2022 수치상대론 및 중력파 여름학교(2022/07/25~29)

일반상대론 기초(III)

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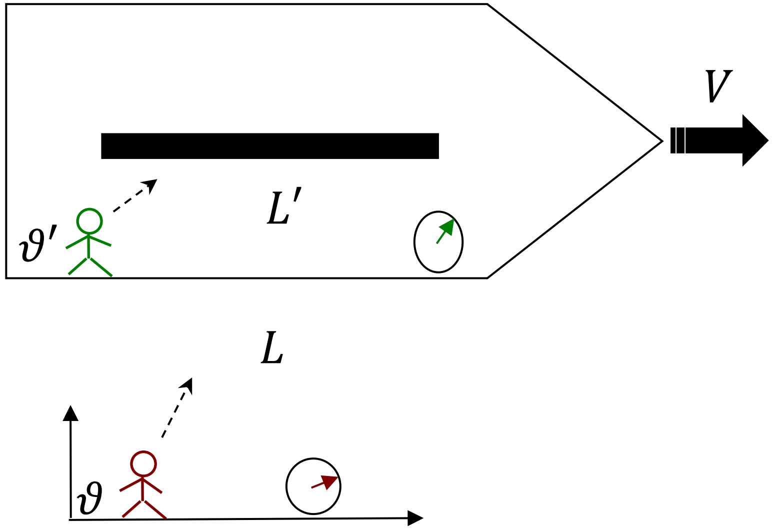
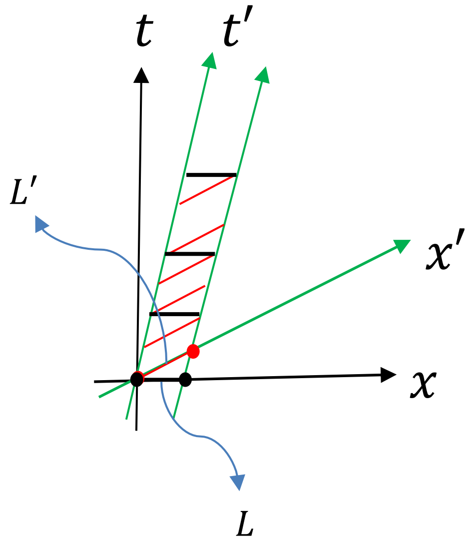
- 길이 수축: ➔ 숙제 문제풀이



- **길이란?:** 동시에 발생한 두 사건의 거리 ➔

  $L' = \Delta x' = x'_2 - x'_1$ w/ $\Delta t' = 0$ (막대 양 끝)
- 마찬가지로 '정지' 관측자(ϑ)도 움직이는 막대의 양 끝을 동시에 측정하여 길이 구함: L
- Note: ϑ' 의 두 사건은 ϑ 에게는 동시에 발생한 두 사건이 아님

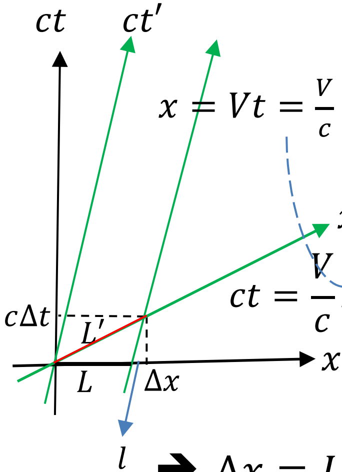


$$\Delta s^2 = -(c\Delta t)^2 + \Delta x^2 = -0^2 + L'^2$$

$$-\left(\frac{V}{c}\Delta x\right)^2 + \Delta x^2 = \Delta x^2\left[1 - \left(\frac{V}{c}\right)^2\right] = L'^2$$

$$\rightarrow \frac{L^2}{[1-(v/c)^2]^2}[1-(v/c)^2] = L'^2$$

$$\boxed{L = \sqrt{1-\left(\frac{V}{c}\right)^2}\,L'}$$



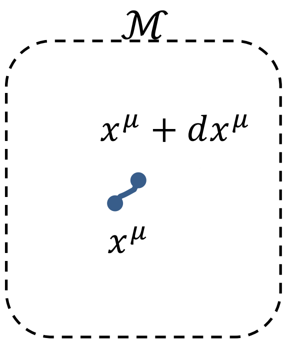
$$x = Vt = \frac{V}{c}ct$$

$$ct = \frac{V}{c}x \rightarrow c\Delta t = \frac{V}{c}\Delta x$$

➔ $\Delta x = L + l = L + \frac{V}{c}c\Delta t = L + \left(\frac{V}{c}\right)^2\Delta x$ ➔ $\Delta x = L/\left[1-\left(\frac{V}{c}\right)^2\right]$

목차

✓ SUMMARY:



$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

dt^2 & $dl^2 = dx^2+dy^2+dz^2$: 뉴튼의 시간과 공간

$\Uparrow$ $c \to \infty$

$ds^2 = -d(ct)^2 + dx^2+dy^2+dz^2$: 특수상대론적 시공간

Curved and Dynamical (interacting w/ matter) $g_{\mu\nu}(x)$

: 일반상대론

$$\boxed{R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi G T_{\alpha\beta}}$$

- 리만 곡률 텐서(Riemann curvature tensor):

$$R_{\alpha\beta\gamma}{}^{\delta} = \partial_\beta\Gamma^\delta_{\alpha\gamma} - \partial_\alpha\Gamma^\delta_{\beta\gamma} + \Gamma^\sigma_{\gamma\alpha}\Gamma^\delta_{\beta\sigma} - \Gamma^\sigma_{\gamma\beta}\Gamma^\delta_{\alpha\sigma} \,,$$

with $\Gamma^\delta_{\alpha\beta} = \frac{1}{2}g^{\delta\mu}\left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right)$

- $R_{\alpha\beta} \equiv R_{\alpha\delta\beta}{}^{\delta}$, $R \equiv R_{\alpha\beta}g^{\alpha\beta}$

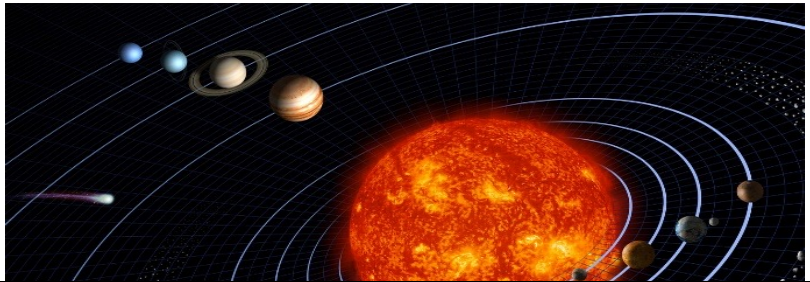
- $R_{\alpha\beta\gamma}{}^{\delta} = 0$ iff ST is FLAT

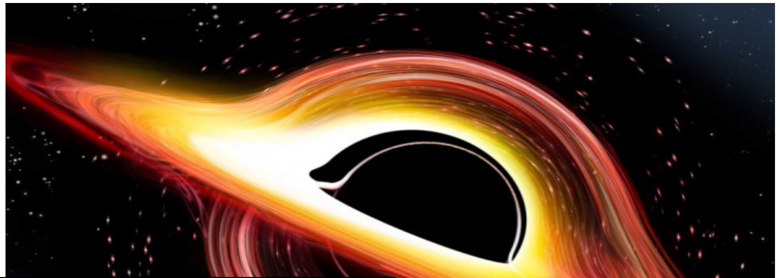
  $\neq 0$ iff ST is CURVED

IV. 블랙홀

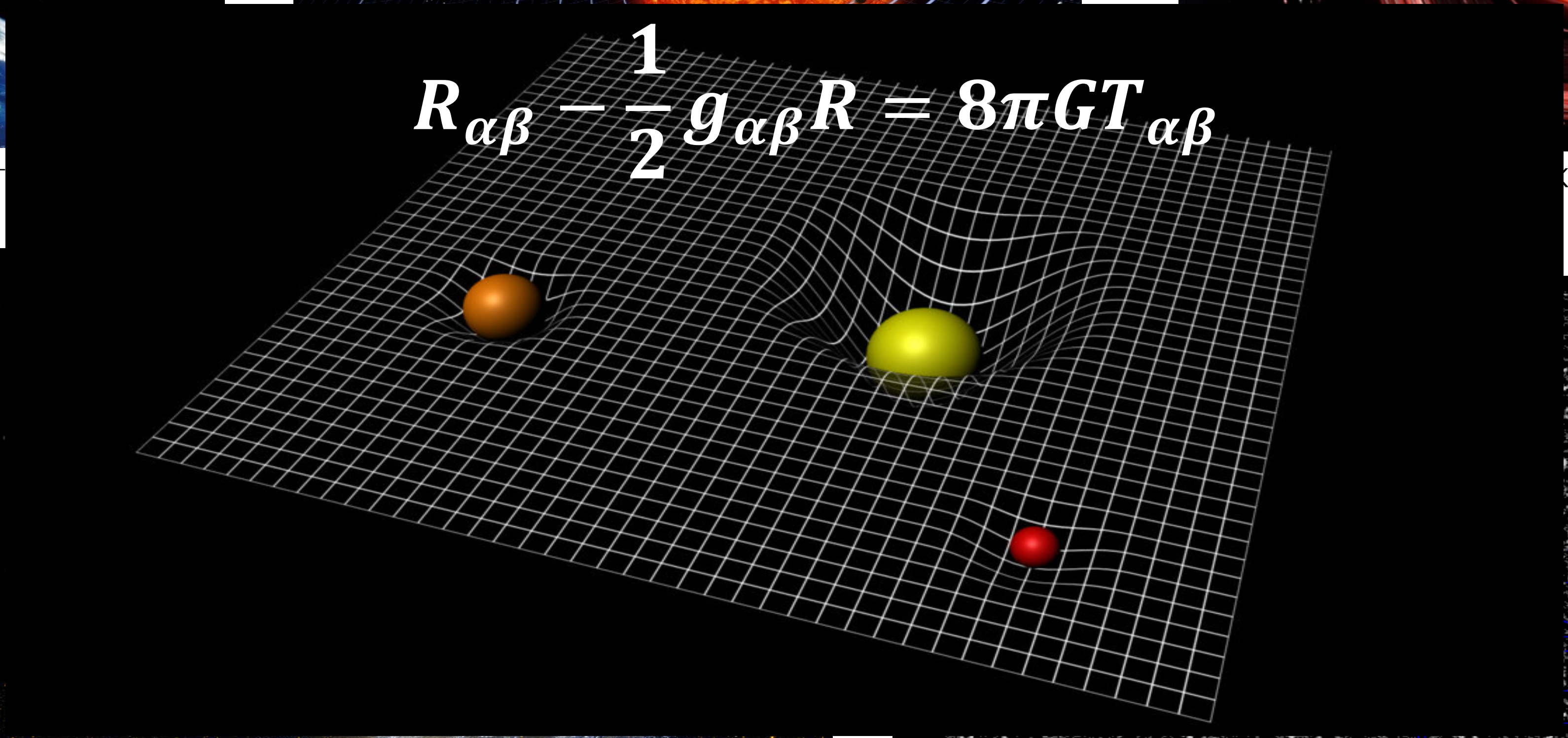
시공간 해

http://www.navcen.uscg.gov/ftp/gps/ggeninfo/gps-

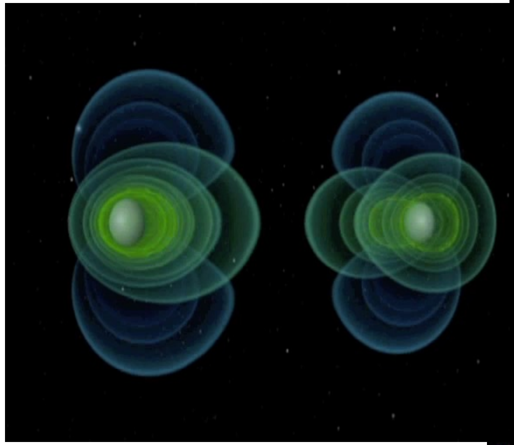




Kelly/Discover

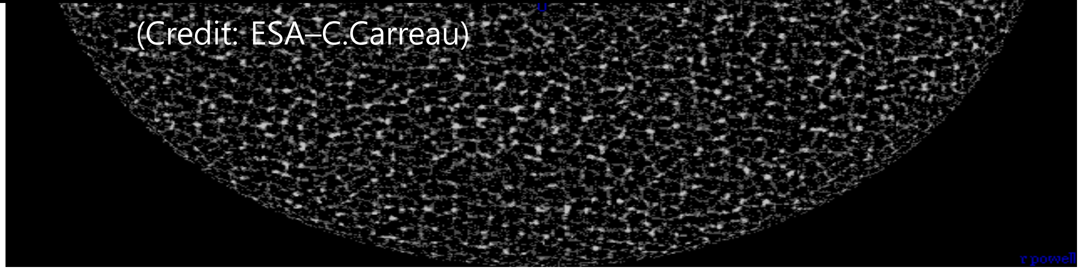


$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$$



Credit: M. Koppitz

Andromeda M31 (~67 kpc, NASA/JPL-Caltech)

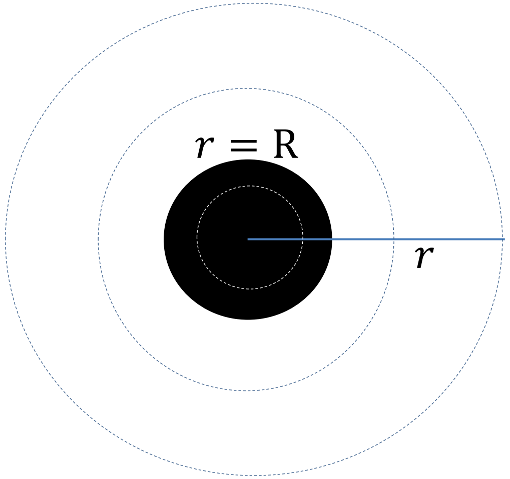


(Credit: ESA–C.Carreau)

Visible Universe (~8.6 Gpc, http://www.atlasoftheuniverse.com/universe.html)

• (정적 상태의) 구대칭 별

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$$



- **메트릭 추정(Metric ansatz):** 적절한 좌표를 도입해서 시간에 따라 변하지 않고(정적 상태) 구대칭을 반영하는 계량텐서를 아래와 같이 추정할 수 있음

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

- 외부($r > \mathrm{R}$): 진공 ➔ $T_{\alpha\beta} = 0$
- 내부($r < \mathrm{R}$): 완전 유체 가정 ➔ $T_{\alpha\beta} = \rho u_\alpha u_\beta + P\left(g_{\alpha\beta} + u_\alpha u_\beta\right)$

  $\rho = \rho(r)$: 질량밀도, $P = P(r)$: 압력, $u^\alpha = \left(\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi}\right) = (u^t, 0,0,0)$: 4-velocity of fluid

  Note: $-1 = u \cdot u = g_{\alpha\beta} u^\alpha u^\beta = -f u^t u^t + 0$ ➔ $u^\alpha = (1/\sqrt{f(r)}, 0,0,0)$

- 진공의 경우: $T_{\alpha\beta} = 0$ ➔ $0 = g^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R\right) = R - \frac{1}{2} \cdot 4 \cdot R = -R$ ➔ $R_{\alpha\beta} = 0$

✓ 별 외부 해(Exterior solutions): $R_{\alpha\beta} = 0$

$$R_{tt} = \frac{1}{4rfh^2}\left[-rhf'^2 + f(-rf'h' + 2h(2f' + rf''))\right] = 0$$

$$R_{rr} = \frac{1}{4rf^2h}\left[f(4f + rf')h' + rh(f'^2 - 2ff'')\right] = 0$$

$$R_{\theta\theta} = R_{\varphi\varphi}/\sin^2\theta = \frac{1}{2fh^2}\left[-rhf' + f(-2h + 2h^2 + rh')\right] = 0$$

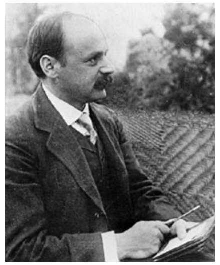
$$\rightarrow \boxed{f = 1 - \frac{C}{r} = h^{-1}}$$

C: 임의의 적분상수

$\updownarrow$ $C \leftrightarrow 2M$

- 슈워츠차일드 메트릭: 1916

출처: 과학동아

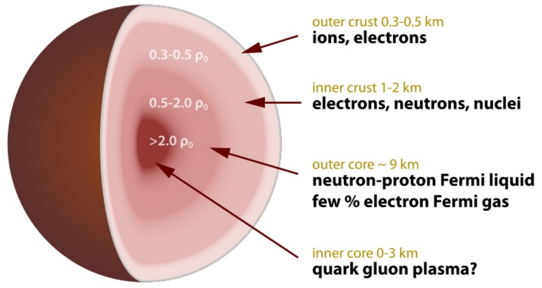


$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}$$

Karl Schwarzschild (1873~1916)

(※ To be continued at the “Global structure ……”.)

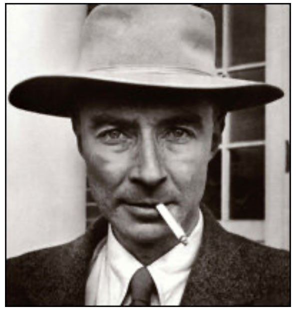
✓ 별 내부 해(Interior solutions):



$f \equiv e^{2\phi(r)}$ and $h \equiv (1-2m(r)/r)^{-1}$ 로 나타내면

$$ds^2 = -e^{2\phi}dt^2 + \frac{dr^2}{1-2m/r} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi G T_{\alpha\beta} = 8\pi G\left[\rho u_\alpha u_\beta + P\left(g_{\alpha\beta} + u_\alpha u_\beta\right)\right]$$



- TOV (Tolman-Oppenheimer-Volkoff) 방정식: 1934, 1939 w/ $G = 1 = c$

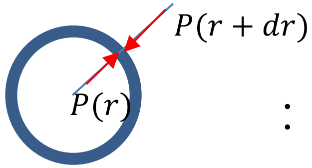
$$\frac{dm}{dr} = 4\pi r^2 \rho \;\Rightarrow\; m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$$

Note: $m(r = R) \equiv M = C/2$

$$\frac{d\phi}{dr} = \frac{m + 4\pi r^3 P}{r(r-2m)} \;\Leftarrow\; \frac{m}{r^2}$$

뉴튼 중력보다 더 강한 “인력“ 작용!!

$$\frac{dP}{dr} = -(\rho + P)\frac{m + 4\pi r^3 P}{r(r-2m)} \;\Leftarrow\; -\frac{\rho m}{r^2}$$



: Balance eq.

