of spurious oscillations near high gradients.

Toro 2009



Toro 2009

Challenges

C2P recovery

Primitive variables

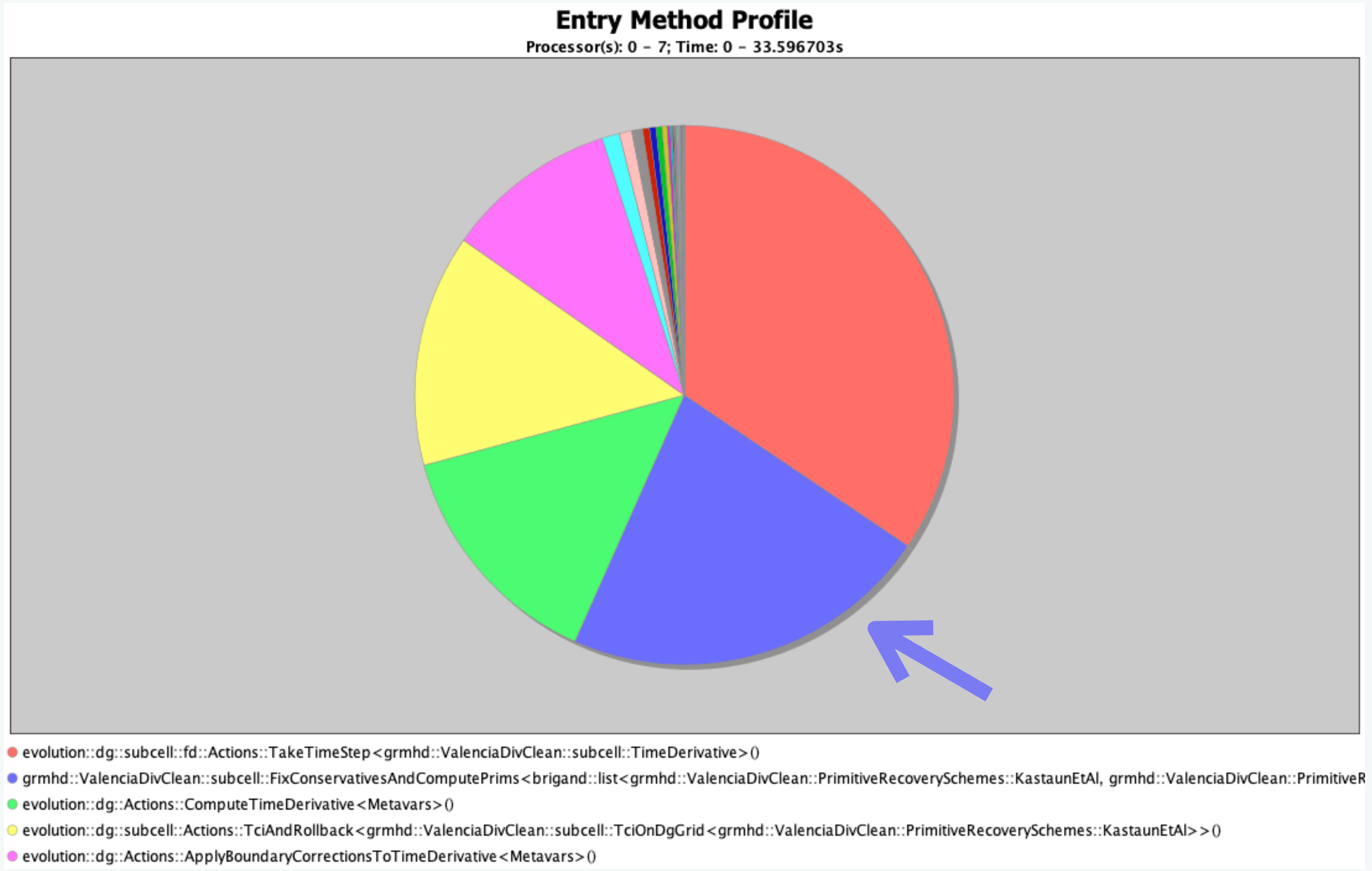
$$\rho, v^i, \epsilon, B^i$$

no closed-form expression exists for inversion

Conserved (evolved) variables

$$\mathbf{U} = \sqrt{\gamma} \begin{bmatrix} \rho W \\ (\rho h + b^2) W^2 v_j - \alpha b^0 b_j \\ (\rho h + b^2) W^2 - (P + b^2/2) - \rho W - (\alpha b^0)^2 \\ B^k \end{bmatrix}$$

- numerical algorithms needed to recover **P** from **C**
- How to efficiently and accurately solve it?



profile of a GRMHD run in SpECTRE

Take-aways

- **Hydrodynamics is hard**
  matter makes everything complex…
- **Rich in physics & astrophysical applications**
  more difficulties, more fun
- **One of the most challenging yet interesting multiphysics problem**
  stay tuned :)