일반상대론 기초

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목 차

III. 일반상대론

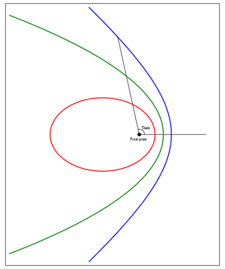
• Modifying Newton's theory of gravity

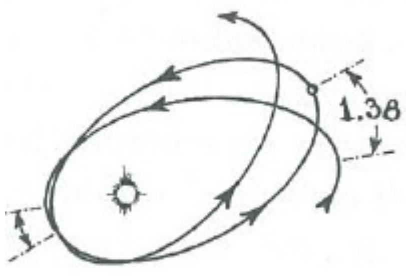
✓ Based on the new understanding of space and time in special relativity, all known physical theories have to be modified to be consistent with it.

✓ Problems in Newtonian gravity:

$$F_G = G\frac{mM}{r^2}$$

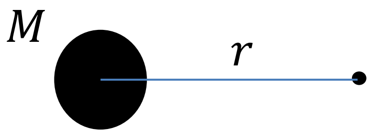
- Action-at-a distance: causality violation
- Violating the principle of relativity under Lorentz transformations
- Mercury's precession: not a closed elliptic orbit





✓ **Scalar gravity theory:** Nordstroem (1914)

$\vec{F}_G = m\vec{g}$ w/ $\vec{g} = -\vec{\nabla}\phi$ ϕ: Gravitational potential



$\phi = -G\dfrac{M}{r}$

$F_G = G\dfrac{mM}{r^2}$

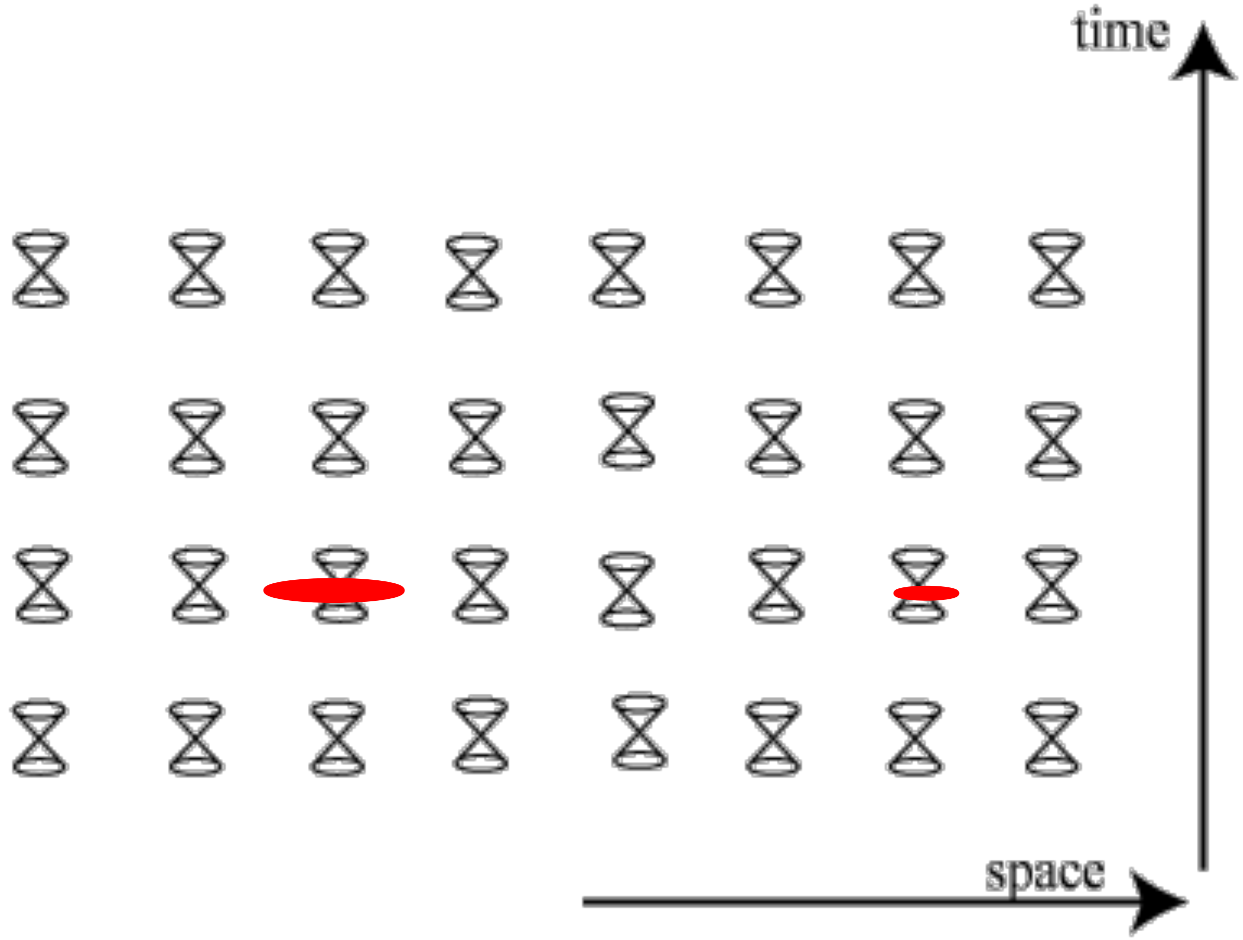
Mass distribution

$\rho(\vec{x}, t)$ ⇨ $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi(\vec{x}, t) = 4\pi G\rho(\vec{x}, t)$$

- Space and Time are not in equalfooting
- ϕ: Scalar, $\rho = T_{00}$: Not a scalar quantity

⇨ $$\left(\boxed{-\frac{\partial^2}{c^2\partial t^2}} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = -\frac{4\pi G}{c^2}T$$ w/ $T = T^{\mu}{}_{\mu}$

- Satisfies the principle of relativity (e.g., covariant form) and the causality
- In failure though: Not in agreement with observation (e.g., negative shift in Mercury's precession)



(Figure credit: http://plato.stanford.edu/entries/spacetime-singularities/lightcone.html)

- Philosophy: Gravitational interaction follows the principle of relativity, but the spacetime itself being intact
- The spacetime is still a rigid background, namely, the special relativistic ST (Minkowski metric)

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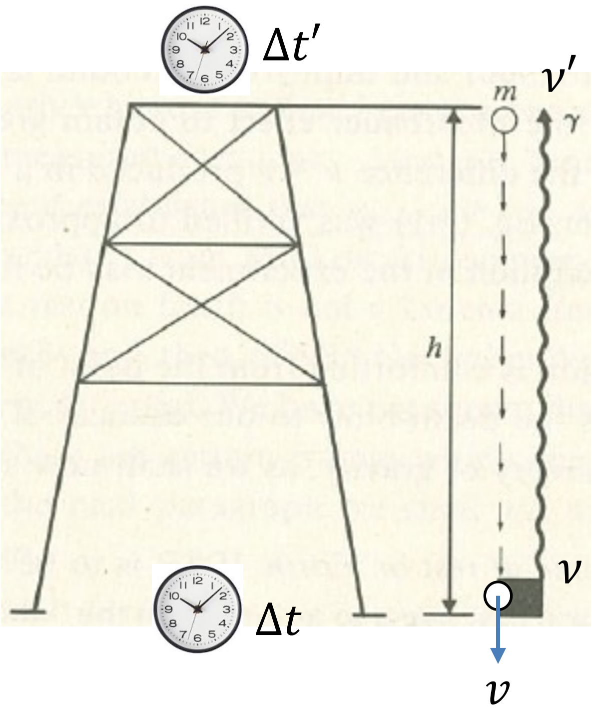
• 중력과 시간: “중력 적색편이와 시간 지연”

$\Delta t' = \Delta t\ ?$



$$E' = h\nu' = mc^2$$

$$\frac{E'}{E} = \frac{h\nu'}{h\nu} = \frac{mc^2}{mc^2 + mgh + \vartheta(v^4)}$$

$$\cong 1 - \frac{gh}{c^2} + \vartheta(v^4)$$

$$\boxed{\nu' \cong \nu\left(1 - \frac{\Delta\phi}{c^2}\right)}$$

$$\begin{aligned} E &= mc^2 + \Delta KE \\ &\cong mc^2 + \frac{1}{2}mv^2 + \vartheta(v^4) \\ &\cong mc^2 + mgh + \vartheta(v^4) \\ &= h\nu \end{aligned}$$

$\nu' < \nu$ or $\lambda' > \lambda$: “적색편이”

그림 출처: Schutz

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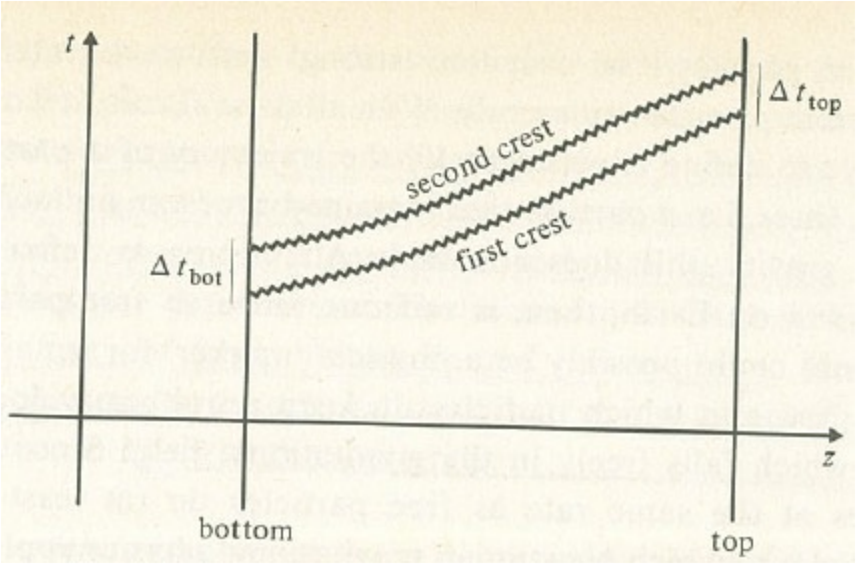
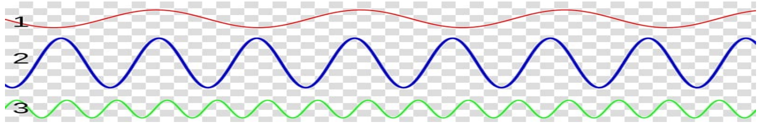




Δt

진동수(ν) = 파동수/1초

➔ $\nu \times \Delta t$ = 총파동수=불변

$$\Delta t' \nu' = \Delta t\, \nu$$

$$\boxed{\Delta t' = \Delta t \frac{\nu}{\nu'} = \Delta t \left(1 + \frac{\Delta\Phi}{c^2}\right)}$$

➔ 시간의 흐름이 중력에 의해 달라진다!?

Ex) 지상 30m: $\frac{\Delta\Phi}{c^2} = \frac{g\,h}{c^2} = \frac{9.8m/s^2 \times 30\,m}{(300000\ \ km/s)^2} \cong 3 \times 10^{-15}$

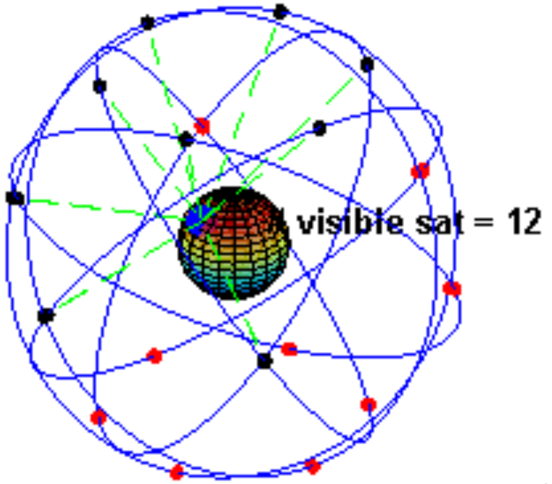
➔ 63빌딩 1층과 63층의 차이(1개층~3m, 80년의 경우):

80년$\times \frac{62}{10} \times 3 \times 10^{-15} \sim 4.7 \times 10^{-5}$ 초 ~47 마이크로초

➔ **인터스텔라:** *“3시간이나 지났을까, 이 거대한 파도와의 싸움을 지나고 보니… 지구에서는 무려 23년 4개월 8일이나 지났었습니다.”*

• 일반상대론과 실생활: GPS의 정밀도

Credit: NASA



Credit: El pak at en.wikipedia

$$T_E = \frac{1}{\sqrt{1-(\frac{V_S}{c})^2}} T_S \cong T_S(1+\frac{1}{2}(\frac{V_S}{c})^2+\ldots)$$

$$\Rightarrow \delta T = T_E - T_S \sim T_S \frac{1}{2}(\frac{V_S}{c})^2$$

$$\frac{\delta T_{특수상}}{T} \sim \frac{1}{2}\left(\frac{V_S}{c}\right)^2 \sim 0.84 \times 10^{-10}$$

$$\frac{\delta T_{GR}}{T} \sim -\frac{GM_{\oplus}}{R_S c^2} \sim -1.6\times10^{-10}$$

$$\Downarrow$$

$$\delta L_{일반상} = c\,\delta T_{일반상} \sim 1.6 \times 10^{-10} cT$$

$$\sim 3\text{m per } 1분$$

✓ 평생 가장 행복한 생각(The happiest thought of my life):

"베른의 특허국 사무실 한 의자에 앉아 있을 때 불현듯 한 생각이 떠올랐다: '만약 한 사람이 자유낙하하고 있으면, 그는 그 자신의 무게를 느끼지 못할 것이다.'" ("*I was sitting in a chair in the patent office at Bern, when all of a sudden a thought occurred to me: 'If a person falls freely, he will not feel his own weight.'"*

- 1907년 11월 어느날

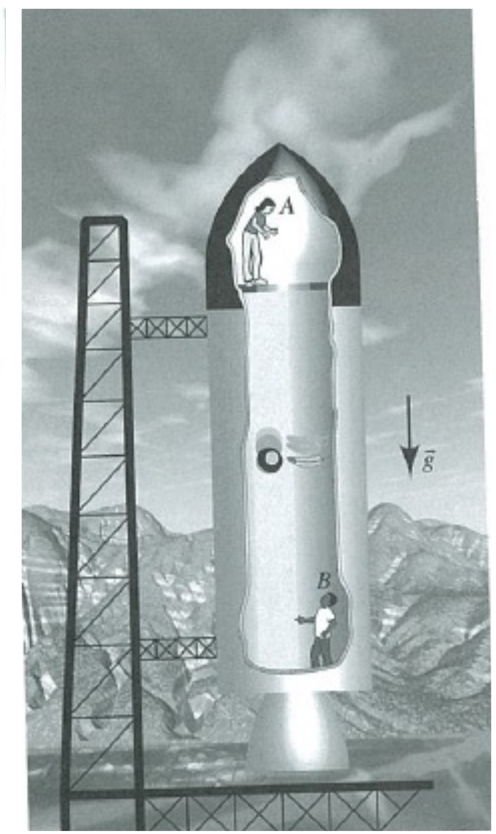
• Equivalence principle (1907):