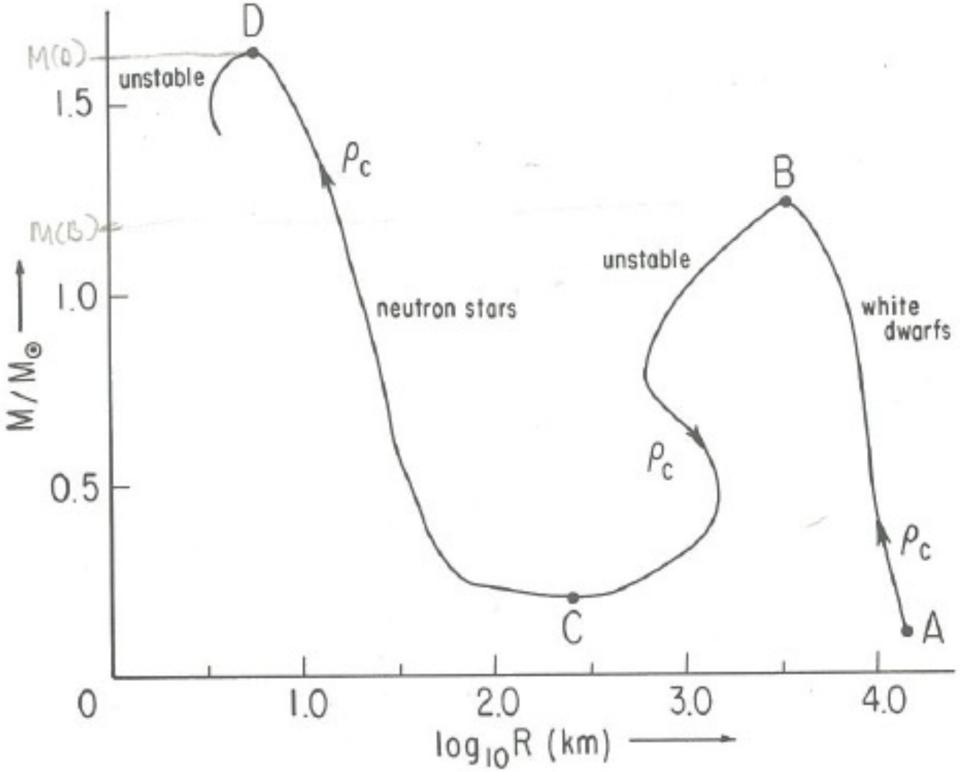
$P = P(\rho)$: EOS (상태방정식)

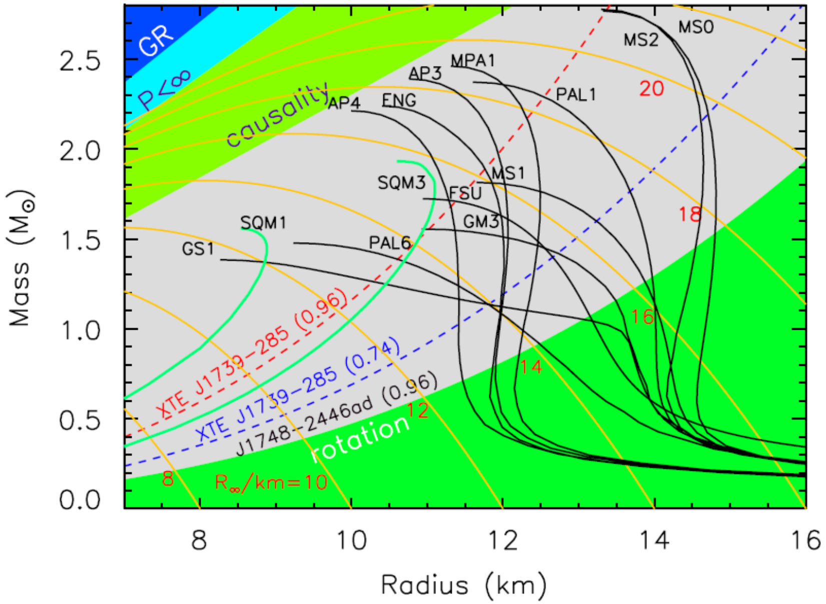
(단위 복구: $\frac{dP}{dr} = -G(\rho + P/c^2)\frac{m + 4\pi r^3 P/c^2}{r(r-2Gm/c^2)} \overset{c\to\infty}{\Longrightarrow} -G\frac{\rho m}{r^2}$)

∞

*

✓ **A class of solutions parameterized by the central density:**





다양한 상태방정식에 따른 별의 해:
Lattimer & Prakash

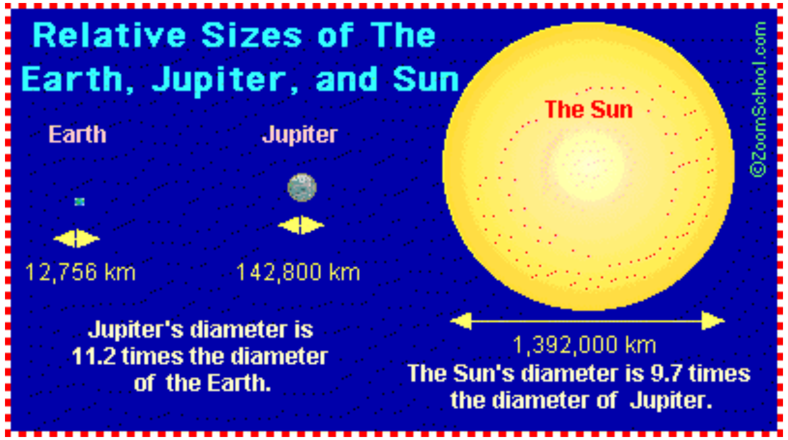
- The maximum mass exists for a given R, for instance, M < 4R/9 for constant density stars!

✓ 뉴튼 중력과 일반상대론에서의 별:

21만, 7만, 109, 9 배

블랙홀 (~1M⊙)

중성자별 (~1.4M⊙)



http://www.enchantedlearning.com/subjects/astronomy/sun/sunsize.shtml

"밀집도(Compactness)" ~ $\frac{\text{질량}}{\text{크기}}$

Ex) 지구: $\frac{M_\oplus}{R_\oplus} = \frac{4.4\times10^{-6}\text{ km}}{6.4\times10^{3}\ km} \sim 10^{-9}$

태양: $\frac{M_\odot}{R_\odot} = \frac{1.5\text{ km}}{7\times10^{5}\ km} \sim 10^{-5}$

은하: $\frac{M_{\mathrm{G}}}{R_{\mathrm{G}}} = \frac{10^{11}M_\odot}{30\sim50\text{ kpc}} \sim \frac{10^{11}\text{ km}}{10^{17}\ km} \sim 10^{-6}$

뉴튼 중력으로 다루어도 크게 틀리지 않음

중성자별: $\frac{M_{\mathrm{NS}}}{R_{\mathrm{NS}}} \sim \frac{1.4\,M_\odot}{10\sim100\text{ km}} \sim (0.1\sim0.01)$

블랙홀: $\frac{M_{\mathrm{BH}}}{R_{\mathrm{BH}}} \sim \frac{M}{2M} = 0.5$

일반상대론으로 다루어야 함:

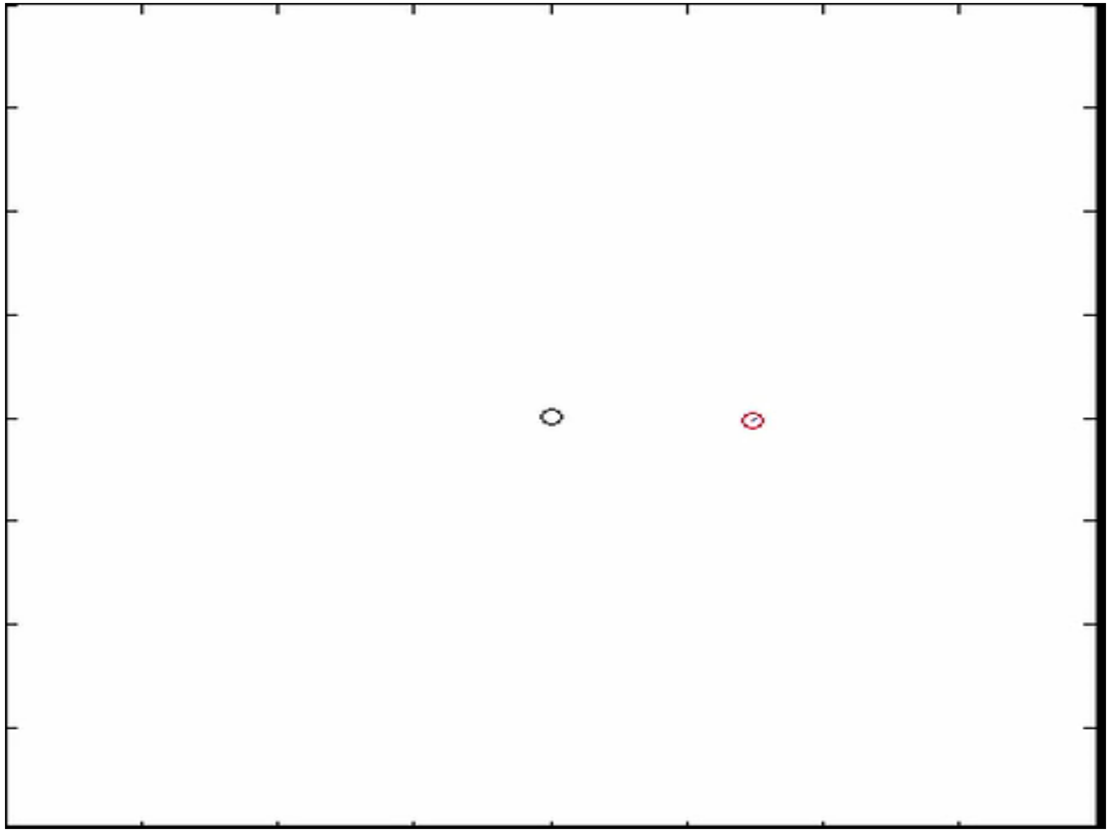
➔ 뉴튼 중력에서는 불가능한 천체

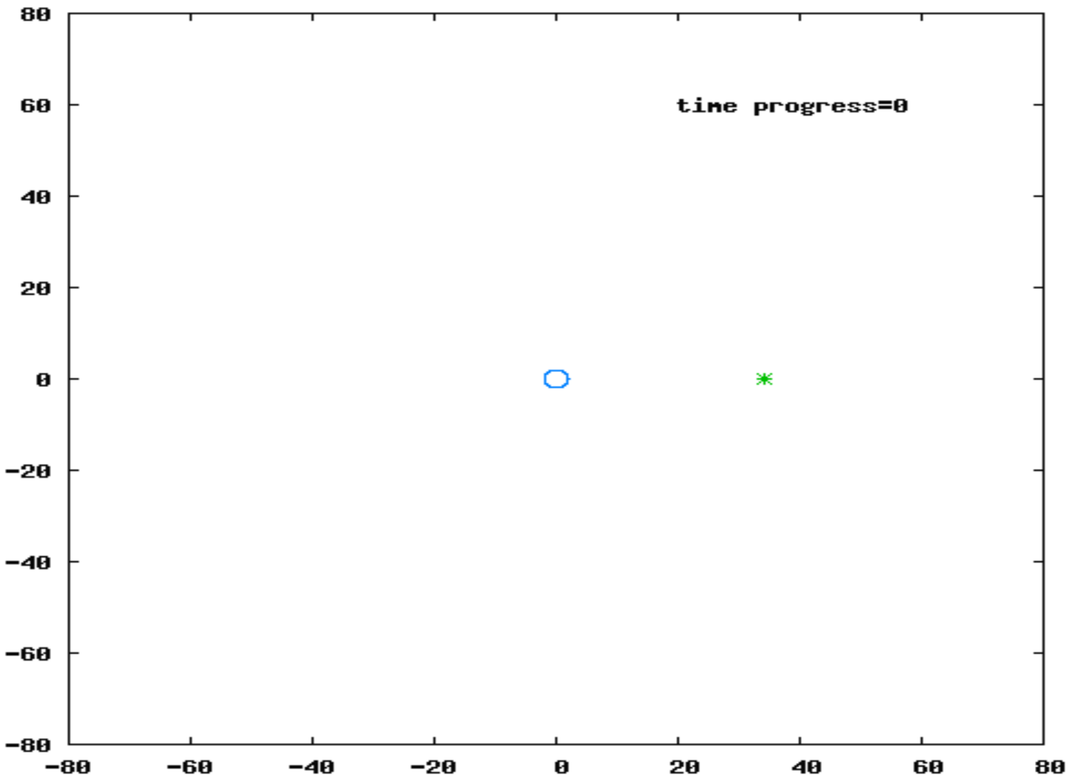
$$\frac{M}{r_{H,Kerr}} = \frac{M}{M+\sqrt{M^2-(J/M)^2}} = 0.5\sim1.0$$

- **Numerical calculations for geodesics:**
  - Schwarzschild BH: 박관호, 박찬 & GK ('13)
  - Kerr BH: 이승환 & GK ('14)

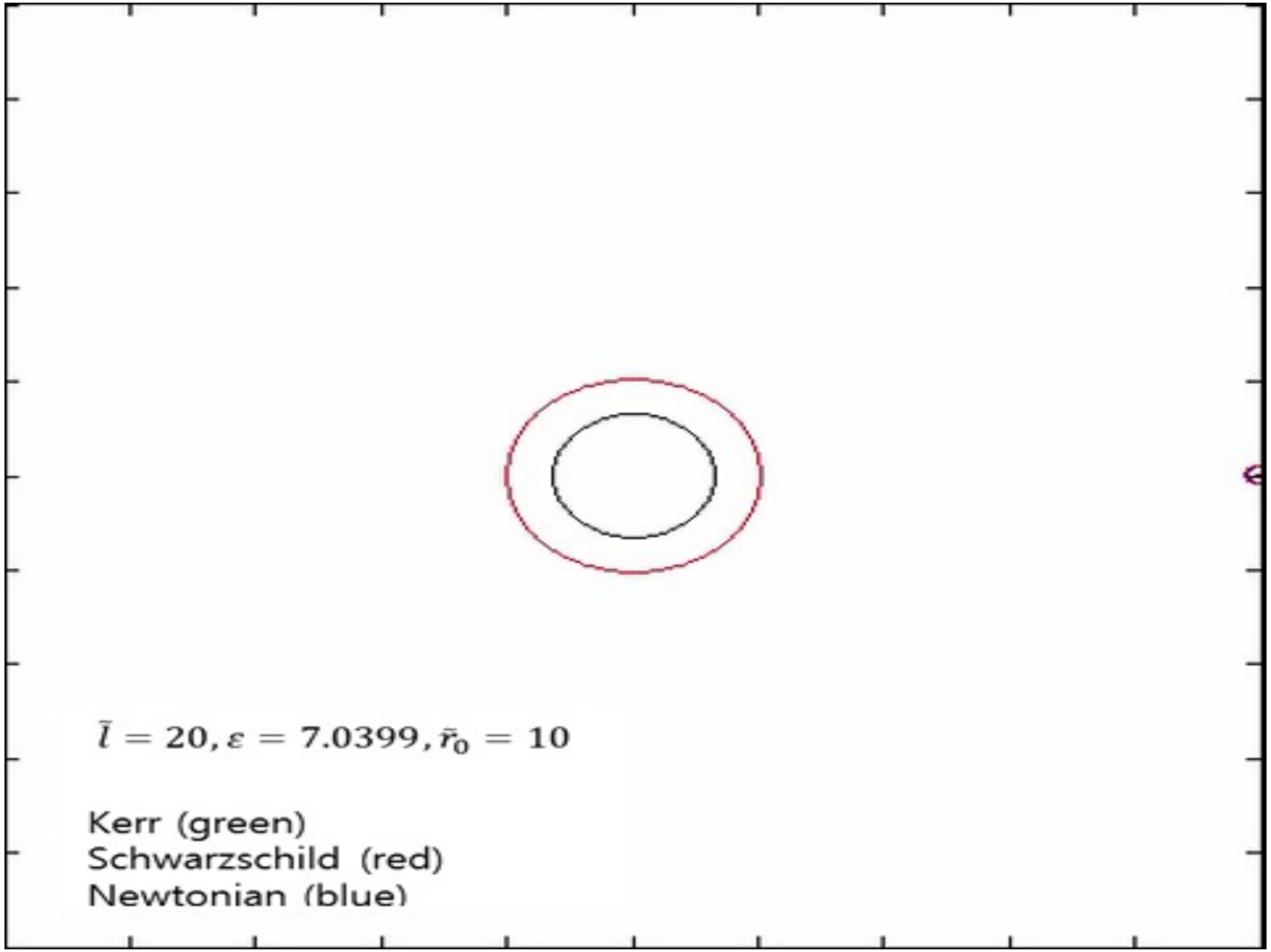
✓ Bound motions

✓ Precessions

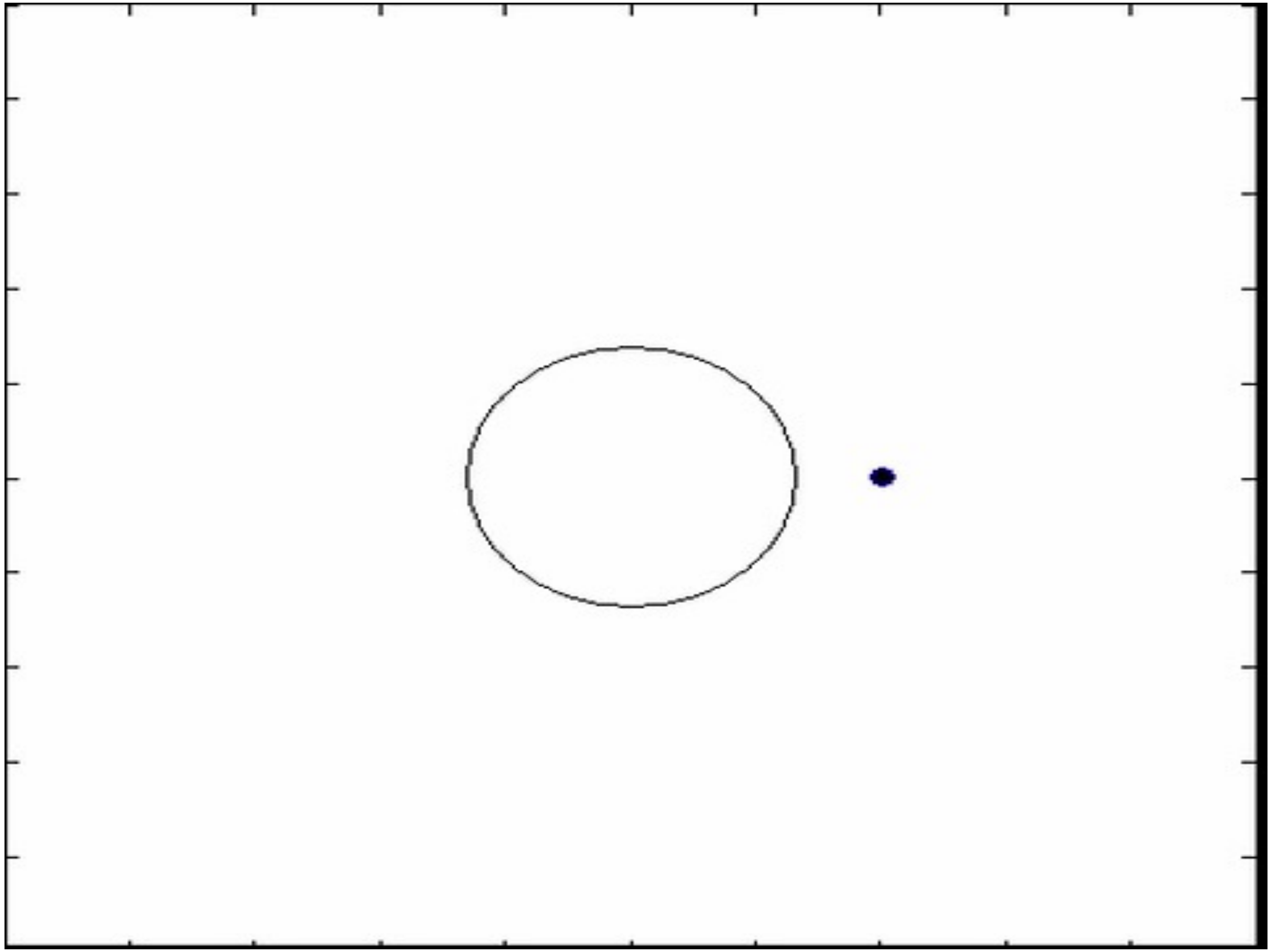




✓ Strength of gravities:

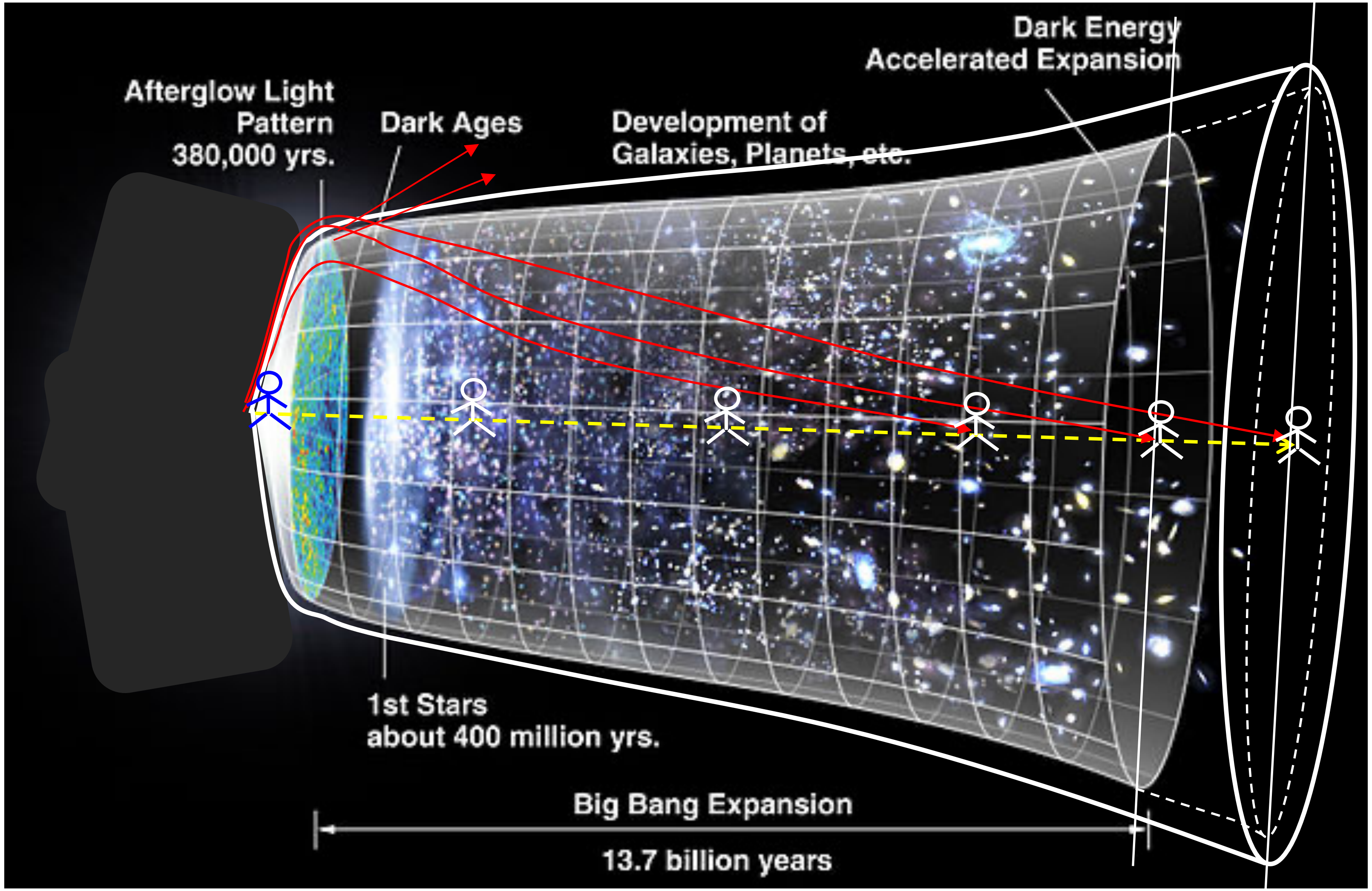


✓ Dragging effect in rotating BH:



- **동적인 우주:** 유한한 과거(시간의 시작), 초기에는 급팽창,가속팽창(암흑 에너지)

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$



$R = ar$

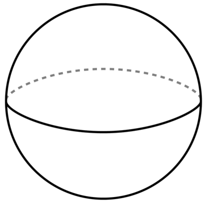
↑ 공간 ↓

$t = 0$ ← 시간 → t

Credit courtesy: NASA/WMAP Science Team

• Global causal structure of a spacetime described by the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$



- For $r \gg 2M$, $ds^2 \cong -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

$= -dt^2 + dx^2 + dy^2 + dz^2$ ➔ FLAT ST

∴ Asymptotically flat ➔ t: $-\infty\sim\infty$, r: $0\sim\infty$, θ: $0\sim\pi$, ϕ: $0\sim2\pi$

- Notice that $g_{\mu\nu}$ is singular (e.g. 0 or ∞) at $r = 0,\ 2M$ & $\theta = 0,\ \pi$
- Curvature squared (Kretschman invariant):

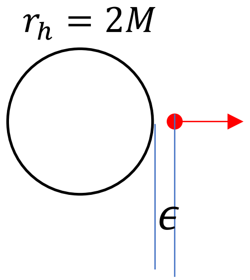
$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{48M^2}{r^6} \quad \rightarrow \quad \infty \quad \text{as} \quad r \rightarrow 0 \quad \text{only}$$

  - $r = 0$ is indeed a curvature singularity.
  - $r = 2M$ may be simply a coordinate singularity.
  - $\theta = 0, \pi$ are coordinate singularities.

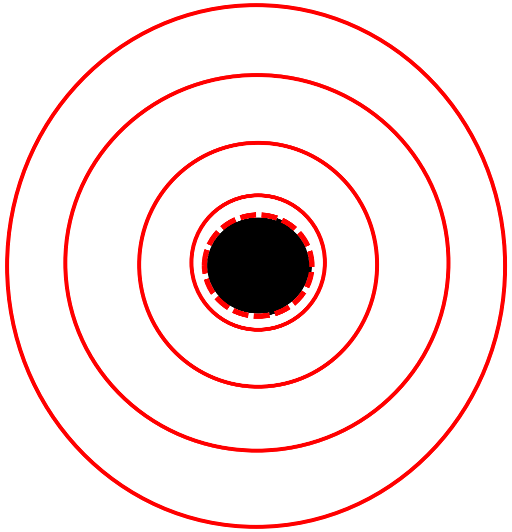
- **Trajectories of light:** $\theta, \varphi =$ fixed for radial motions

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1-2M/r} = 0$$

➔ $\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right) \xRightarrow{r \gg 2M} \pm 1$ ➔ + : Out-going light, - : In-going light



$dv \equiv dt + \frac{dr}{1-2M/r}$ ➔ $ds^2 = -\left(1 - \frac{2M}{r}\right) dv \left(dv - \frac{2dr}{1-2M/r}\right) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$



- Radially outgoing light: $ds^2 = 0 = -\left(1 - \frac{2M}{r}\right) dv \left(dv - \frac{2dr}{1-2M/r}\right)$

➔ $V = \int_0^V dv = \int_{r_h+\epsilon}^R \frac{2}{1-2M/r} dr = 2(R - r_h + r_h \ln(R - r_h) - \epsilon - r_h \ln \epsilon) \to \infty$ as $r \to r_h$

- What does it mean?
- Maybe light CAN NOT escape from the surface at $r = r_h$ …?!

➔ “**EVENT HORIZON**”

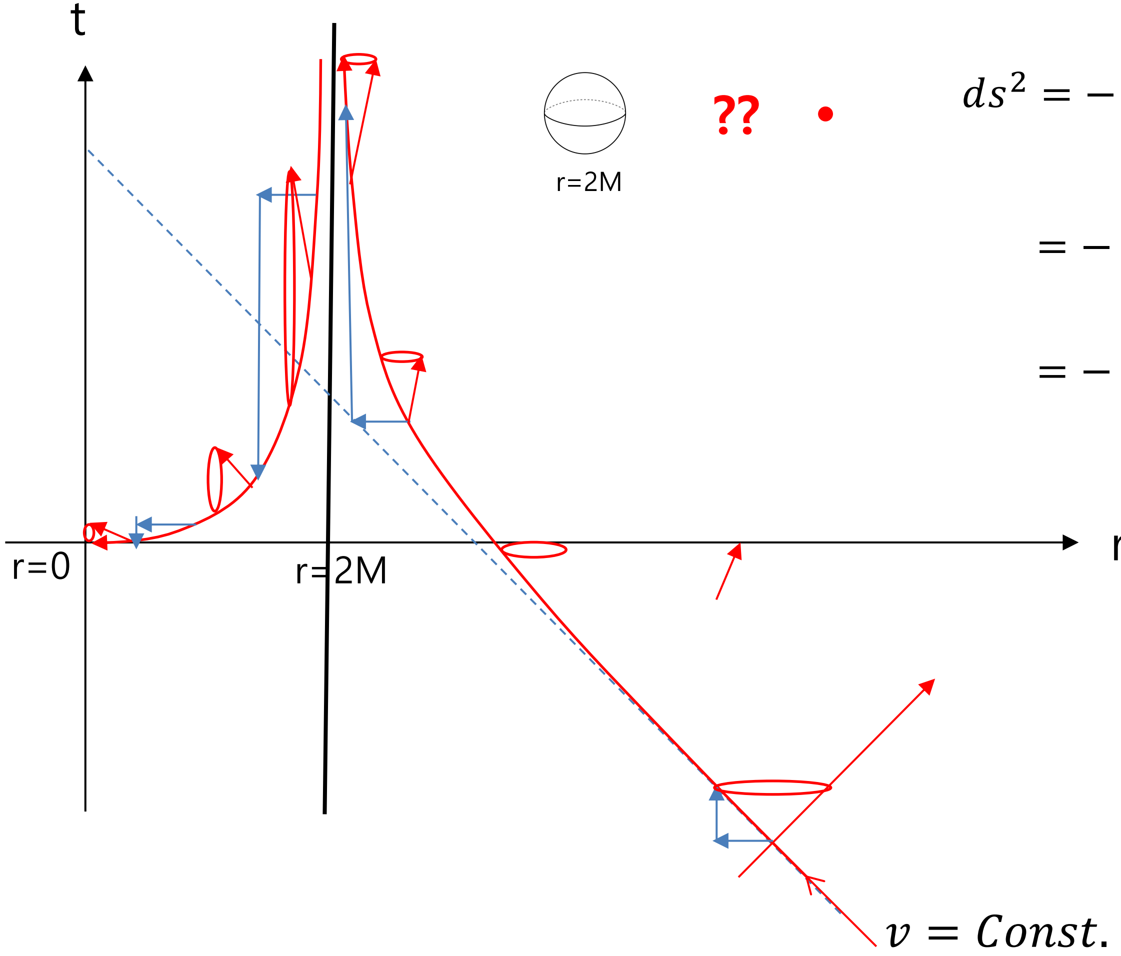
- Horizon radius: $r_h = \frac{2GM}{c^2} \sim 3 \frac{M}{M_\odot}$ km

*

- **In-going light:** $0 \le r < \infty$

$$\frac{dr}{dt} = -\left(1-\frac{2M}{r}\right) \xRightarrow{r \gg 2M} -1, \quad \xRightarrow{r\sim 2M^+} -0, \quad \xRightarrow{r\sim 2M^-} +0, \quad \xRightarrow{r\sim 0} +\infty$$

$t \to \infty$?



광경로 좌표(Null coordinate)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r}$$

$$= -\left(1-\frac{2M}{r}\right)\left[dt^2 - \left(\frac{dr}{1-2M/r}\right)^2\right]$$

$$= -\left(1-\frac{2M}{r}\right)\underbrace{\left(dt+\frac{dr}{1-2M/r}\right)}_{\stackrel{\text{def}}{=} dv}\left(dt-\frac{dr}{1-2M/r}\right)$$

➔ $ds^2 \sim dv(dv - \blacksquare dr) \Rightarrow 0$

for a path of $v = \text{Const.}$

$$v = t + r + 2M\ln\left|\frac{r}{2M}-1\right|$$

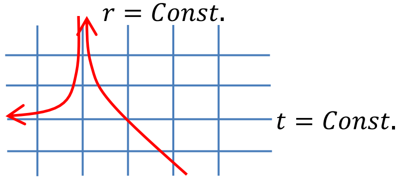
* - (v,r) **coordinate system:** Edington-Finkelstein c.s.

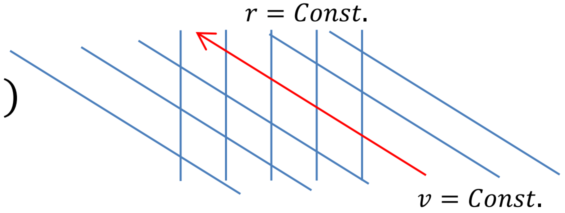
$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2+\sin^2\theta d\varphi^2) \quad \text{in } (t,r,\theta,\varphi)$$

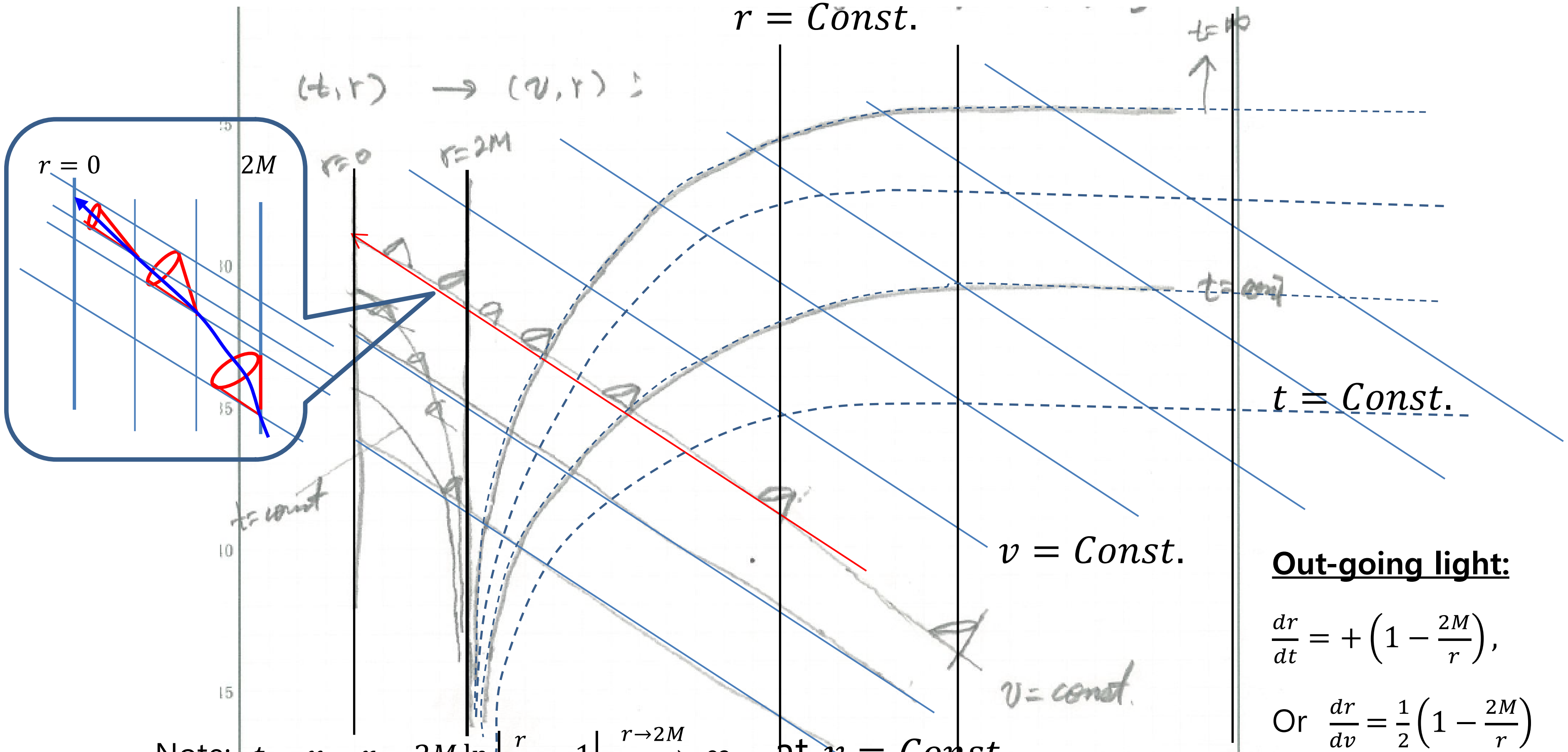
$$= -\left(1-\frac{2M}{r}\right)dv\left(dv - \frac{2dr}{1-\frac{2M}{r}}\right) + r^2(d\theta^2+\sin^2\theta d\varphi^2)$$

$$= -\left(1-\frac{2M}{r}\right)dv^2 + 2dv\,dr + r^2(d\theta^2+\sin^2\theta d\varphi^2) \quad \text{in } (v,r,\theta,\varphi)$$

$$v = t + r + 2M\ln\left|\frac{r}{2M}-1\right| \xrightarrow{r\gg 2M} t+r \quad \text{or} \quad t = -r+v$$







