2022 수치상대론 및 중력파 여름학교(2022/07/25~29)

일반상대론 기초

강궁원(중앙대)
문의: gwkang@cau.ac.kr

여름학교 주제

- 일반상대론
- 별의 진화와 중성자별/블랙홀

- **수치상대론**
  - 3+1 형식론/블랙홀 쌍성 모델링
  - 블랙홀 시뮬레이션(블랙홀 쌍성의 형성 및 진화)
  - 상대론적 유체역학과 불연속 갤러킨 기법

- **중력파**
  - 중력파 기초
  - 중력파 검출기 원리
  - 중력파 데이터 분석
  - CBC Search
  - 중력파 우주론
  - 중력파 천체물리학
  - 다파장천문학과 다중신호천문학

목 차

I. 서 론

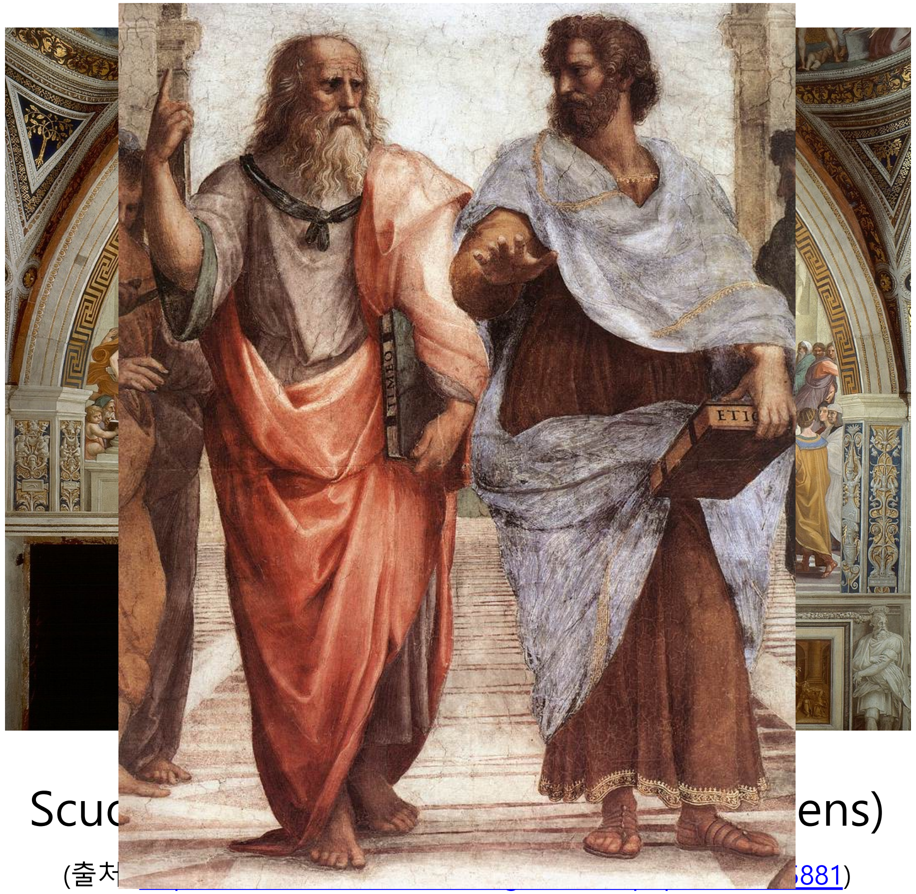
(사진 출처: http://www.rickety.us)

Why things fall?

✓ Aristotle (~B.C. 4th):

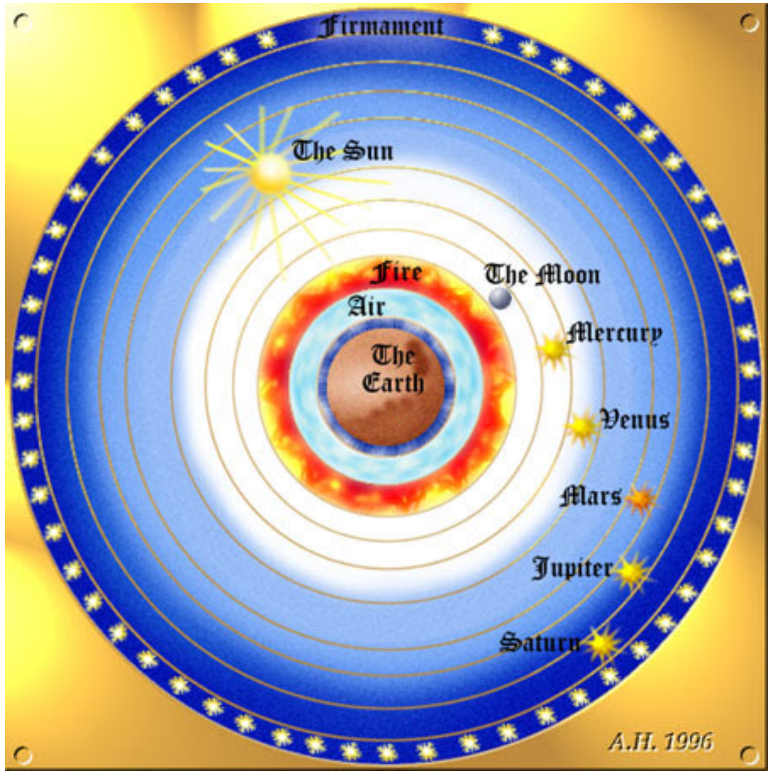
– 만물은 세상의 중심으로 움직이려는 성향 有

– 지상계 vs 천상계



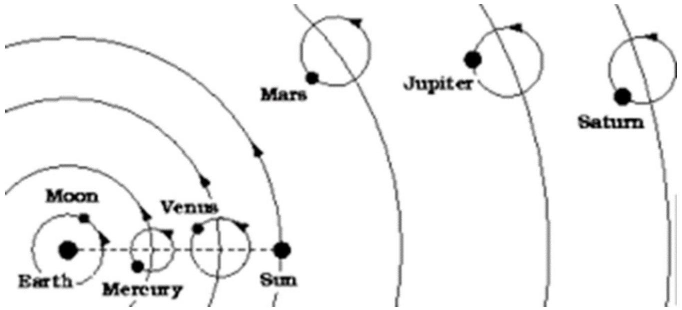
Scuc … ens)

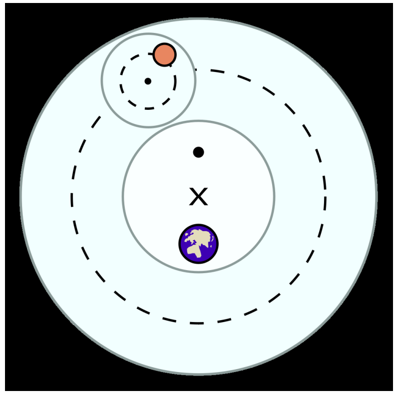
(출처 … 881)



• “프톨레마이오스” 체계(BC3th~16세기):

Almagest





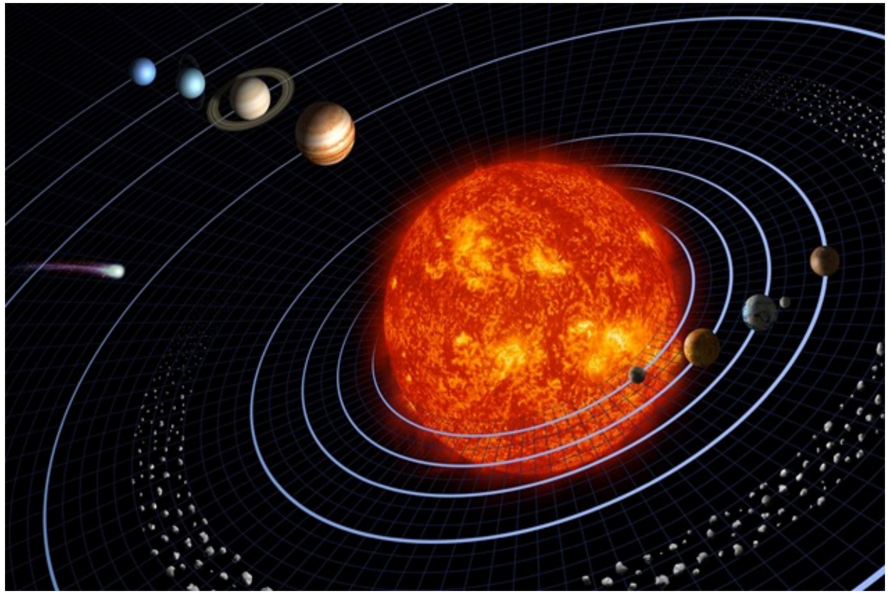
- 코페르니쿠스(1543): “천구의 혁명에 관하여”

(출처: 강궁원)

✓ 뉴튼(1687):

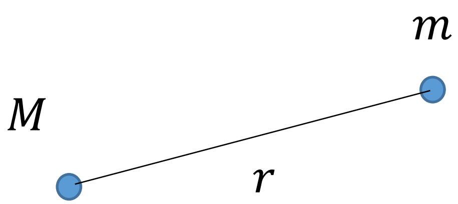
– 운동의 법칙

– 만유인력의 법칙



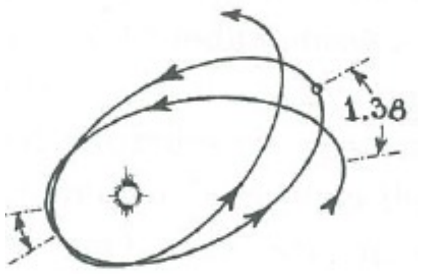
(Credit: Harman Smith and Laura Generosa)

(출처: Scott Berkun (2010))



- $\boldsymbol{F}_{Gravity} = G\frac{mM}{r^2}(-\hat{\boldsymbol{r}})$
- $\mathrm{m}\ddot{\mathbf{r}} = \mathbf{F}_{\mathrm{Sun}} + \mathbf{F}_{\mathrm{Jupiter}} + \mathbf{F}_{\mathrm{Earth}} + \mathbf{F}_{\mathrm{Saturn}} + \cdots$

- **뉴튼 중력 이론의 문제점:**
  - 수성의 세차운동: $\sim 1.38''$ per century
  - 특수상대성이론과 부합하지 않음
    - Action-at-distance ←→ 속도의 유한성
    - 상대성 원리 비만족

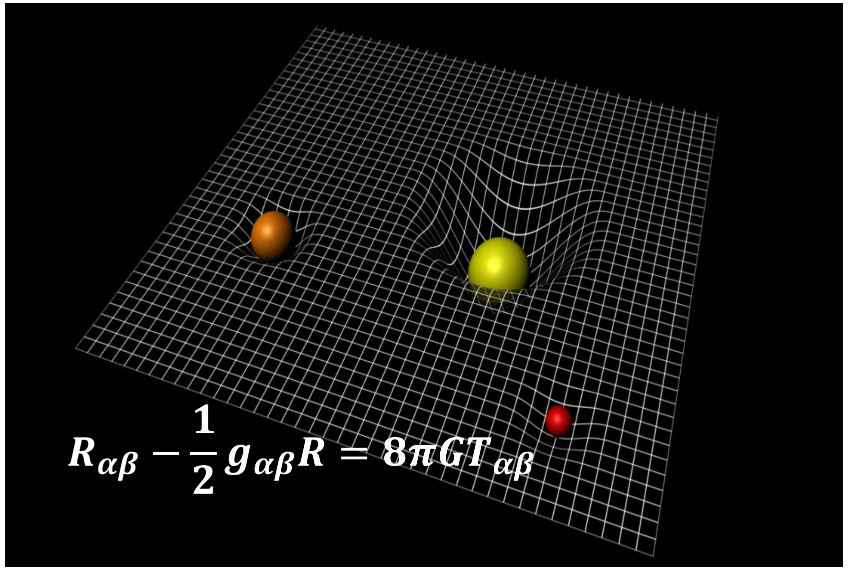


$\phi(t,\vec{x})$

$\vec{g}(t,\vec{x}) \equiv -\vec{\nabla}\phi$

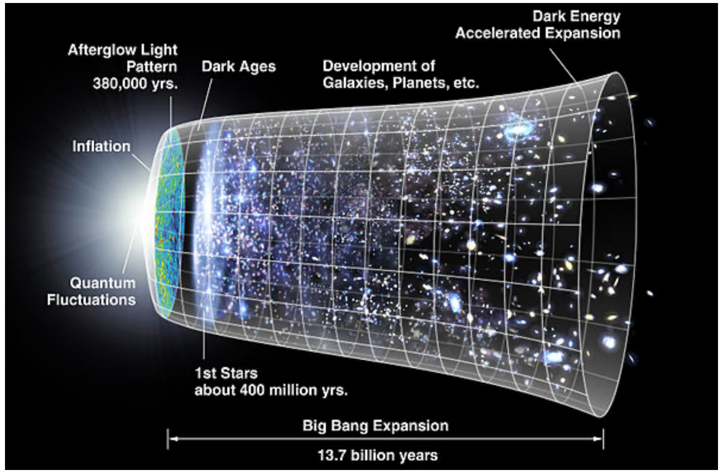
$$\text{“}\cdots\partial_t^2\text{”} + \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\phi(t,\vec{x}) = 4\pi G\rho(t,\vec{x})$$

✓ **아인슈타인(1915):** 일반상대론



(Credit: ESA–C.Carreau)

(Credit courtesy: NASA/WMAP Science Team)



➔ **“역동적인 시공간”**

II. 특수상대론

BOX 4.2 Railway Trains in Spacetime : Paris – Lyon (1885(?))

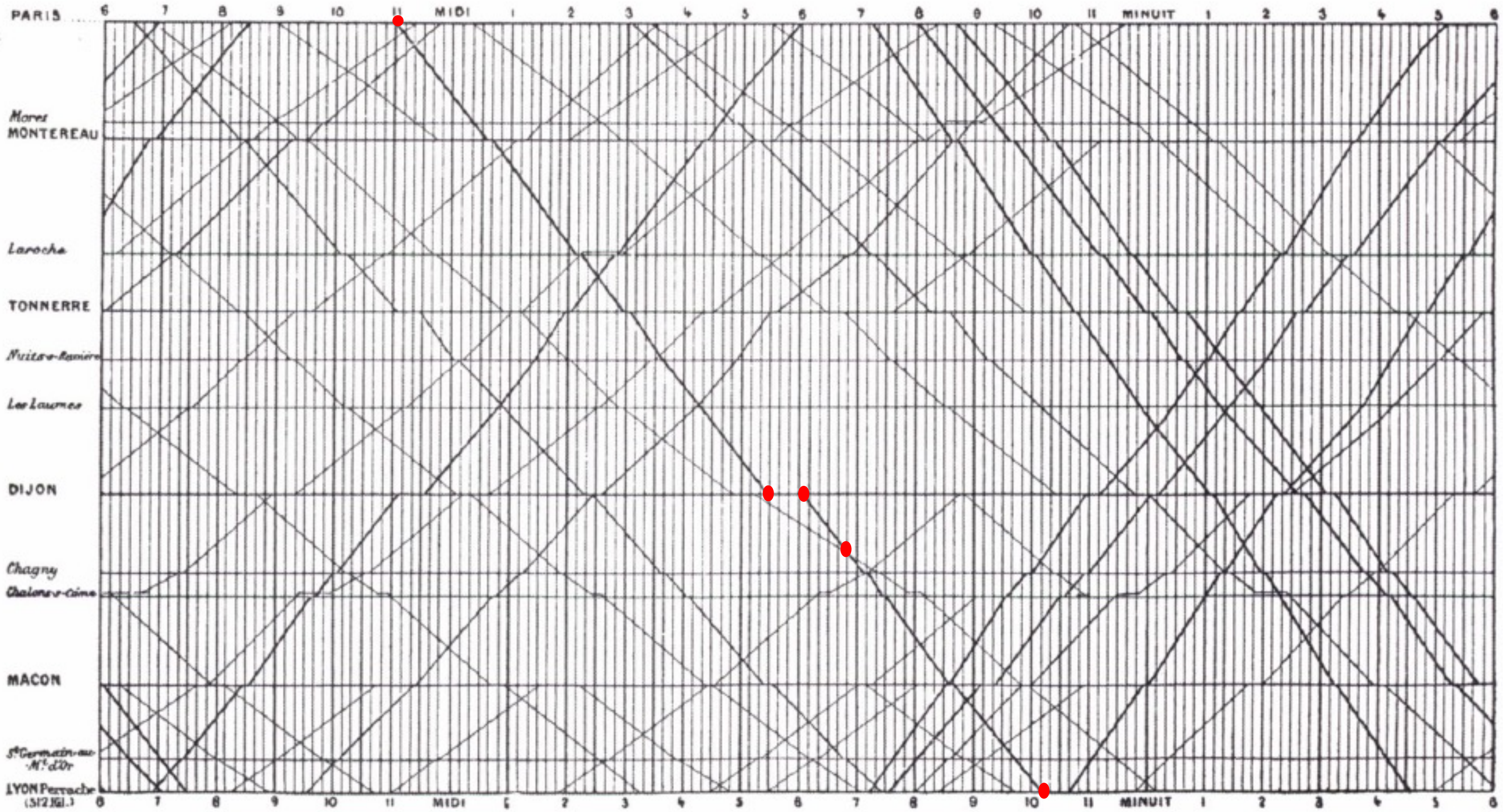
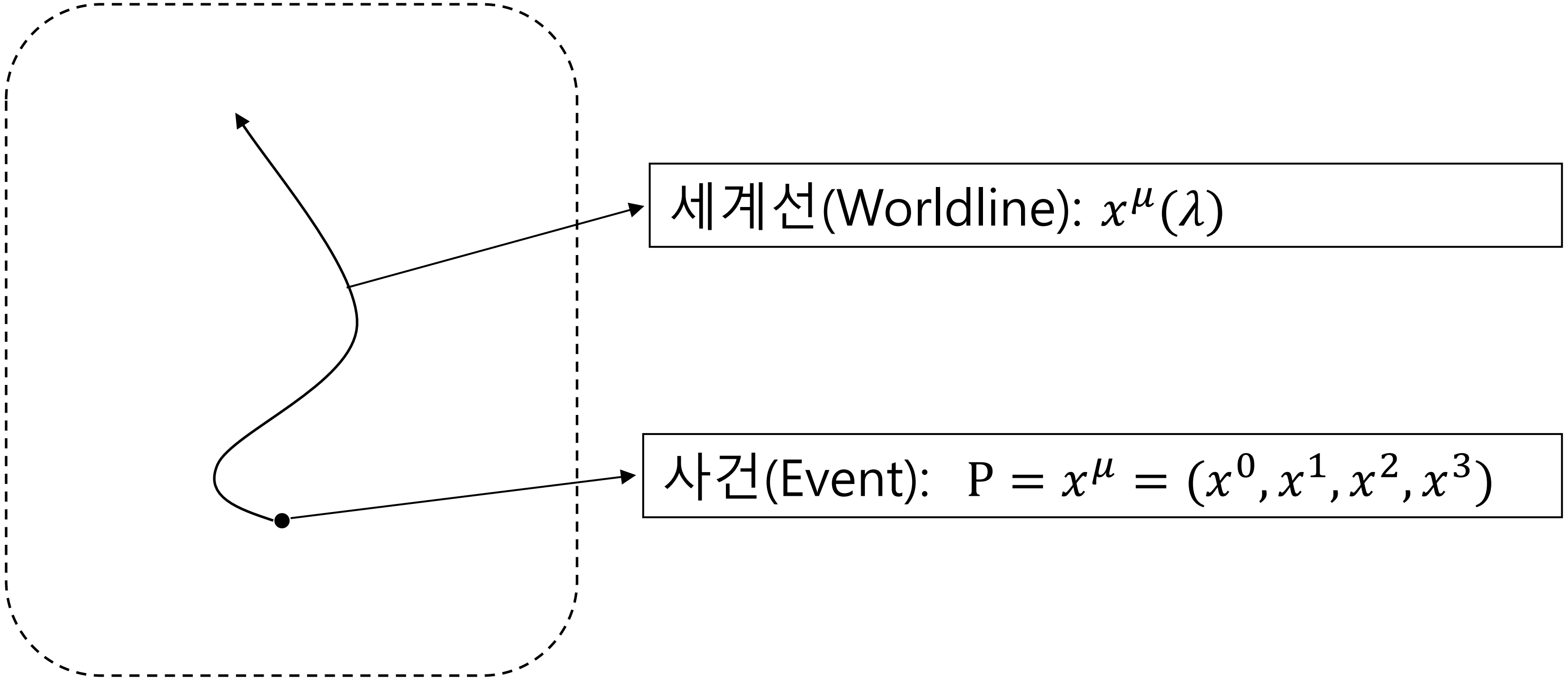