$m_I a = F$

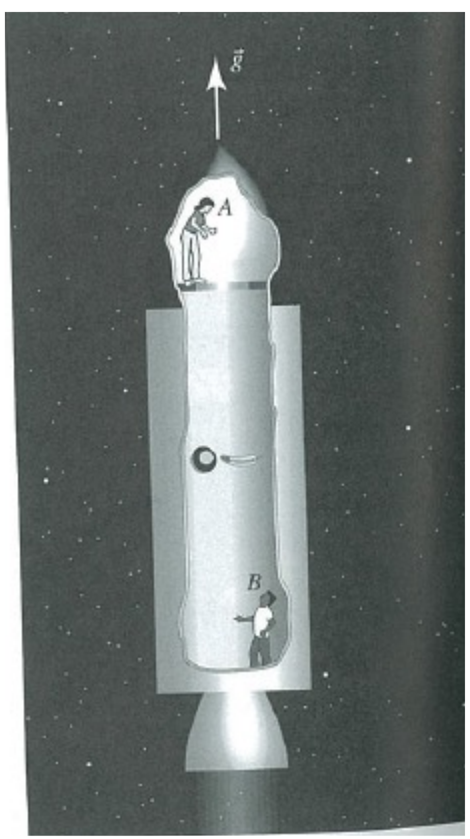
$F_G \sim m_G M_G / r^2$

$\Rightarrow$ $m_I = m_G$

Inertial mass = Gravitational mass



With gravity



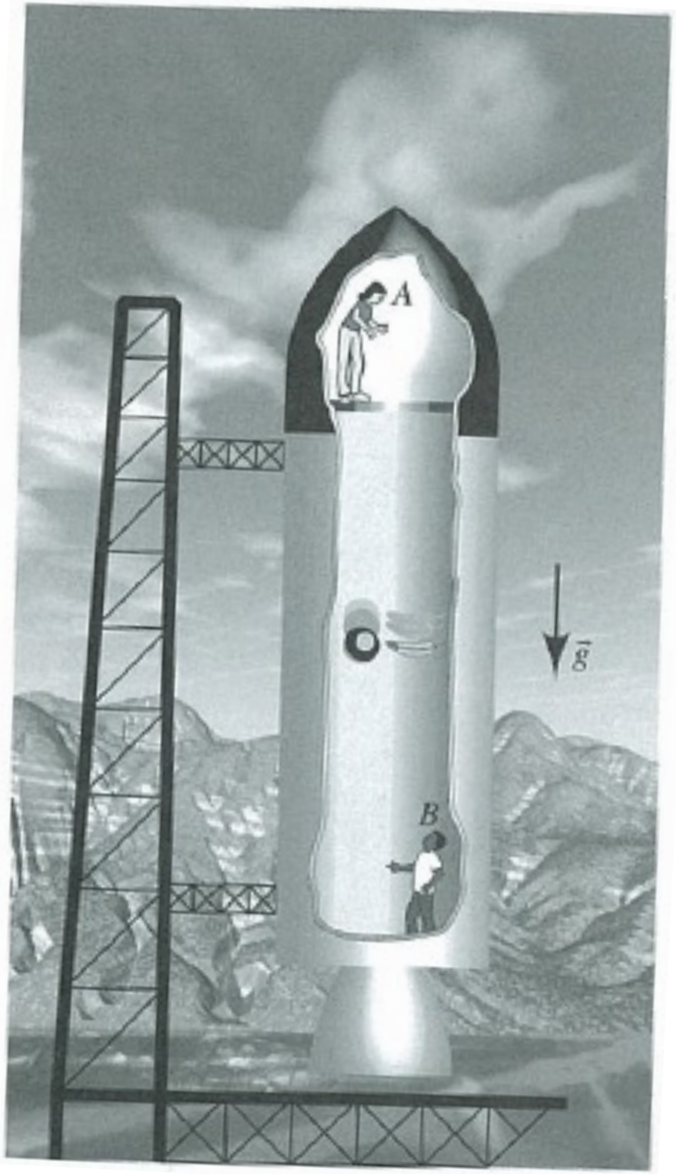
Without gravity

(Figure credit: Hartle)

- Question: 균일 중력장 하에서 뛰어내리면(자유낙하) "중력 " 을 느낄 수 있을까? (Does a freely falling person feel the gravity?)

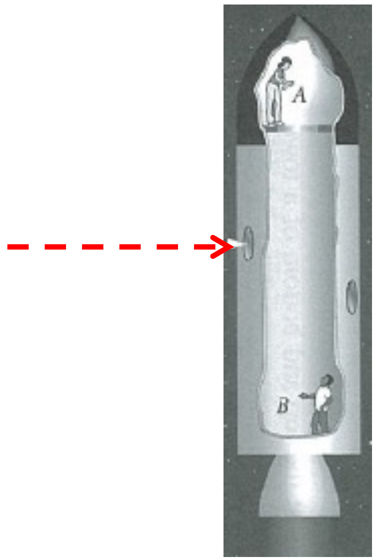
  Ex) $m_I \vec{a} = m_G \vec{g}$ $\Rightarrow$ $\vec{a} = \vec{g}$

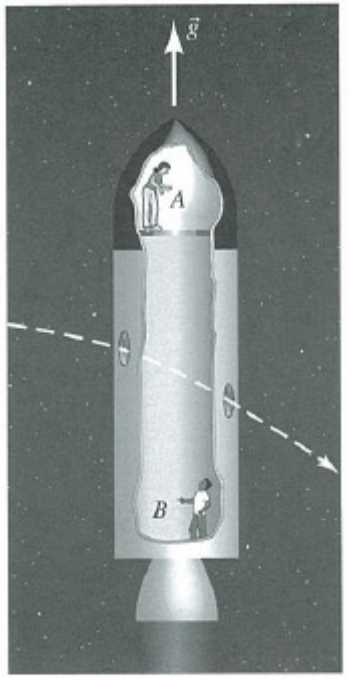
- Motions are independent of masses: $m_I = m_G$
- A steel ball and a feather will fall in the same way provided that they had same initial velocities.
- 일정한 크기의 중력이 작용하는 계(왼쪽 그림)와 편평한 시공간에서 크기는 같으며 방향은 반대로 등가속하는 계(오른쪽 그림)는 구별 불가능 (A frame with gravity is equivalent to a frame without gravity but being accelerated in the opposite direction.)
- A freely falling person on the left corresponds to the ball at rest on the right. ➔ …wouldn't feel his weight (gravity)!!



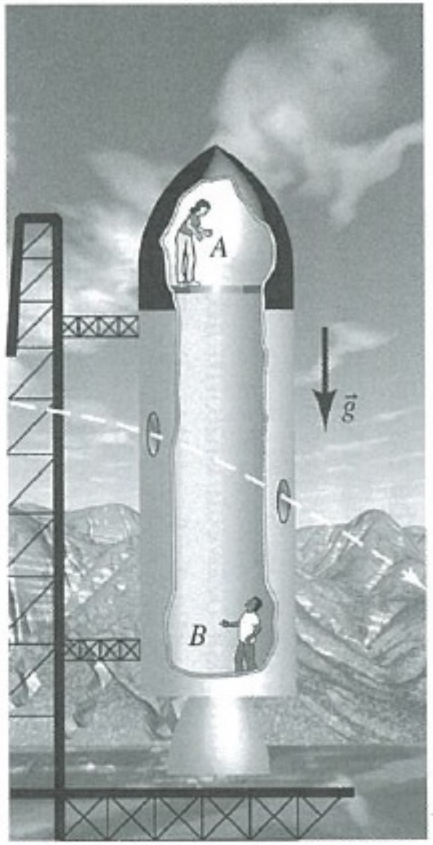
(?)

그림 출처: Hartle





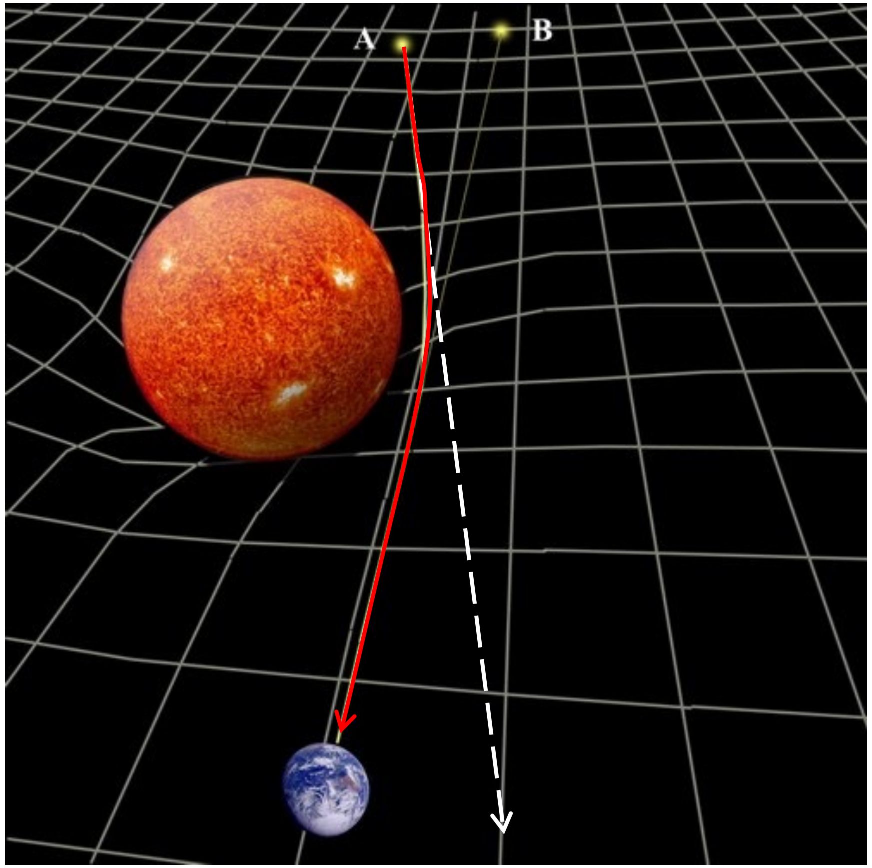
rocket frame



gravitational field

- **The path of a light is bent in the presence of gravity ….!!???**

그림 출처: Hartle

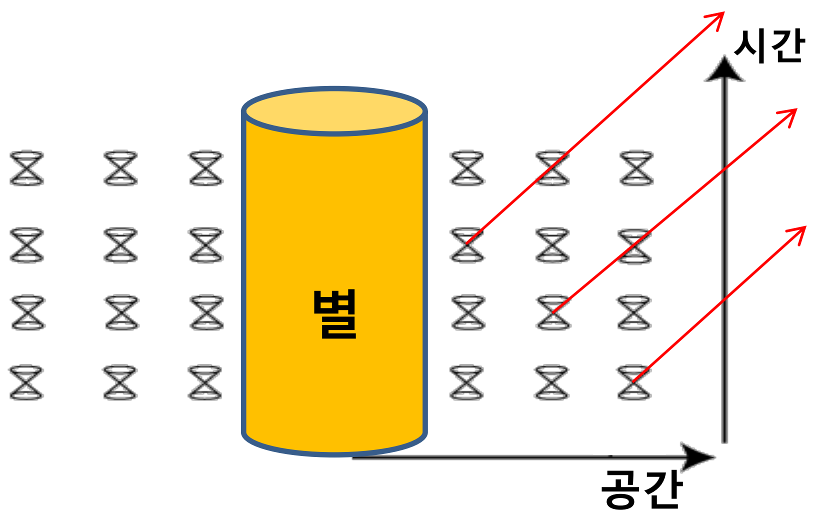


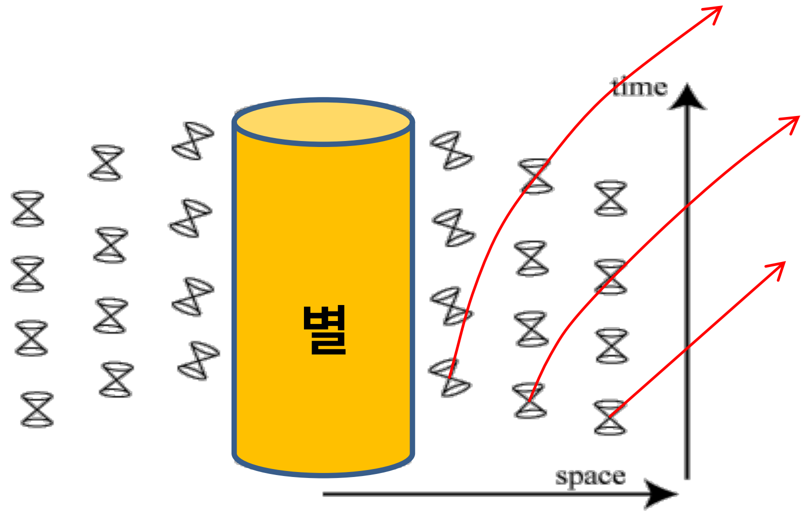
출처: http://www.zamandayolculuk.com/cetinbal/HTMLdosya1/RelativityFile.htm

- Light path bends...?
- Light does not have mass, and so wouldn't be attracted by gravity......
- Then... the space itself is curved?

- **중력의 존재 혹은 물체의 출현이 공간(시간)을 휘게 한다??!!**

- **그렇다면 특수상대론에서 시공간은 고정시켜 놓고 뉴튼 중력을 수정하려는 것은 잘못된 방향...??!!!**





$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$= \eta_{\mu\nu} dx^\mu dx^\nu$$

- Flat ST ➔ Curved ST

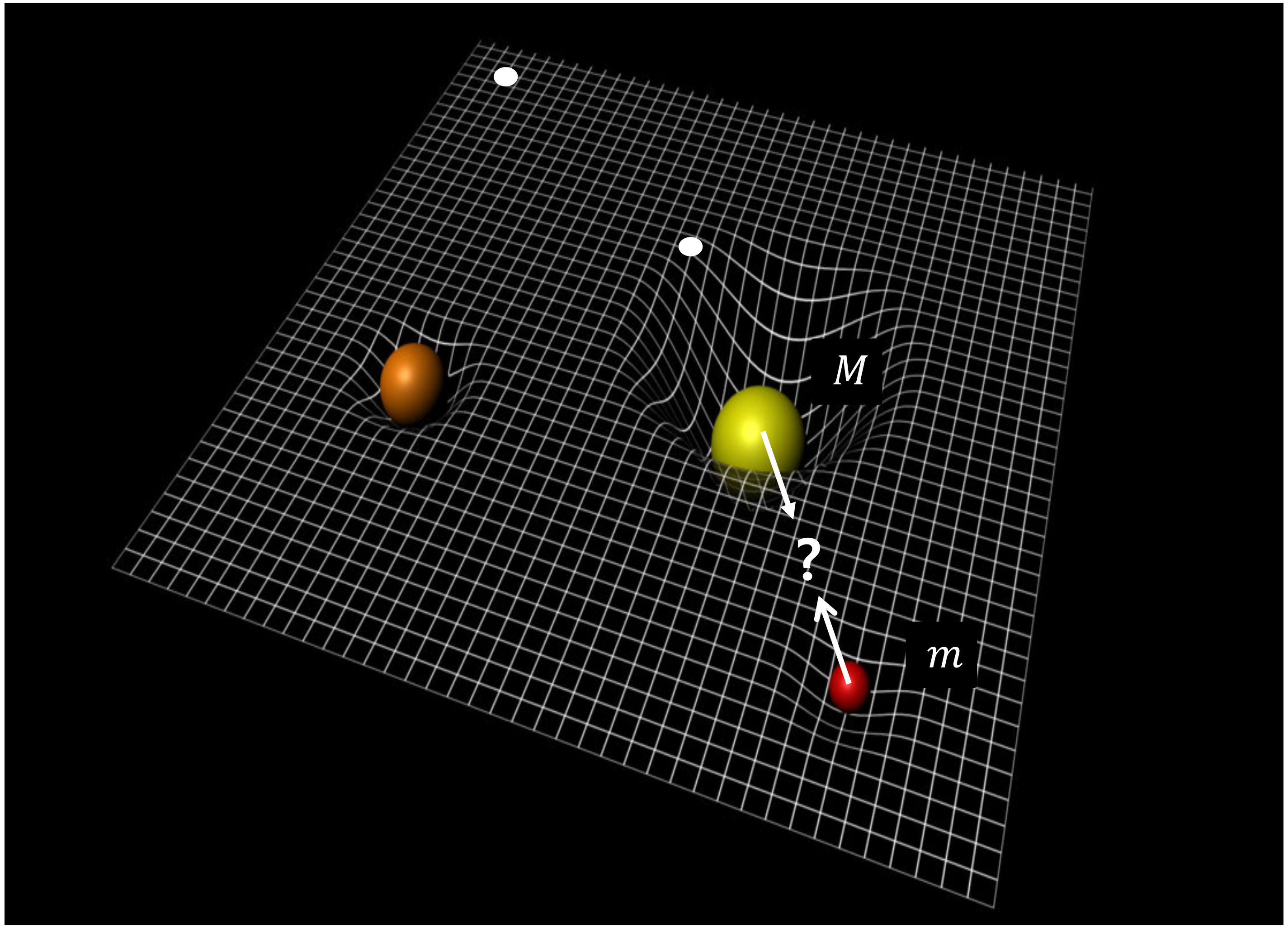
$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- How does gravity work in this curved ST?