Out-going light:

$$\frac{dr}{dt} = +\left(1-\frac{2M}{r}\right),$$

Or $\frac{dr}{dv} = \frac{1}{2}\left(1-\frac{2M}{r}\right)$

Note: $t = v - r - 2M\ln\left|\frac{r}{2M}-1\right| \xrightarrow{r\to 2M} \infty$ at $v = Const.$

✓ Causal properties around $r = 2M$:

In ("v",r) coordinate system,

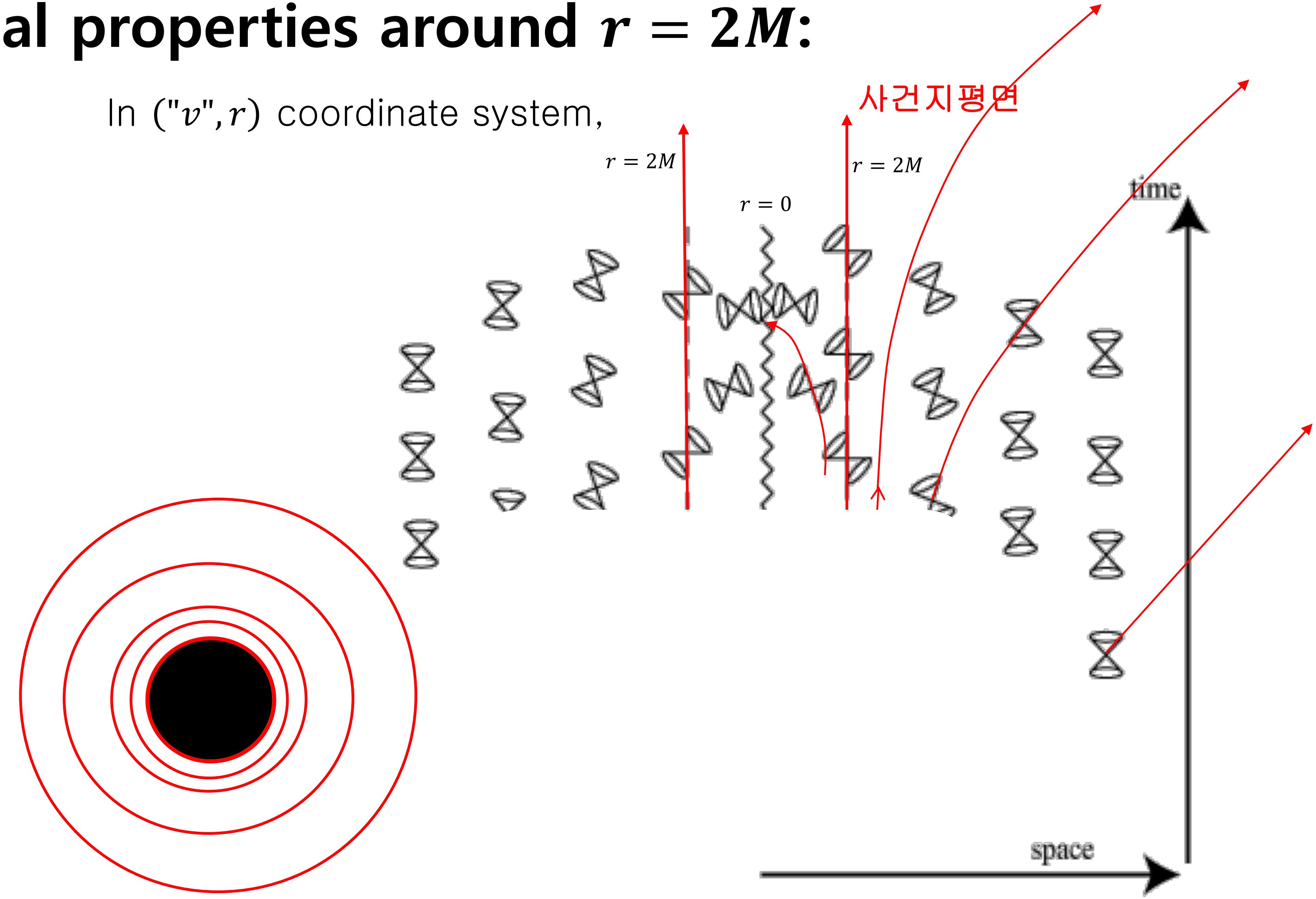


그림 출처: http://plato.stanford.edu/entries/spacetime-singularities/lightcone.html

- Global structure:

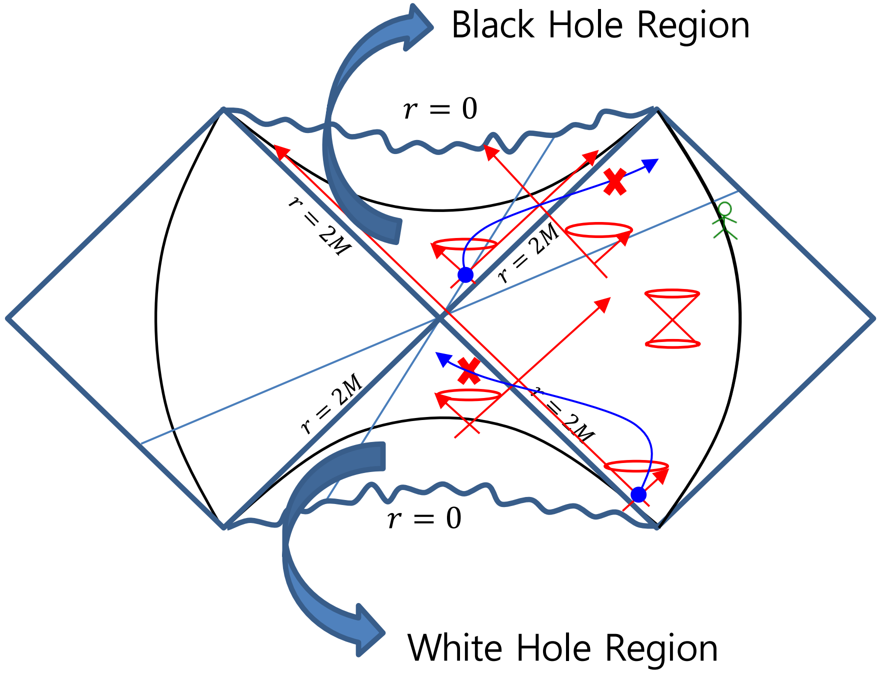
w/ $\left(\frac{r}{2M}-1\right)e^{r/2M}=\tilde{x}^2-\tilde{t}^2$ & $\frac{t}{2M}=2\tanh^{-1}(\tilde{t}/\tilde{x})$

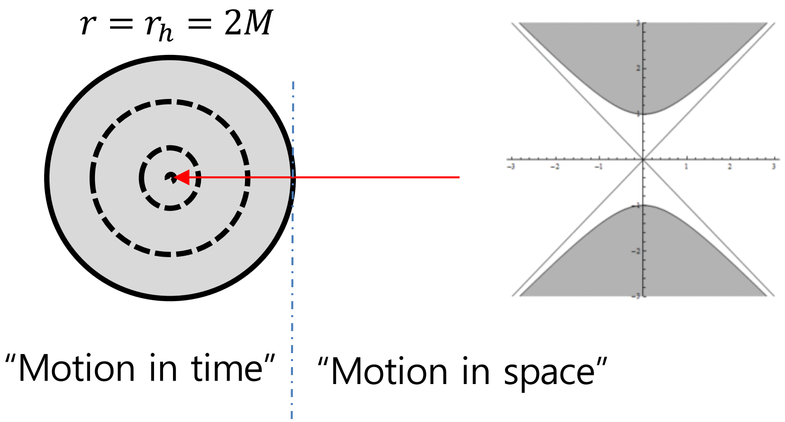
$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-2M/r}=\frac{32M^3}{r}e^{-r/2M}(-d\tilde{t}^2+d\tilde{x}^2)$$

- Light cone: $ds^2=0$
- Conformal trans.:

$$d\tilde{s}^2=\Omega(x)^2\,ds^2$$

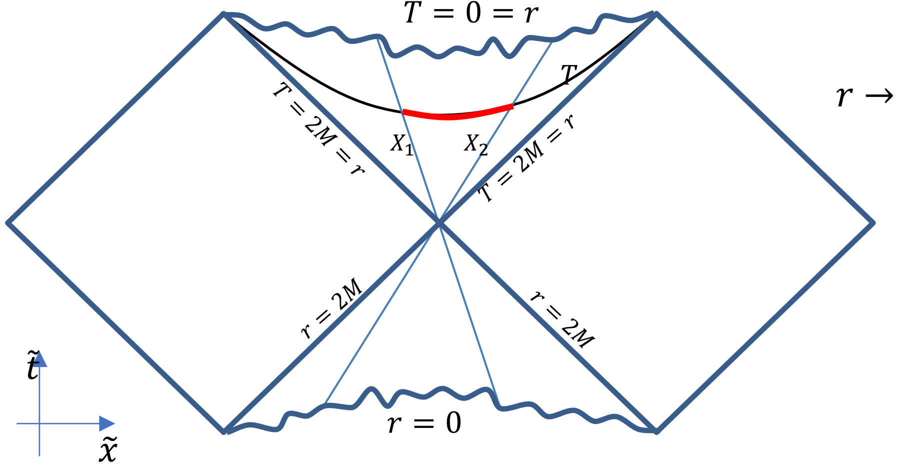
$$\tilde{g}_{\mu\nu}=\Omega(x)^2 g_{\mu\nu}$$





(See my youtube lectures for more details: https://www.youtube.com/watch?v=-y-XJZzprK0&t=705s)

- Interior of the black hole: $0 \le r < 2M$



$$ds^2 = -\frac{dr^2}{2M/r-1} + \left(\frac{2M}{r}-1\right)dt^2 + r^2 d\Omega^2$$ w/ $t: -\infty\sim\infty, r: 0\sim 2M$

$r \to T, t \to X$: $$= -\frac{dT^2}{2M/T-1} + \left(\frac{2M}{T}-1\right)dX^2 + T^2 d\Omega^2$$

- Time-dependent spacetime!

At $T =$ Constant,

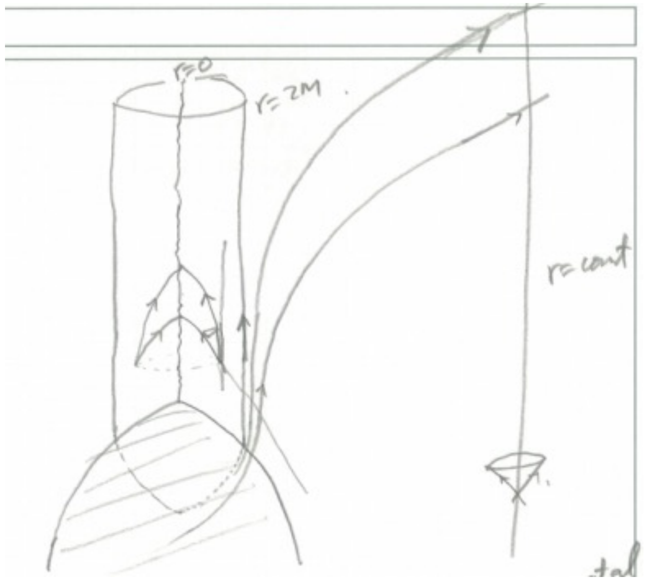
$$\Delta L = \sqrt{\frac{2M}{T}-1}\int_{X_1}^{X_2} dX = \sqrt{\frac{2M}{T}-1}\,\Delta X : \ 0\sim\infty$$

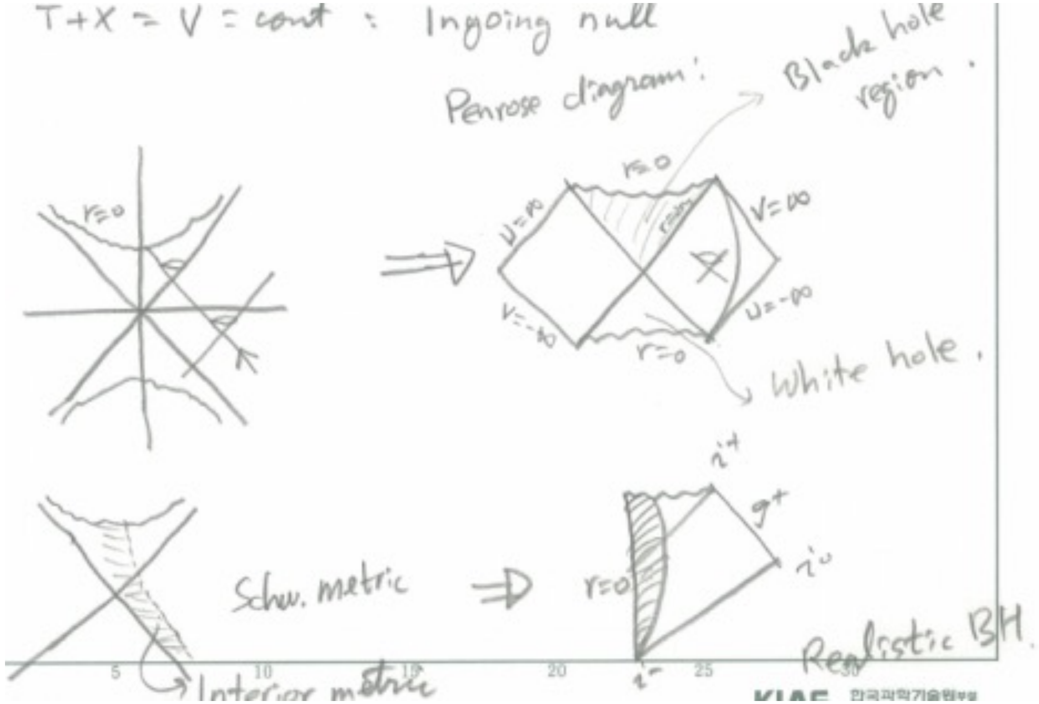
$$A = 4\pi T^2 : 16\pi M^2 \sim 0$$

➔ $\Delta V = 4\pi T^2 \times \sqrt{\frac{2M}{T}-1}\,\Delta X : \ 0\sim 3\sqrt{3}\pi M^2 \Delta X \sim 0$ as T: $2M\sim 0$

- Thus, the BH interior is isomorphic to $ds^2 = -d\tau^2 + a(\tau)^2(d\chi^2 + \sin\chi^2\, d\Omega^2)$.
- **시간 흐름의 끝이 있다!? (Time flow ends!?)**

- **Realistic astrophysical black hole solution:**
  - A final end state of a gravitational collapse of a confined matter





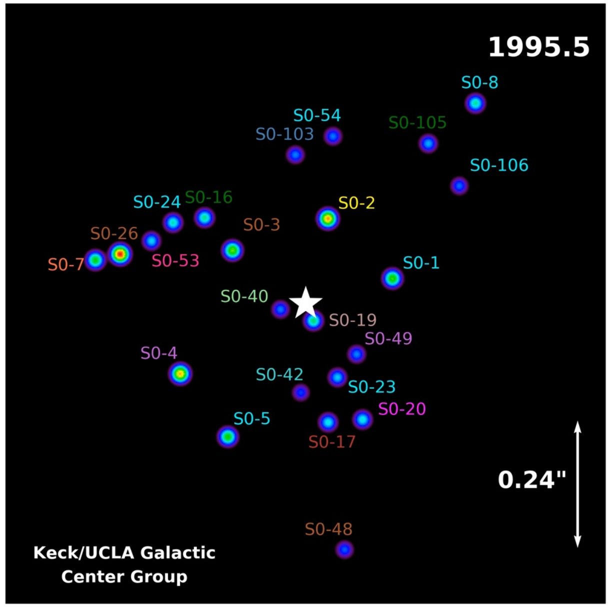
- See the slide for singularity theorem!