그림 출처: Hartle

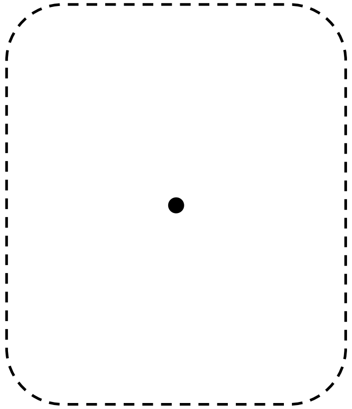
✓ 시공간 다양체(Spacetime manifold) $\mathcal{M}$: 사건의 4차원 연속체

$$\mathcal{M} = \{\text{Event} \mid x^\mu = (x^0, x^1, x^2, x^3) \in R^4\}$$

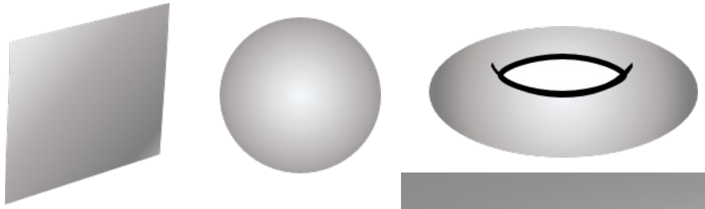


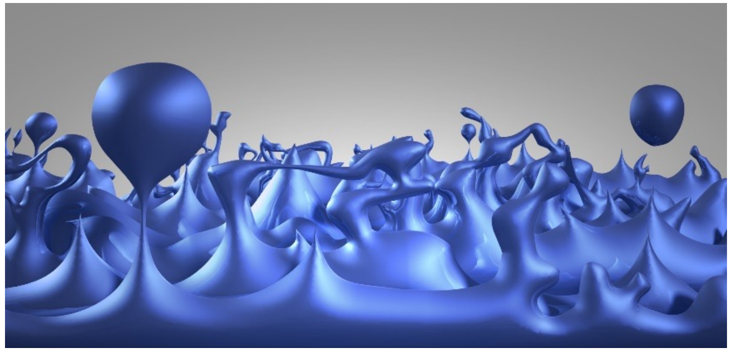
- Topological structure:

  ➔ $\equiv R^4$



https://www.maths.ox.ac.uk/node/36782

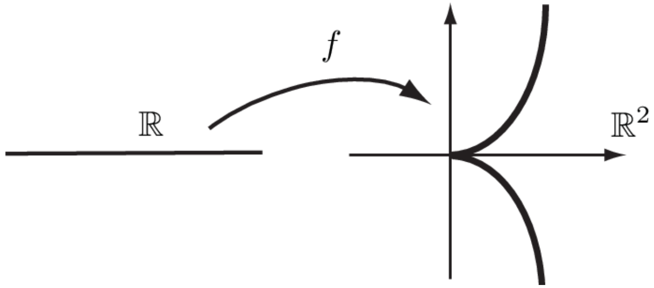




https://chandra.harvard.edu/blog/node/558

- Differentiable structure:

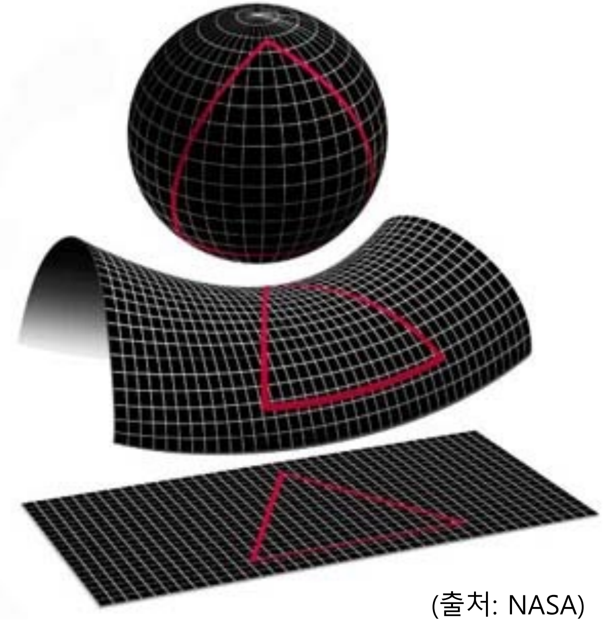
  ➔ 미분 가능



https://www.goodhumor.com/us/en

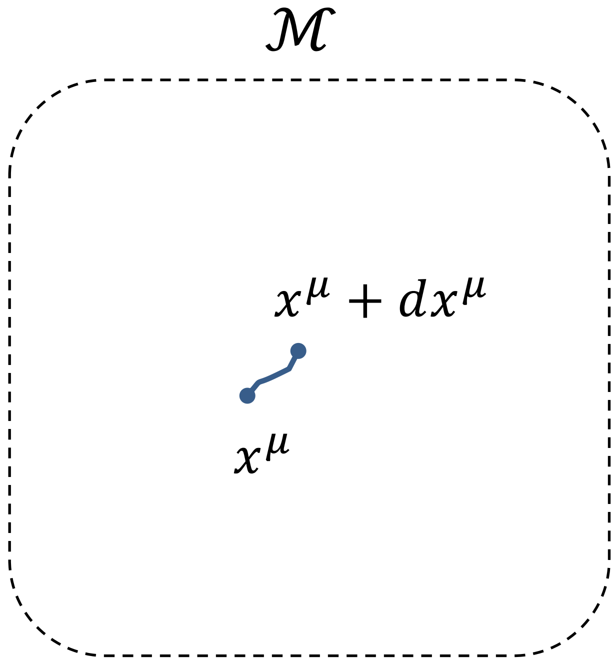
- **Metric structure:**

  ➔ Absolute (Newtonian), Special relativistic (Minkowskian), or Curved



(출처: NASA)

✓ Absolute (Newtonian) spacetime: Newton (1687 in Principia)



- Event in a Cartesian coordinate system:

  $x^{\mu} = (x^0, x^1, x^2, x^3) = (t, x, y, z)$

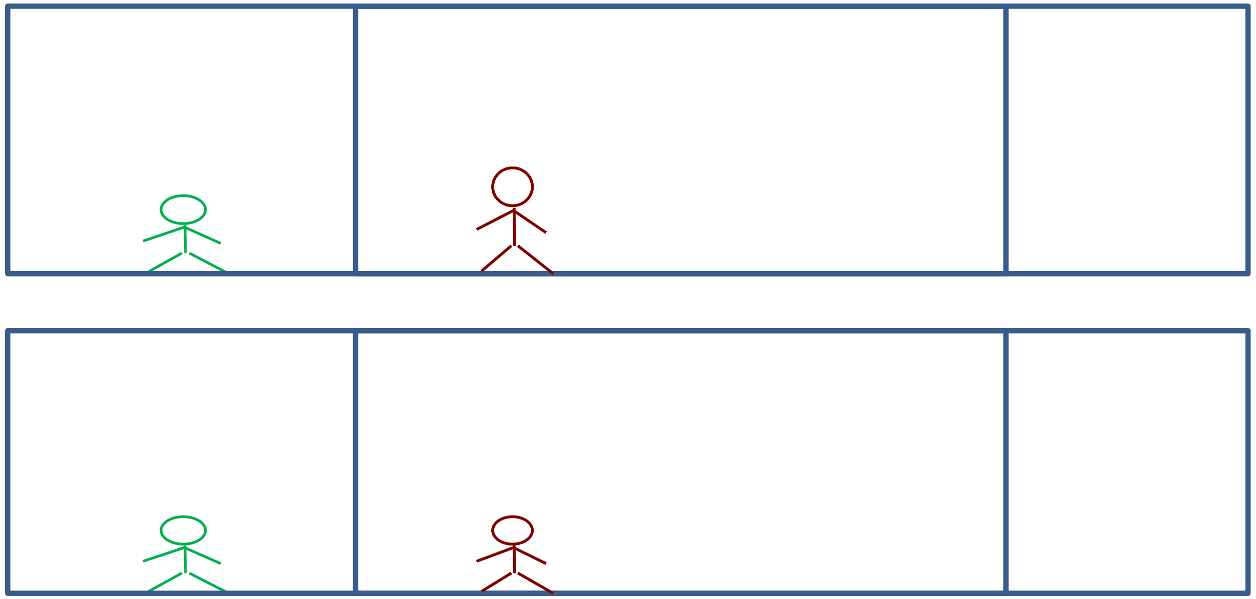
- Metric structure: 인접한 두 사건간의 거리

  For infinitesimally neighboring two events $\mathrm{P} = x^{\mu} = (t, x, y, z)$

  and $\mathrm{Q} = x^{\mu} + dx^{\mu} = (t + dt, x + dx, y + dy, z + dz)$

  dt^2 & $\mathrm{d}l^2 = dx^2 + \mathrm{d}y^2 + \mathrm{d}z^2$: 불변

- **관성계** ≡ 절대 공간에 대해 “정지”해 있는 관측계
  - 관성계는 유일한가?

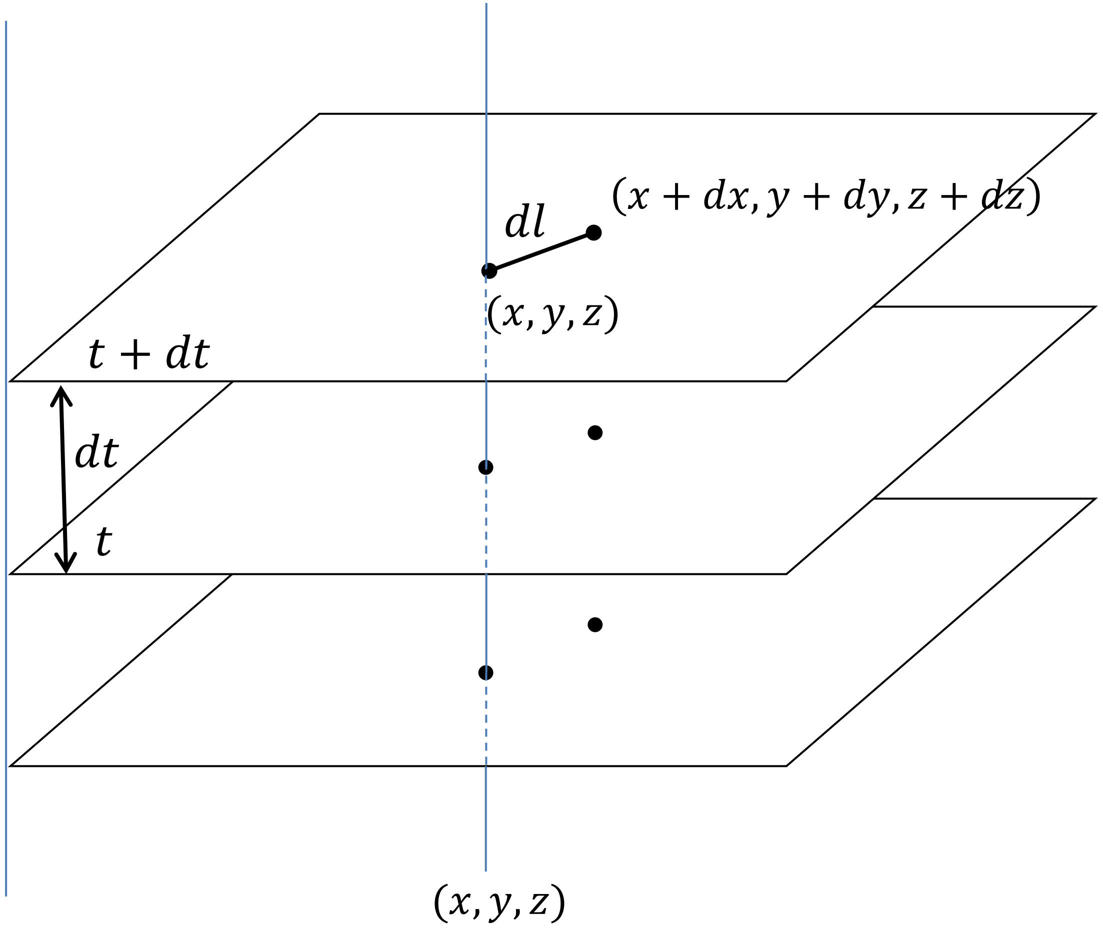


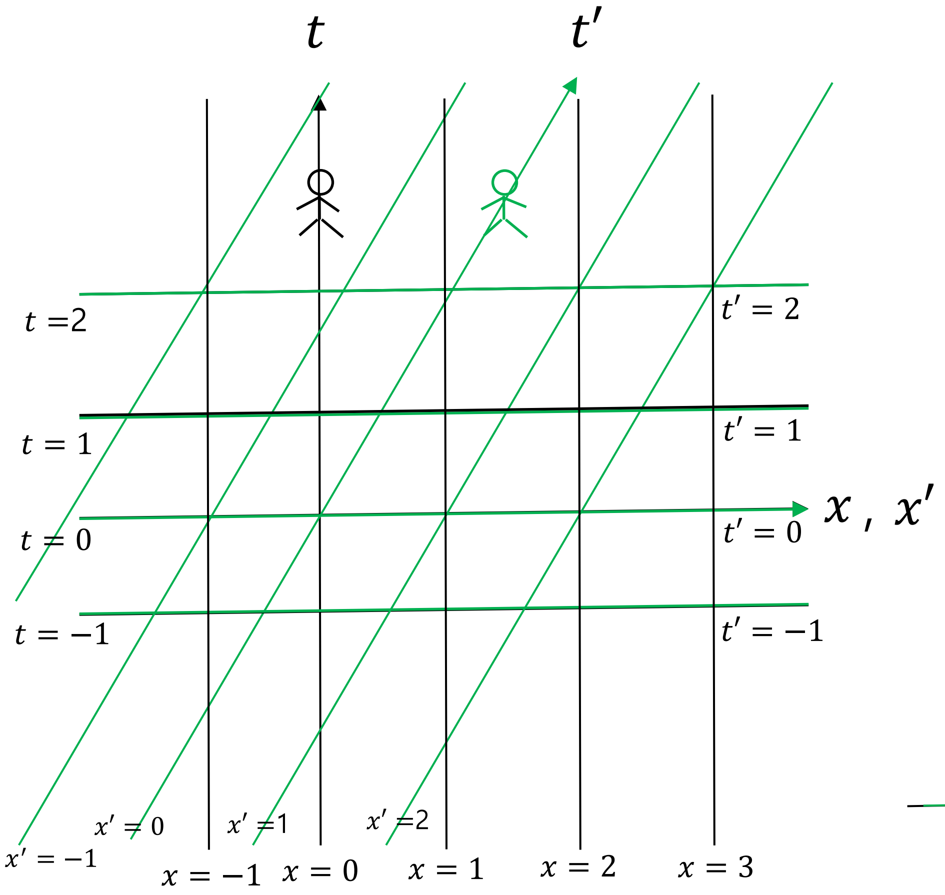
- 유일하지 않으며 등속 운동계는 모두 관성계: 갈릴레오 변환,

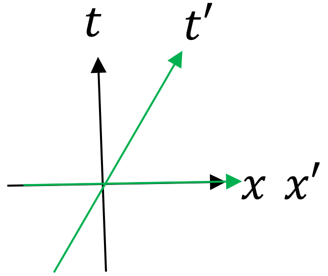
➔ $t = t', \ x = x' - vt', \ y = y', \ z = z'$

- 관성계는 구별 불가능하며, 물리법칙은 모든 관성계에서 동일한 형태

➔ “(갈릴레오의) 상대성 원리”

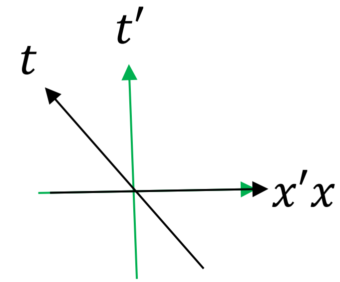






- $dt = dt'$ & $dl^2 = dx^2 + dy^2 + dz^2 = dx'^2 + dy'^2 + dz'^2 = dl'^2$ SEPARATELY

  ➔ Absolute space and time!