그림 출처: http://plato.stanford.edu/entries/spacetime-singularities/lightcone.html

✓ 질량을 갖는 물체는 주위의 공간(/시간)을 휘게 한다…?



(Credit: ESA–C.Carreau)

- 그렇다면 이 휘어진 공간에 놓여 있는 물체의 움직임은?

➔ "측지선(Geodesic) 운동": $\boldsymbol{u} \cdot \boldsymbol{\nabla u} = \mathbf{0}$

Ex) $\ddot{r}(0) \sim \partial_r g_{tt}(r) \neq 0$: Moves toward …

- 여기에 두 질량 사이의 "중력" 은 여전히 추가적으로 작용하고 있을 것인가?
- 아니면… 그런 "중력" 은 더 이상 없고 휘어진 공간에 의한 움직임만이 …????
- **"Gravitation" is NOT a force, but simply a natural motion (Geodesic or free-fall motion) in a curved spacetime.**
- **J. Wheeler: "Matter tells how to curve, and ST tells how to move."**

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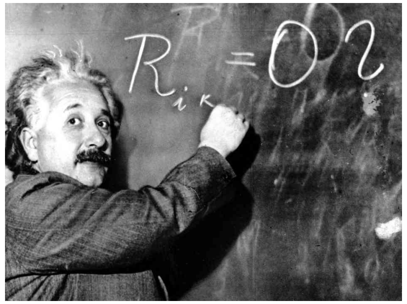
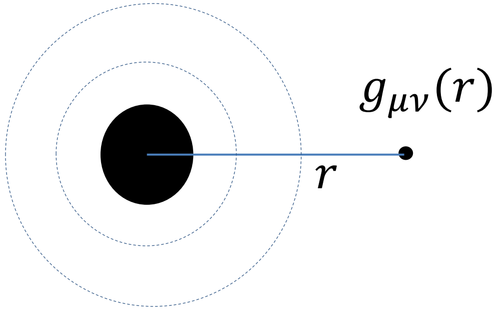
✓ "Free motion (freely falling) in a curved spacetime:



Ex) A spherically symmetric mass:



$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
$$= -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2\, d\phi^2$$

$$\Downarrow$$

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{\theta\theta}(r)(d\theta^2 + \sin\theta^2\, d\phi^2)$$

- A test mass at rest: $u^\mu(\tau=0) = \frac{dx^\mu}{d\tau} = \left(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}\right) = (u^t, 0,0,0)$ ➔ $u^\mu(\tau) = ?$
- Free motion: a motion in a curved ST without external forces/influences ➔ "Inertial motion" ➔ Geodesic motion in a curved ST, i.e., $\boldsymbol{u}\cdot\boldsymbol{\nabla u} = 0$

Note: $u^\mu \frac{\partial u^\alpha}{\partial x^\mu} = \frac{dx^\mu}{d\tau}\frac{\partial u^\alpha}{\partial x^\mu} = \frac{du^\alpha}{d\tau} = \frac{d}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) = \frac{d^2x^\alpha}{d\tau^2}$

- $\boldsymbol{u}\cdot\boldsymbol{\nabla u} = u^\mu \nabla_\mu u^\alpha = u^\mu\left(\frac{\partial u^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\mu\nu}u^\nu\right) = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu = 0$ ➔ $\ddot{r} = -\Gamma^r_{\mu\nu}u^\mu u^\nu$
- At $\tau = 0$, $\ddot{r} = -(\Gamma^r_{tt}u^t u^t + 0) = -\frac{1}{2}g^{rr}(\partial_t g_{tr} + \partial_t g_{tr} - \partial_r g_{tt})(u^t)^2 = \frac{1}{2}g^{rr}\partial_r g_{tt}\dot{t}^2$

➔ $\ddot{r}(0) \sim \partial_r g_{tt}(r) \neq 0$ Moves toward the center!!

"A curved spacetime can make an object move" ➔ "Gravity" is a bending of spacetime itself (?)

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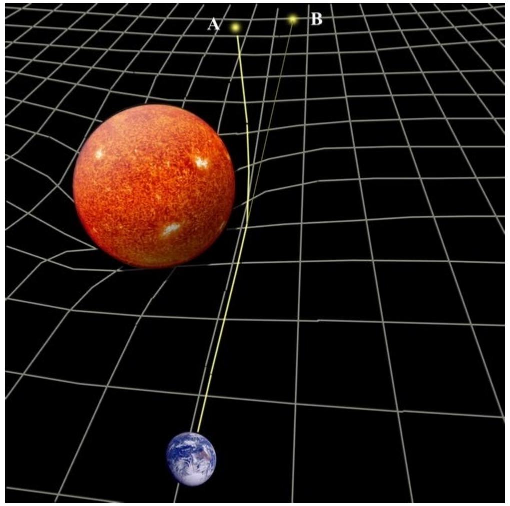
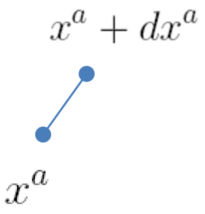


? ! ? ! ?
? ?
! ?

(1912)

- Gravity as a "Curved spacetime"
- "A new theory of gravity"
  ➔ "A theory describing interactions between matter and spacetime"
- ~1912

Spacetime+Matter



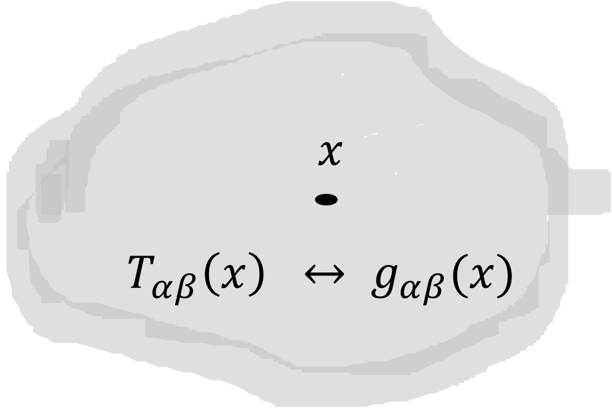
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$: FLAT SPACETIME

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \equiv \sum_{\mu,\nu} g_{\mu\nu}(x) dx^\mu dx^\nu$$ **: CURVED ST OR <u>DYNAMICAL ST</u>**

$$= g_{tt}(x) dt^2 + 2 g_{tx} dt dx + g_{xx} dx^2 + \cdots + dz^2$$

"Metric tensor": $\boldsymbol{g_{\mu\nu}(x)}$ How should it be determined?

- 중력 장방정식:
  - Equation(s) determining the metric tensor, $g_{\alpha\beta}(x)$, for a given mass distribution
  - Newton's gravity equation: $\vec{\nabla}^2\phi = 4\pi G\rho(\vec{x}, t)$: Any hint?



$\rho \sim T_{00} = T_{\alpha\beta}u^\alpha u^\beta$
$\to T_{\alpha\beta}$ (?)

ST curvature:
"Curvature tensor"

Mass-distribution: "Energy-momentum-stress tensor"

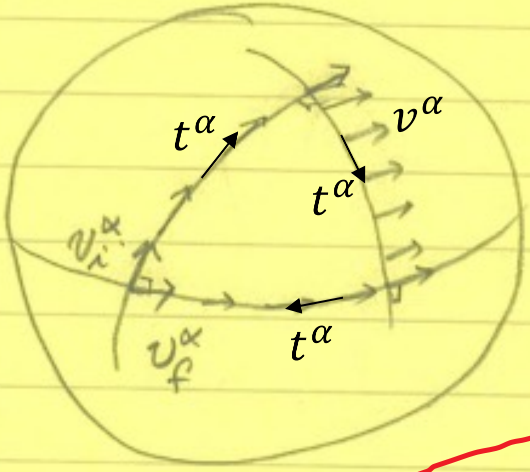
$$(??)_{\alpha\beta} \sim T_{\alpha\beta}$$

$$T_{\alpha\beta} = \begin{pmatrix} T_{00} & T_{0i} \\ T_{i0} & T_{ij} \end{pmatrix} = \left(\begin{array}{c|c} \begin{matrix} Energy \\ density \end{matrix} & \begin{matrix} Momentum \\ flux \end{matrix} \\ \hline \begin{matrix} Momentum \\ flux \end{matrix} & Stress\ tensor \end{array}\right)$$

✓ 시공간의 휘어짐을 기술하는 물리적/수학적 량은 무엇일까?

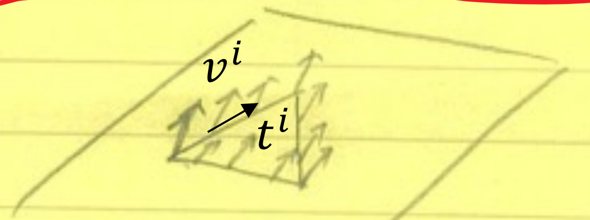
What quantities can measure the curvedness of a spacetime?

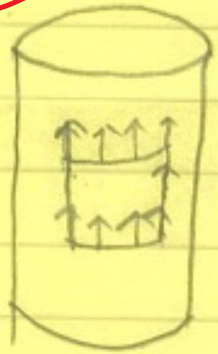
Ex) On the surface of a 2-sphere, the parallel transport of a vector along a closed curve does not give the original vector!



$v_{final} \neq v_{initial}$

This is due to the curvature or non-flatness of the surface.





평행이동된 $\vec{v}$ ⟷ $t^j \partial_j v^i = 0$이

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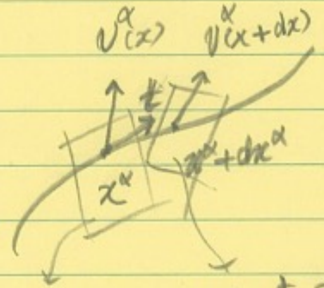
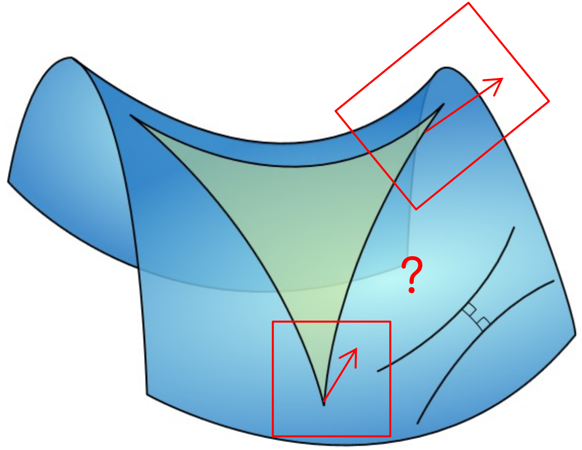


$$t\cdot\nabla v^\alpha = t^\beta \nabla_\beta v^\alpha$$

$$\equiv \lim_{\epsilon\to 0} \frac{\text{"}v^\alpha(x+\epsilon t) - v^\alpha(x)\text{"}}{\epsilon}$$

$$\equiv \lim_{\epsilon\to 0} \frac{v^\alpha_{pt}(x) - v^\alpha(x)}{\epsilon}$$



V_p at x^α V_p at $x^\alpha+dx^\alpha$

In order for the subtraction above to be defined well, we need to move the vector $v(x+\epsilon t)$ into V_p at x.

Let

$$v^\alpha_{\parallel}(x) \equiv v^\alpha(x+\epsilon t) + \Gamma^\alpha_{\beta\gamma}(x)\,\epsilon t^\beta v^\gamma(x)$$

↳ Parallel-transported ↳ Connection or christoffel symbol

$$v^\alpha(x+\epsilon \mathbb{t}) = v^\alpha(x) + \frac{\partial v^\alpha}{\partial x^\beta}t^\beta\epsilon + \frac{1}{2}\frac{\partial^2 v^\alpha}{\partial x^\beta \partial x^\mu}t^\beta t^\mu \epsilon^2 + \cdots$$

$$v^\alpha(x) + \frac{\partial v^\alpha}{\partial x^\beta}\epsilon t^\beta + \cdots$$

Then,

$$t^\beta \nabla_\beta v^\alpha = \lim_{\epsilon\to 0} \frac{v^\alpha(x+\epsilon t) - v^\alpha(x) + \Gamma^\alpha_{\sigma\delta}\,\epsilon t^\sigma v^\delta}{\epsilon}$$

$$= \lim_{\epsilon\to 0} \frac{\left[v^\alpha(x+dx) + \Gamma^\alpha_{\mu\nu}(x)v^\nu(x)t^\mu(x)\epsilon\right] - v^\alpha(x)}{\epsilon}$$

$$= t^\beta \frac{\partial v^\alpha}{\partial x^\beta} + t^\sigma \Gamma^\alpha_{\sigma\delta} v^\delta$$

$$= t^\beta\left(\frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\sigma} v^\sigma\right) \quad \text{for any } t.$$

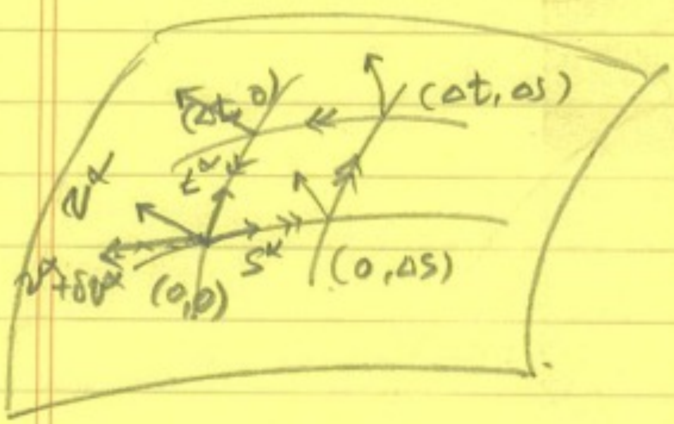
$$\Rightarrow \boxed{\nabla_\beta v^\alpha \equiv \partial_\beta v^\alpha + \Gamma^\alpha_{\beta\sigma} v^\sigma}$$

✓ Parallel transport equation:

$$\boldsymbol{t}\cdot\boldsymbol{\nabla}\boldsymbol{v} = 0 \text{ or } t^\beta \nabla_\beta v^\alpha = 0$$

$$\boldsymbol{t}\cdot\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{w}) = \boldsymbol{0} \rightarrow \Gamma^\delta_{\alpha\beta} = \frac{1}{2}g^{\delta\mu}\left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right)$$

Let us consider the parallel transport of a vector v^α around a small closed loop.



$$t^\alpha \nabla_\alpha v^\beta = 0 = s^\alpha \nabla_\alpha v^\beta .$$

where t & s are coordinate basis vectors (i.e., $t^\alpha = \left(\frac{\partial}{\partial t}\right)^\alpha$, $s^\alpha = \left(\frac{\partial}{\partial s}\right)^\alpha$)

For an arbitrary dual vector field w_α, $v \cdot w = v^\alpha w_\alpha$ is a scalar function.

$$v \cdot w\Big|_{(0,\Delta s)} \simeq v \cdot w\Big|_{(0,0)} + \int_{(0,0)}^{(0,\Delta s)} \underbrace{\frac{\partial (v \cdot w)}{\partial s}}_{s^\beta \nabla_\beta (v^\alpha w_\alpha) \,=\, \cancel{(s^\beta \nabla_\beta v^\alpha)}^{\,0} w_\alpha \,+\, v^\alpha s^\beta \nabla_\beta w_\alpha} ds$$

$$\simeq v \cdot w\Big|_{(0,0)} + \int_{(0,0)}^{(0,\Delta s)} v^\alpha s^\beta \nabla_\beta w_\alpha \, ds$$

*

$$\Rightarrow \quad \delta(v\cdot w)\Big|_{(0,0)} = v\cdot w\Big|_{\substack{(0,0)\\ \text{final}}} - v\cdot w\Big|_{(0,0)}$$

$$\simeq \Delta t\,\Delta s\; v^\alpha \Big[\underbrace{s\cdot\nabla(t\cdot\nabla W_\alpha) - t\cdot\nabla(s\cdot\nabla W_\alpha)}_{①} \Big] .$$

$$s\cdot\nabla(t\cdot\nabla W_\alpha) = s^\beta \nabla_\beta (t^\gamma \nabla_\gamma W_\alpha)$$

$$= t^\gamma s^\beta \nabla_\beta\nabla_\gamma W_\alpha + s^\beta \nabla_\beta t^\gamma\, \nabla_\gamma W_\alpha$$

$$① = t^\gamma s^\beta (\nabla_\beta\nabla_\gamma W_\alpha - \nabla_\gamma\nabla_\beta W_\alpha) + \cancel{(s^\beta\nabla_\beta t^\gamma - t^\beta\nabla_\beta s^\gamma)}^{\,0}\,\nabla_\gamma W_\alpha$$

$$= t^\gamma s^\beta\, \underline{R_{\beta\gamma\alpha}{}^{\delta}}\, W_\delta .$$

$\because\ [t, s] = 0$

*

$\because R_{\alpha\beta\gamma}{}^{\sigma}\omega_\sigma \equiv 2\nabla_{[\alpha}\nabla_{\beta]}\omega_\gamma \equiv (\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\omega_\gamma$ w/ $\nabla_\alpha\omega_\gamma = \partial_\alpha\omega_\gamma - \Gamma^\delta_{\alpha\gamma}\omega_\delta$

$= -2\partial_{[\alpha}\Gamma^{\sigma}_{\beta]\gamma}\omega_\sigma + 2\Gamma^{\sigma}_{\gamma[\alpha}\Gamma^{\delta}_{\beta]\sigma}\omega_\delta$ for all ω_σ.

$$\therefore \boxed{R_{\alpha\beta\gamma}{}^{\delta} = \partial_\beta\Gamma^\delta_{\alpha\gamma} - \partial_\alpha\Gamma^\delta_{\beta\gamma} + \Gamma^\sigma_{\gamma\alpha}\Gamma^\delta_{\beta\sigma} - \Gamma^\sigma_{\gamma\beta}\Gamma^\delta_{\alpha\sigma}}$$

$$\Gamma^\delta_{\alpha\beta} = \frac{1}{2}g^{\delta\mu}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta})$$

Note:
- Defined locally.
- Function of $g_{\alpha\beta}$, $\partial_\alpha g_{\beta\gamma}$ & $\partial_\alpha\partial_\beta g_{\gamma\delta}$.
- Linear in $\partial_\alpha\partial_\beta g_{\gamma\delta}$.
- Even in a LIF in which $\partial_\alpha g_{\beta\gamma} = 0$, it does not vanish in general. — $R_{\alpha\beta\gamma}{}^{\delta} = 0$ for a flat ST.