Observation of black holes:

- **Indirect obs.:** Observe lights such as x-ray in the vicinity of a BH, or orbits of nearby luminous stars

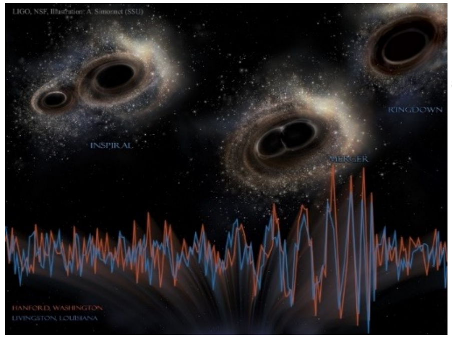
  Ghez, Genzel

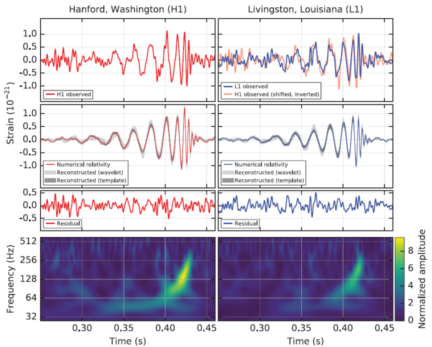
Keck/UCLA observations: 1995~2016

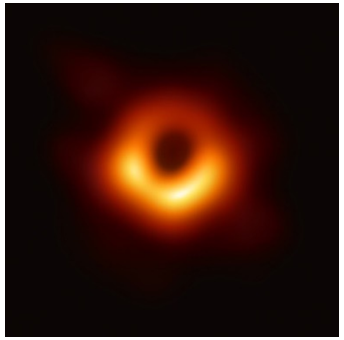


출처: http://www.astro.ucla.edu/~ghezgroup/gc/animations.html

- **Direct obs.:**
  - Obs. Of gravitational waves (2015): Coalescing binary BHs
  - EHT (2019): Supermassive BH (~400억 $M_{\odot}$) at the center of M87







* • Rotating charged black holes: Newman *et. al.* (1965)

$$
\boxed{\begin{aligned}
ds^2 \; &= -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a\sin^2\theta\,(r^2+a^2-\Delta)}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 \\
&\quad + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \Sigma d\theta^2, \\
A_a &= (-\frac{er}{\Sigma}, 0, 0, \frac{er}{\Sigma}a\sin^2\theta).
\end{aligned}}
$$

M: ADM mass
J: Angular momentum
(a=J/M)
e: Charge

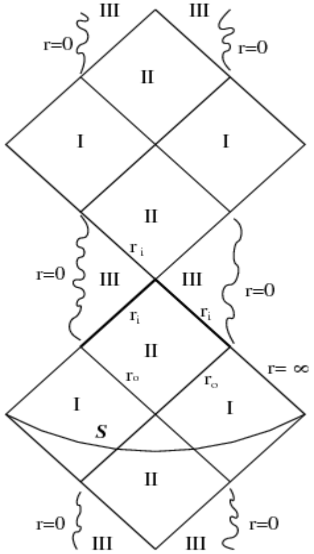
where $\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2 + e^2.$

i) Charged BH, *i.e.,* a=0: Reissner-Nordstrom (1916, 1918)

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - 2M/r + e^2/r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

ii) Rotating BH, *i.e.,* e=0: Kerr (1963)

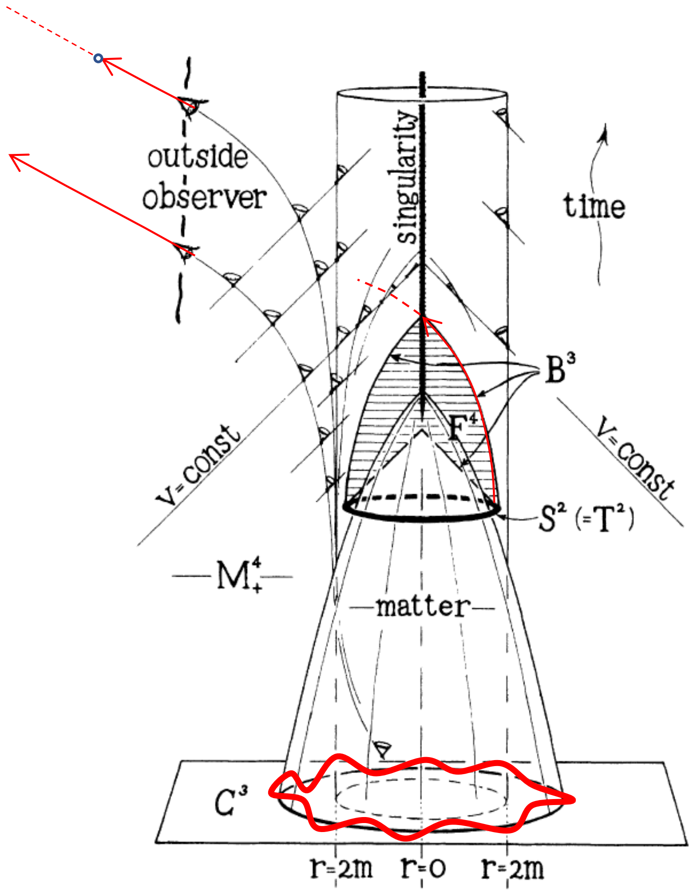
$$
\begin{aligned}
ds^2 \; &= -\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)dt^2 - \frac{4Mar\sin^2\theta}{r^2+a^2\cos^2\theta}dtd\phi + \frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2}dr^2 \\
&\quad + \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\sin^2\theta d\phi^2 + \left(r^2 + a^2\cos^2\theta\right)d\theta^2.
\end{aligned}
$$



http://inspirehep.net/record/841642/files/rnpd2.png

*

• Singularity theorem:



GRAVITATIONAL COLLAPSE AND SPACE-TIME SINGULARITIES

Roger Penrose
Department of Mathematics, Birkbeck College, London, England
(Received 18 December 1964)

Let (M, g_{ab}) be a connected hyperbolic spacetime such that

i) there is a noncompact Cauchy surface Σ,
ii) $R_{ab}k^a k^b \geqq 0$ for all null k^a,
iii) M contains a trapped surface T.

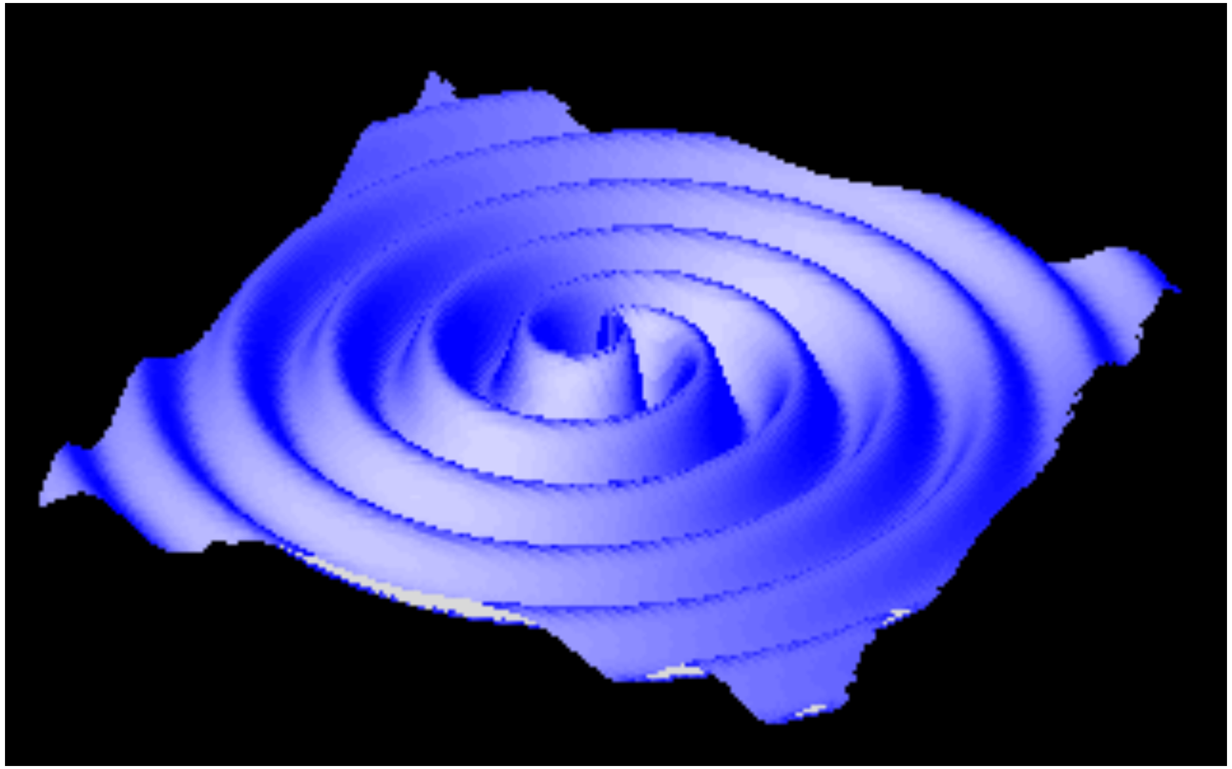
Then M is future null geodesically incomplete.

Sketch of the proof:

- Def. of singularity: Existence of an incomplete timelike or null geodesic
- $\left(R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}\right) k^a k^b$ ➔ $R_{ab}k^a k^b \geqq 0$ iff $T_{ab}k^a k^b \geqq 0$ *i.e.,* null energy condition

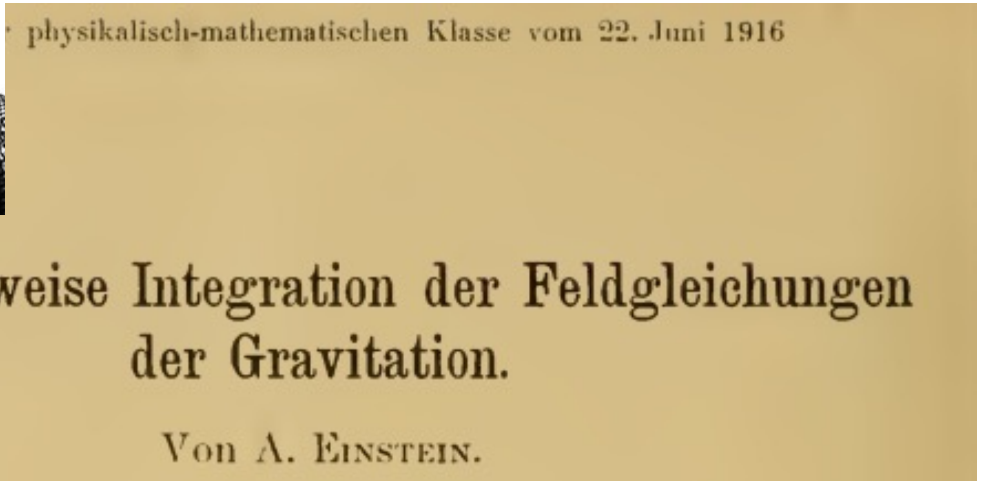
➔ Once a trapped surface is formed, there exists at least one inextendible future directed null geodesic, e.g., a singularity must occur!

V. 중력파, 블랙홀 쌍성계



- 가속하는 전하가 전자기파를 발생시키듯이 질량의 요동이 중력파 발생
- 시공간 자체의 파동
- 물질의 요동이 주위의 시공간을 변형시키고 이 시공간의 주름이 빛의 속도로 퍼져 나감

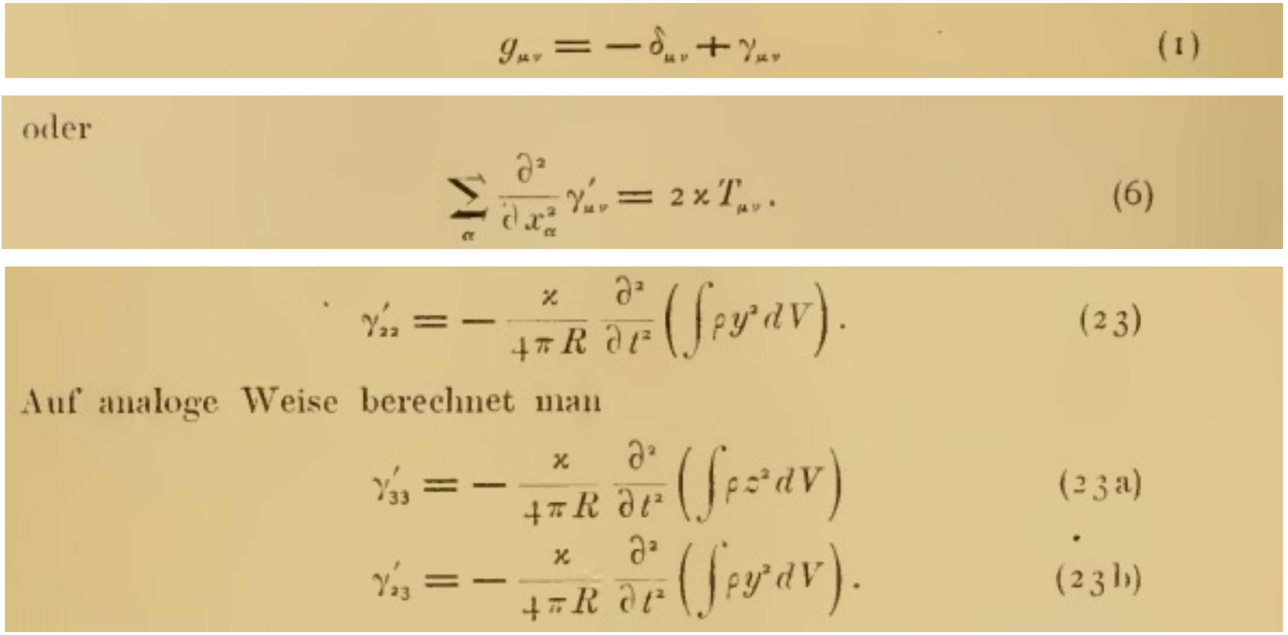
✓ 1916년 아인슈타인이 예측



physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. EINSTEIN.



$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \qquad (1)$$

oder

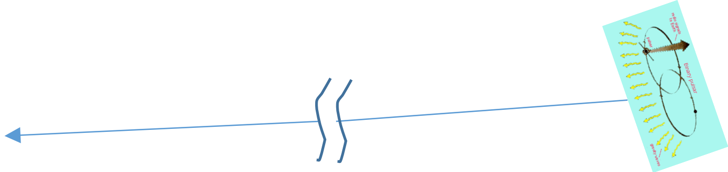
$$\sum_{\alpha} \frac{\partial^2 \gamma'_{\mu\nu}}{\partial x_\alpha^2} = 2\varkappa T_{\mu\nu}. \qquad (6)$$

$$\gamma'_{22} = -\frac{\varkappa}{4\pi R}\frac{\partial^2}{\partial t^2}\left(\int \rho y^2 dV\right). \qquad (23)$$

Auf analoge Weise berechnet man

$$\gamma'_{33} = -\frac{\varkappa}{4\pi R}\frac{\partial^2}{\partial t^2}\left(\int \rho z^2 dV\right) \qquad (23a)$$

$$\gamma'_{23} = -\frac{\varkappa}{4\pi R}\frac{\partial^2}{\partial t^2}\left(\int \rho y^2 dV\right). \qquad (23b)$$



g_{ab}

$= \eta_{ab} + h_{ab}$

$with\ \ |h_{ab}| \ll 1$

- **선형 근사:** "편평한 시공간의 미약한 섭동"

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi G}{c^4} T_{ab}$$

$$\Rightarrow \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right) h_{ab} = 0$$

in vacuum with TT- gauge

- **우주론적 중력파:**

$$ds^2 = -dt^2 + a^2(t)\left(\delta_{ij} + h_{ij}\right) dx^i dx^j$$

- In general, $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$

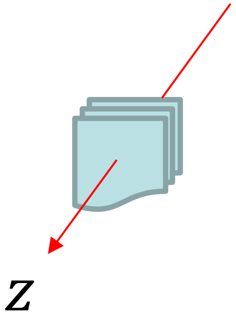
(※ 박찬 박사님 강의 참조)

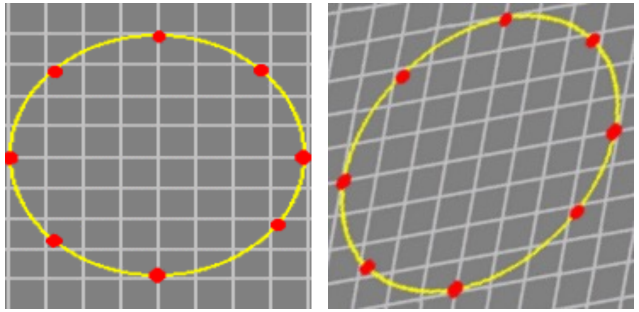
✓ 중력파가 지나가면 어떻게 되는가?

- 시공간 자체가 변함: $ds^2 = -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + 2h_\times dxdy + dz^2$
- 물체의 변형 및 길이 변화 일으킴

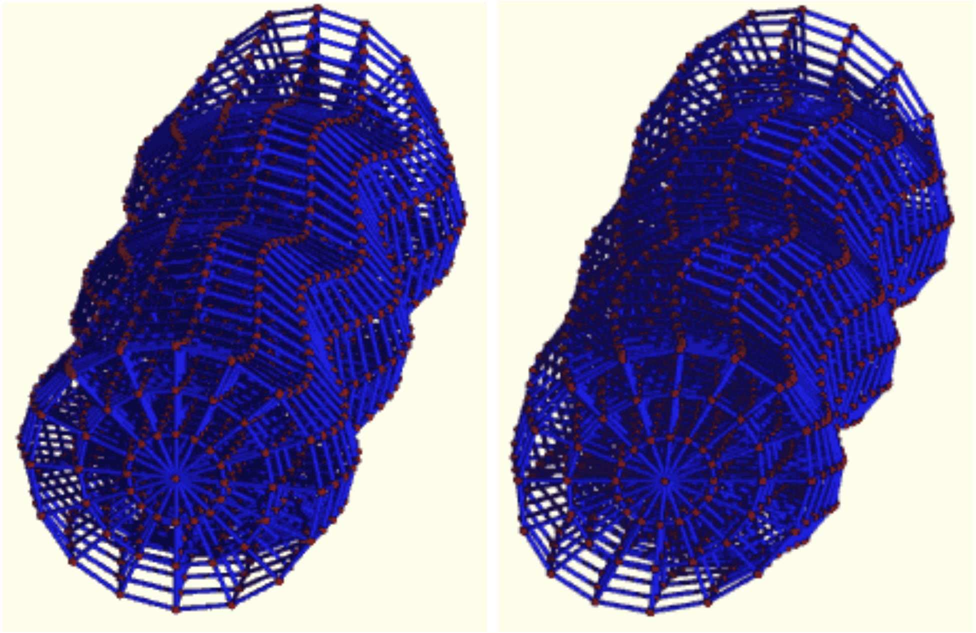
$$\begin{aligned} L'_x &= \int \sqrt{1 + h_{xx}}dx \\ &\sim \int (1 + \frac{1}{2}h_{xx})dx \\ &= (1 + \frac{1}{2}h\sin(\omega t)) \int dx \\ &= L + \frac{1}{2}Lh\sin(\omega t) \end{aligned}$$

✓ Strain: $\frac{\Delta L}{L} \cong h_{GW}(t)$





플러스(좌), 크로스(우) 편극된 중력파



Credit: Ravikumar Kopparapu

- Lots of confusions and contraversals for the reality of GWs till 1950s
- **"Sticky bead" argument:** Chapel Hill meeting in 1957

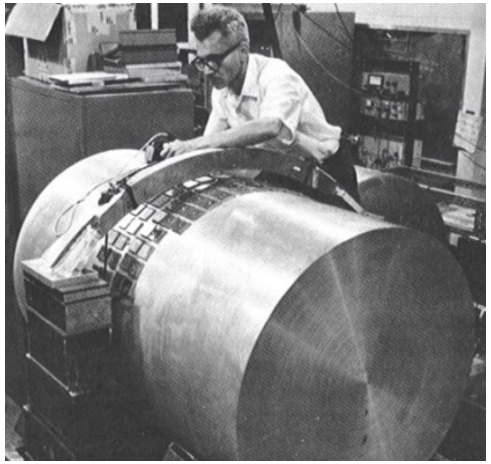
$$\frac{d^2\eta^i}{d\tau^2} = -R_{0j0}{}^{i}\eta^j = \frac{1}{2}\frac{d^2h_{ij}^{\mathrm{TT}}}{dt^2}\xi^j.$$

(Pirani '57)

앗 뜨거~

- It was **<u>J. Weber ('63)</u>** who tried for the first time the detection experiment by using a resonant-mass cylindrical bar **at 1660Hz**.
- It was very sensitive $\boldsymbol{h\sim 10^{-16}}$, but still far from $\sim 10^{-22}$.