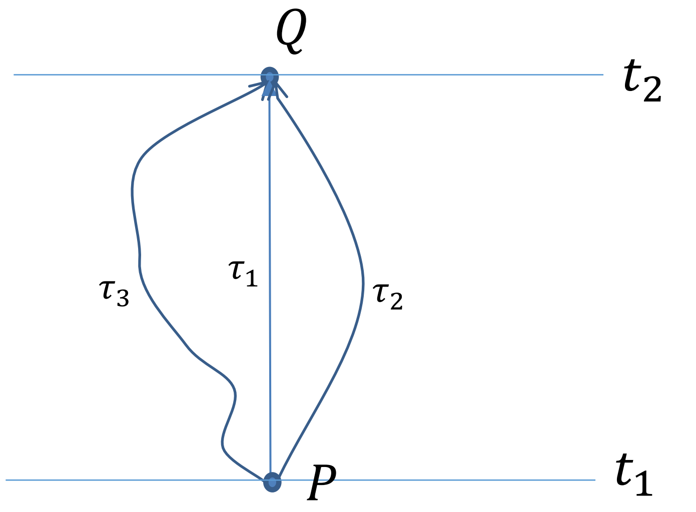


- Flat metric both in space and time: 휘어져 있지 않음, *i.e.*, “곡률 (Curvature)” = 0

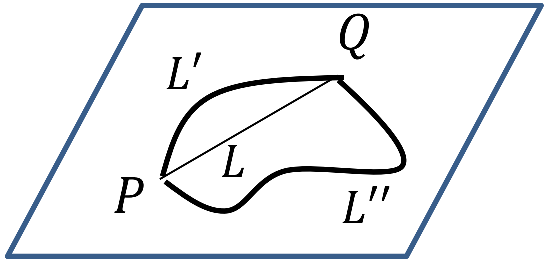
  ➔ Flat Euclidean space and time

• 절대 시공간의 성질:

- 무한개의 관성계: Galilean 변환 $t' = t,\ x' = x - vt,\ y' = y,\ z' = z$
- Galilean 속도의 합
- 시간 간격: 경로, 관측자에 무관,
- 공간 간격(길이): 경로에 의존하나 관측자에 무관
- 동시성
- 인과관계
- Three-velocity
- ......
- 상대성 원리



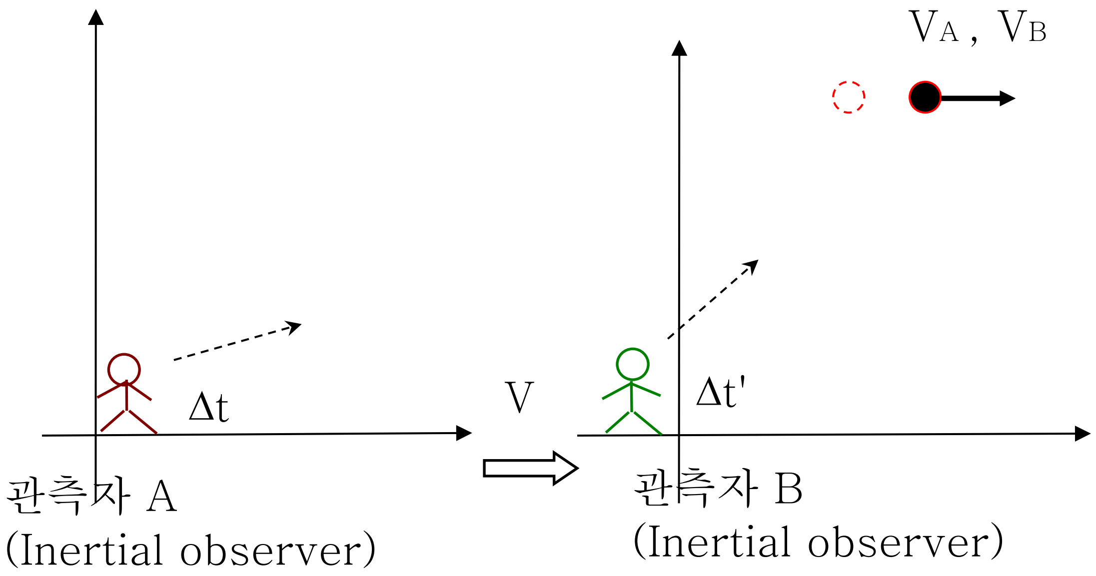
시간 간격: $\tau_A = t_2 - t_1 = \tau_B = \tau_C$



공간 거리: $L < L' < L''$

• 갈릴레오 변환과 속도의 합

$V_A \sim V_B$? ➡ $V_A = V_B + V$



$$t' = t$$
$$X' = X - Vt$$
$$Y' = Y$$

$$V_A = \frac{\Delta X}{\Delta t} = \frac{\Delta X' + V\Delta t}{\Delta t} = \frac{\Delta X'}{\Delta t} + V = \frac{\Delta X'}{\Delta t'} + V$$

$\Delta t = \Delta t'$

Ex) 지면에서 본 야구공 속도는?

= 기차에서 본 야구공 속도 + 기차의 속도

= 150 km/h + 300 km/h

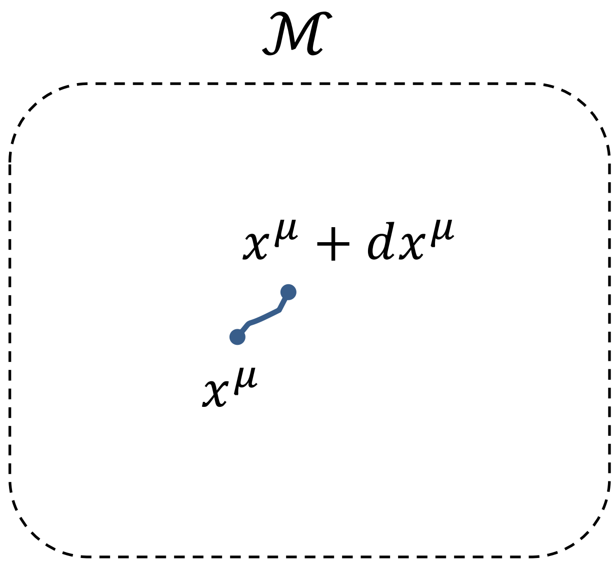
= 450 km/h

✓ Why is it so?

✓ What are the assumptions or conditions for “time” and “space” behind?

✓ V≠0 이면 VA ≠ VB ➔ 운동 상태가 다른 두 관측자에게 동일한 속도를 갖는 것 불가

✓ Special relativistic spacetime:



$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$
$$= \eta_{00}(dx^0)^2 + \eta_{01} dx^0 dx^1 + \cdots$$

Minkowski metric:

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Cartesian　　　　　　　　Polar

- Metric structure:

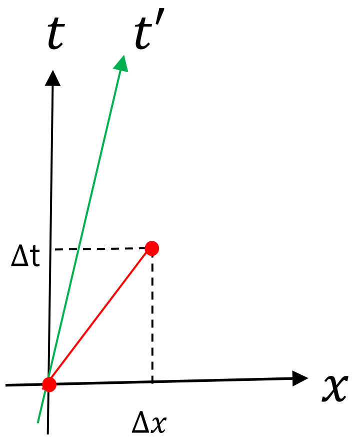
$$ds^2 = -d(\alpha t)^2 + dx^2 + dy^2 + dz^2$$

- i.e, 밍코프스키 (Minkowski) spacetime
- Flat metric: 휘어져 있지 않음, i.e., “곡률 (Curvature)” = 0
- By definition, ds^2 is independent of observers
- Feature: Invariant light speed, relative simultaneity, time dilation, length contraction, maximum speed, Lorentz transformation, relativistic velocity addition, etc.
- Lorentz trans.: $t' = \frac{1}{\sqrt{1-(v/\alpha)^2}}\left(t - \frac{v}{\alpha} x/\alpha\right)$

  $x' = \frac{1}{\sqrt{1-(v/\alpha)^2}}(x - vt)$, $y' = y$, $z' = z$
- Recovering the Newtonian ST: $\alpha \to \infty$

- 관측자에 무관한 속도:

- 임의의 관성계: $ds^2 = -d(\alpha t)^2 + dx^2 + dy^2 + dz^2 = ds'^2 = ? = -d(\alpha t')^2 + dx'^2 + dy'^2 + dz'^2$
- "특별한" 경로: $ds^2 = 0 \rightarrow -\left(d(\alpha t)\right)^2 + dx^2 = 0 = -(d(\alpha t'))^2 + dx'^2$



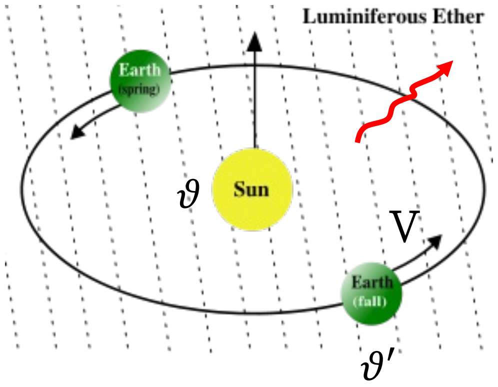
$$c_\vartheta = \frac{dx}{dt} = \frac{\alpha dt}{dt} = \alpha$$

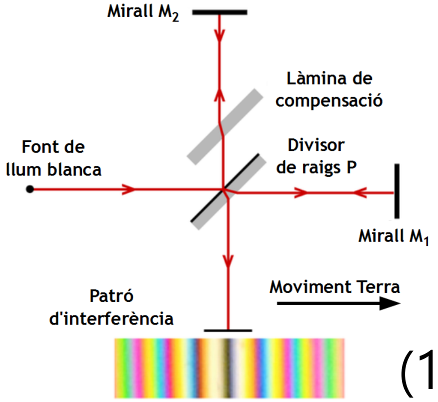
$$c_{\vartheta'} = \frac{dx'}{dt'} = \frac{\alpha dt'}{dt'} = \alpha = c_\vartheta$$: 모든 관측자에 대해 동일!

Michelson-Morley: 빛 ➔ 광속 불변 ➔ $\alpha = c = 3\times10^5$ km/s

관측자 ϑ: $(\Delta t, \Delta x, 0{,}0)$

관측자 ϑ': $(\Delta t', \Delta x', 0{,}0)$





(1887)

- 동시 사건:

- 두 사건: $\mathrm{P}_1 = (t_1, x_1)$ & $P_2 = (t_2, x_2)$

  $dt = t_2 - t_1 = 0$ $\rightarrow$ "동시에 발생한 사건"

- $\mathrm{P}_1 = (t'_1, x'_1)$ & $P_2 = (t'_2, x'_2)$

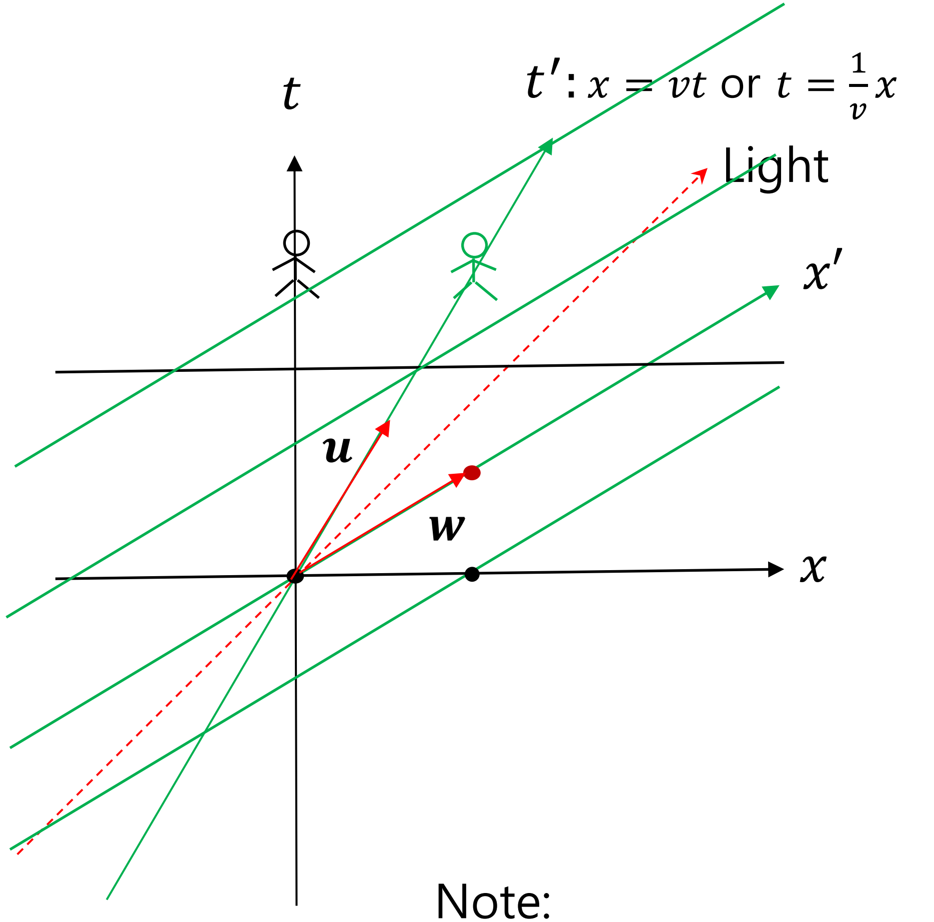
  $dt' = t'_2 - t'_1 = 0$?

- Note: $ds^2 = -dt^2 + dx^2 = dx^2 = -dt'^2 + dx'^2$

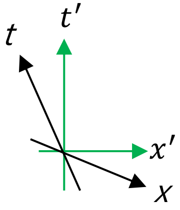
  ➔ Not necessarily $dt' = 0$ $(t'_1 = t'_2)$ even if $dt = 0$.

  ➔ $dt'^2 = dt^2 - dx^2 + dx'^2 = dx'^2 - dx^2 \neq 0$ in general!