*

$$\therefore \delta(v \cdot w)\Big|_{(0,0)} \simeq \Delta t \Delta s \, v^\alpha t^\gamma s^\beta R_{\beta\gamma\alpha}{}^{\delta} w_\delta .$$

$$= \delta(v^\delta w_\delta)\Big|_{(0,0)} = v^\delta_{final} w_\delta - v^\delta_{initial} w_\delta$$

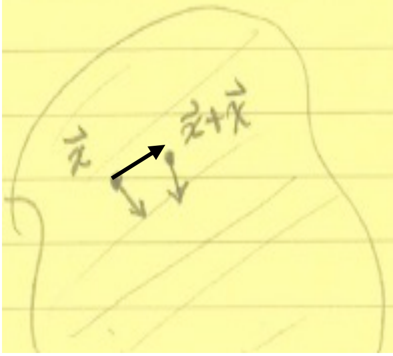
$$= \delta v^\delta w_\delta .$$

$$\boxed{\therefore \delta v^\alpha \simeq \Delta t \Delta s \, t^\gamma s^\beta R_{\beta\gamma\sigma}{}^{\alpha} v^\sigma}$$

$\rightarrow$ FLAT$(\delta v^\alpha = 0)$ iff $R_{\beta\gamma\sigma}{}^{\alpha} = 0$

Therefore, the Riemann curvature tensor measures the failure of returning its initial value when parallel transported around a small closed loop.

* • 조석력(Tidal force):



$$\ddot{\vec{x}} = \frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\Phi$$

↳ Newtonian potential

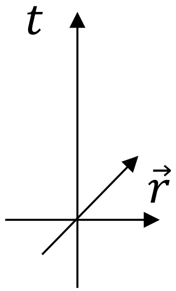
$$\frac{d^2x^i}{dt^2} = -\frac{\partial\Phi}{\partial x^i} = -\delta^{ij}\frac{\partial\Phi(x)}{\partial x^j}$$

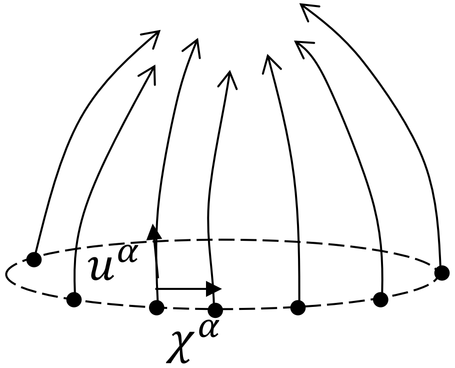
$$\frac{d^2(x^i+\chi^i)}{dt^2} = -\delta^{ij}\frac{\partial\Phi(x^k+\chi^k)}{\partial x^j}$$

$$= -\delta^{ij}\left[\frac{\partial\Phi(x)}{\partial x^j} + \frac{\partial}{\partial x^k}\left(\frac{\partial\Phi}{\partial x^j}\right)\cdot\chi^k + \cdots\right]$$

⇨ $a^i \equiv \frac{d^2\chi^i}{dt^2} = -\partial^i\partial_\gamma\Phi\,\chi^\gamma$

⇨ $a^\alpha \simeq -\partial^\alpha\partial_\gamma\Phi\,\chi^\gamma$





✓ Geodesic deviation Eq.:

$$a^\alpha \equiv u\cdot\nabla(u\cdot\nabla\chi^\alpha) = -R_{\beta\gamma\delta}{}^{\alpha}\chi^\gamma u^\beta u^\delta$$

$$R_{\beta\gamma\delta}{}^{\alpha}u^\beta u^\delta \simeq \partial^\alpha\partial_\gamma\Phi \quad (?)$$

$R_{\beta\delta} \equiv R_{\beta\alpha\delta}{}^{\alpha}$: Ricci curvature tensor

$$R_{\beta\delta}u^\beta u^\delta \simeq \partial^\alpha\partial_\alpha\Phi \simeq \vec{\nabla}^2\Phi = 4\pi G\rho(x)$$

*

* Stress-energy-momentum tensor:

$$T_{\alpha\beta} = \left(\begin{array}{c|c} T_{00} & T_{0i} \\ \hline T_{i0} & T_{ij} \end{array} \right) = \left(\begin{array}{c|c} \text{Energy density} & \text{Momentum flux} \\ \hline & \text{stress tensor} \end{array} \right)$$

$$\rho = T_{00} \simeq T_{\alpha\beta} u^\alpha u^\beta .$$

$$\Rightarrow R_{\alpha\beta} u^\alpha u^\beta \simeq 4\pi G \rho = 4\pi G \, T_{\alpha\beta} u^\alpha u^\beta$$

✓ $$\boxed{\therefore \; R_{\alpha\beta} = 4\pi G \, T_{\alpha\beta} \quad (?)}$$

*

However, we have

$\nabla^\alpha T_{\alpha\beta} = 0$: Local energy conservation law.

Since $\nabla^\alpha R_{\alpha\beta} \neq 0$, we cannot have $R_{\alpha\beta} = 4\pi G T_{\alpha\beta}$ in general.

Bianchi identity : $\nabla^\alpha G_{\alpha\beta} = \nabla^\alpha (R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R) = 0$ always!

✓

w/ $R \equiv g^{\alpha\beta} R_{\alpha\beta} = R^\alpha{}_\alpha$: Ricci scalar

So, let us postulate

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \lambda T_{\alpha\beta}$$

Trace: $g^{\alpha\beta} G_{\alpha\beta} = R - \frac{1}{2} \cdot 4 \cdot R = -R = \lambda T$

$$\Rightarrow R_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} (-\lambda T) + \lambda T_{\alpha\beta}$$

$$= \lambda (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T)$$

*

In the Newtonian limit,

$$T \equiv T^{\alpha}_{\ \alpha} = g^{\alpha\beta} T_{\alpha\beta} \simeq -T_{00} + T_{ii} \simeq -T_{00} = -\rho .$$

$$R_{\alpha\beta} u^\alpha u^\beta \simeq \lambda \Big[T_{\alpha\beta} u^\alpha u^\beta - \tfrac{1}{2} \underbrace{(u \cdot u)}_{=-1} \big(\overbrace{-T_{\alpha\beta} u^\alpha u^\beta}^{=-T_{\alpha\beta} u^\alpha u^\beta} \big) \Big]$$

$$= \lambda \tfrac{1}{2} T_{\alpha\beta} u^\alpha u^\beta$$

$$\simeq 4\pi G\, T_{\alpha\beta} u^\alpha u^\beta .$$

$\therefore \quad \lambda = 8\pi G$!

✓ $$\boxed{\therefore \quad R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = 8\pi G\, T_{\alpha\beta}}$$: Einstein equation 1916.

- "Curvature of a spacetime" = "Matter distribution energy"

- Coupled non-linear 2^{nd}-order partial differential equations for $g_{\alpha\beta}(x)$;

*

$$R_{\alpha\beta} = \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\alpha \Gamma^\mu_{\mu\beta} + \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\mu} - \Gamma^\nu_{\mu\beta}\Gamma^\mu_{\nu\alpha}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right).$$

- Linear in $\partial_\mu\partial_\nu g_{\alpha\beta}$ though.
- 10 equations for 10 unknown $g_{\alpha\beta}$.
- Local equation.
- Have a good initial value formulation
- Hyperbolic in character

– One must solve for both the spacetime metric and the matter distribution simultaneously.

– Cosmological constant term: Λ

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \underline{\Lambda g_{\alpha\beta}} = 8\pi G T_{\alpha\beta}.$$

$$T_{\alpha\beta} \rightarrow T_{\alpha\beta} - \frac{\Lambda}{8\pi G} g_{\alpha\beta}$$

“암흑에너지”

–

	Newtonian gravity	General relativity
Mass	Gravitational force: $\vec{F} = -m\vec{\nabla}\Phi$	Curves ST: $ds^2 = g_{\alpha\beta}(x)\,dx^\alpha dx^\beta$.
Motion	2^{nd} law: $\frac{d^2x^i}{dt^2} = -\delta^{ij}\partial_j\Phi$	Geodesic: $\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma$
Field eq.	$\nabla^2\Phi = 4\pi G\rho$	$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$.

✓ Other gravitational dynamics or theories of gravity?:

$$I = \int dx^4 \sqrt{-g}\, [\frac{1}{16\pi G} R + \mathcal{L}_{matter} + \cdots]$$

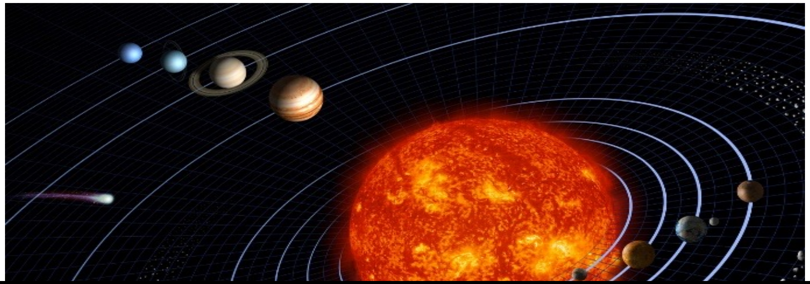
$$a_1 R^2 + a_2 R_{\alpha\beta} R^{\alpha\beta} + a_3 R^3 + \cdots$$

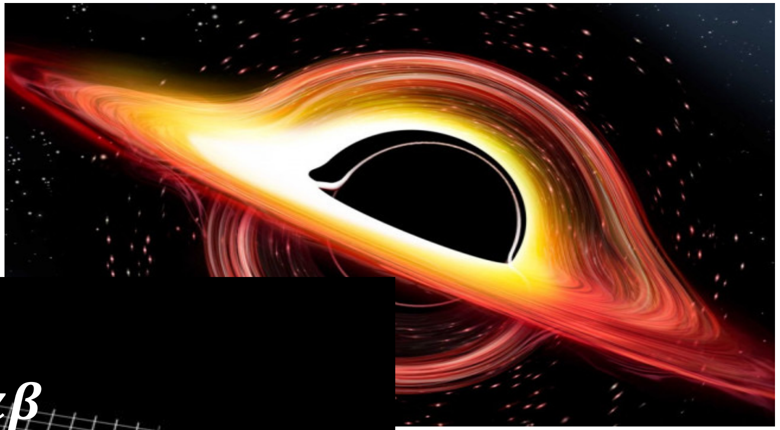
✓ Well-posed initial value formulation?:

- YES ➔ 3+1 formulation (See the lecture by 현영환 박사님)