UMD

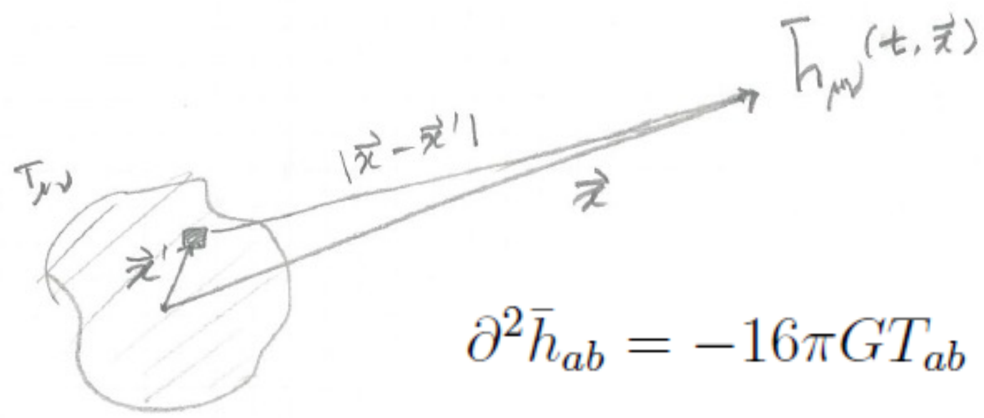


➔ ALEGRO: $h\sim 10^{-19}$ in 1990s

LSU ('09)

✓ 중력파의 발생원과 세기는?

$$I^{ij} = \int \rho x^i x^j d^3x$$



$$\partial^2 \bar{h}_{ab} = -16\pi G T_{ab}$$

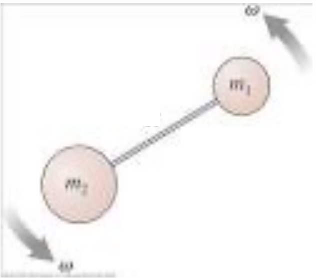
$$h_{ij}(t,\vec{x}) = \boxed{\frac{2G}{c^4}}\frac{1}{r}\ddot{I}_{ij}\left(t-\frac{r}{c}\right)$$

$$\sim \frac{1}{r}\times\omega^2\times M_{source}\times R_{source}^2\times\vartheta(1)$$

$$\sim 1.6\times 10^{-44}\,\frac{s^2}{kg\,m}$$

$M_{\odot} \sim 10^{30}\ kg$!!

- 주먹을 흔들어도 발생하나 지극히 약함:



$1\ ton \times 2,\ 2\ m\ \ \&\ \ 1\ kHz$

➔ $\boldsymbol{h_{GW} \sim 9\times 10^{-39}}$

at $r \sim \lambda = 300\ km$

$\sim 10^{11}$ protons at $v \sim 0.999999991c$

➔ $\boldsymbol{h_{GW} \sim 10^{-43}}$

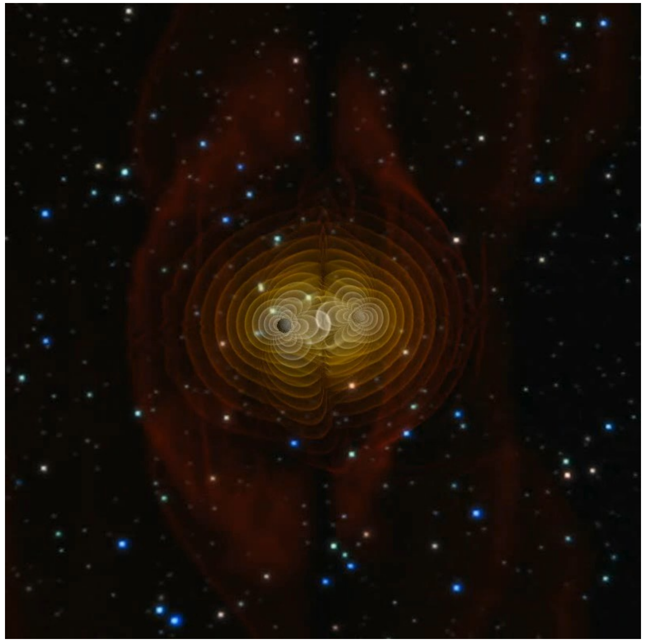
- 격렬한 천체현상에서 강력한 중력파 발생:

$$h_{\mathrm{BM}} \sim 10^{-20} \frac{\mathrm{Mpc}}{r} \frac{M}{\mathrm{M}_\odot} \left(\frac{M}{\mathrm{M}_\odot} \frac{f}{\mathrm{kHz}}\right)^{2/3}$$

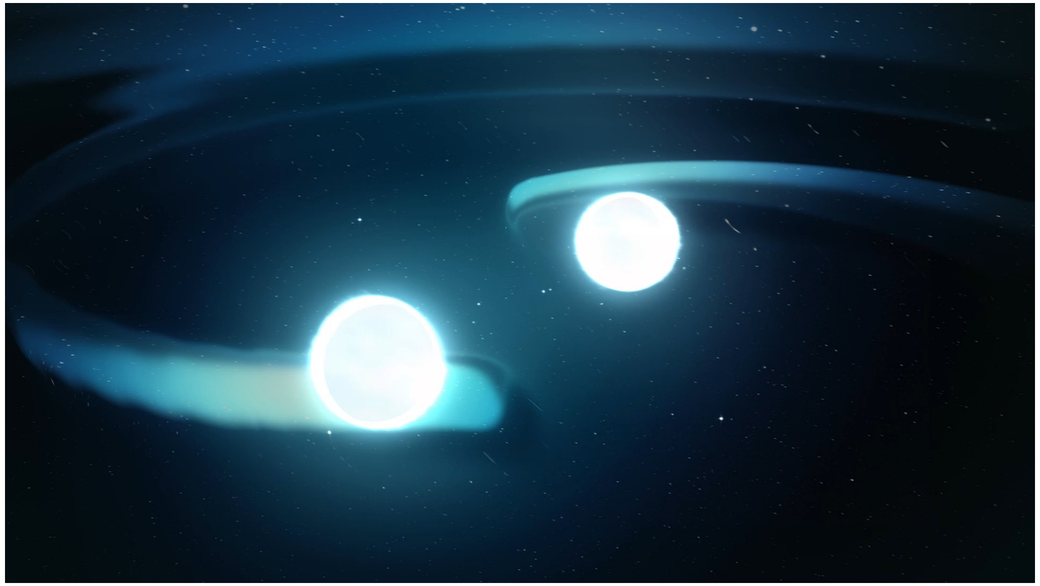
Ex) Black hole binary of $10M_\odot$

➔ $h \sim 5 \times 10^{-21}$

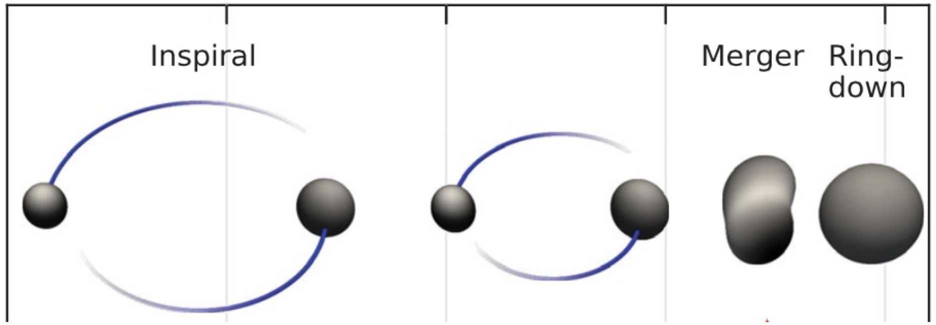
$(\gg 10^{-39})$



블랙홀 병합과 중력파 발생 (Credit: NASA/C. Henze)

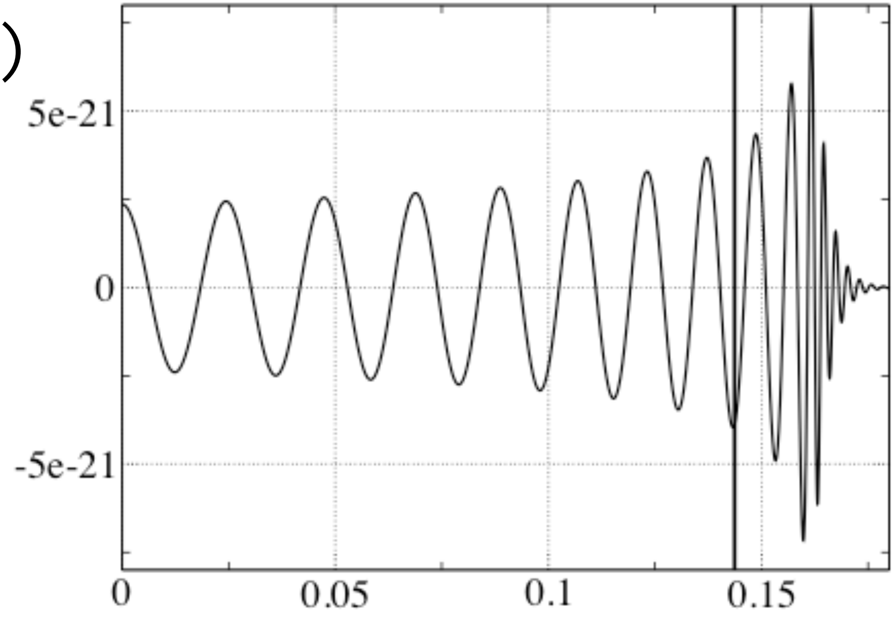


중성자별 병합과 중력파-전자기파 발생
(Credit: LIGO/SXS/R.Hurt and T. Pyle)



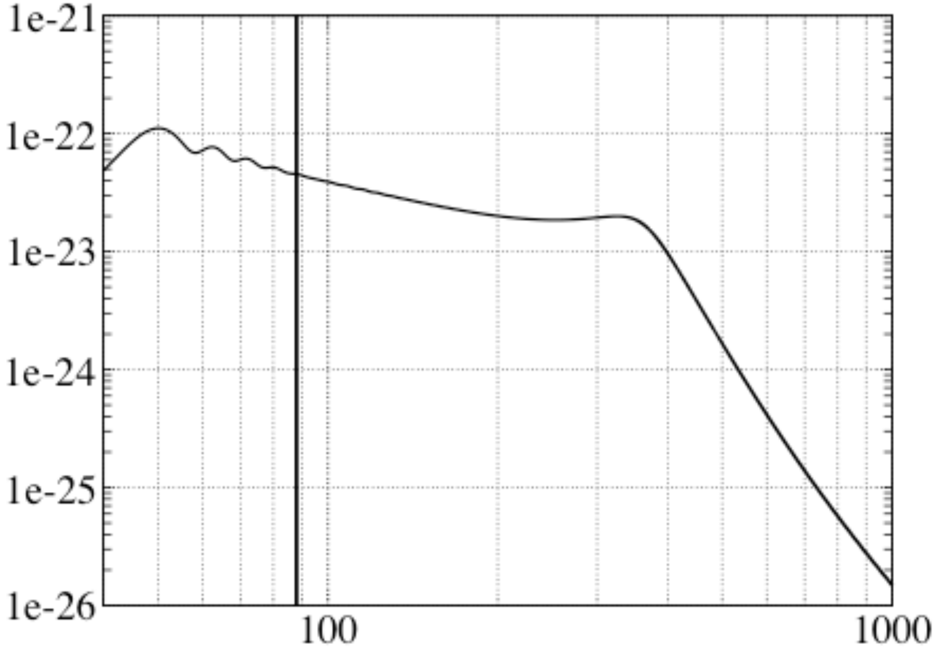
- Binaries emit GWs, resulting in decays of orbit.
- Eventually collide or merge
- Quickly becomes quite, e.g., a stationary single spinning BH which is probably described by the Kerr metric
- PN gives waveforms for inspiral and ringdown phases

h(t)



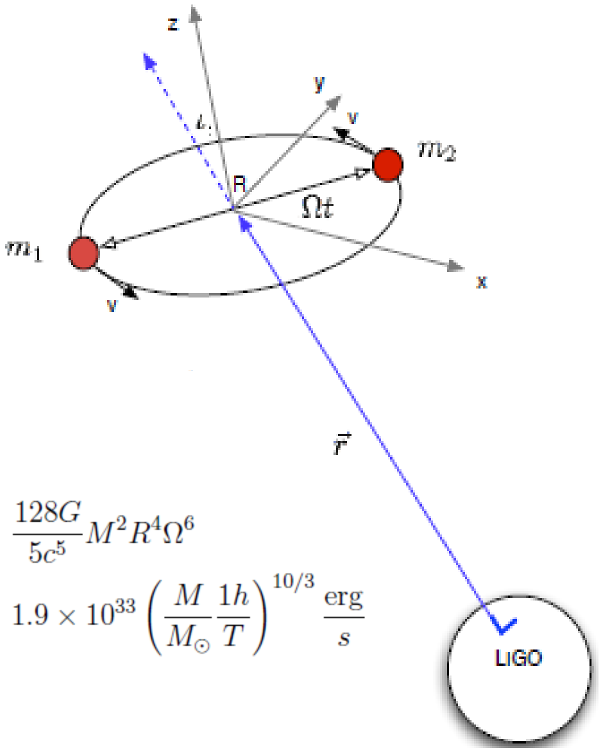
t-ime

h(f)



f-requency

Abadie et al. arXiv:1102.3781



$$P = \frac{128G}{5c^5} M^2 R^4 \Omega^6$$
$$= 1.9 \times 10^{33} \left(\frac{M}{M_\odot} \frac{1h}{T} \right)^{10/3} \frac{\text{erg}}{s}$$

$$\bar{h}^{ij}_{\mathrm{TT}} \sim -\frac{G}{c^4} \frac{8\Omega^2 M R^2}{r} \begin{pmatrix} \cos[2\Omega(t-r)] & \sin[2\Omega(t-r)] & 0 \\ \sin[2\Omega(t-r)] & -\cos[2\Omega(t-r)] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h \sim \frac{1}{r} M R^2 f^2 \cos 2\pi f (t-r) \quad \text{with} \quad 2\Omega = \Omega_{GW} = 2\pi f$$

$$f \to f(t) : ? \qquad \frac{1}{R^2} \sim \frac{v^2}{R} \sim f^2 R \to f^2 \sim \frac{1}{R^3} \to \frac{\dot{f}}{f} = -\frac{3}{2} \frac{\dot{R}}{R}$$

$$E \sim -\frac{G m_1 m_2}{2R} \sim f^{\frac{2}{3}} \to \dot{E} \sim f^{-1/3} \dot{f} \sim -P \sim -f^{10/3}$$

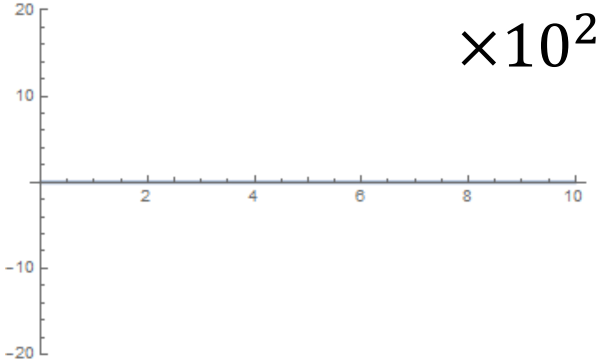
➔ $f(t) \sim (t_{coal} - t)^{-3/8}$

$$h_{+(t)} \sim -\frac{G\mathcal{M}/c^2}{r} \frac{1+\cos^2 \iota}{2} \left(\frac{5G\mathcal{M}/c^3}{t_c - t} \right)^{1/4} \cos\left[\left(\frac{t_c - t}{5G\mathcal{M}/c^3} \right)^{5/8} - 2\phi_c \right]$$

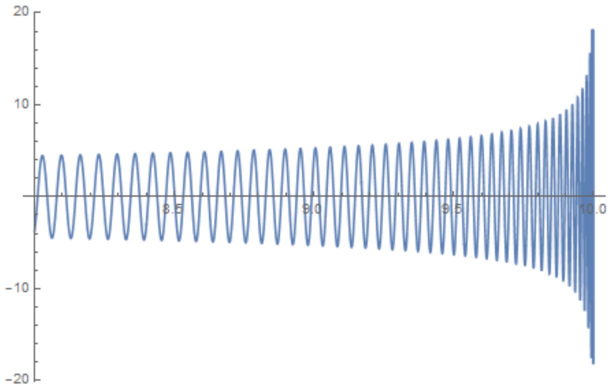
Ex) m1=36Msun, m2=29Msun, r=410Mpc

➔ $M_{Chirp} = \dfrac{(\mathrm{m1} * \mathrm{m2})^{3/5}}{(\mathrm{m1} + \mathrm{m2})^{1/5}}$ ~ 28 Msun

$$h(f) \sim e^{i\psi(f)} f^{-7/6}$$



$\times 10^{22}$ ➔



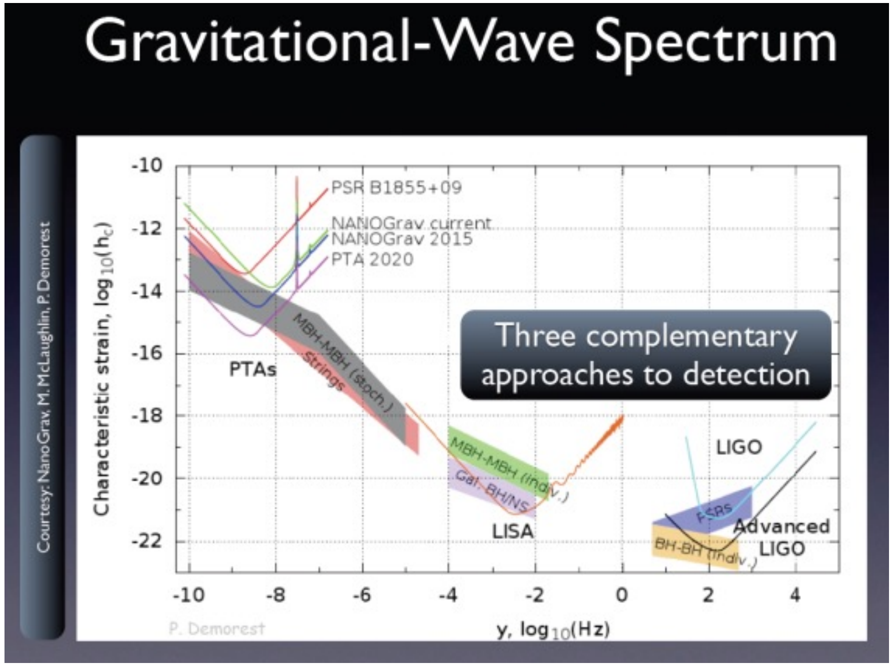
*

- Astronomical sources:
  - BH-BH, BH-NS, NS-NS coalescences
  - Supernova explosions: GW+Neutrino+…
  - Stochastic signals
  - Cosmic string kinks
  - Etc.