  ➔ **$t'_1 \neq t'_2$ for other observers!**



Note:



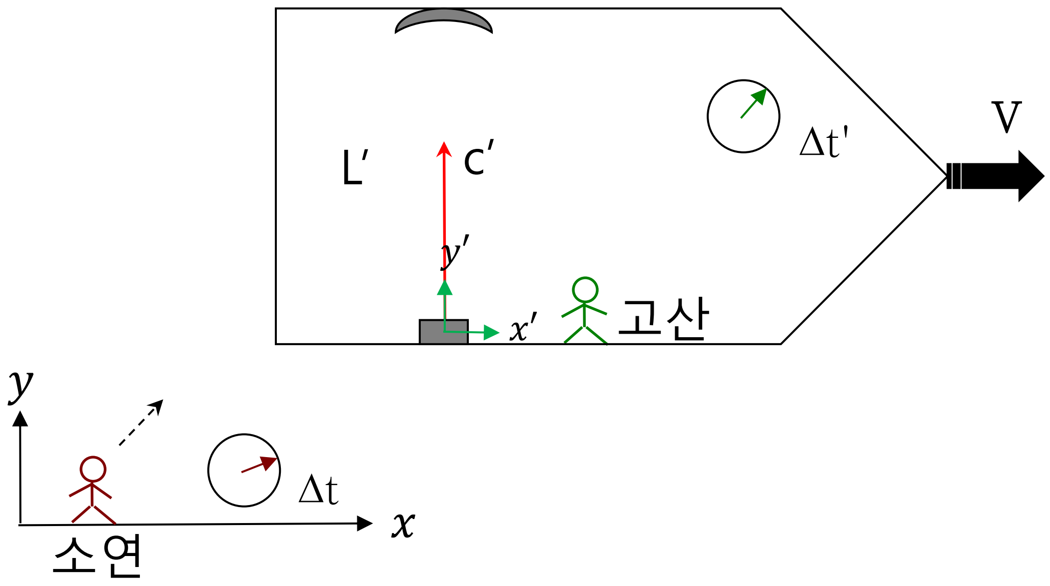
$\boldsymbol{u} = (u^t, u^x) = u^t(1/v, 1)$

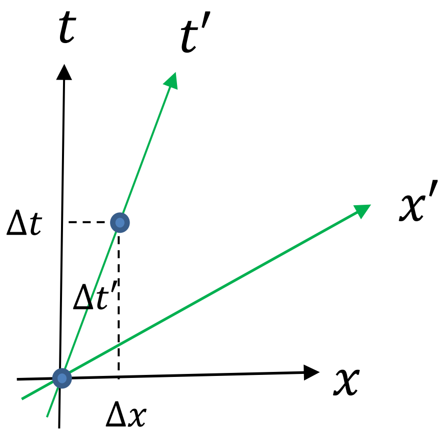
$\boldsymbol{u} \cdot \boldsymbol{w} = 0 \sim -\frac{1}{v} w^t + w^x$

$\rightarrow w^t = v w^x$

➔ x' 축: $t = vx \neq 0$!!

- 시간 지연:





$\left(\frac{\Delta x}{\Delta t} = V \right)$

$$\Delta s^2 = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

$$\Delta s'^2 = -(c\Delta t')^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

두 사건: 빛의 방출과 도착

- 고산: "동일 위치" $\Delta x' = 0 = \Delta y' = \Delta z'$
- 소연: "다른 위치" $\Delta x \neq 0 = \Delta y = \Delta z$

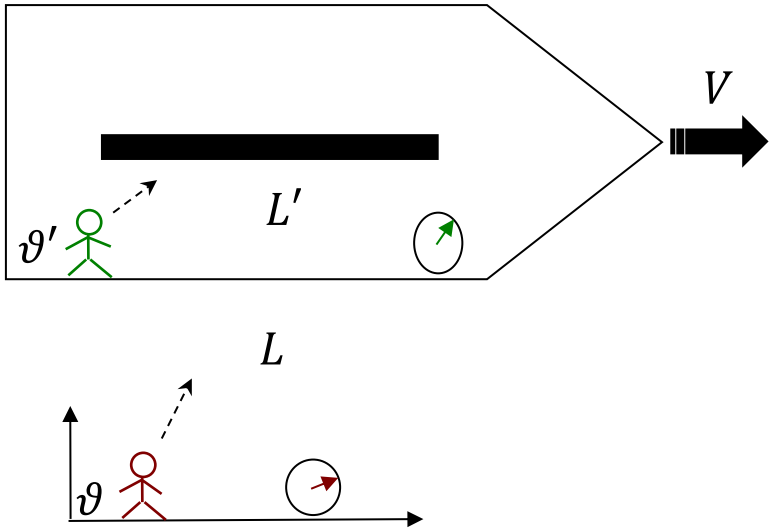
$$\Delta s^2 = \Delta s'^2 \Rightarrow -(c\Delta t)^2 + \Delta x^2 = -(c\Delta t')^2 + 0^2$$

$$(c\Delta t)^2 [1 - (\frac{\Delta x}{c\Delta t})^2] = (c\Delta t)^2 [1 - (\frac{V}{c})^2] = (c\Delta t')^2$$

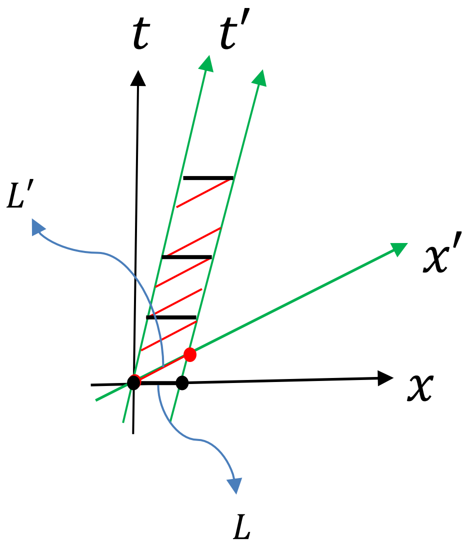
$$\Rightarrow \boxed{\Delta \mathrm{t} = \frac{1}{\sqrt{1 - (\mathrm{V}/\mathrm{c})^2}} \Delta \mathrm{t}'}$$

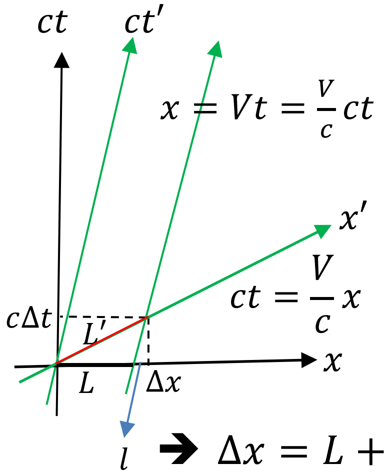
(※ 역으로 시간 지연으로부터 위 4차원적 거리가 불변임을 보일 수 있음)

- 길이 수축: ➔ 숙제(Deadline: 7/26 9:00)



- **길이란?**: 동시에 발생한 두 사건의 거리 ➔
  $L' = \Delta x' = x'_2 - x'_1$ w/ $\Delta t' = 0$ (막대 양 끝)
- 마찬가지로 '정지' 관측자(ϑ)도 움직이는 막대의 양 끝을 동시에 측정하여 길이 구함: L
- Note: ϑ' 의 두 사건은 ϑ 에게는 동시에 발생한 두 사건이 아님





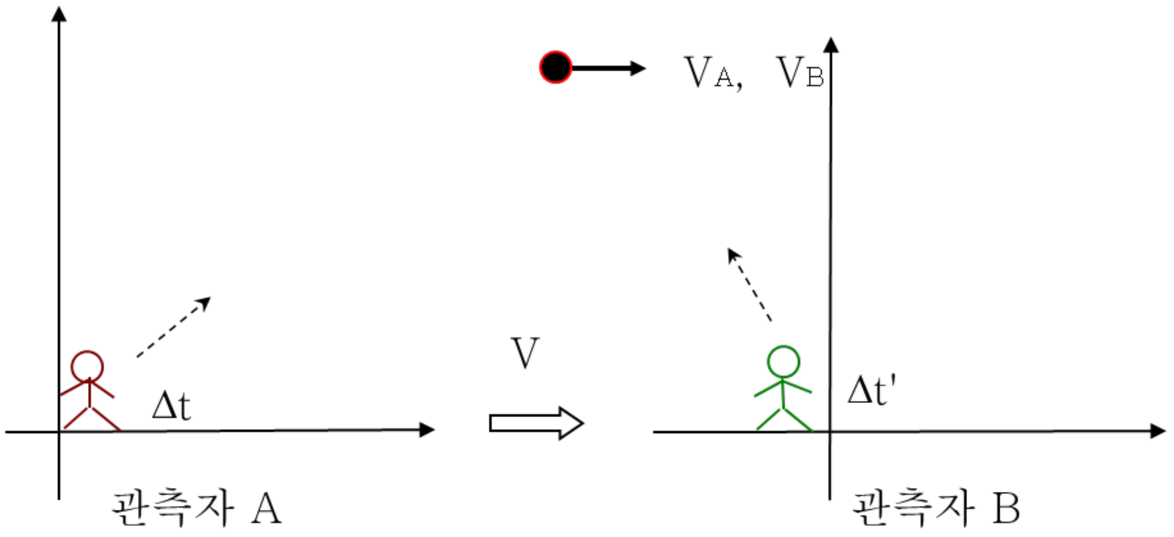
➔ $\Delta x = L + l$

$$\Delta s^2 = -(c\Delta t)^2 + \Delta x^2 = -0^2 + L'^2$$

$$L = \sqrt{1 - \left(\frac{V}{c}\right)^2}\, L'$$

임을 보여라.

- 속도 합성:



$$\Delta t = \frac{1}{\sqrt{1-(v/c)^2}}\left(\Delta t' + \frac{v}{c}\frac{\Delta x'}{c}\right)$$

$$\Delta x = \frac{1}{\sqrt{1-(v/c)^2}}\left(\Delta x' + v\,\Delta t'\right)$$

$$\Delta y = \Delta y' = 0$$

$$\Delta z = \Delta z' = 0$$

$$v_A = \frac{\Delta x}{\Delta t} = \frac{\frac{1}{\sqrt{1-(-v/c)^2}}(\Delta x' + v\,\Delta t')}{\frac{1}{\sqrt{1-(-v/c)^2}}(\Delta t' + \frac{v\Delta x'}{c\ \ c})} = \frac{\Delta x' + v\,\Delta t'}{\Delta t' + \frac{v}{c}\frac{\Delta x'}{c}} = \frac{\frac{\Delta x'}{\Delta t'} + v}{1 + \frac{v}{c}\frac{\Delta x'/\Delta t'}{c}} = \frac{v_B + v}{1 + \frac{v}{c}\frac{v_B}{c}}$$

$$\boldsymbol{V_A = \frac{V_B + V}{1 + V\,V_B/c^2}}$$

Note:

i) $V_A < c$ for $V_B, V < c$

ii) $V_A = c$ for $V_B = c$ (or $V = c$)

It never exceed the speed of light!

*

- 차고 패러독스:

- 정지 상태의 자동차와 차고:

$$L_{Car} > L_{Garage}$$

- 움직이는 자동차와 차고:

$$L'_{Car} < L_{Garage}$$

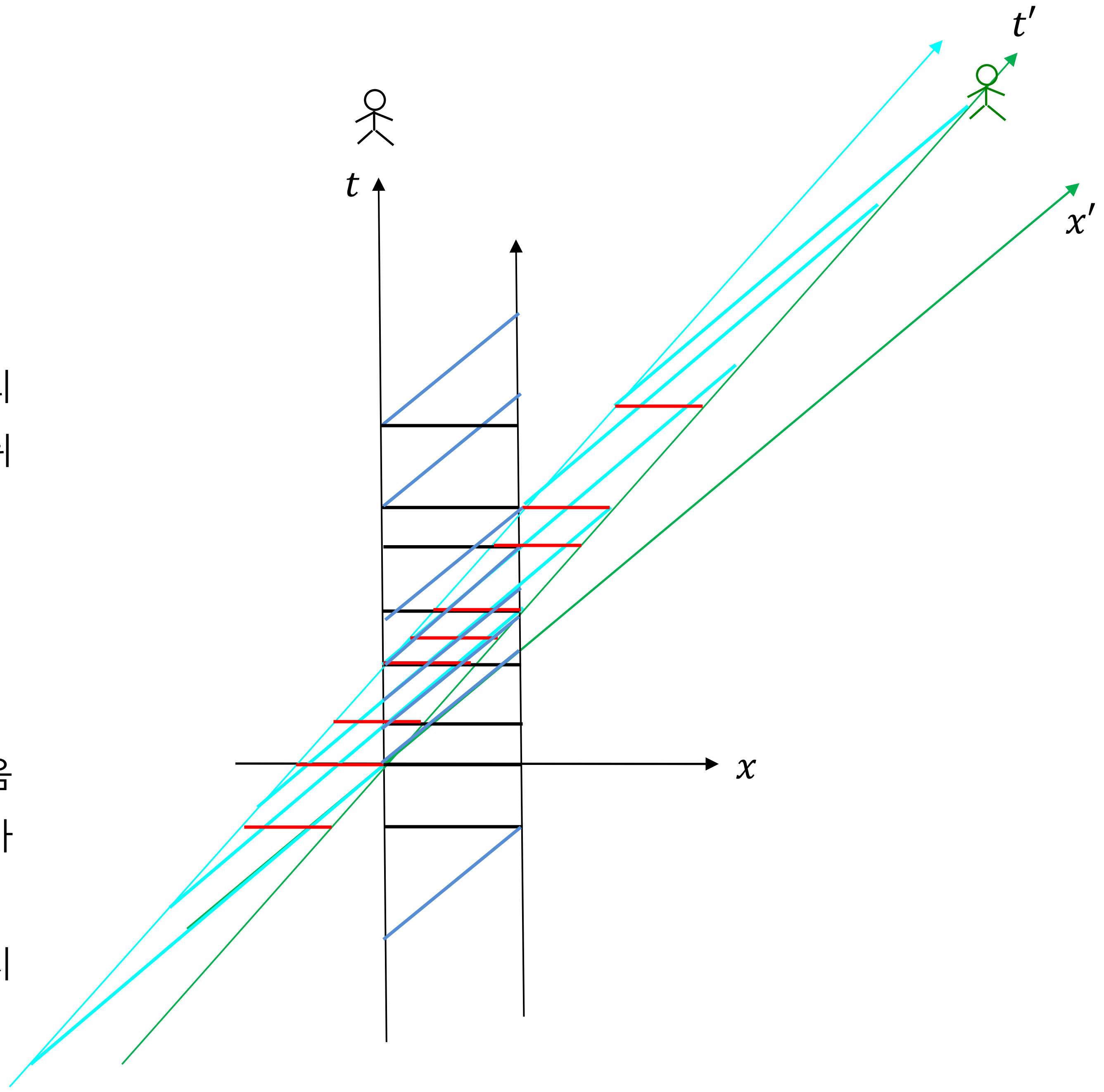
차고에 서있는 관측자가 보면 충분히 빨리 달리는 자동차는 길이 수축에 의해 차고 안에 앞뒤 모두 들어갈 수 있음

- 운전자가 본 자동차와 차고:

$$L_{Car} > L'_{Garage}$$

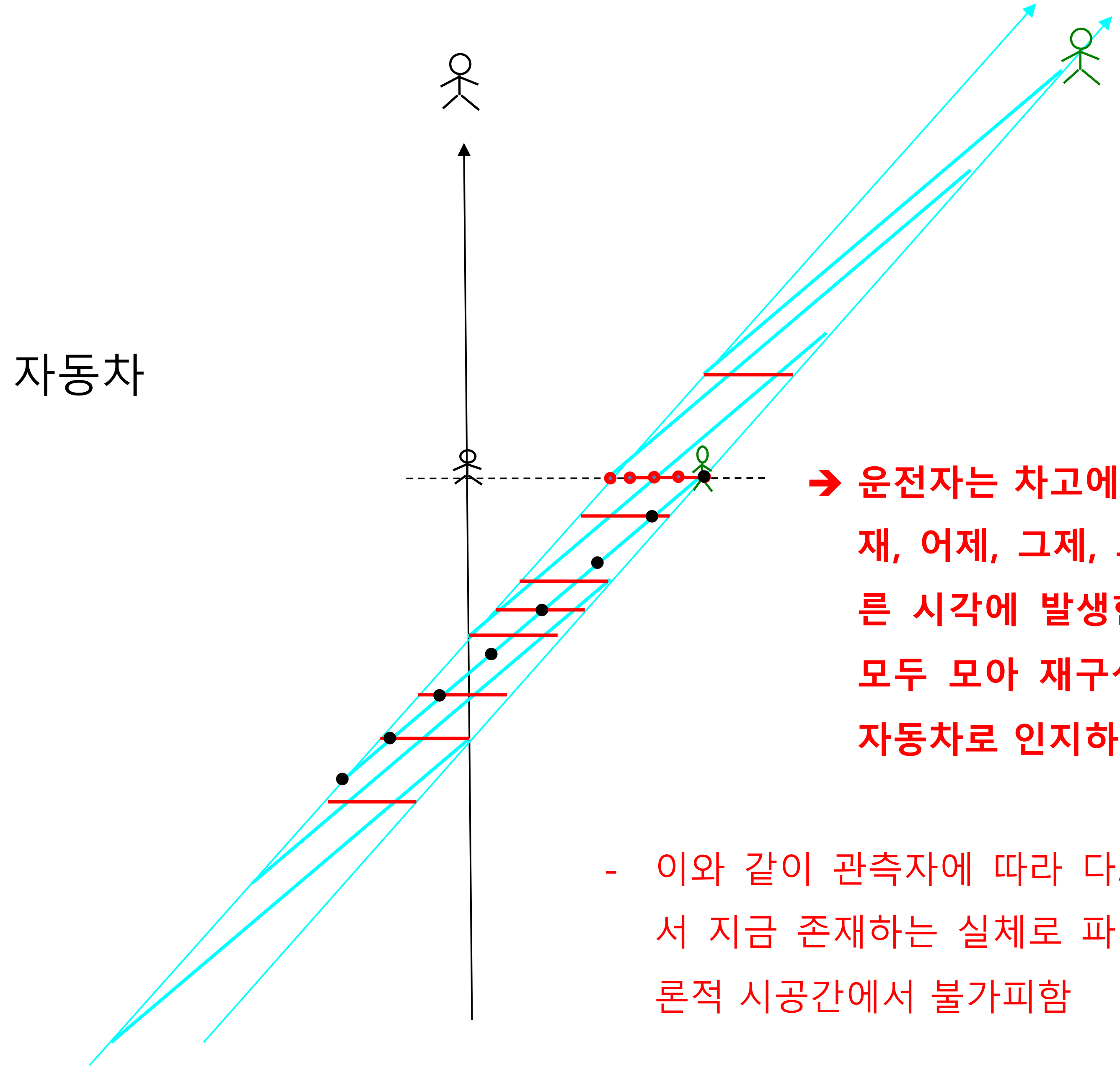
한편 달리는 운전자의 입장에서 보면 차고가 움직이는 것이므로 차고는 더욱 줄어들어 자동차가 차고 안에 완전히 들어가는 것은 불가능

➔ 어떻게 차고에 완전히 갇히는 상황과 갇히지 못하는 상황이 둘 다 가능할 수 있겠는가?