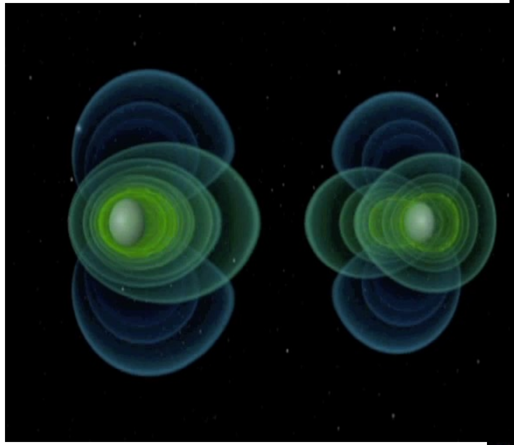
시공간 해

http://www.navcen.uscg.gov/ftp/gps/ggeninfo/gps-

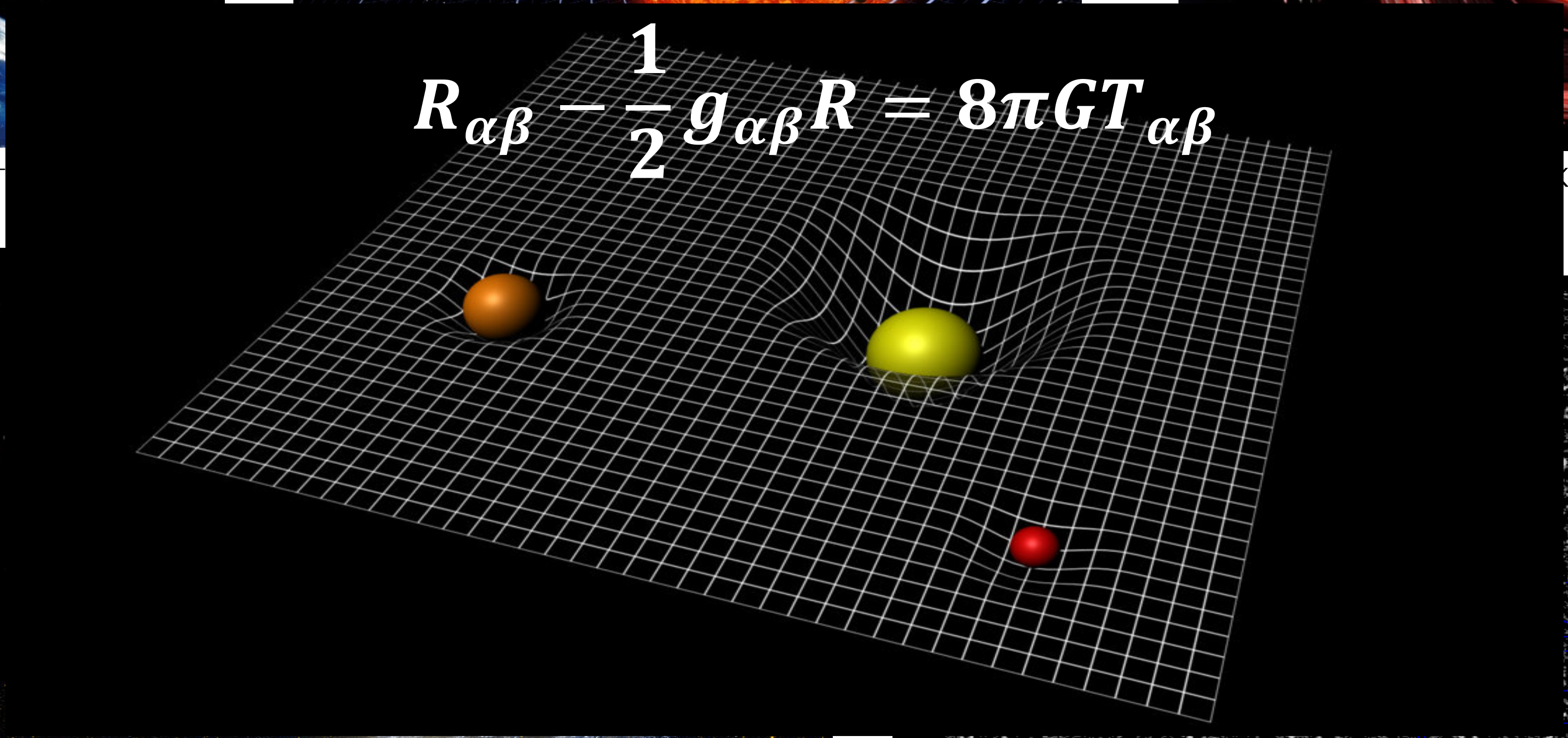




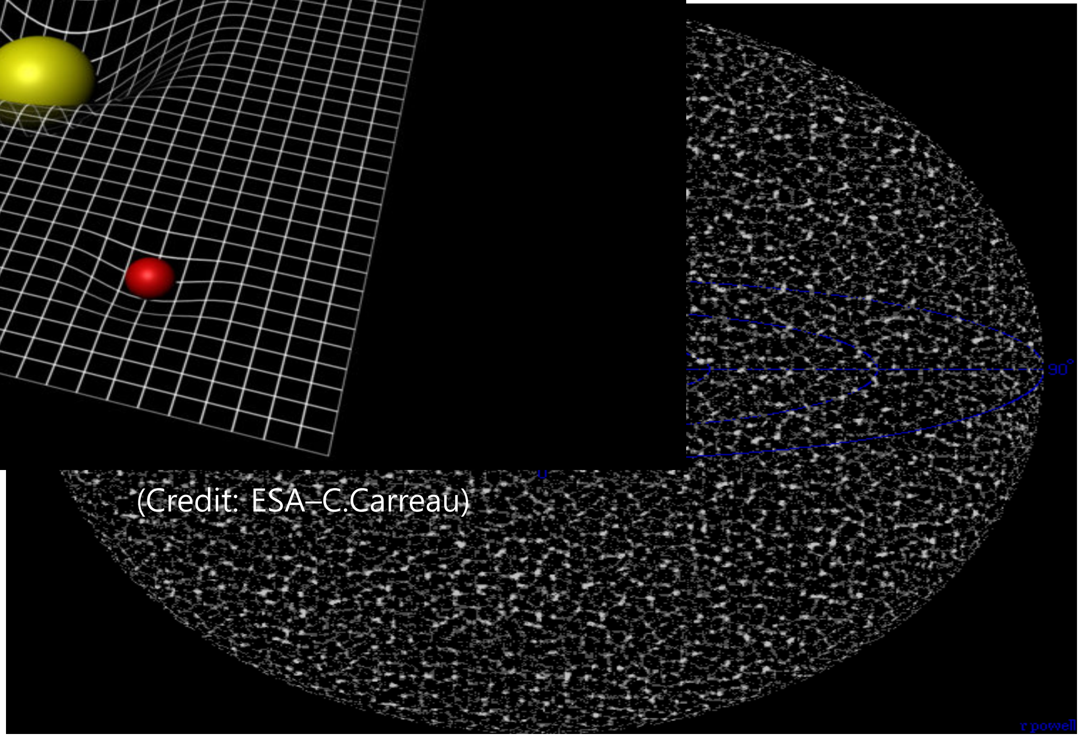
Kelly/Discover



Credit: M. Koppitz



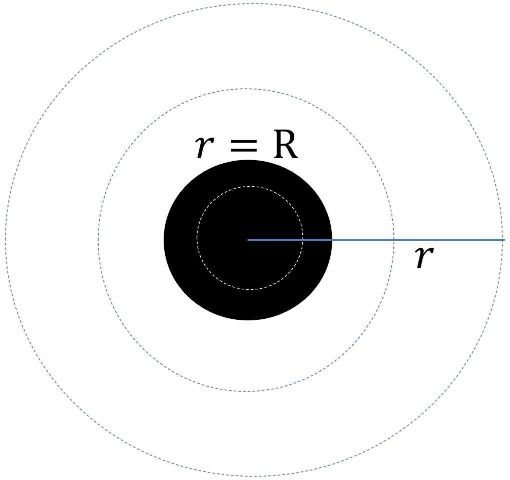
Andromeda M31 (~67 kpc, NASA/JPL-Caltech)



(Credit: ESA–C.Carreau)

Visible Universe (~8.6 Gpc, http://www.atlasoftheuniverse.com/universe.html)

• (정적 상태의) 구대칭 별



$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$$

- **메트릭 추정(Metric ansatz):** 적절한 좌표를 도입해서 시간에 따라 변하지 않고(정적 상태) 구대칭을 반영하는 계량텐서를 아래와 같이 추정할 수 있음

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

- 외부($r > \mathrm{R}$): 진공 ➔ $T_{\alpha\beta} = 0$
- 내부($r < \mathrm{R}$): 완전 유체 가정 ➔ $T_{\alpha\beta} = \rho u_\alpha u_\beta + P\left(g_{\alpha\beta} + u_\alpha u_\beta\right)$

  $\rho = \rho(r)$: 질량밀도, $P = P(r)$: 압력, $u^\alpha = \left(\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi}\right) = (u^t, 0,0,0)$: 4-velocity of fluid

  Note: $-1 = u \cdot u = g_{\alpha\beta} u^\alpha u^\beta = -f u^t u^t + 0$ ➔ $u^\alpha = (1/\sqrt{f(r)}, 0,0,0)$

- 진공의 경우: $T_{\alpha\beta} = 0$ ➔ $0 = g^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R\right) = R - \frac{1}{2} \cdot 4 \cdot R = -R$ ➔ $R_{\alpha\beta} = 0$

✓ 별 외부 해(Exterior solutions): $R_{\alpha\beta} = 0$

$$R_{tt} = \frac{1}{4rfh^2}\left[-rhf'^2 + f(-rf'h' + 2h(2f' + rf''))\right] = 0$$

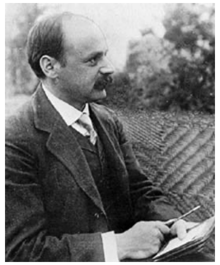
$$R_{rr} = \frac{1}{4rf^2h}\left[f(4f + rf')h' + rh(f'^2 - 2ff'')\right] = 0$$

$$R_{\theta\theta} = R_{\varphi\varphi}/\sin^2\theta = \frac{1}{2fh^2}\left[-rhf' + f(-2h + 2h^2 + rh')\right] = 0$$

➔ $\boxed{f = 1 - \frac{C}{r} = h^{-1}}$ C: 임의의 적분상수

$C \leftrightarrow 2M$

- 슈워츠차일드 메트릭: 1916

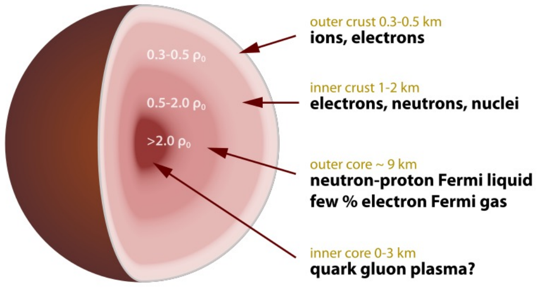


Karl Schwarzschild (1873~1916)

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}$$

출처: 과학동아

✓ 별 내부 해(Interior solutions):

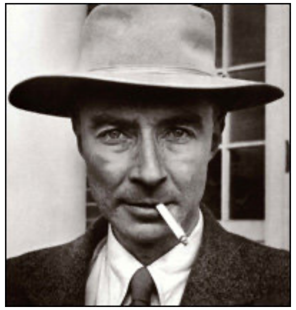


$f \equiv e^{2\phi(r)}$ and $h \equiv (1-2m(r)/r)^{-1}$ 로 나타내면

$$ds^2 = -e^{2\phi}dt^2 + \frac{dr^2}{1-2m/r} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi G T_{\alpha\beta} = 8\pi G\left[\rho u_\alpha u_\beta + P\left(g_{\alpha\beta} + u_\alpha u_\beta\right)\right]$$

- TOV (Tolman-Oppenheimer-Volkoff) 방정식: 1934, 1939 w/ $\mathrm{G} = 1 = c$



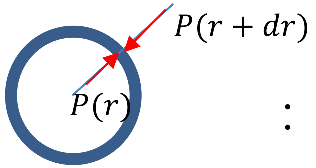
$$\frac{dm}{dr} = 4\pi r^2\rho \quad \Rightarrow \quad m(r) = 4\pi\int_0^r \rho(r')r'^2 dr'$$

Note: $m(\mathrm{r}=\mathrm{R}) \equiv M = C/2$

$$\frac{d\phi}{dr} = \frac{m + 4\pi r^3 P}{r(r-2m)} \quad \Leftarrow \quad \frac{m}{r^2}$$

뉴튼 중력보다 더 강한 "인력" 작용!!

$$\frac{dP}{dr} = -(\rho + P)\frac{m + 4\pi r^3 P}{r(r-2m)} \quad \Leftarrow \quad -\frac{\rho m}{r^2}$$



: Balance eq.

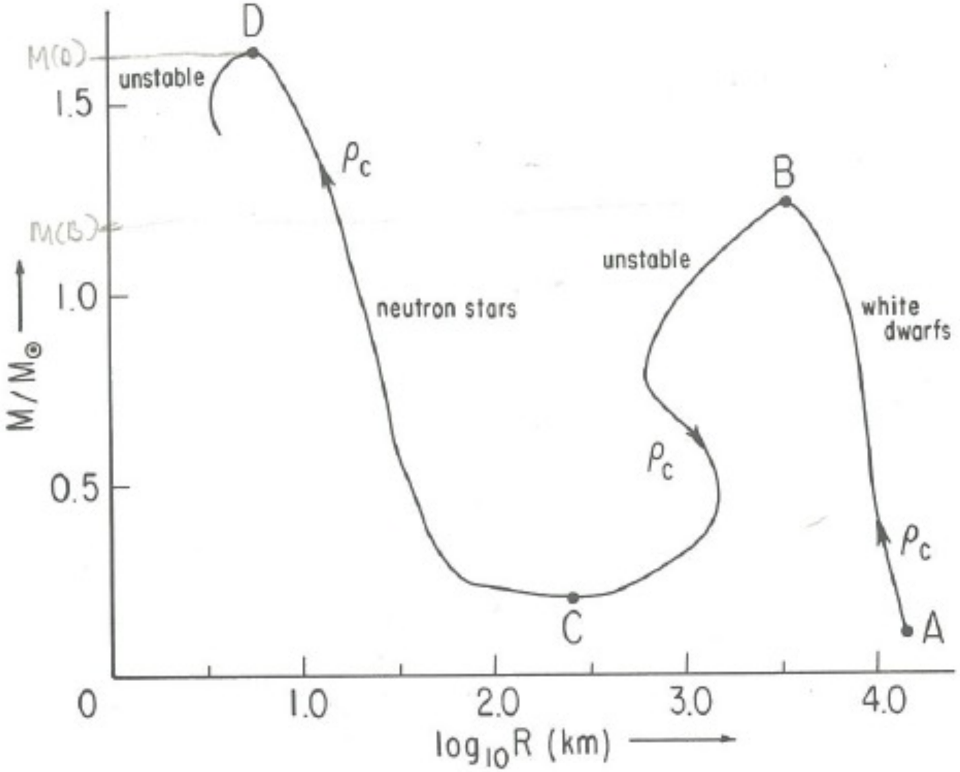
$P = P(\rho)$: EOS (상태방정식)

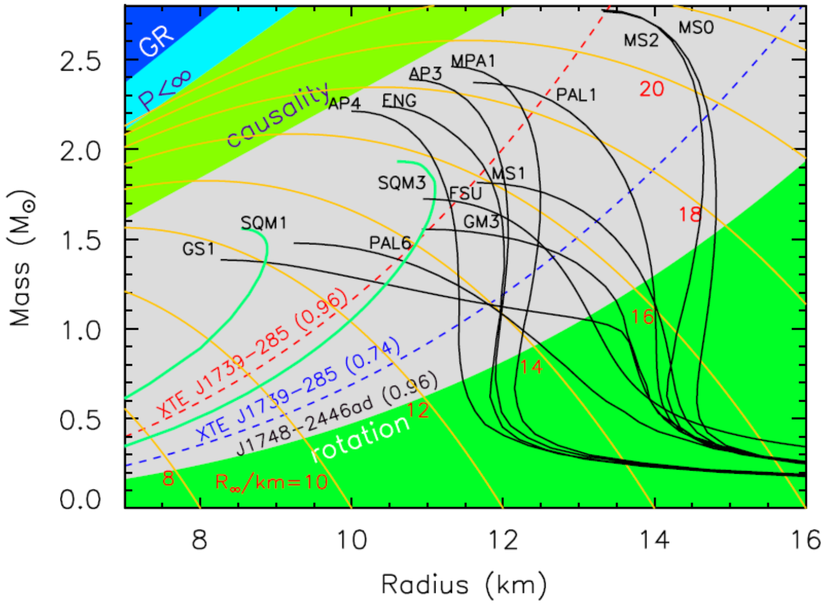
(단위 복구: $\frac{dP}{dr} = -G(\rho + P/c^2)\frac{m+4\pi r^3 P/c^2}{r(r-2Gm/c^2)} \xRightarrow{c\to\infty} -G\frac{\rho m}{r^2}$)

∞

*

✓ **A class of solutions parameterized by the central density:**





다양한 상태방정식에 따른 별의 해:
Lattimer & Prakash

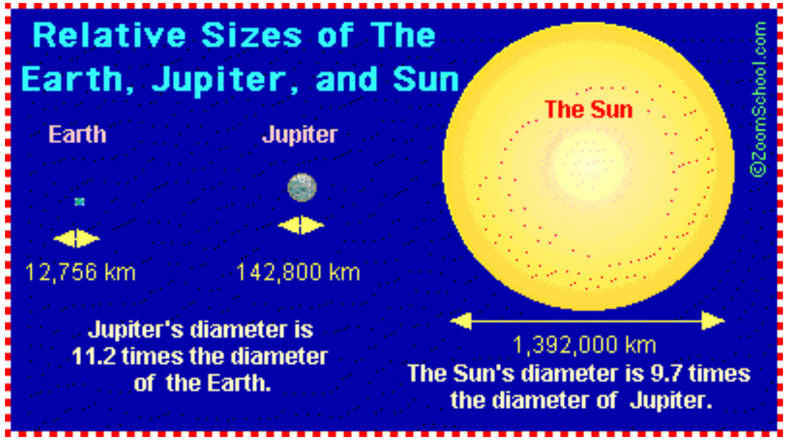
- The maximum mass exists for a given R, for instance, M < 4R/9 for constant density stars!