  - Galaxies ~1,000억 개/Universe. Stars ~1,000억 개/Galaxy.

* Bandwidths and significances of sources: (Cutler & Thorne ‘02)

- Extremely Low Freq. band (ELF, $\mathbf{10^{-15}\sim10^{-18}}\ \boldsymbol{Hz}$):
  - Primordial GWs
  - Imprint on the polarization of CMB radiations
  - Quantum origin at big bang subsequently amplified by inflation
  - Great potential for probing the physics of inflation
- Very Low Freq. band (VLF, $\mathbf{10^{-7}\sim10^{-9}}\ \boldsymbol{Hz}$):
  - Emitted by pulsars (e.g., Hulse-Taylor ‘75)
  - via pulsar timing array, or indirectly by pulses at earth
  - Extremely massive BH binary or violent processes in 0.1 second of the early universe
- Low Freq. band (LF, $\mathbf{10^{-4}\sim0.1}\ \boldsymbol{Hz}$):
  - From massive (10^5~10^7M⊙) BH binaries out to cosmological distances (CD)
  - From small BHs, NSs and WDs spiraling into massive BHs out to CDs
  - From orbital motions of WDB, NSB, and stellar-mass BHB in our own galaxy
  - And possibly from violent processes in the very early universe
  - To be observed by the space-based detector, LISA



- **High Freq. band (HF, $\mathbf{10\sim10^{3}}\ \boldsymbol{Hz}$):**
  - From a spinning slightly deformed NS in our Milky Way galaxy
  - From a variety of sources in the more distance:
    - Final inspiral and collisions of NSB and stellar-mass BHB (up to ~100M⊙)
    - Tearing apart of a NS by a companion BH
    - Supernovae, Triggers of GRBs, etc.
  - **To be measured by earth-based detectors such as LIGO, Virgo, KAGRA, and resonant-mass bar**

*

✓ 중력파에 의한 지구 직경의 변화: h_{GW} ~ ΔL/L

$$\frac{\Delta L}{L} = \mathrm{h}$$

$$= \frac{1}{1000000000000000000000}$$

ΔL ~ h L ~ 10^(-21) x 6,400km x 2

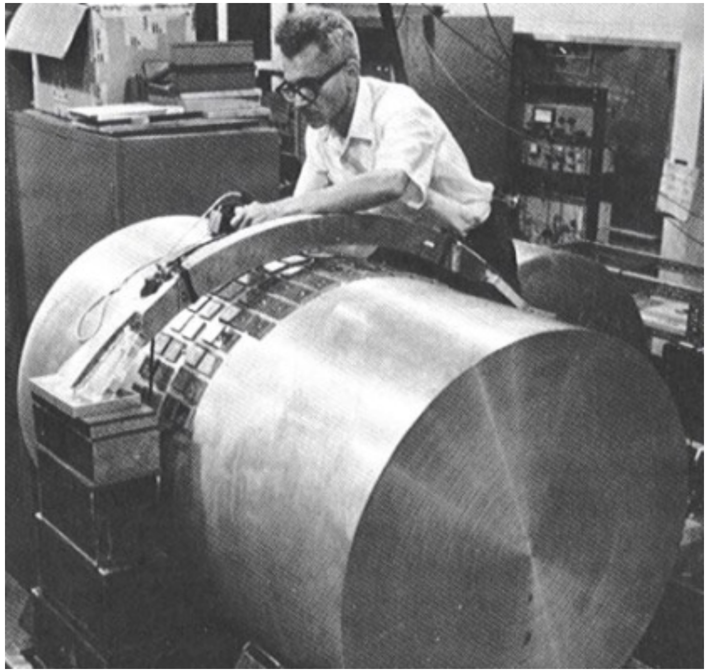
~ 10^(-14) m

~ Size of a proton

➔ 극도로 약한 효과!!

*

✓ 중력파는 어떻게 검출하나?



Weber at U. of Maryland (1960s)

(※ 박준규 박사님 강의 참조)

- First challenge: J. Weber (1963)
- Cylindrical Aluminum bar, Piezo sensor, Resonant freq. ~1,660Hz

- **However, its sensitivity is $h{\sim}10^{-16}$, which is far from a feasible signal $h{\sim}10^{-21}$.**

*

✓ 새로운 형태의 안테나: 간섭계

1970년대 말 MIT
실험실에서의
레이너 와이스
(Rainer Weiss)

The very *idea* for LIGO came to Rainer Weiss in the early 1970's when, as associate professor of physics at MIT, he had to find a way to explain gravitational waves (a prediction of general relativity) to his students. In an interview with MIT news writer, Jennifer Chu, Weiss recalled his revelation:

> "That was my quandary at the time, and that's when the invention was made. I said, **'What's the simplest thing I can think of to show these students that you could detect the influence of a gravitational wave? … The obvious thing to me was, let's take freely floating masses in space and measure the time it takes light to travel between them. The presence of a gravitational wave would change that time.** [Later] knowing what you could do with lasers, I worked it out: Could you actually detect gravitational waves this way? And I came to the conclusion that yes, you could detect gravitational waves…"

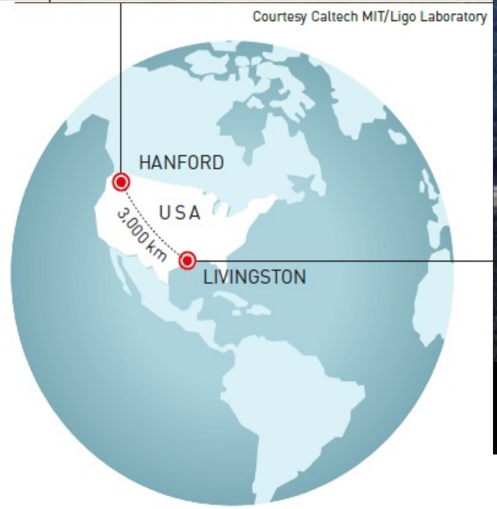
Sometime later, in 1972, Weiss carefully thought through and wrote down his idea, subsequently publishing it as a paper titled, *"Electromagnetically Coupled Broadband Gravitational Antenna"* . *In this paper, Weiss described in great detail,* the design and promise of using laser interferometry to detect gravitational waves. Within its 22 pages, the paper laid out the blueprint for the Laser Interferometer Gravitational-Wave Observatory (at the time, Weiss called it an antenna.)

*

라이고 (LIGO, Laser Interferometer Gravitational-wave Observatory)

핸퍼드, 워싱턴

Courtesy Caltech MIT/Ligo Laboratory



리빙스턴, 루이지애나

Illustration: ©Johan Jarnestad/The Royal Swedish Academy of Sciences

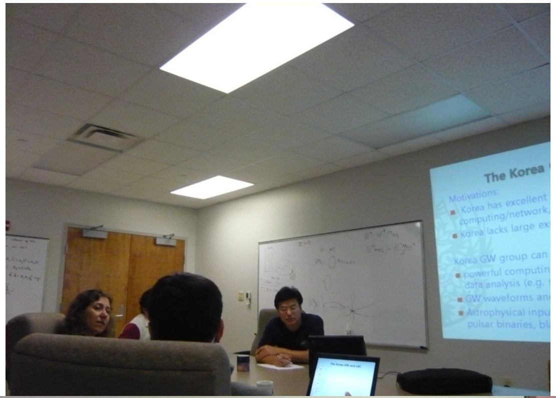
1987년 LIGO제안서
1991년 예산 의회승인
2002년 건설완료/ 가동시작
- **최종 공사비용: 약 3억달러 소요**

2002~2010: Initial LIGO, S1~S6
2005년: Advanced LIGO 예산 승인

2010~2015: 업그레이드
2015.09.12~2016.01.12: aLIGO O1, 검출(2)
2016.11.30~2017.08.25: O2, 검출(>1?)
- **종비용: 약1조원**
- **NSF 약 40년 지속적 지원**

*

✓ 라이고 리빙스톤 방문 사진: 2009년 2월

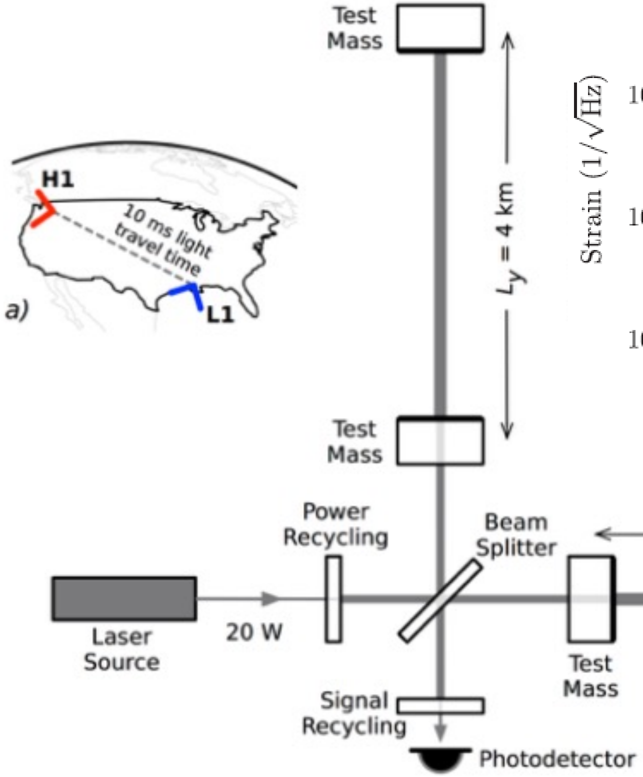


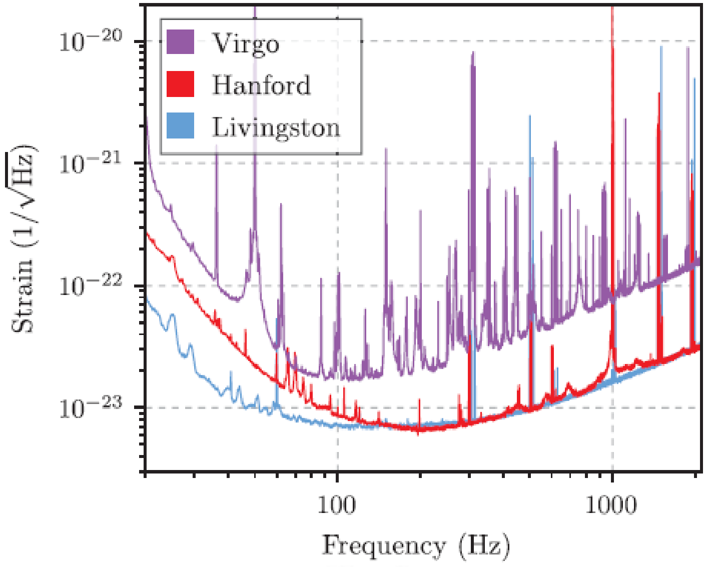
ALEGRO

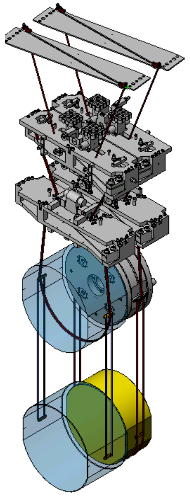
- 2.3 ton Al bar @ 4.2 K
- 897~920 Hz
- Sensitivity: $h \sim 10^{-19}$
- Operations: ~1990s

- 각종 진동에 의한 길이 변화와 중력파에 의한 길이 변화가 섞여 있음
- 잡음을 최소화
- 감도 향상: 팔의 길이 최대화, 패브리-페롯 공진기, 고출력의 빛, ……

- **고성능 라이고의 구조와 감도:**







LIGO-G1600341

Phys. Rev. Lett. 119, 141101 (2017)

Simple Estimates

- Detectable strain with laser

$$h \equiv \frac{\Delta l}{l} = \frac{\lambda_{laser}}{l} = \frac{10^{-6}\mathrm{m}}{10^{3}\mathrm{m}} = 10^{-9}$$

- Optical path length can be significantly increased by adopting optical cavity, but should be smaller than GW wavelength

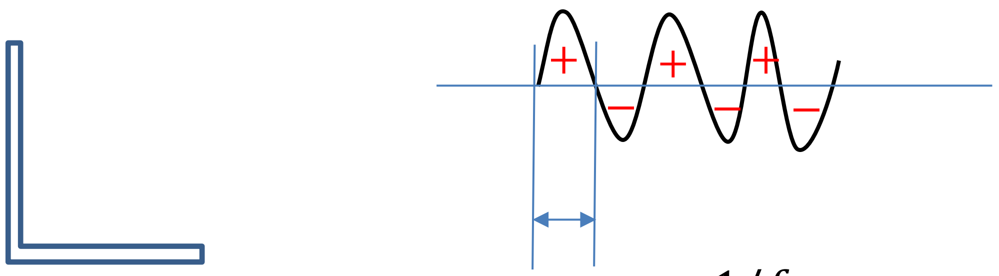
$$h \sim \frac{\Delta l}{l_{eff}} \sim \frac{\lambda_{laser}}{\lambda_{GW}} \sim \frac{10^{-6}\mathrm{m}}{10^{6}\mathrm{m}} = 10^{-12}$$

(See the next slide!)

- With sensitive photodiode, better result can be achieved. Smallest change in the amount of light that is detectable.

$$\Delta l \sim \frac{N_{photons}^{1/2}}{N_{photons}} \lambda_{laser}$$

✓ Effective arm length:



$$\tau_{max} = \frac{1/f}{2}$$

What will be the best arm length?

For a GW of $f \sim 100\ Hz$, $\lambda = \frac{c}{f} \sim 3000\ km$

Optimally: $\mathrm{T}_{\mathrm{travel}} = \frac{2L}{c} \sim \tau_{max} = \frac{1/f}{2} = \frac{\lambda/c}{2}$ ➔ Desired arm length: $L \sim \frac{\lambda}{4} \sim 1000\ km$

Practically, however, $L \sim 4\ km$.

A way to overcome this is to put a mirror between Mirror and BS:

B = (# of round trips) ~ $\frac{1000\ km}{4\ km}$ ~ 200

Quantum Limit

- Collect photons for a time of the order of the period of GW wave $\tau \sim 1/f_{GW}$

$$N_{photons} = \frac{P_{laser}}{hc/\lambda_{laser}}\tau \sim \frac{P_{laser}}{hc/\lambda_{laser}}\frac{1}{f_{GW}}$$

- For 1W laser with λ_{laser}=1 μm, f_{GW}=300Hz, $N_{\mathrm{photons}}=10^{16}$

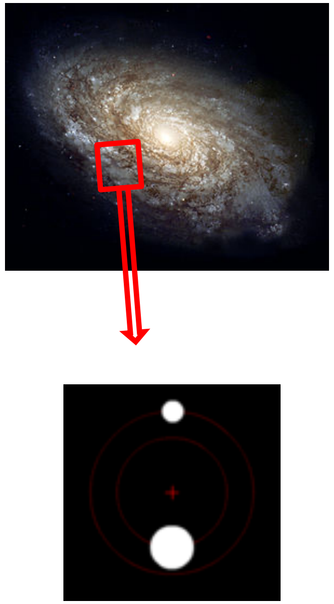
$$h \sim \frac{\Delta l}{l_{eff}} \sim \frac{N_{\mathrm{photons}}^{-1/2}\lambda_{\mathrm{laser}}}{\lambda_{\mathrm{GW}}} \sim \frac{10^{-8}\times 10^{-6}\mathrm{m}}{10^{6}\mathrm{m}} = 10^{-20}$$

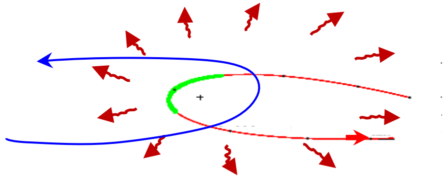
- By adopting high power laser and power recycling, we can reach ‘astrophysical sensitivity’ of ~10^{-22} or 10^{-23}. Further improvement can be achieved by signal recycling.

- **Recent topics in my study**
  - **BH encounters**
  - **SOGRO**

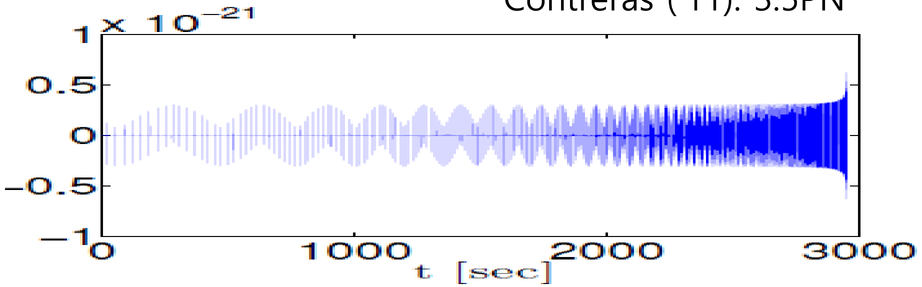
✓ The whole life of a BBH system:

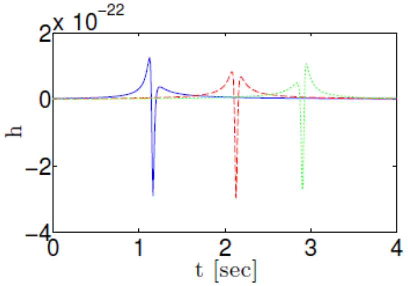
- "Inspiral-Merger-Ringdown" is just a tiny part at the last moment of binary coalescences!