*

- # 그러면 실체란 무엇인가?



➔ **운전자는 차고에 서 있는 관측자가 본 현재, 어제, 그제, 그끄저께, … 등 각자 다른 시각에 발생한 "자동차의 사건들"을 모두 모아 재구성해서 현재라는 시각의 자동차로 인지하고 있음**

- 이와 같이 관측자에 따라 다르게 사건들을 재구성해서 지금 존재하는 실체로 파악하는 방식은 특수상대론적 시공간에서 불가피함

*

- 인과 관계:

- **절대(뉴튼) 시공간:**

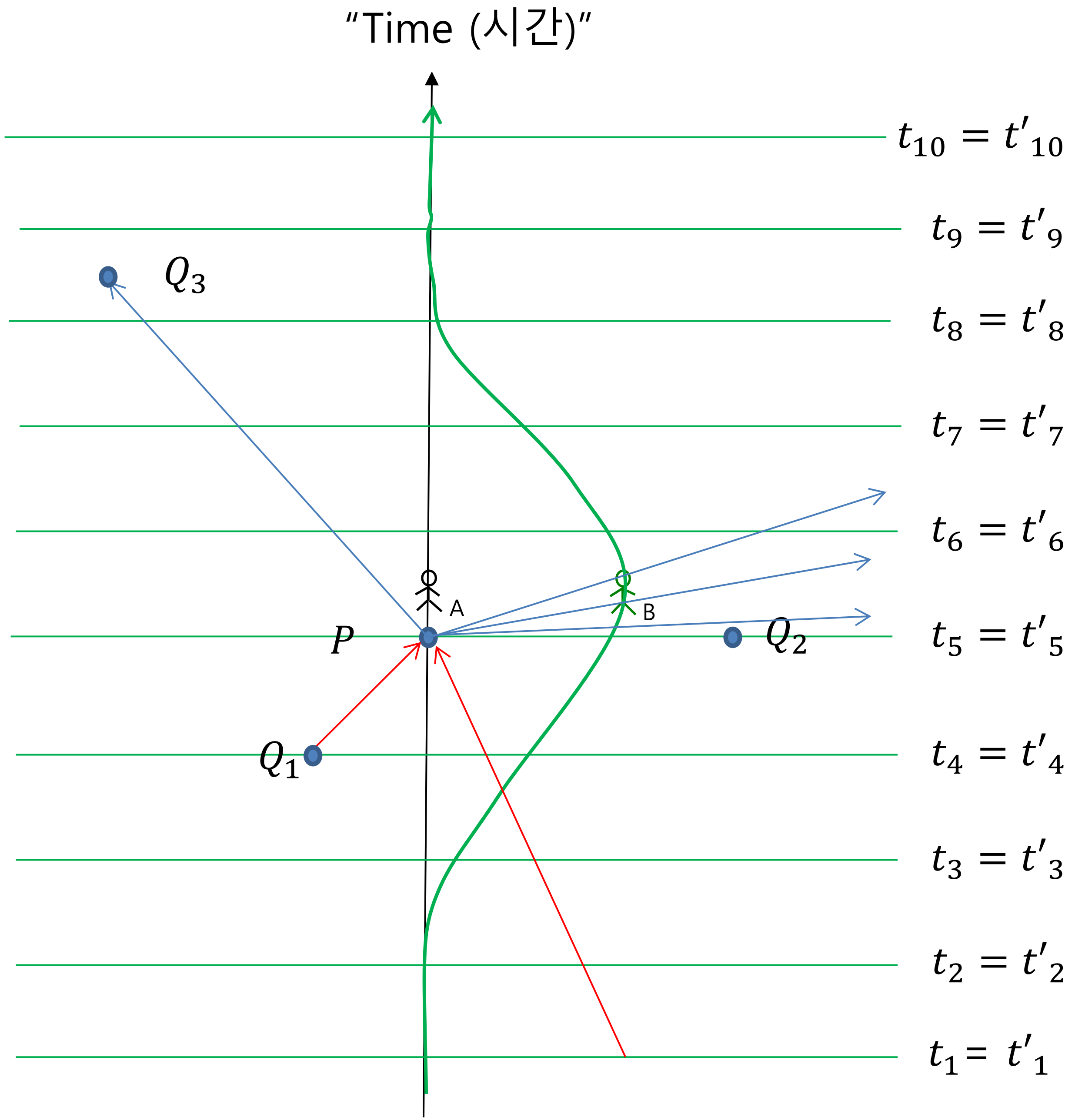
- 시간 흐름의 순서는 모든 관측자에게 동일
- **속도크기에 제한이 없음: $\mathbf{0 \leq v \leq \infty}$**
- P 사건은 t_5 이전의 모든 사건으로부터 영향을 받을 수 있고, 이후의 모든 사건한테 영향을 미칠 수 있음
- 동시에 발생한 Q_2 사건한테는 물리적인 정보를 받거나 줄 수 없음

➔ **Q_1: P 사건의 과거,**

Q_2: 현재,

Q_3: 미래

"동시 발생 사건면"



*

• 특수상대론적(밍코프스키) 시공간:

- 시간 흐름의 순서는 관측자에 따라 상대적

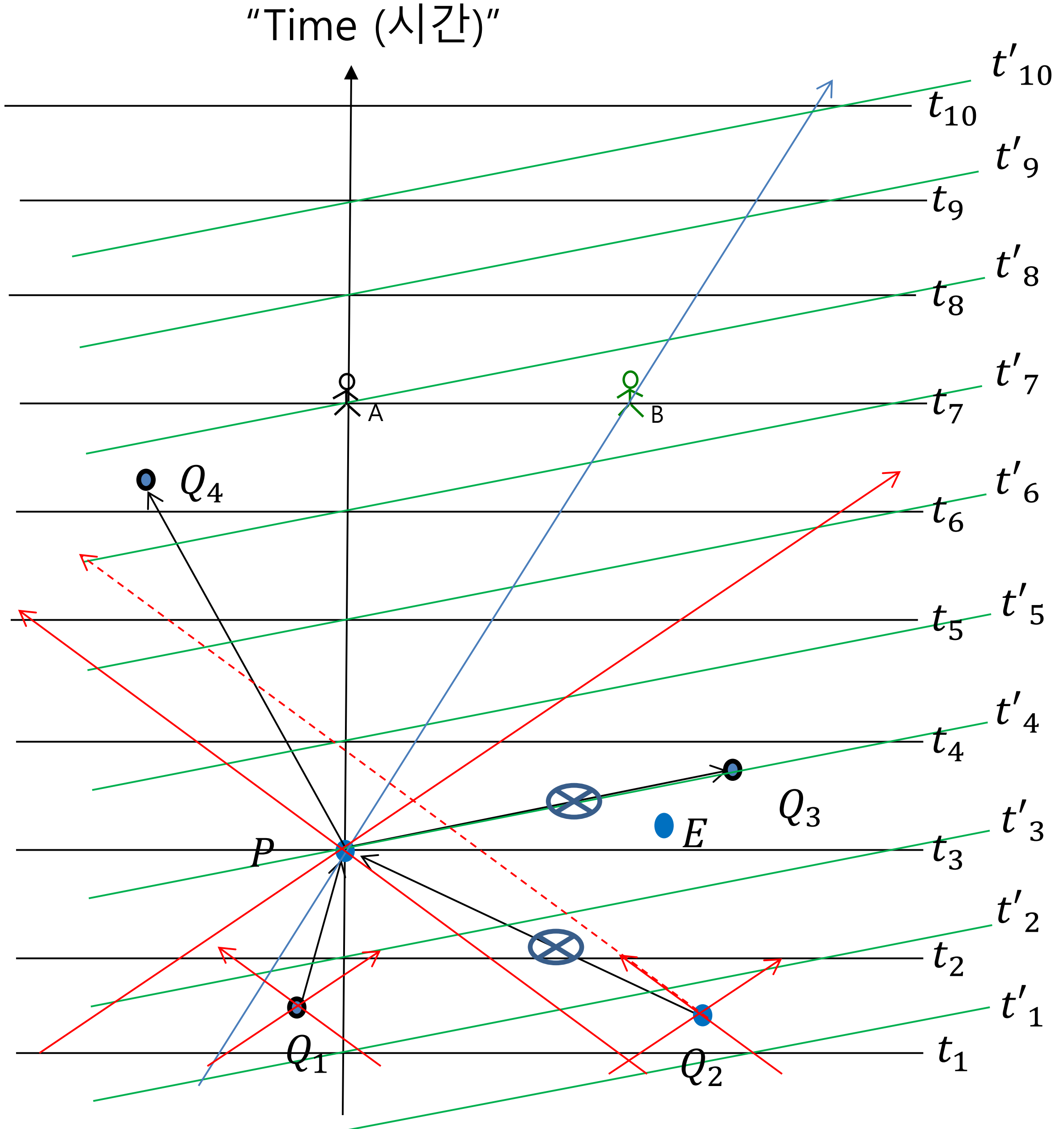
  Ex) 사건 E

  - 관측자 A: P의 미래
  - 관측자 B: P의 과거

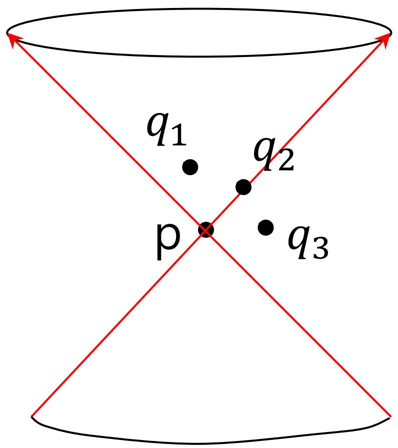
- 한 관성계의 관측자에 한해서는 순서 바뀌지 않음

- **정상적인 물체는 빛보다 빠르지 못함: $0 \le v \le c$**

➔ **물리적 정보전달은 빛의 경로에 의해 결정됨**



• Light-con and causality:



$$\Delta s^2 = -(c\Delta t)^2 + \Delta x^2$$
$$= -(c\Delta t)^2 \left[1 - \frac{\Delta x^2}{(c\Delta t)^2}\right]$$
$$= -(c\Delta t)^2 \left[1 - (\frac{v}{c})^2\right]$$

- **광 간격(Null-like separation):** $\overline{pq_2}$

$$ds^2 = 0$$

Ex) $ds^2 = -(cdt)^2 + dx^2 = 0 \rightarrow dx = \pm cdt$

➔ $v \equiv \frac{dx}{dt} = \pm c$: all events connected to p by a trajectory of light

➔ determins 'light-con'

- **시간적 간격(Time-like separation):** $\overline{pq_1}$

$$ds^2 < 0$$

$ds^2 = -(cdt)^2 + dx^2 < 0 \rightarrow \left(\frac{dx}{dt}\right)^2 < c^2$ ➔ $|v| < c$

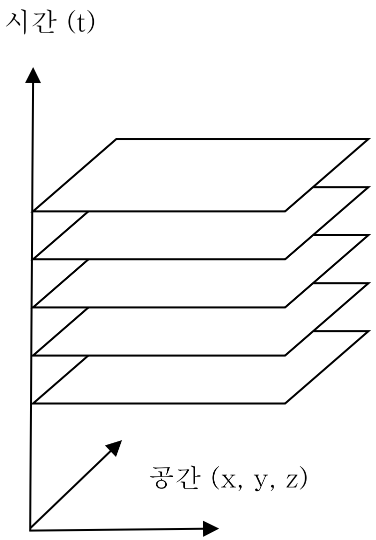
- **공간적 간격(Space-like separation):** $\overline{pq_3}$

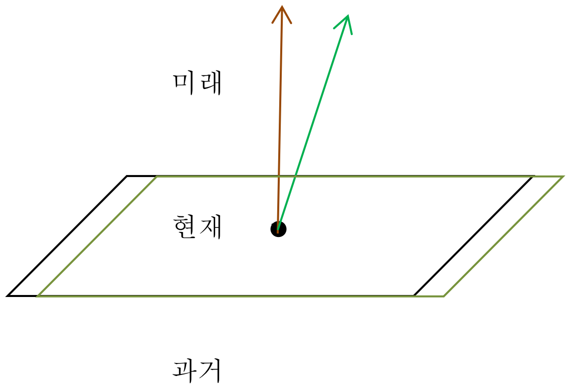
$$ds^2 > 0$$

$\left(\frac{dx}{dt}\right)^2 > c^2$ ➔ $|v| > c$: "타키온(Tachyon)"

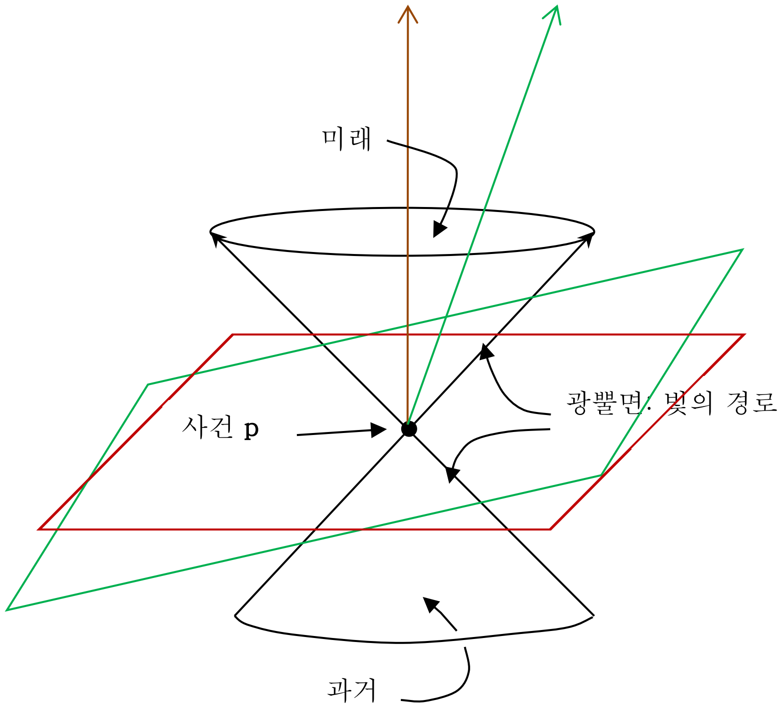
*

- **뉴튼적 시공간의 인과구조 (Newtonian ST):**



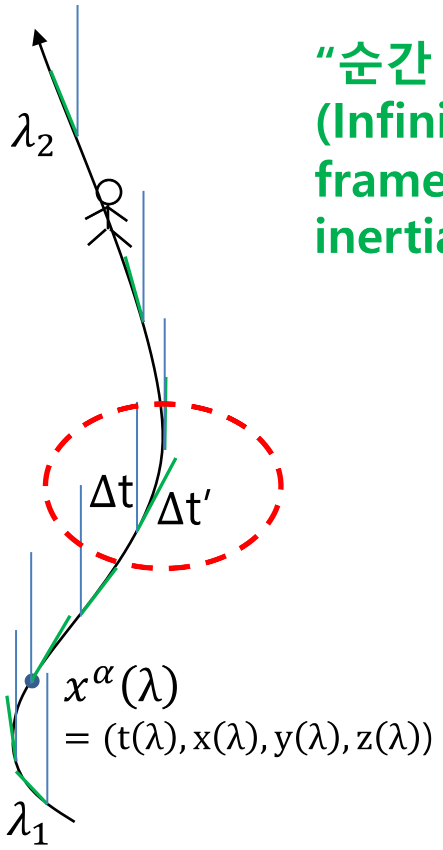


- **특수상대론적 시공간의 인과구조 (Special relativity):**



- 고유시간(Proper time): defined for a time-like curve

- "Time intervals" depend on observers!
- Time measured by the co-moving observer along that trajectory



"순간 등속계" (Infinitesimal frame or locally inertial frame)

$$\Delta\tau = \Delta t' = \sqrt{1-\left(\frac{v}{c}\right)^2}\,\Delta t$$
$$= \sqrt{c^2-\left(\frac{\Delta x}{\Delta t}\right)^2}\,\frac{\Delta t}{c}$$
$$= \sqrt{-[-(c\Delta t)^2+\Delta x^2]}\,\frac{1}{c}$$
$$= \frac{1}{c}\sqrt{-\Delta s^2}$$