✓ 뉴튼 중력과 일반상대론에서의 별:

21만, 7만, 109, 9 배

6km 20km

블랙홀 (~1M⊙) 중성자별 (~1.4M⊙)



http://www.enchantedlearning.com/subjects/astronomy/sun/sunsize.shtml

"밀집도(Compactness)" ~ $\dfrac{\text{질량}}{\text{크기}}$

Ex) 지구: $\dfrac{M_\oplus}{R_\oplus} = \dfrac{4.4\times10^{-6}\ \mathrm{km}}{6.4\times10^{3}\ km} \sim 10^{-9}$

태양: $\dfrac{M_\odot}{R_\odot} = \dfrac{1.5\ \mathrm{km}}{7\times10^{5}\ km} \sim 10^{-5}$

은하: $\dfrac{M_\mathrm{G}}{R_\mathrm{G}} = \dfrac{10^{11} M_\odot}{30\sim50\ \mathrm{kpc}} \sim \dfrac{10^{11}\ \mathrm{km}}{10^{17}\ km} \sim 10^{-6}$

뉴튼 중력으로 다루어도 크게 틀리지 않음

중성자별: $\dfrac{M_\mathrm{NS}}{R_\mathrm{NS}} \sim \dfrac{1.4\ M_\odot}{10\sim100\ \mathrm{km}} \sim (0.1\sim0.01)$

블랙홀: $\dfrac{M_\mathrm{BH}}{R_\mathrm{BH}} \sim \dfrac{M}{2M} = 0.5$

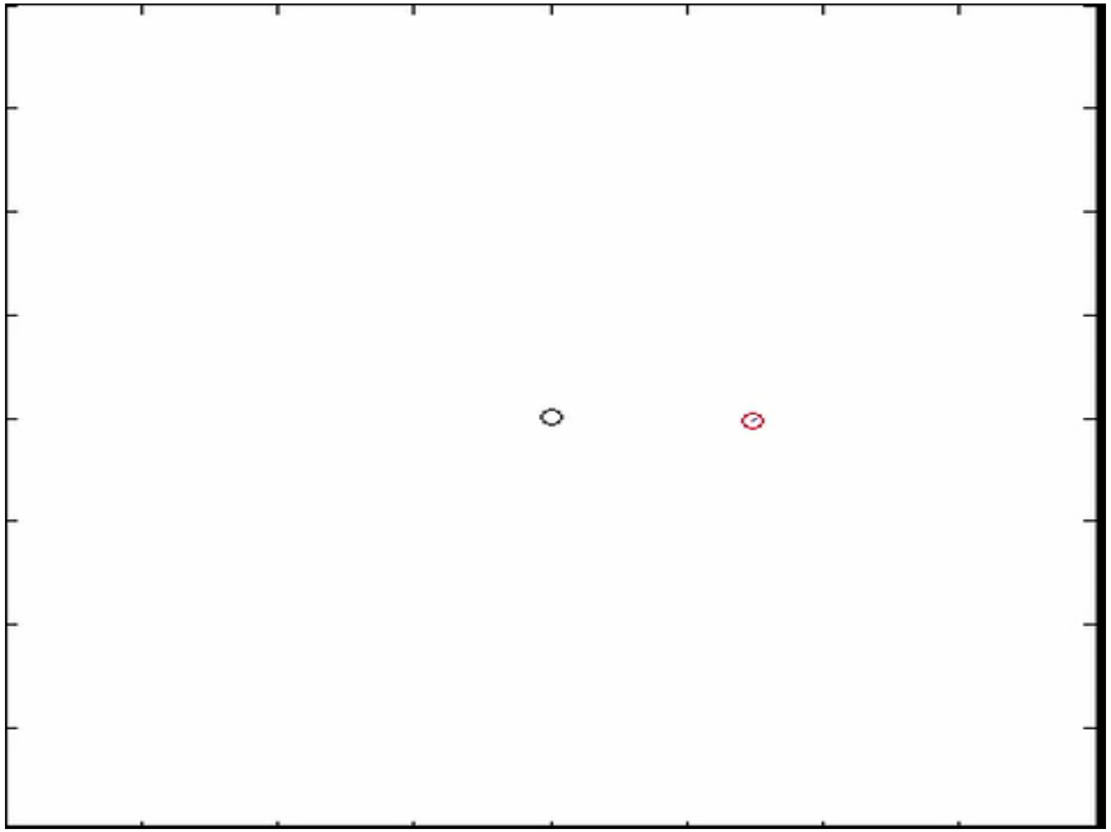
일반상대론으로 다루어야 함:

➔ 뉴튼 중력에서는 불가능한 천체

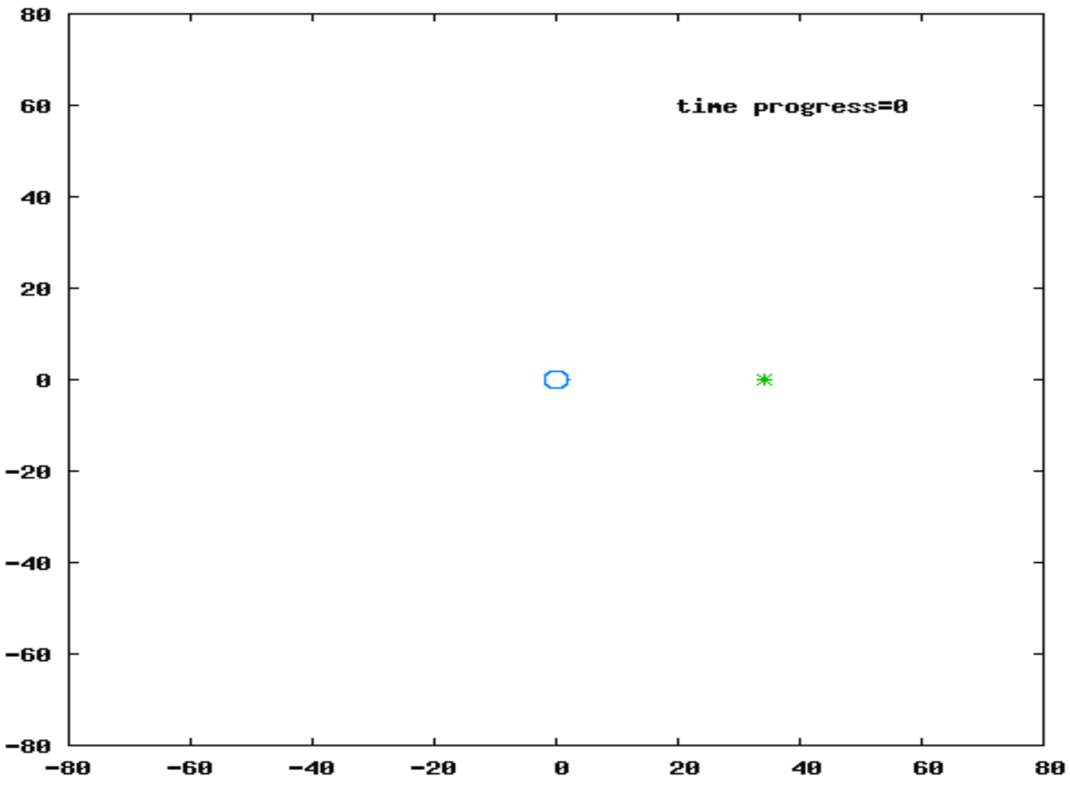
$$\frac{M}{r_{H,Kerr}} = \frac{M}{M+\sqrt{M^2-(J/M)^2}} = 0.5\sim1.0$$

- **Numerical calculations for geodesics:**
  - Schwarzschild BH: 박관호, 박찬 & GK ('13)
  - Kerr BH: 이승환 & GK ('14)

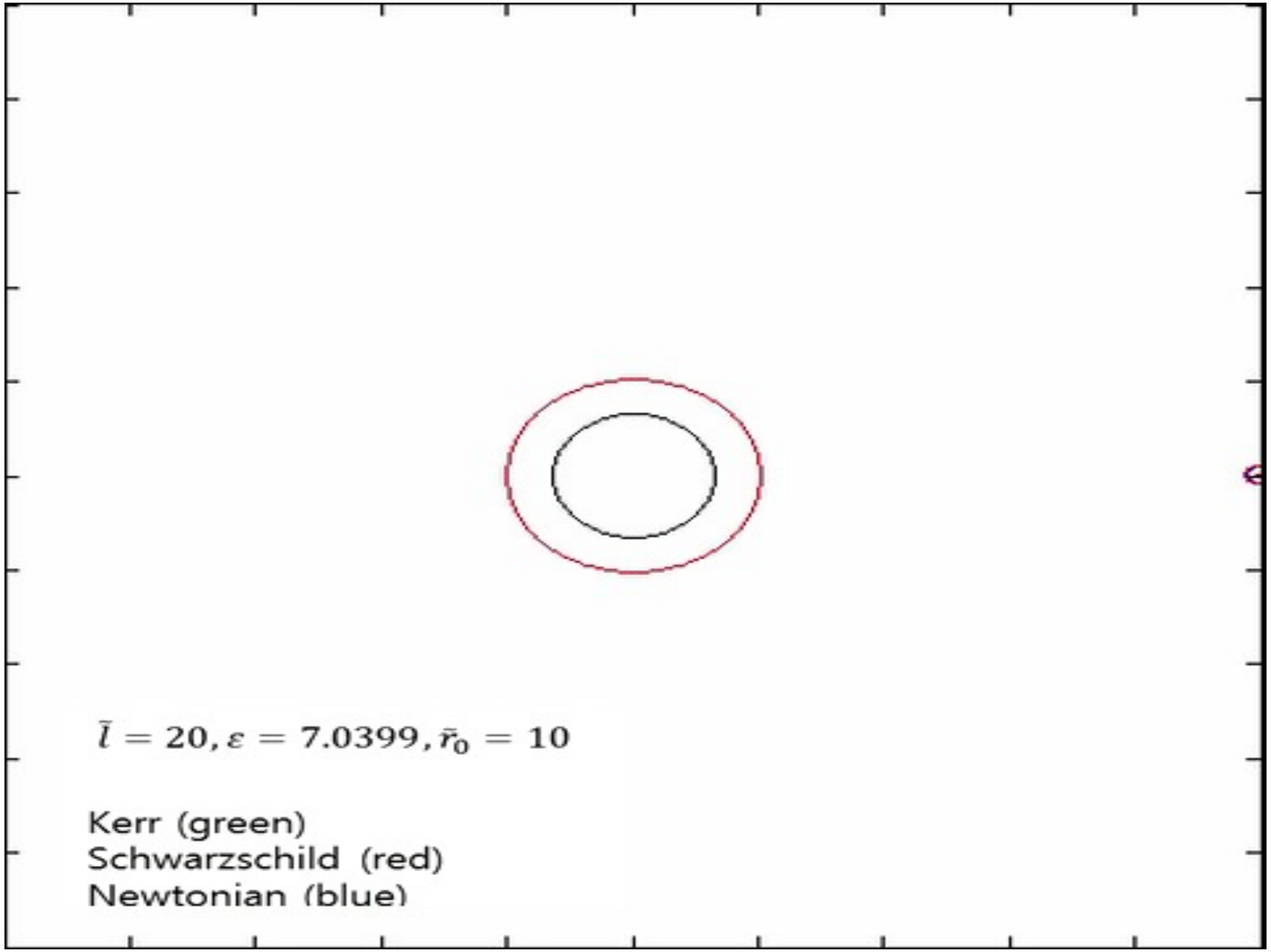
✓ Bound motions



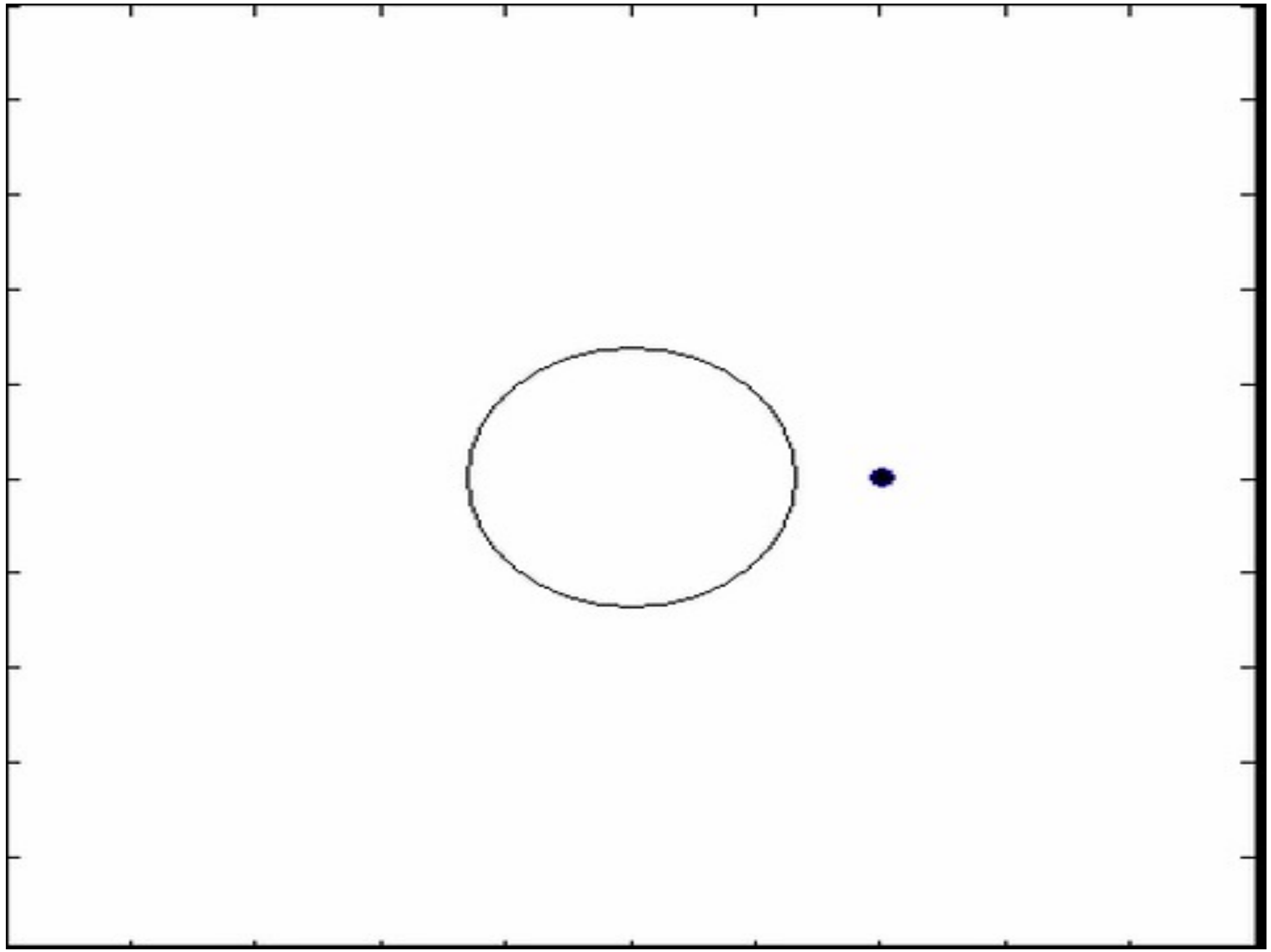
✓ Precessions



✓ Strength of gravities:

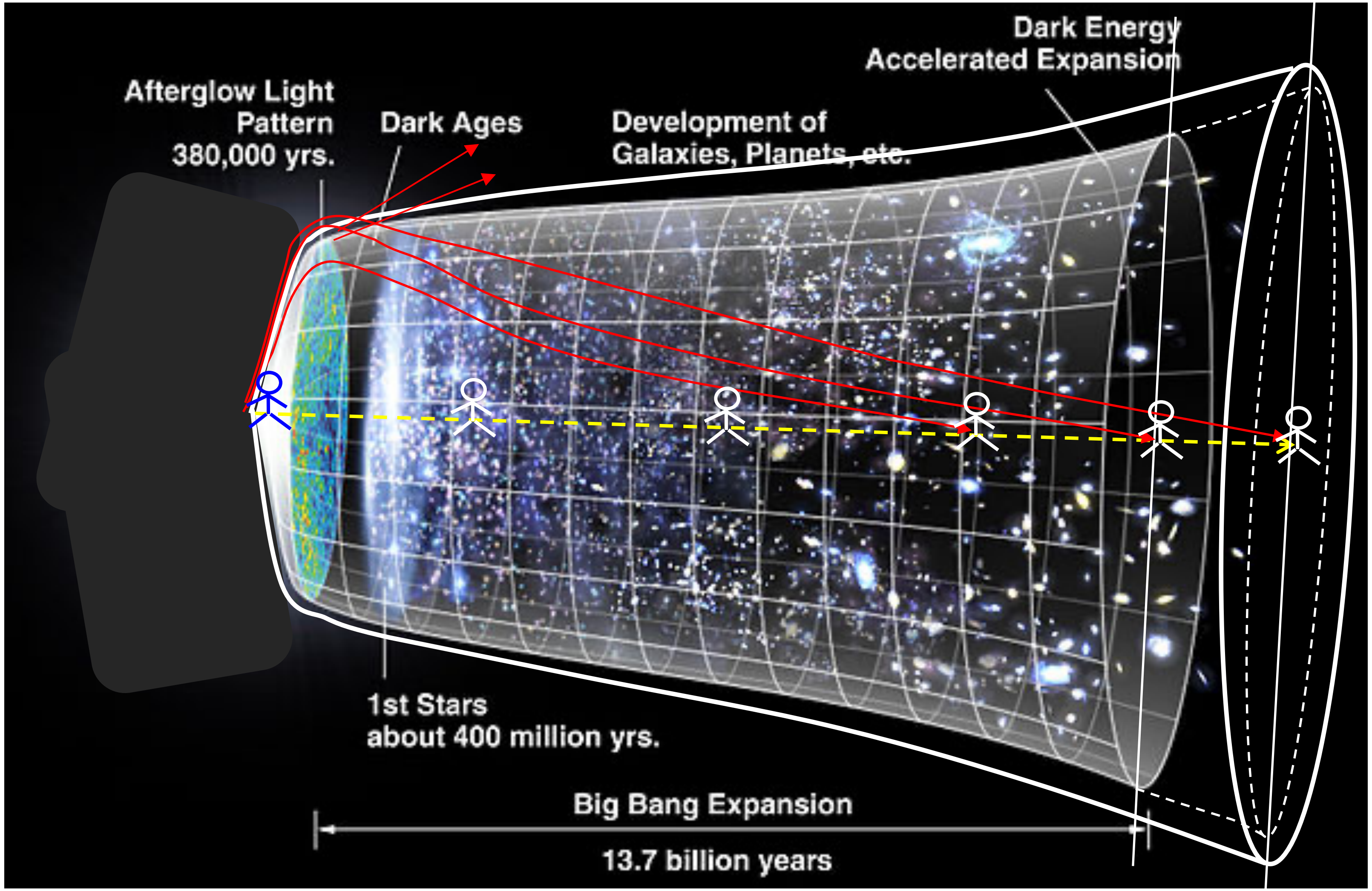


✓ Dragging effect in rotating BH:



- **동적인 우주:** 유한한 과거(시간의 시작), 초기에는 급팽창,가속팽창(암흑 에너지)

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$



$R = ar$

↑ 공간 ↓

$t = 0$ ← 시간 → t

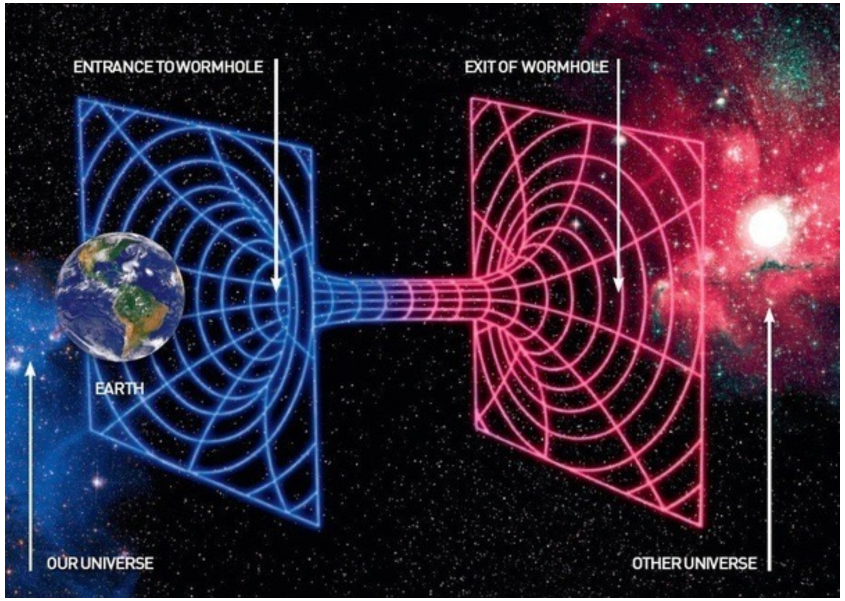
Credit courtesy: NASA/WMAP Science Team

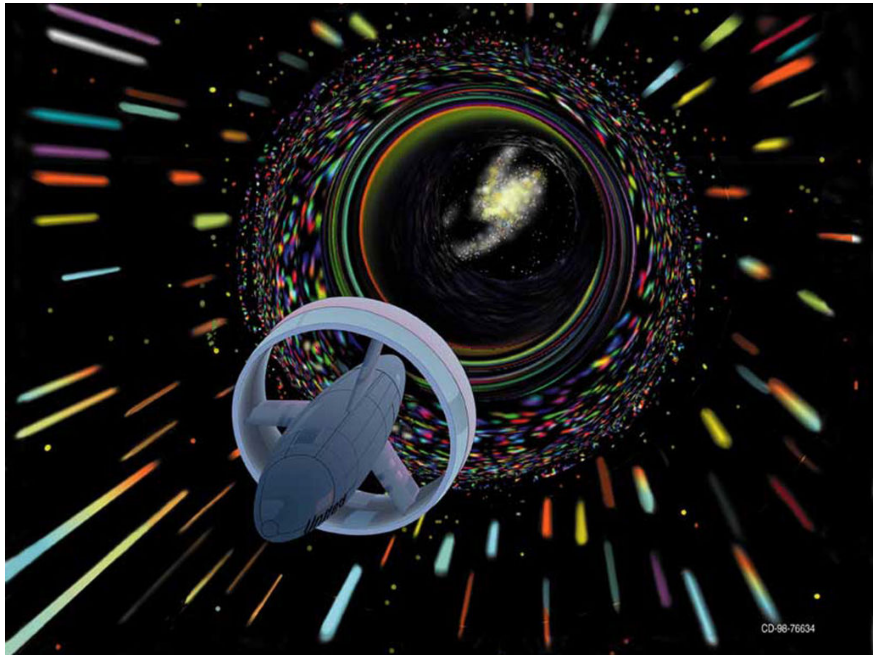
*

• 웜홀(Wormhole):

Morris & Thorne: 1988

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$





$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi G T_{\alpha\beta}$$

"Exotic matter" 필요!?

Credit: Les Bossinas (1998)

*

• 시간 여행(Time Travel)

(어느 SF 작가의 질문)

타임 패러독스 관련 질문

1. 질문 포인트

주인공이 불가사의한 현상으로 미래의 죽음을 알게 된 후, 죽음의 원인이 되는 행동을 하지 않는다면 주인공은 미래의 죽음을 피할 수 있을까? 아니면 어떤 다른 방식으로라도 죽게 되는 것일까?

2. 질문 내용 - 사례 중심

사례 1) 미래 뉴스

1) 주인공은 유명인이거나 영화배우다.

2) 주인공은 우연한 기회에 미래의 뉴스가 나오는 라디오를 얻게 된다.

3) 그 라디오에서 자신이 내일 교통사고를 당해 죽게 됨을 알게 된다. 라디오를 들은 날은 4월 2일이고, 죽인공이 죽는 날은 4월 3일이다.

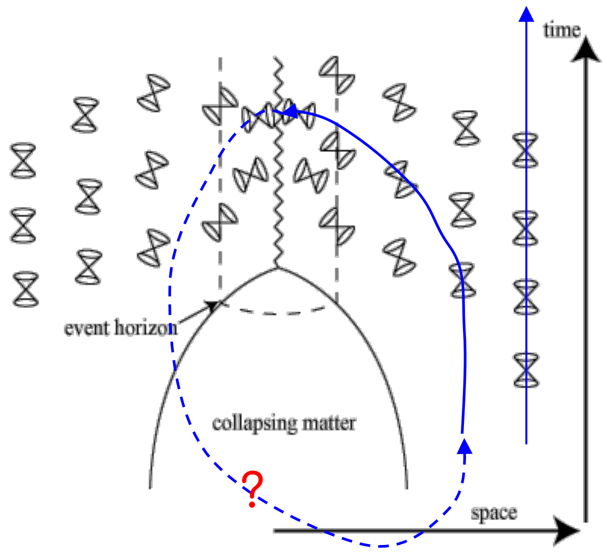
3) 주인공은 4월 3일 차를 몰고 갔어야 할 곳을 가지 않기로 하고, 집에만 있어 사고를 피했다.

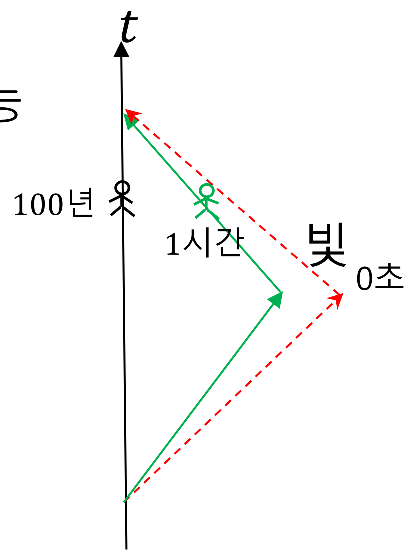
이 경우에서 타임 패러독스는 일어나지 않는가?

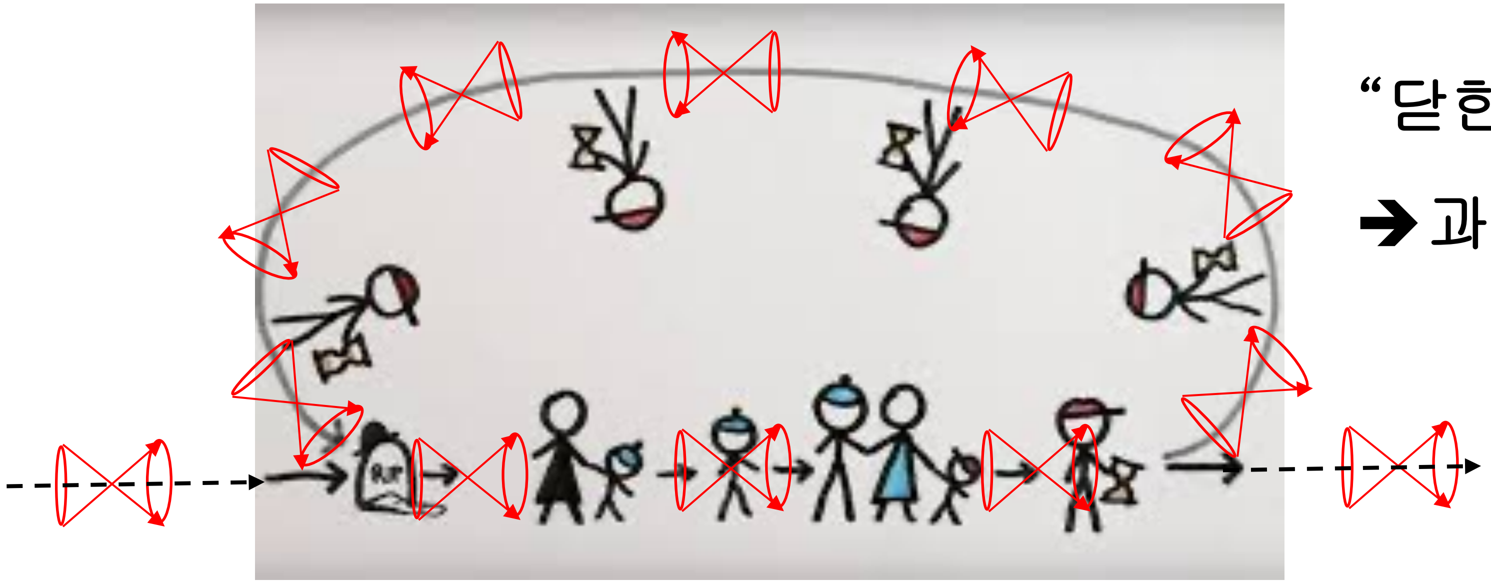
주인공이 다음 날 사고를 피했다면, 결과적으로 4월 3일에는 아무 일도 일어나지 않은 것이 된다. 그렇다면 미래의 뉴스가 나오는 라디오에서는 주인공의 죽음에 대한 뉴스를 방송하지 않을 것이고, 그렇다면 4월 2일의 주인공은 자신이 교통사고를 당한다는 사실을 알지 못하게 된다.

이러한 타임 패러독스에 대해 현재 시점에서는 선택의 변화로 A에서 A'의 세계로 간 것이다, 라는 평행우주 가설을 적용한다고 들은 적이 있다. 그렇게 평행우주의 관점으로 이 문제를 풀 수밖에 없을까? 다른 과학적인 설명이 가능한지...

- **미래로의 시간여행:** 빠른 우주선만 있으면 얼마든지 가능
- **과거로의 시간여행:** ??







"닫힌 인과 경로"

➔과거로의 여행 가능

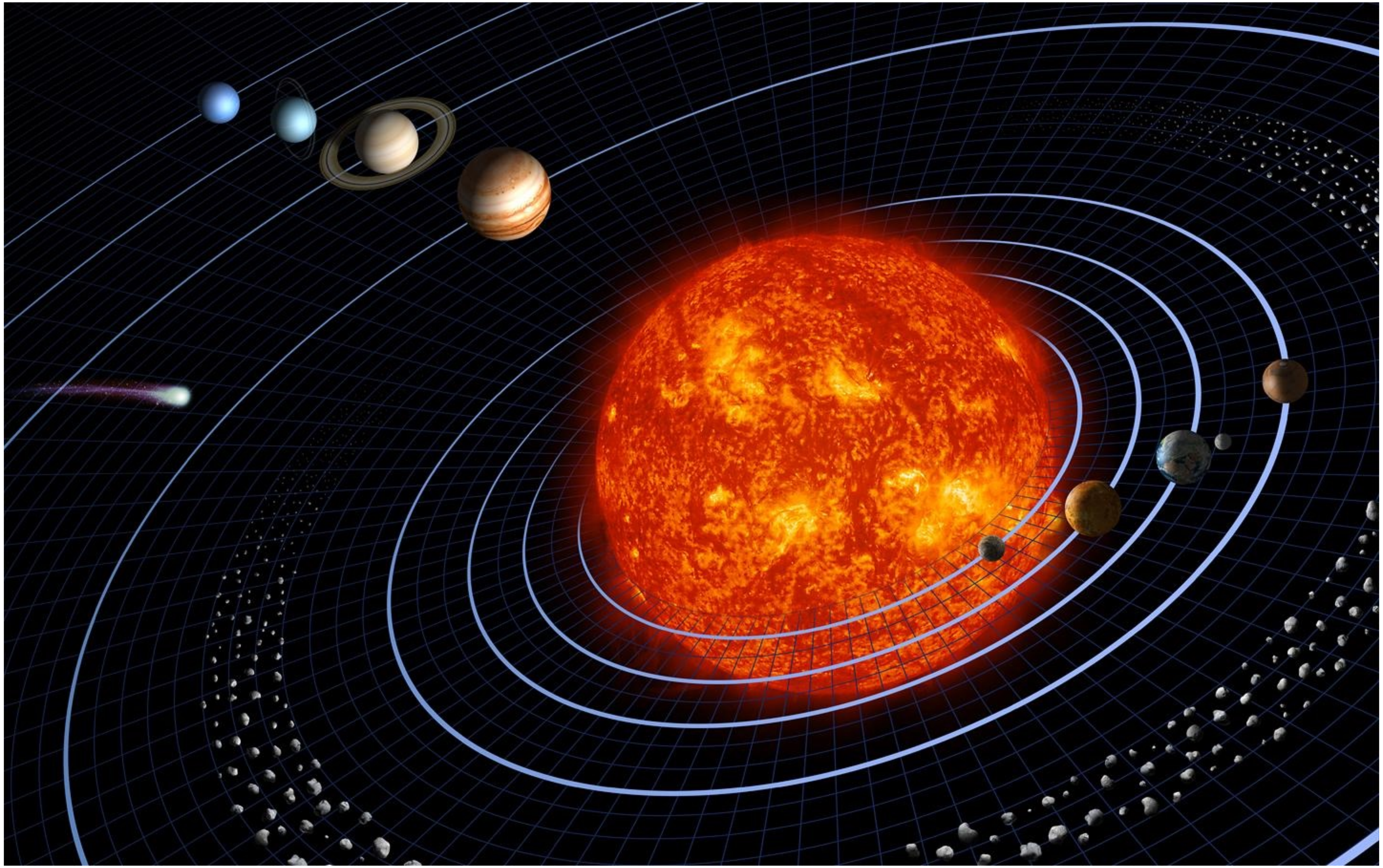
Credit: Lwp Kommunikacio

Stephen Hawking (2009/06/28): "What a shame. I was hoping a future Miss Universe was going to step through the door."