Levin, McWilliams & Contreras ('11): 3.5PN





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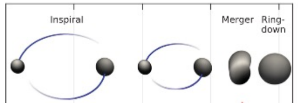
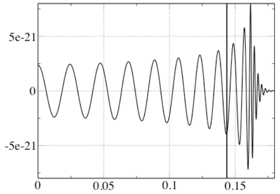




Formation of binary:

Unbound ➔ Bound

(Hyperbolic ➔ Elliptic)

Encounters

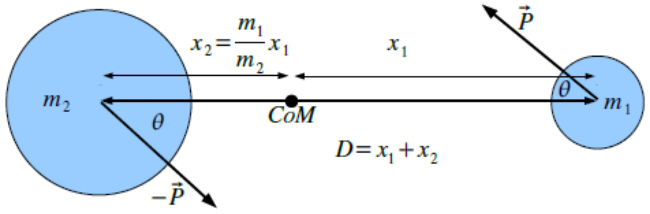
Precessions

Coalescence-IMR

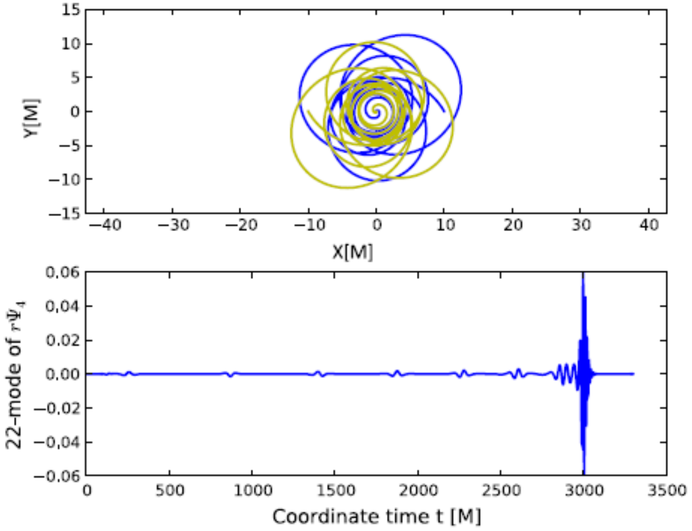
"Construct a waveform model covering all of it, in particular, highly eccentric phases!"

- **3PM EOB Hamiltonian**
- **Collaboration with Ik-Siong Heng's group at Glasgow U.**

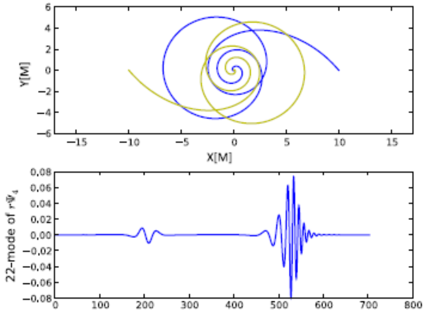
- **Gold & Brugmann ('13):**



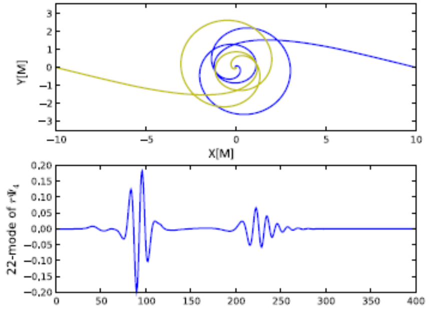
$P = P_{qc}$, $\Theta = 60°$



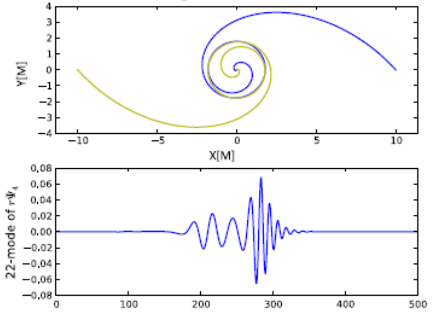
$P = P_{qc}$, $\Theta = 50°$:



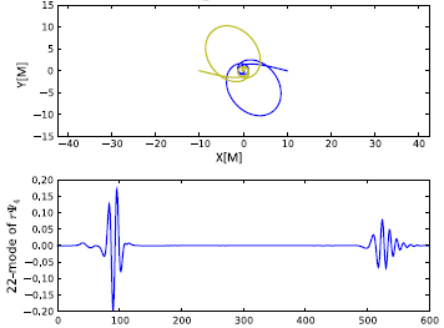
$P = 5P_{qc}$, $\Theta = 14.15°$



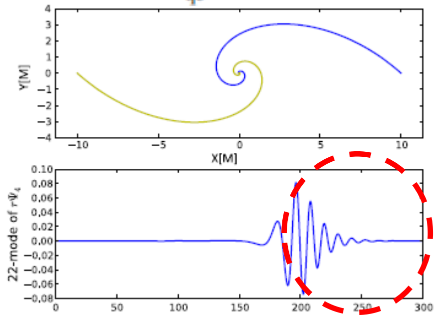
$P = P_{qc}$, $\Theta = 48°$



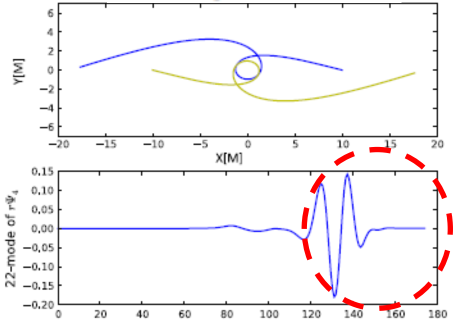
$P = 5P_{qc}$, $\Theta = 14.20°$:

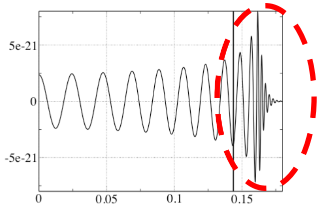


$P = P_{qc}$, $\Theta = 42°$



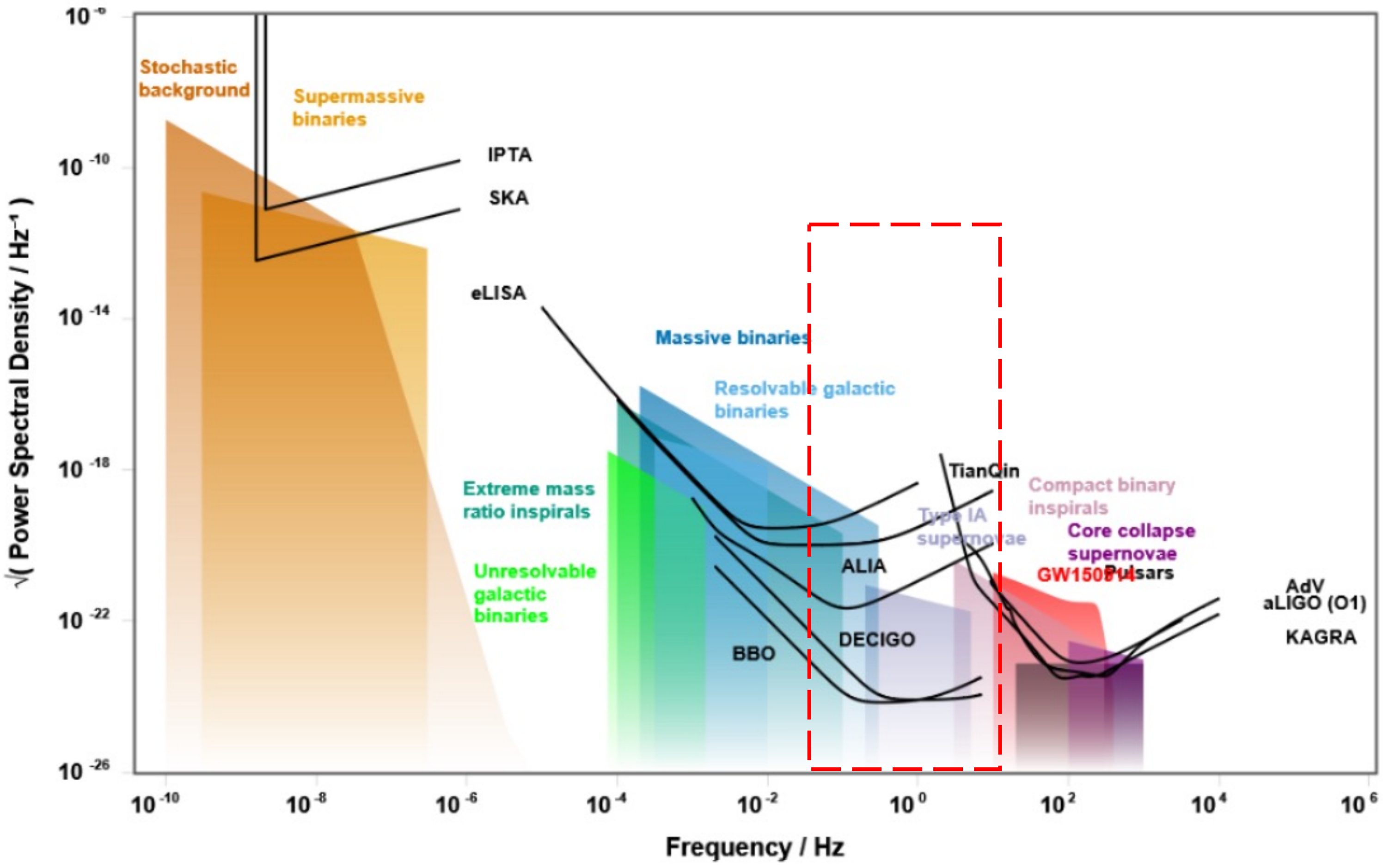
$P = 5P_{qc}$, $\Theta = 14.30°$





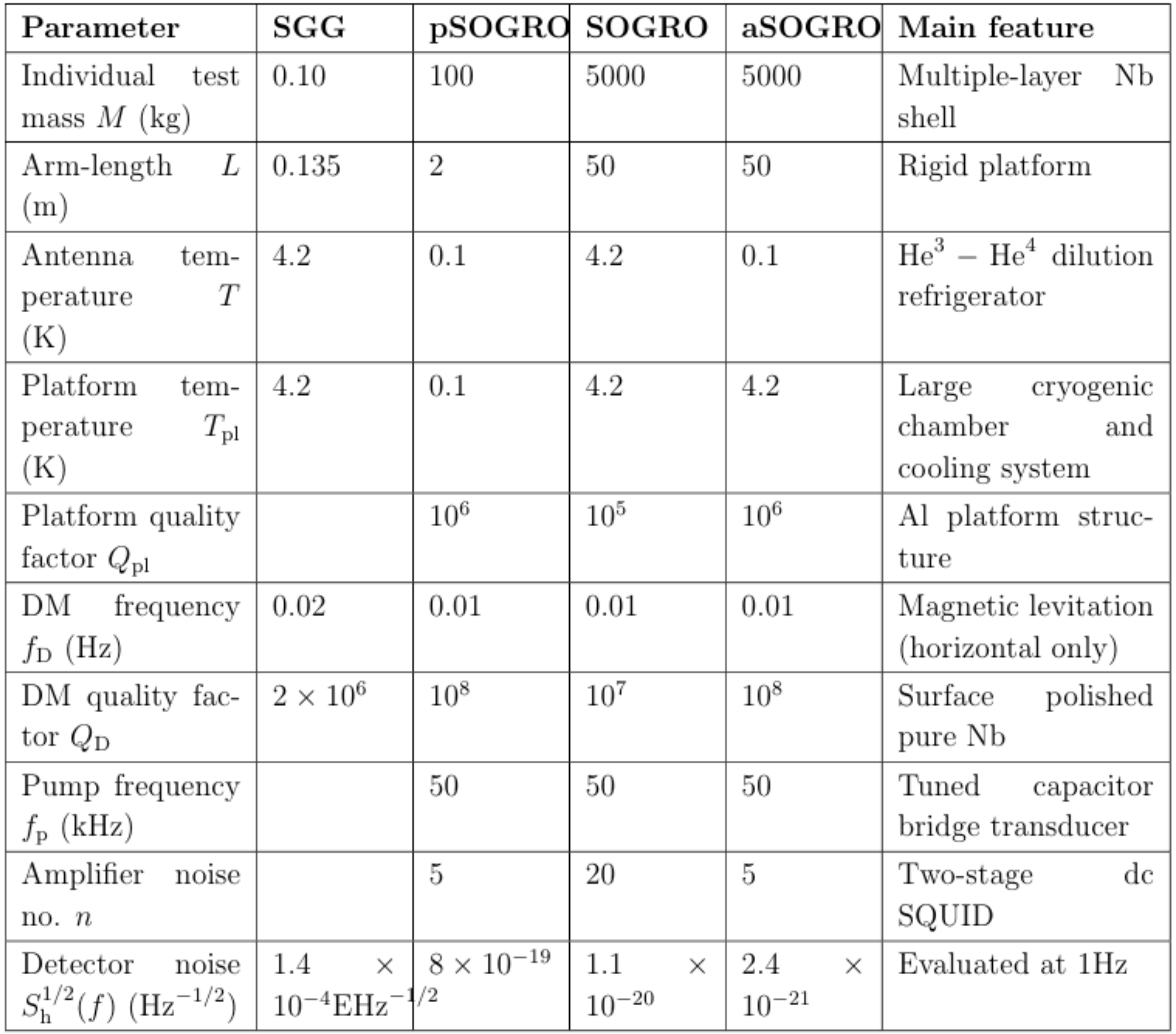
• SOGRO

✓ Gravitational Wave Spectrum:

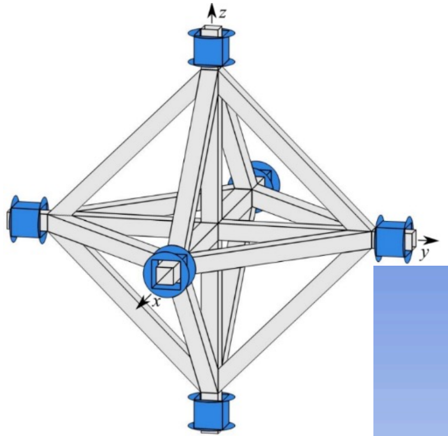


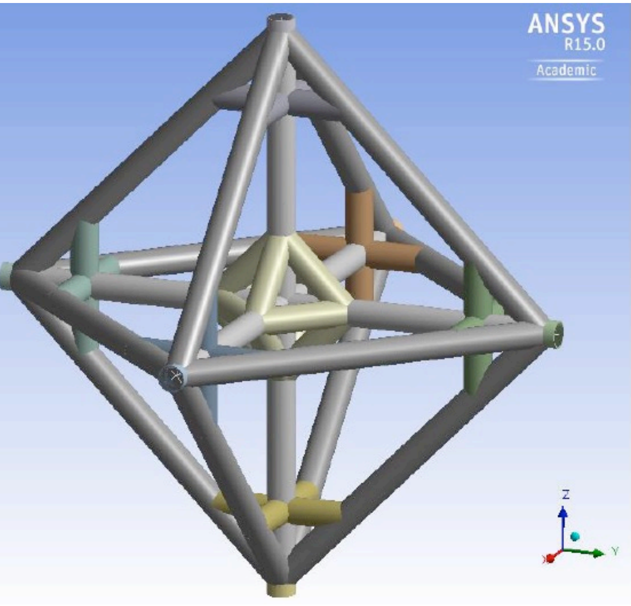
[Based on http://rhcole.com/apps/GWplotter/](http://rhcole.com/apps/GWplotter/) by Moore, Cole & Berry

✓ Design parameters



Parameter	SGG	pSOGRO	SOGRO	aSOGRO	Main feature
Individual test mass M (kg)	0.10	100	5000	5000	Multiple-layer Nb shell
Arm-length L (m)	0.135	2	50	50	Rigid platform
Antenna temperature T (K)	4.2	0.1	4.2	0.1	$\mathrm{He}^3 - \mathrm{He}^4$ dilution refrigerator
Platform temperature T_{pl} (K)	4.2	0.1	4.2	4.2	Large cryogenic chamber and cooling system
Platform quality factor Q_{pl}		10^6	10^5	10^6	Al platform structure
DM frequency f_{D} (Hz)	0.02	0.01	0.01	0.01	Magnetic levitation (horizontal only)
DM quality factor Q_{D}	2×10^6	10^8	10^7	10^8	Surface polished pure Nb
Pump frequency f_{p} (kHz)		50	50	50	Tuned capacitor bridge transducer
Amplifier noise no. n		5	20	5	Two-stage dc SQUID
Detector noise $S_{\mathrm{h}}^{1/2}(f)$ ($\mathrm{Hz}^{-1/2}$)	$1.4 \times 10^{-4} \mathrm{EHz}^{-1/2}$	8×10^{-19}	1.1×10^{-20}	2.4×10^{-21}	Evaluated at 1Hz

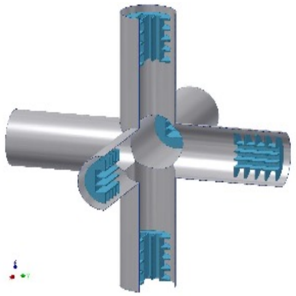




SOGRO/aSOGRO

(30 m, 50 m, 100 m)

- (100~250) tons
- ≤ 4 K
- Underground: ~200 m



pSOGRO

2 m, 1 ton

✓ Other topics in interest:

- Black hole physics
  - Final fate of the Gregory-Lafflamme instability
  - Overtone modes in Ring-down waveforms
  - Singularity
  - .....

Weak because of

$\sim G\,/\hbar$

- Quantum gravity

$\sim \vartheta(1)$

- 중력파와 양자현상의 상호작용

GWs+QS

✓ Welcome anybody to join!

결 론

- **특수상대론, 일반상대론, 중력파에 대한 개요**

- 중력파는 우주를 탐색하는 매우 독특하고 강력한 도구
- 장기적인 과학사적 관점에서 보면 여전히 태동기
- 과학적 성취의 잠재력이 매우 높고 도전할 미개척지 많음

(사진 출처: http://www.rickety.us)

Why things fall?

✓ Question & Answer:

경청해 주셔서

감사합니다!