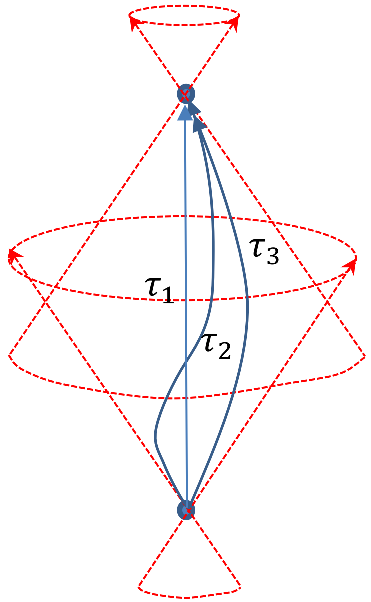
➔ $$\tau = \int d\tau = \frac{1}{c}\int\sqrt{-ds^2} = \frac{1}{c}\int\sqrt{-\eta_{\alpha\beta}dx^\alpha dx^\beta}$$

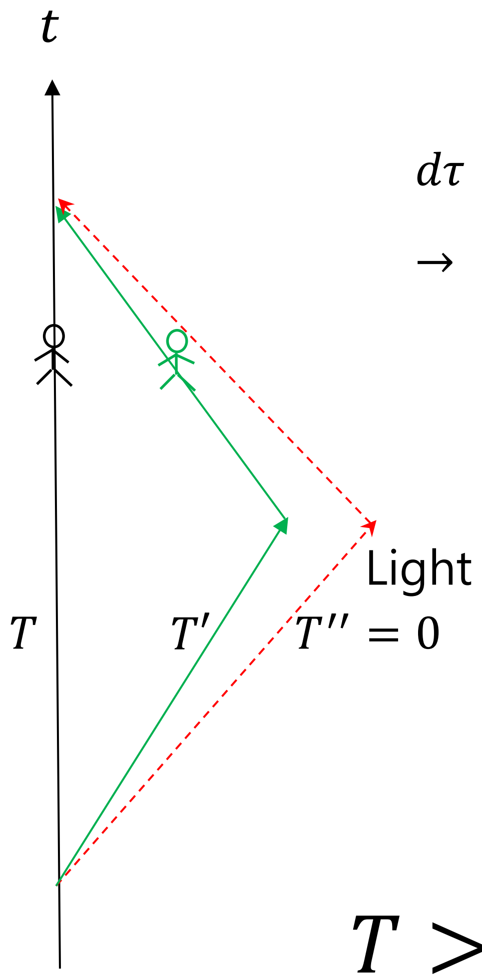
$$= \frac{1}{c}\int_{\lambda_1}^{\lambda_2}\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha(\lambda)\dot{x}^\beta(\lambda)}\,d\lambda$$

where $\dot{x}^\alpha(\lambda) = dx^\alpha/d\lambda$.

- 고유시간은 정의상 스칼라량이어서 어느 관측자가 측정해도 동일 (Invariant for any observer)

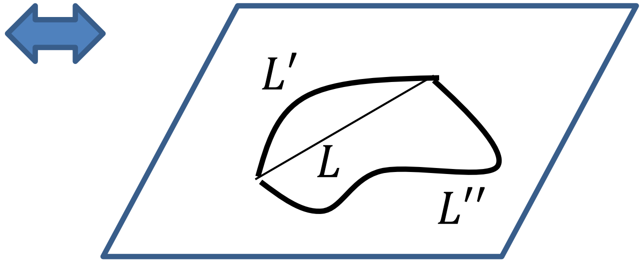
- 두 사건을 잇는 경로에 따라 시간흐름 달라짐: $\boldsymbol{\tau_1 \neq \tau_2 \neq \tau_3 \neq \cdots}$





$$d\tau = \sqrt{-ds^2} = \sqrt{(c\,dt)^2 - dx^2}$$

$\rightarrow$ 0 for the path of a light



공간 거리: $L < L' < L''$

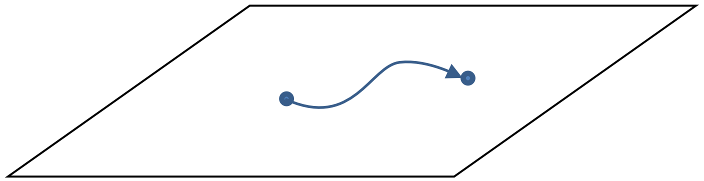
Note: 뉴튼의 시공간에서 두 사건의 시간 간격은 경로에 무관

$$T > T' > T'' = 0$$

*

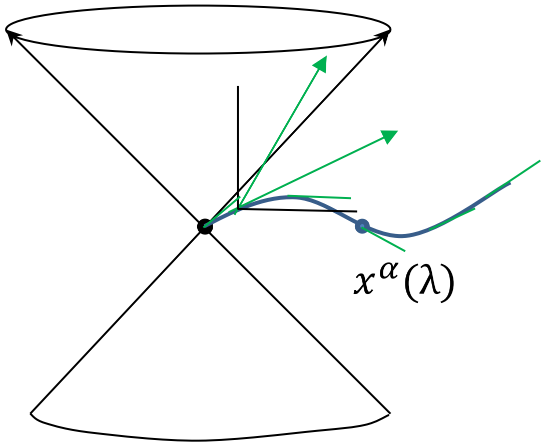
- 고유길이(Proper length): 공간적 경로(space-like curve)에 대해 정의

- 뉴튼 시공간



"동시 공간"

- 특수상대론



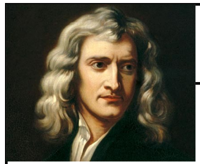
"극소 막대에 앉아 있는 순간 등속계"

$$\Delta l = \Delta x' = \sqrt{-0^2 + \Delta x'^2} = \sqrt{\Delta s^2}$$

$$L = \int dl = \int \sqrt{+ds^2} = \int \sqrt{\eta_{\alpha\beta} dx^\alpha dx^\beta / d\lambda^2}\, d\lambda$$

$$= \int \sqrt{-\left(c\frac{dt}{d\lambda}\right)^2 + \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2}\, d\lambda$$

Spacetime

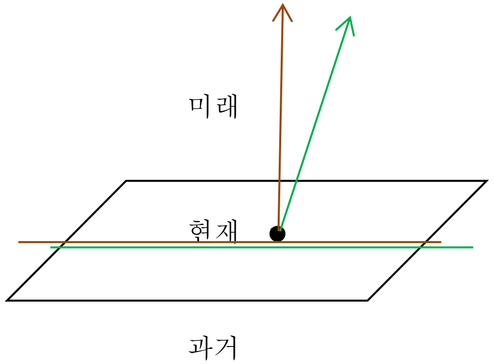


Newton (1687)	Einstein (Special relativity, 1905)

✓ 4-dimensional continuum consisted of events
✓ Uniform, without boundary, infinite …
✓ Not affected by the presence of matter

Newton (1687)

- Space and time are separated
- Causality:



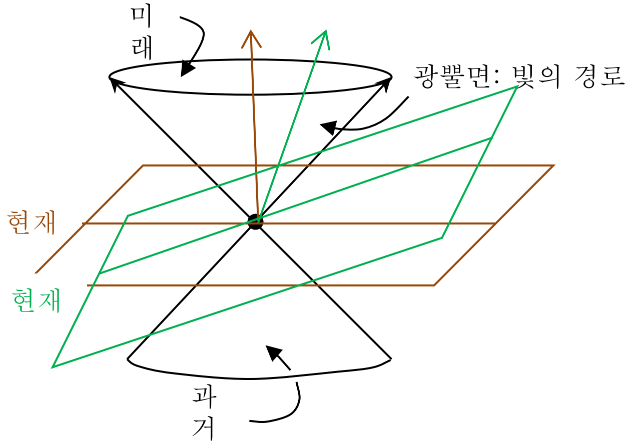
- Metric structure:

$(\Delta l)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$,
$(\Delta t)^2$

$\Delta l = \Delta l'\ ,\ \Delta t = \Delta t'$

Einstein (Special relativity, 1905)

- Spacetime continuum
- Causality: Light-cone, observer-dependent



- Metric structure:

$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

$\Delta s = \Delta s'\ \ (\Delta l \neq \Delta l'\ ,\ \Delta t \neq \Delta t')$

✓ **상대성 원리:** All physical theories should be consistent with this special relativistic background spacetime.

➔ "물리법칙은 모든 관성계에서 동일한 형태이어야 함 "

뉴튼 역학: $m\vec{a} = \vec{F}$ Or, $m\dfrac{d^2x^i}{dt^2} = F^i,\ i = 1,2,3$

- 갈릴레오 변환에 대해서는 동형이지만 로렌츠 변환에 대해 동형이 아님

$$m\frac{d^2x'^i}{dt'^2} = F'^i\ ?$$ ➔ **NO!**

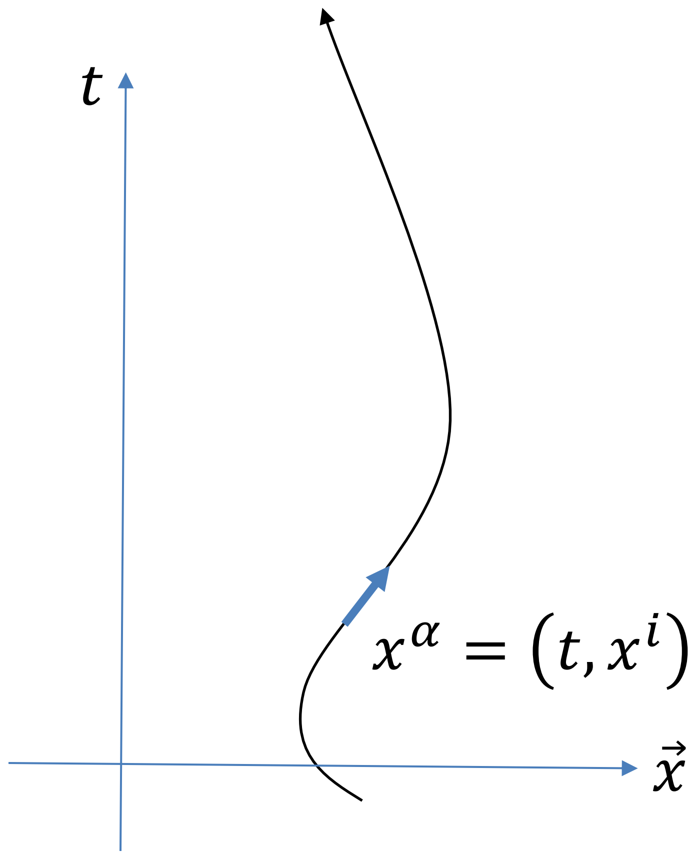
Note: 갈릴레오 변환의 경우 $x = x' + Vt\ \&\ t = t'$

➔ $\dfrac{d^2x^i}{dt^2} = F^i \rightarrow$ 左 $= \dfrac{d^2(x'^i + Vt\delta^i_x)}{dt^2} = \dfrac{d^2x'^i}{dt^2} = \dfrac{d^2x'^i}{dt'^2}$,

右 $= \dfrac{\partial x^i}{\partial x'^j}F'^j = \delta^i_j F'^j = F'^i$ ➔ $\dfrac{d^2x'^i}{dt'^2} = F'^i$: 동형

$$\frac{d^2x^i}{dt^2} = \frac{d^2[\gamma(x'^i + vt'\delta^i_x)]}{dt^2} = \gamma\frac{d^2x'^i}{dt^2} + \cdots = \frac{\partial x^i}{\partial x'^\mu}F'^\mu = \frac{\partial x^i}{\partial t'}F'^0 + \frac{\partial x^i}{\partial x'^j}F'^j$$

✓ 특수상대론적 역학:



$$x^\alpha = \left(t, x^i\right) = x^\alpha(\tau)$$

- For a time-like trajectory, the four-velocity is defined as follows;

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right) : \text{ a 4-vector}$$

$$d\tau \equiv \sqrt{-ds^2} = \sqrt{-\eta_{\alpha\beta}dx^\alpha dx^\beta}$$

Or, $(d\tau)^2 = -\eta_{\alpha\beta}dx^\alpha dx^\beta$

- Three-velocity: $\vec{V} \equiv \frac{d\vec{x}}{dt}$

- Note: $u^t = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-V^2}} \equiv \gamma,\ u^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \gamma V^i$

  ➔ $u^\alpha = (\gamma, \gamma\vec{V})$

- Four-acceleration: $a^\alpha \equiv \frac{du^\alpha}{d\tau}$

- Four-momentum: $p^\alpha \equiv mu^\alpha = \left(\frac{m}{\sqrt{1-V^2}}, \frac{m\vec{V}}{\sqrt{1-V^2}}\right) = (E, \vec{p})$

- **Note:** (스칼라곱) $u \cdot v \equiv \eta_{\alpha\beta}u^\alpha v^\beta = -u^0 v^0 + \delta_{ij}u^i v^j$
  - $u \cdot u \equiv \eta_{\alpha\beta}u^\alpha u^\beta = \eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{\eta_{\alpha\beta}dx^\alpha dx^\beta}{(d\tau)^2} = \frac{-(d\tau)^2}{(d\tau)^2} = -1 \rightarrow u \cdot u = -1,$
  - $0 = \frac{d(u \cdot u)}{d\tau} = 2u \cdot \frac{du}{d\tau} \rightarrow a \cdot u = 0$ (Orthogonal), $p \cdot p = -E^2 + \vec{p}^2 = -m^2 \rightarrow E^2 = (mc^2)^2 + \vec{p}^2$

- **상대론적 운동방정식:**

$$ma^{\mu} \equiv m\frac{d^2x^{\mu}}{d\tau^2} = f^{\mu} \quad (\mu = 0, 1, 2, 3)$$

- Note:

- $m\frac{d^2x\prime^{\mu}}{d\tau^2} = ma'^{\mu} = f'^{\mu}$

- $f^{\alpha} = \left(f^0, \vec{f}\right) = (\gamma\vec{F}\cdot\vec{V}, \gamma\vec{F})$ where $\vec{F} \equiv \frac{d\vec{p}}{dt}$

• Three-fource : $\vec{F} \equiv \frac{d\vec{p}}{dt}$. $\left(\vec{p} = m\frac{d\vec{u}}{d\tau} \neq m\frac{d\vec{V}}{dt} \right)$

$$\vec{f} = \frac{d\vec{p}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{p}}{dt} = \gamma\vec{F}.$$

$$\Rightarrow \quad \mathbf{f} = (f^0, \vec{f}) = (f^0, \gamma\vec{F})$$

$$\mathbf{f}\cdot\mathbf{u} = 0 \Rightarrow -f^0\gamma + (\gamma\vec{F})\cdot(\gamma\vec{V}) = \gamma(-f^0 + \gamma\vec{F}\cdot\vec{V}) = 0.$$

$$\therefore \mathbf{f} = (\gamma\vec{F}\cdot\vec{V}, \gamma\vec{F}).$$

$$\frac{d\mathbf{p}}{d\tau} = \mathbf{f} \Rightarrow \left(\frac{dE}{d\tau}, \frac{d\vec{p}}{d\tau}\right) = (\gamma\vec{F}\cdot\vec{V}, \gamma\vec{F})$$

$$\frac{dE}{d\tau} = \frac{dt}{d\tau}\frac{dE}{dt} = \gamma\frac{dE}{dt}.$$

$$\Rightarrow \boxed{\frac{dE}{dt} = \vec{F}\cdot\vec{V}}$$ = "Work done per second"

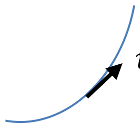
$$\mathbf{f} = 0 \Rightarrow \frac{d\mathbf{p}}{d\tau} = 0 \Rightarrow \boxed{\mathbf{p} \text{ is conserved!}}$$ { Energy & Momentum

- **특수상대론적 운동에너지 (Relativistic energy):**

Four-momentum

$V = 0$

$\vec{V}\ \left(V^i = \frac{dx^i}{dt}\right)$



$u^\alpha = \frac{dx^\alpha}{d\tau}$

Four-velocity

$$
\begin{aligned}
p^\alpha \equiv m u^\alpha &= m\left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) \\
&= m\left(\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{dx}{dt}, \frac{dt}{d\tau}\frac{dy}{dt}, \frac{dt}{d\tau}\frac{dz}{dt}\right) \\
&= \left(m\frac{1}{\sqrt{1-\left(\frac{V}{c}\right)^2}}, \frac{1}{\sqrt{1-\left(\frac{V}{c}\right)^2}}\mathrm{m}\vec{V}\right) = (m\gamma, m\gamma\vec{v})
\end{aligned}
$$

운동에너지: KE = 0 → $\frac{1}{2}mV^2$ (뉴튼 역학)

$$
\tilde{m} \equiv m\frac{1}{\sqrt{1-\left(\frac{V}{c}\right)^2}} = m[1-(V/c)^2]^{-1/2} = m\left[1 + \left(-\frac{1}{2}\right)(-)\left(\frac{V}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \cdots\right]
$$

$$
\rightarrow \quad \tilde{m}c^2 = mc^2 + \frac{1}{2}mV^2 + \frac{3}{8}mc^2\left(\frac{v}{c}\right)^4 + \cdots
$$

$$
\boxed{E = \tilde{m}c^2 = \frac{mc^2}{\sqrt{1-\left(\frac{V}{c}\right)^2}}}
$$

- 빛의 속도로 증가시키려면($V \rightarrow c$) 무한의 에너지 필요!!
- 물체의 속도는 빛의 속도를 넘지 못한다.
- $p^\alpha = \left(E, p^i\right) = (E, \vec{p})$: $\quad E^2 = m^2 + \vec{p}\cdot\vec{p}$