✓ Question & Answer:

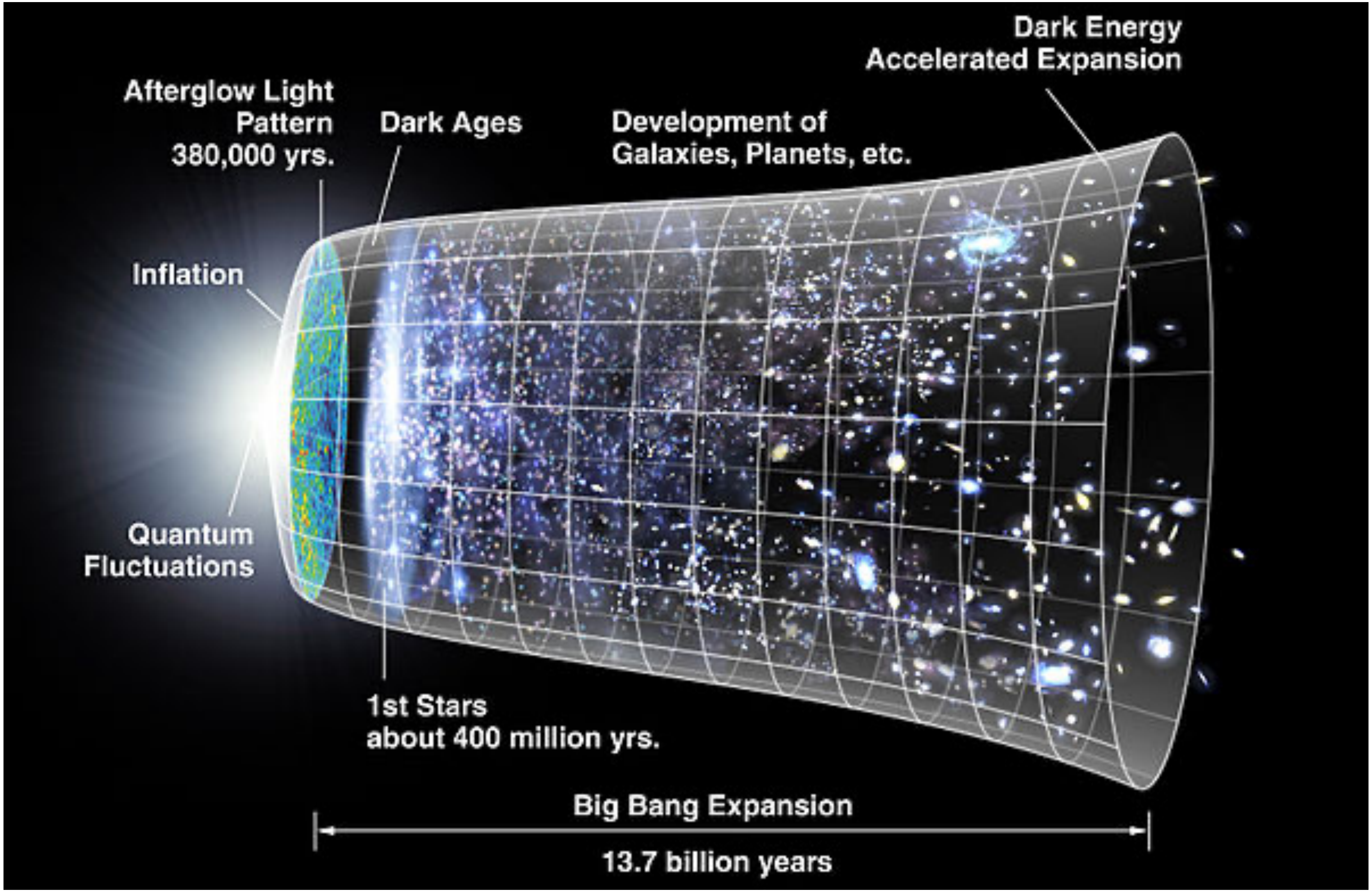
BACK-UP SLIDES

- **중력과 관련되는 자연현상은?**

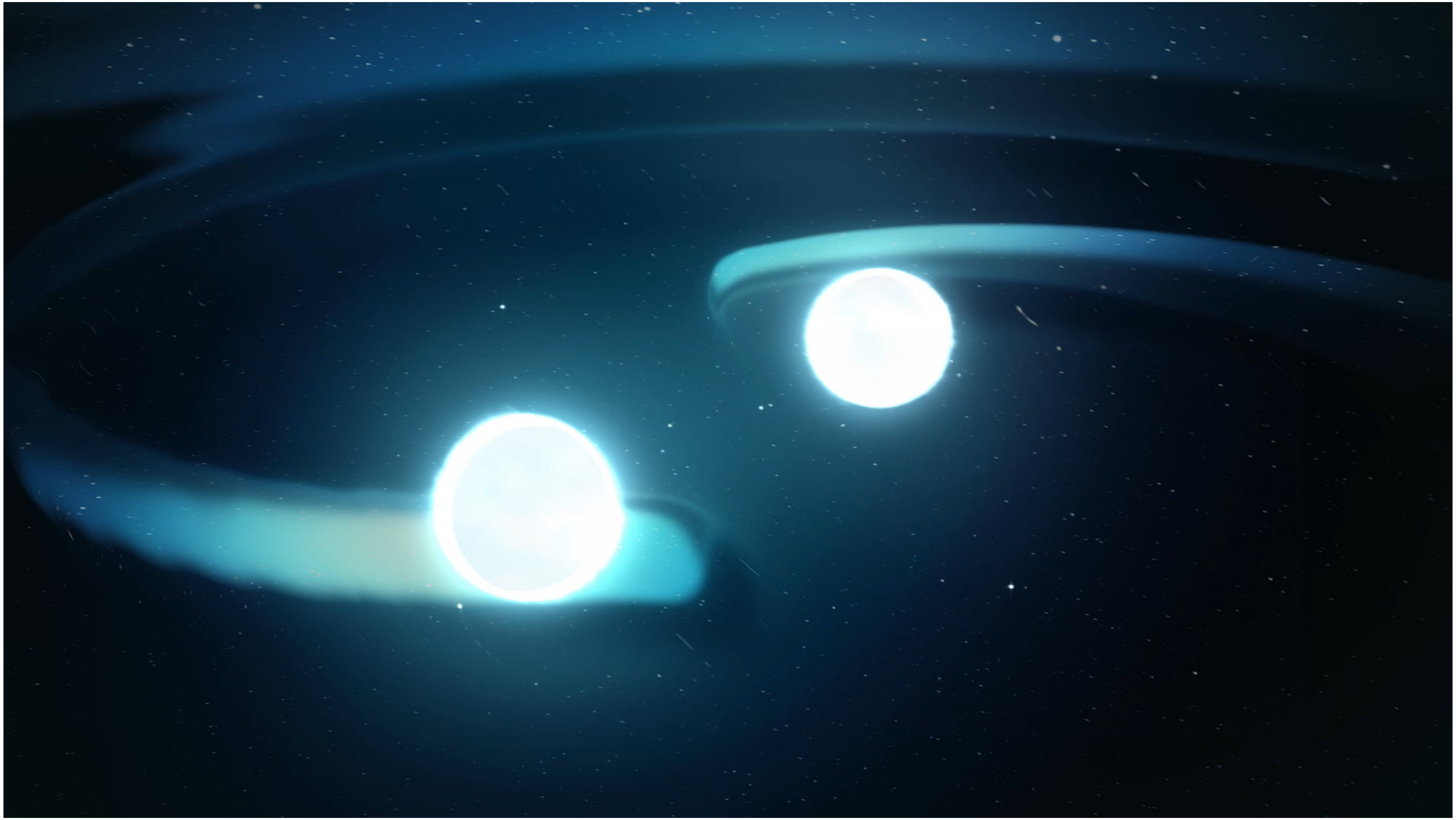


http://en.wikipedia.org/wiki/Image:Solar_sys8.jpg

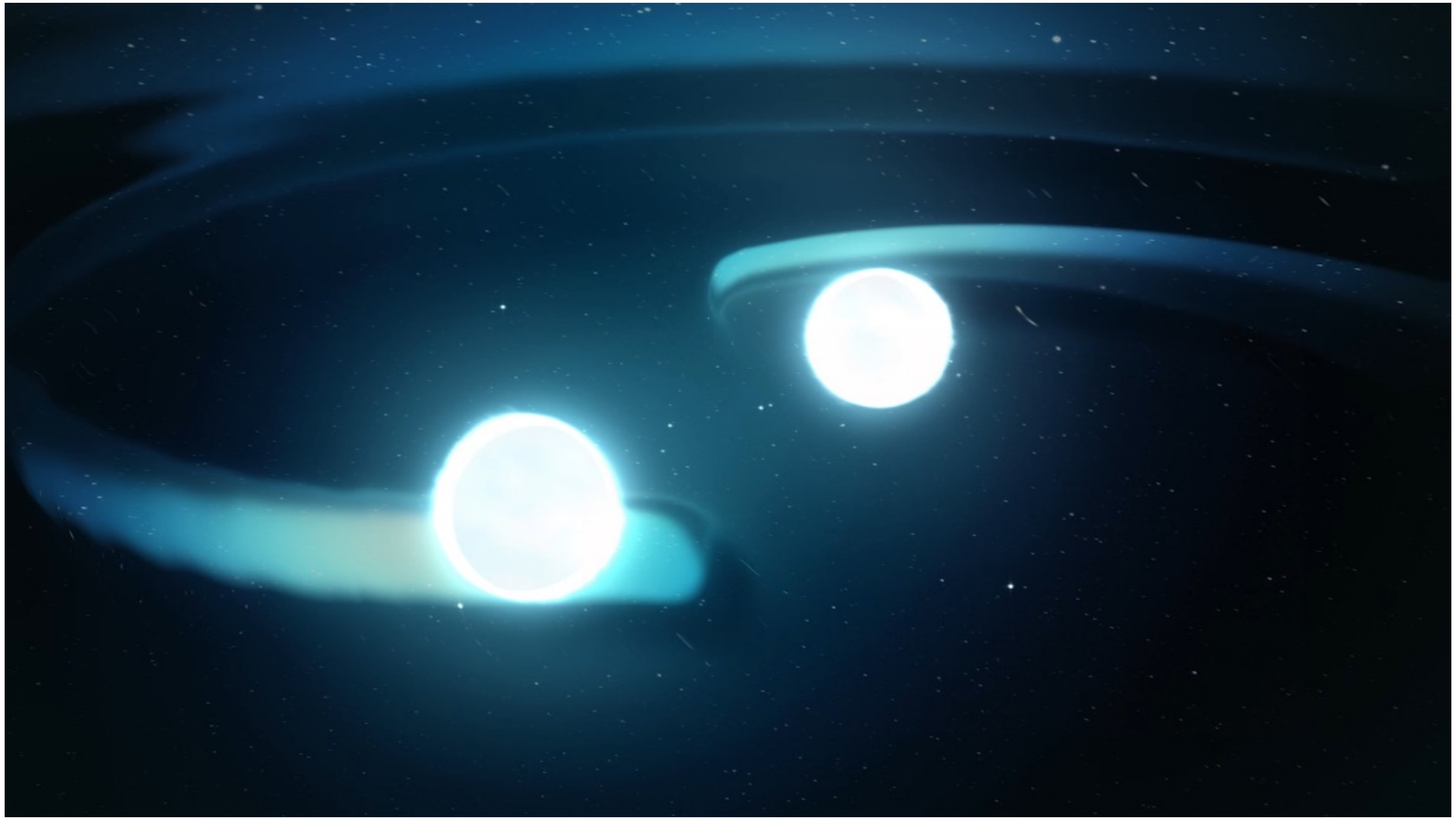
Mars, left, and the Milky Way are visible in the clear night sky as photographed near Salgotarjan, some 110 kms northeast of Budapest, Hungary, Aug. 03, 2018. (MTVA - Media Service Support and Asset Management Fund)



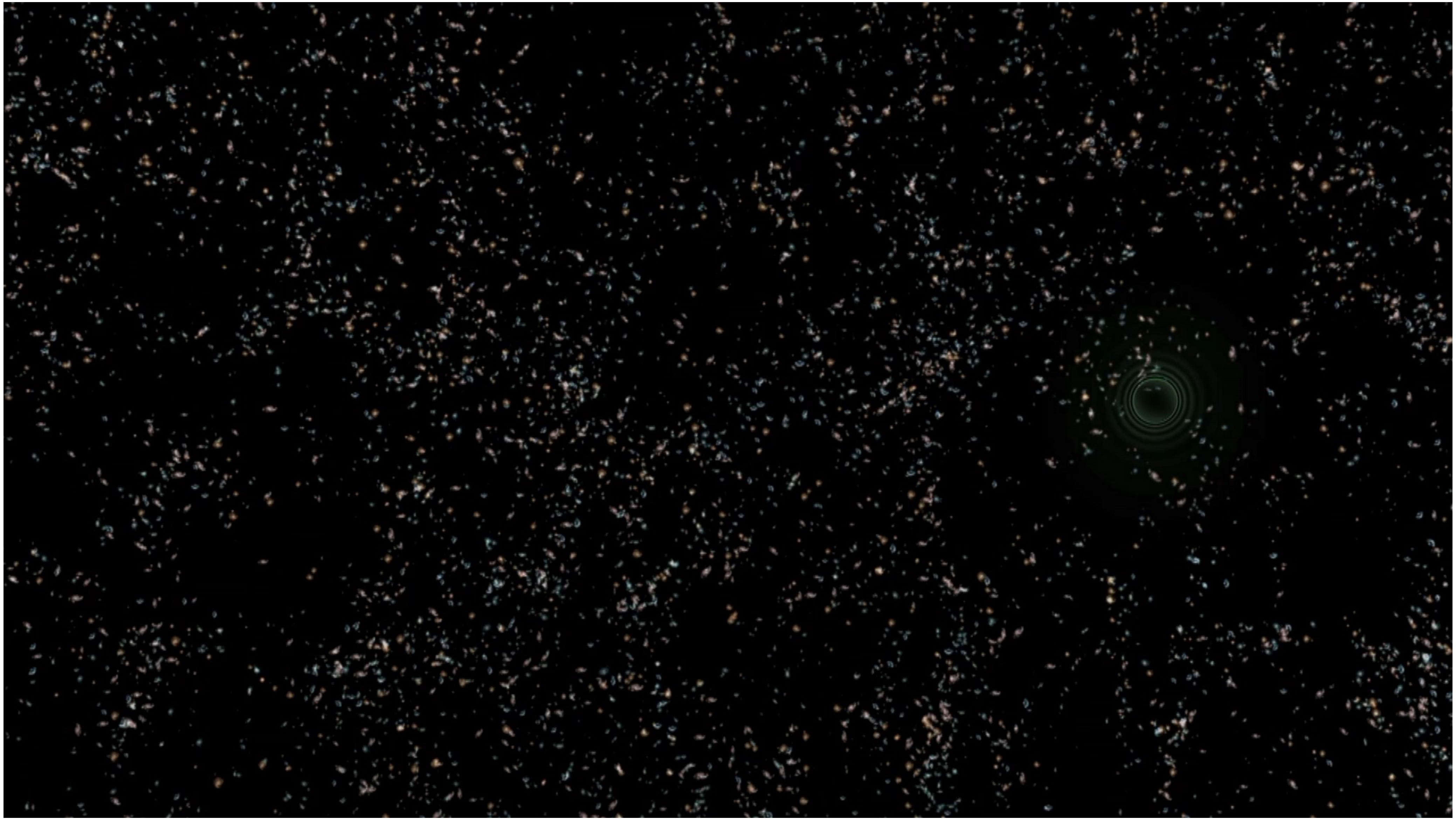
Credit courtesy: NASA/WMAP Science Team



중성자별 병합과 중력파-전자기파 발생(Credit: LIGO/SXS/R.Hurt and T. Pyle)



시공간 곡률의 파동



(Credit: LIGO/SXS/R.Hurt and T. Pyle)