Note: $p\cdot p = \eta_{\alpha\beta}p^\alpha p^\beta = -(p^0)^2 + \eta_{ij}p^i p^j = -E^2 + \vec{p}\cdot\vec{p} = \eta_{\alpha\beta}\left(m\frac{dx^\alpha}{d\tau}\right)\left(m\frac{dx^\beta}{d\tau}\right) = \frac{m^2\eta_{\alpha\beta}dx^\alpha dx^\beta}{(d\tau)^2} = \frac{m^2(-d\tau^2)}{(d\tau)^2} = -m^2$

*

✓ Electromagnetism:

$$\nabla \cdot \mathbf{B} = 0, \quad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

➡

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0$$

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

with

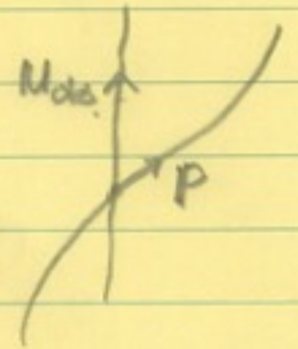
$$\begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} = F^{\mu\nu}.$$

$$J^\alpha = (c\rho, \mathbf{J})$$

- **A tensor equation in Minkowski ST**

  ➔ Guarantees the principle of relativity

3.5 Observers and energy measurement



$$\mathbb{P} \equiv m \frac{dx^\alpha}{d\tau} . \qquad u_{obs} \equiv \frac{dx^\alpha_{obs}}{d\tau_{obs}} .$$

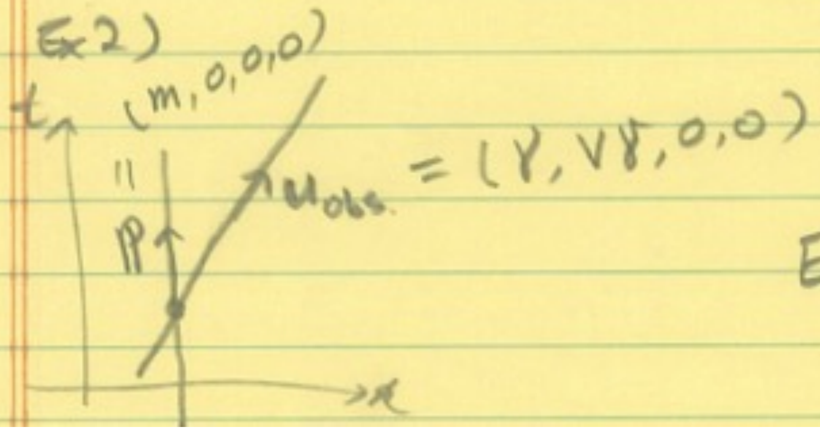
$$E = -\mathbb{P} \cdot \mathbf{e}_0 = -\mathbf{p} \cdot u_{obs} .$$

Ex1). For the observer comoving with the particle,

$$u_{obs.} = u .$$

$$E = -\mathbb{P} \cdot u_{obs} = -m u \cdot u_{obs} = -m u \cdot u = m = mc^2 \text{ : Rest mass energy.}$$

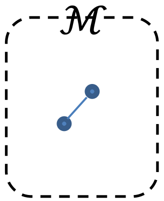
Ex2)



$$E = -\mathbb{P} \cdot u_{obs} = -(-m\gamma + 0)$$
$$= m\gamma = \frac{m}{\sqrt{1-v^2}} .$$

*

✓ Newtonian limit:



- What is the relationship between the special relativistic ST and the Newtonian ST?
- "Invariant light speed"($c = c'$) $\leftrightarrow$ "Existence of the maximum speed or the upper bound"
- Lorentz metric: $ds^2 = 0 \rightarrow -(cdt)^2 + dx^2 = 0 = -(cdt')^2 + dx'^2$
- Even if $v_{light} \neq c$ (*e.g.,* massive photon), the special relativistic ST would be same. There would be some physical reality having $v = c$ instead of the light.

- **Newtonian limit:** $\boldsymbol{c} \rightarrow \infty$

i) Lorentz transformation ➔ Galilean transformation

$$t' = \frac{1}{\sqrt{1-v^2/c^2}}\left(t - \frac{v}{c}\frac{x}{c}\right)$$
$$x' = \frac{1}{\sqrt{1-v^2/c^2}}\left(x - \frac{v}{c}\,ct\right)$$
$$y' = y,\ z' = z$$

⇩

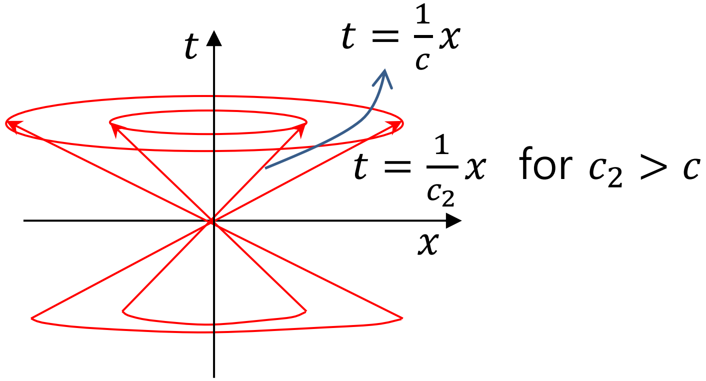
$$t' = t$$
$$x' = x - vt$$
$$y' = y,\ z' = z$$

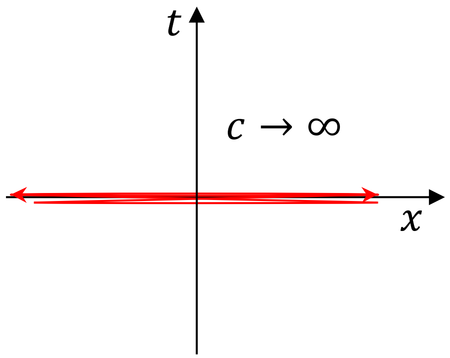
ii) Absolute time

$$\Delta s^2 = -(c\Delta t)^2 + (\Delta l)^2 \cong -(c\Delta t)^2 = \Delta s'^2 \cong -(c\Delta t')^2$$

➔ $\Delta t = \Delta t'$. Consequently, we also obtain $\Delta l = \Delta l'$

iii) Causal structure



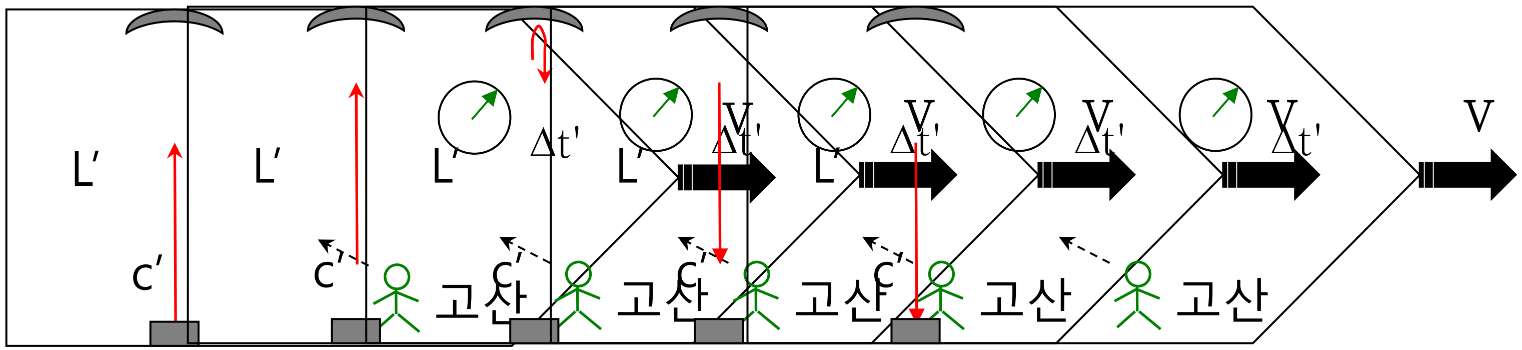


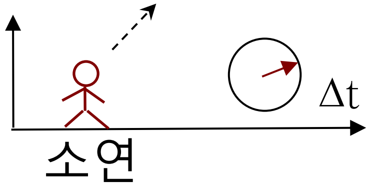
✓ Question and Answer:

BACK-UP SLIDES

• 시간의 흐름

✓ 시간 흐름: “두 사건 사이의 시간 간격”



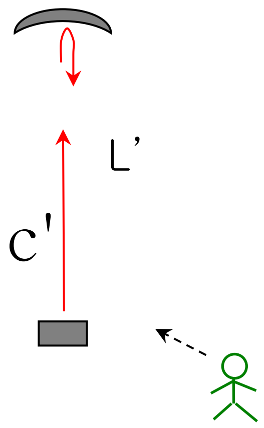


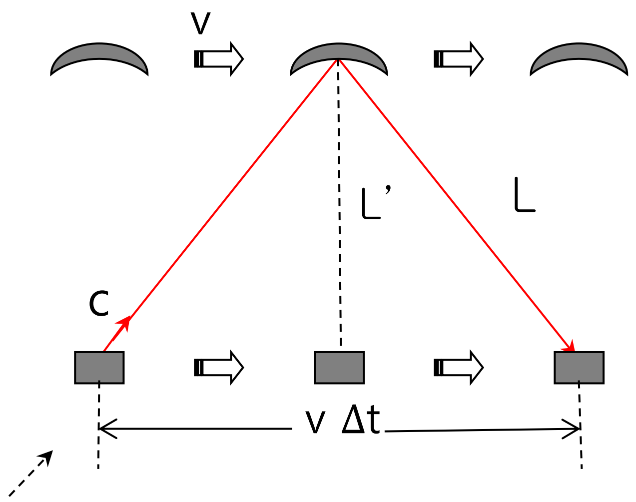
두 사건: 빛의 방출과 도착

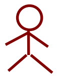
i) 뉴튼의 시공간: $\Delta t = \Delta t'$

ii) 광속 불변을 가정할 경우: ??

i) 뉴튼:







고산: $\Delta t' = 2L'/c'$

소연: $\Delta t = 2L/c$

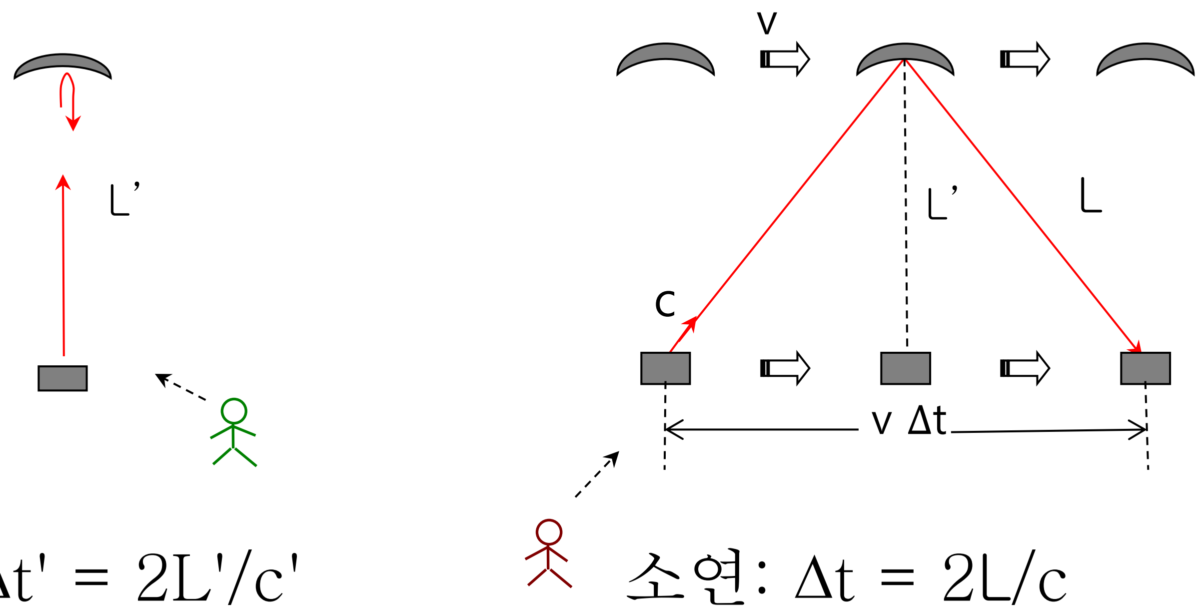
✓ 소연의 빛 속도: $c=\sqrt{c'^2 + v^2}$ ➔ $\Delta t= \frac{2\sqrt{L'^2+(v\Delta t/2)^2}}{\sqrt{c'^2+v^2}}$

$$(c'^2 + v^2)(\Delta t)^2 = 4(L'^2 + (v\Delta t/2)^2)$$

➡ $c'^2(\Delta t)^2 = 4L'^2$ ➡

$$\Delta t = \frac{2L'}{c'} = \Delta t'$$

ii) 광속 불변:



고산: $\Delta t' = 2L'/c'$

소연: $\Delta t = 2L/c$

✓ 빛 속도 불변: c=c' ➔ $\Delta t > \Delta t'$ ➔ "시간 지연"

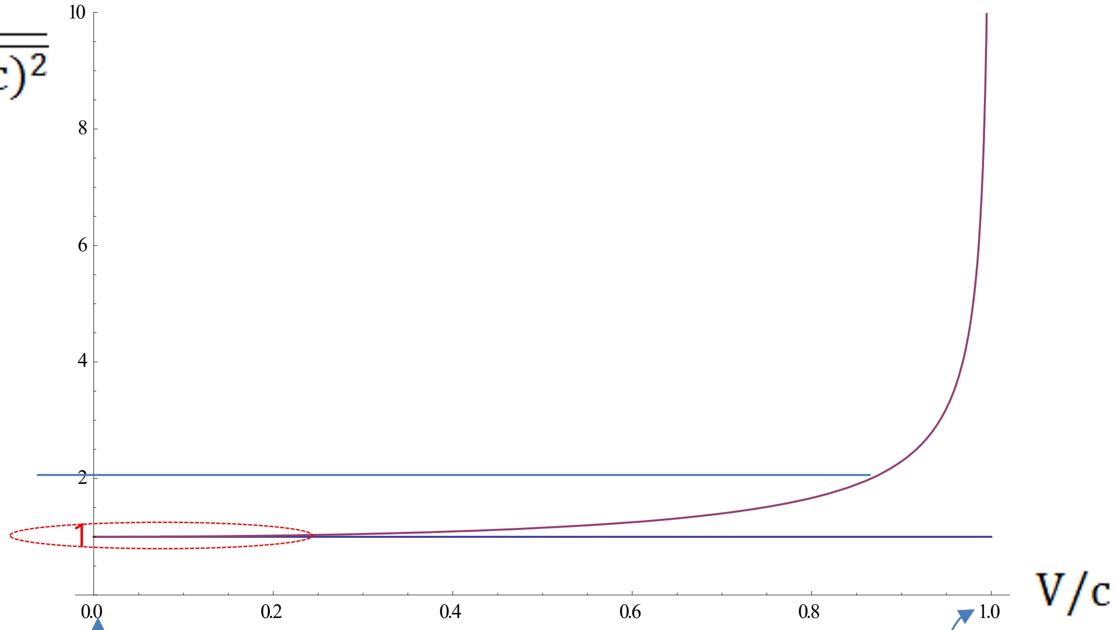
$$\Delta t = \frac{2\sqrt{L'^2 + (V\Delta t/2)^2}}{c'} \Rightarrow \Delta t = \frac{1}{\sqrt{1-(V/c)^2}}\Delta t'$$

✓ 빛 대신 다른 물체를 사용해도 결론은 같음

시간의 흐름은 동일하지 않음...? ➔ "절대적 시간" 아님!

$$\Delta t = \frac{1}{\sqrt{1-(V/c)^2}}\,\Delta t' = \Gamma\,\Delta t'$$

$$\Gamma = \frac{1}{\sqrt{1-(V/c)^2}}$$



$c = 300{,}000$ km/s

$V_{KTX} \sim 0.08$ km/s ~ 0.0000003 c

$\Gamma_{KTX} = 1 + 3.9\times10^{-14}$

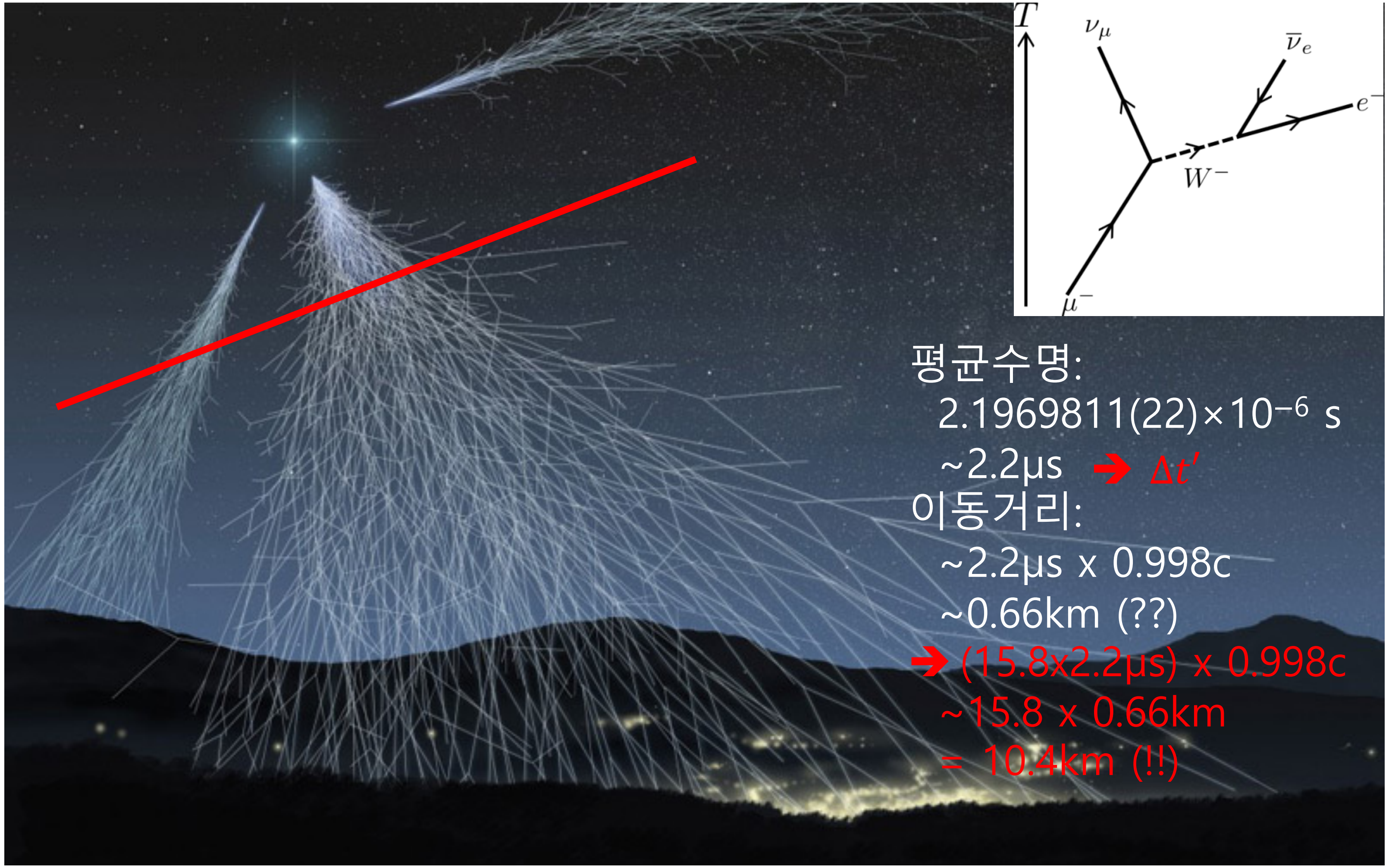
$V_{비행기} \sim 0.28$ km/s ~ 0.0000009 c

$V_{위성} \sim 8.14$ km/s ~ 0.00003 c

$\Gamma_{뮤온} \sim 15.8$ ⇨ $\Delta t = \Gamma_{뮤온}\Delta t' \sim 15.8\,\Delta t'$

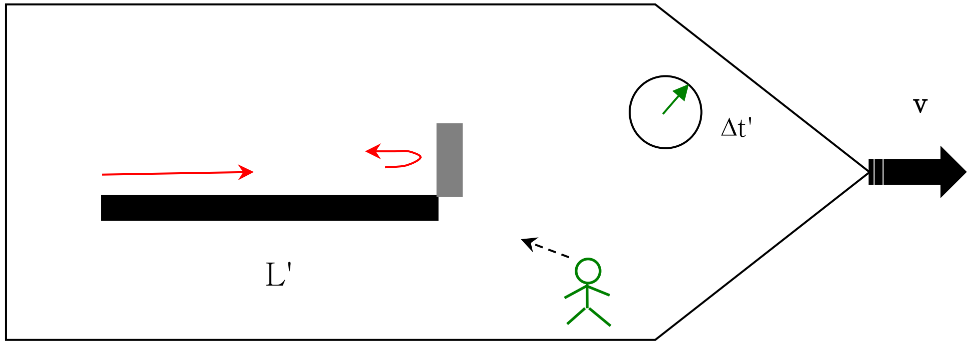
$V_{뮤온} \sim 0.998$ c

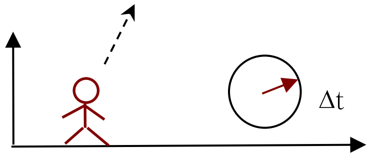
❖ 뮤온 검출의 미스터리:

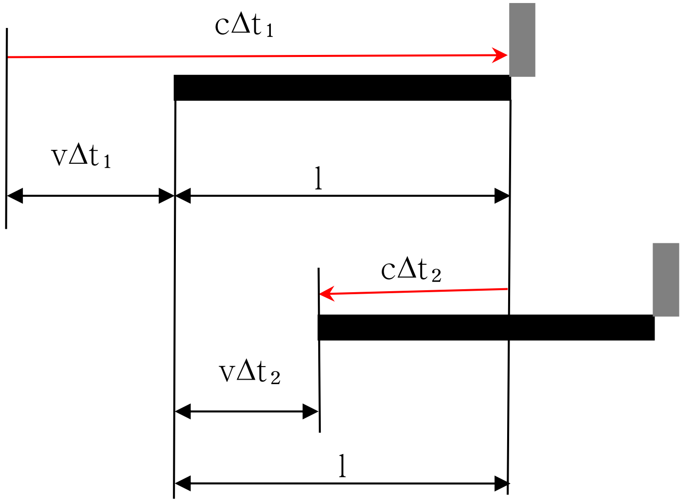


$V_{뮤온} \sim 0.998\ c$ $\Gamma_{뮤온} \sim 15.8$ $\Delta t = \Gamma_{뮤온} \Delta t' \sim 15.8\ \Delta t'$

• 막대의 길이







$L' = (c\Delta t')/2 \quad \Rightarrow \quad \Delta t' = 2L'/c$

$c\Delta t_1 = v\Delta t_1 + L \quad \Rightarrow \quad \Delta t_1 = L/(c - v)$

$c\Delta t_2 = L - v\Delta t_2 \quad \Rightarrow \quad \Delta t_2 = L/(c + v)$

$$\Delta t = \frac{1}{\sqrt{1-(V/c)^2}}\Delta t'$$

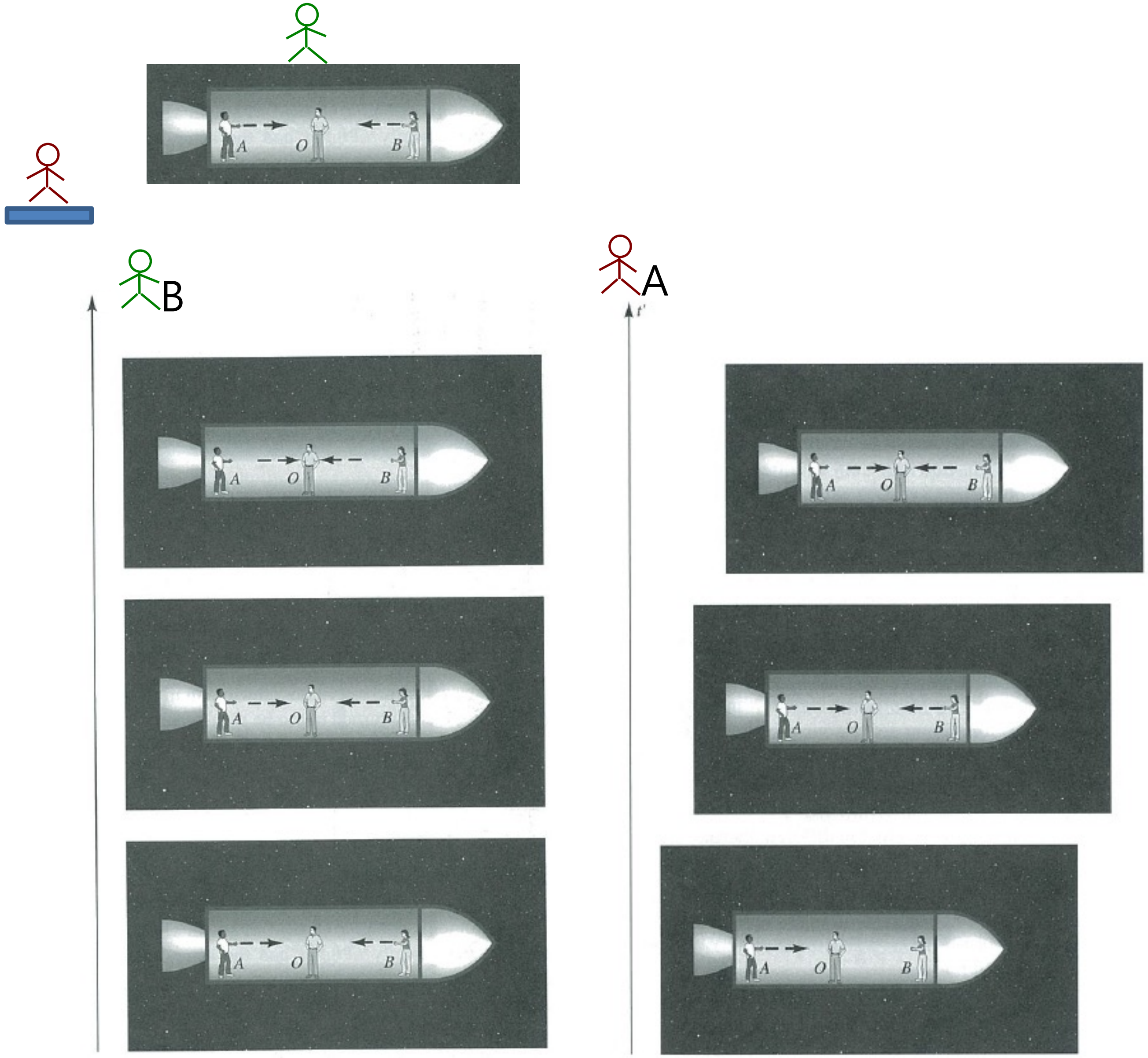
$$\begin{aligned}\Delta t &= \Delta t_2 + \Delta t_1 \\ &= L/(c - v) + L/(c + v) \\ &= \frac{2L}{c}\frac{1}{1-(v/c)^2}\end{aligned}$$

$$\frac{2L}{c}\frac{1}{1-(v/c)^2} = \frac{1}{\sqrt{1-(v/c)^2}}\frac{2L'}{c}$$

$$L = \sqrt{1-(v/c)^2}\,L'$$

"길이 수축"

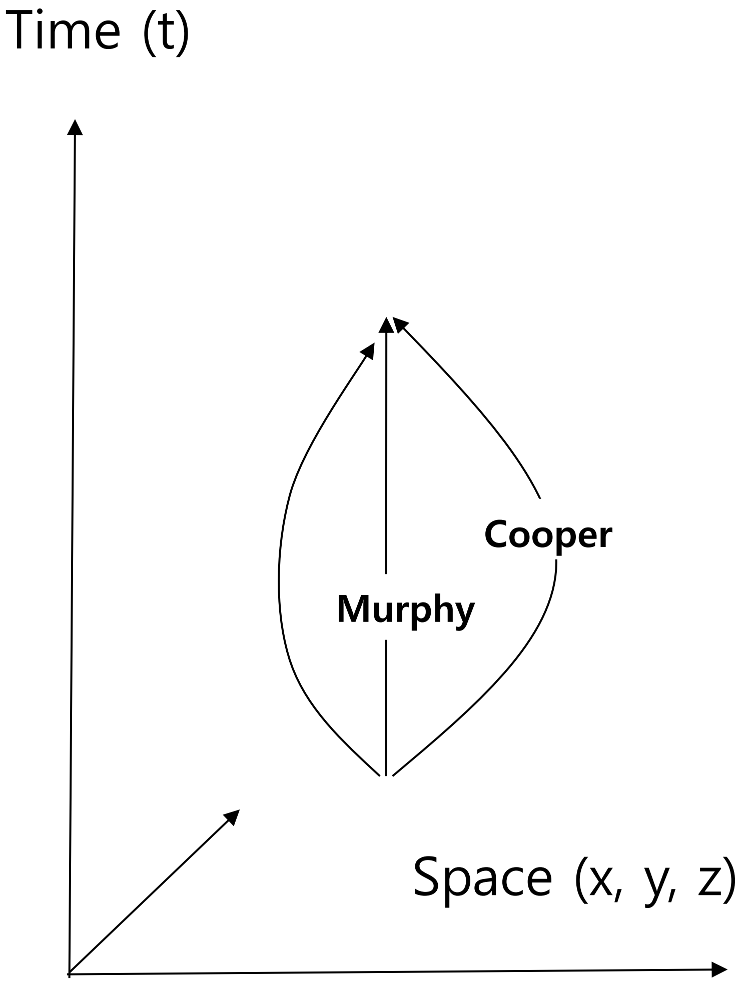
• 동시에 발생한 사건



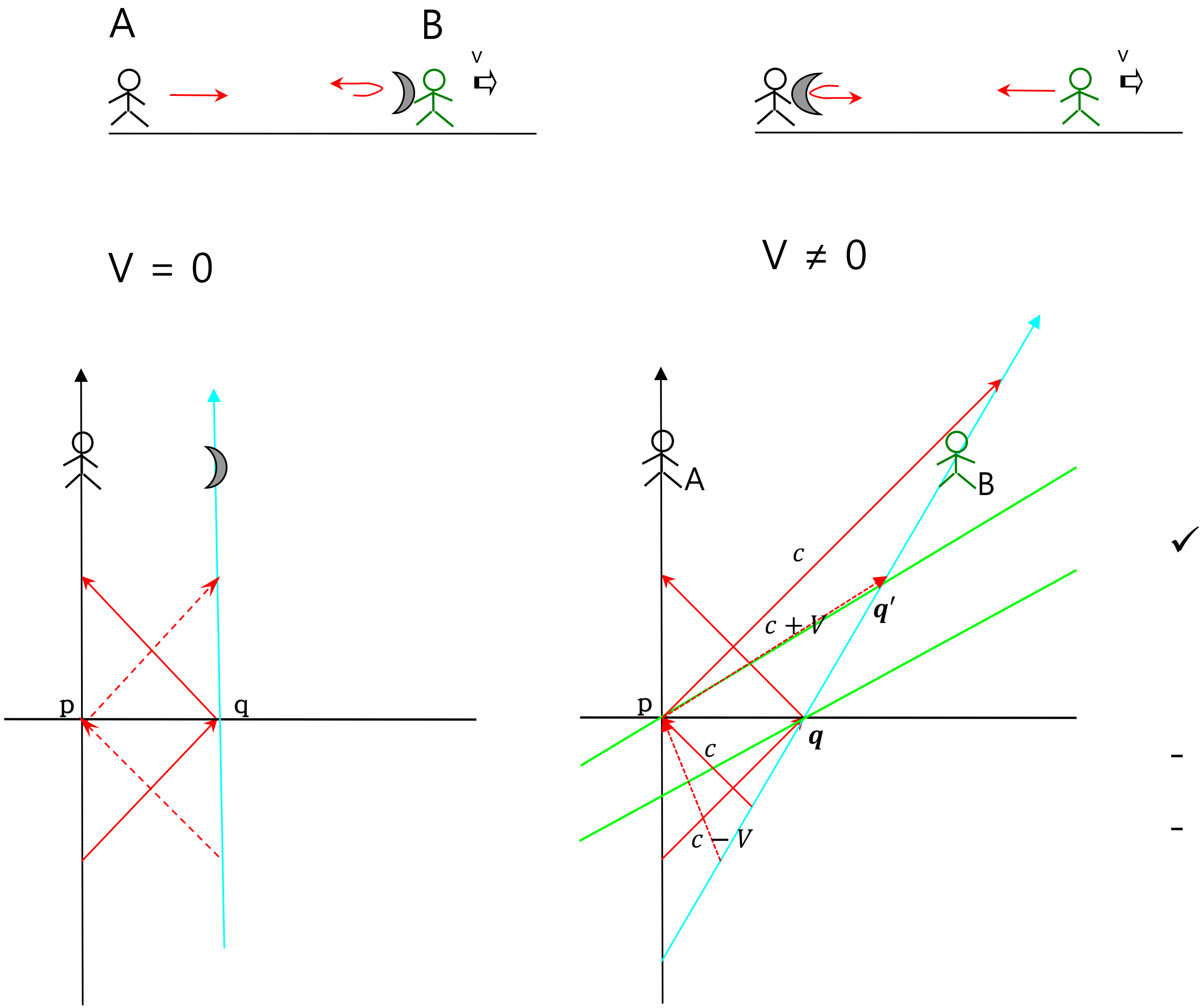
- 한 관측자에게 동시에 일어난 두 사건이 다른 관측자에게는 다른 시간에 발생한 사건!!??

- 동시성의 상대성

그림 출처: Hartle


Time (t)
Cooper
Murphy
Space (x, y, z)

• Simultaneous events and spacetime coordinates



✓ 사건 $\boldsymbol{p}$ 와 동시에 일어난 사건은? (Simultaneous events with the event $\boldsymbol{p}$?)

- Observer A: $\boldsymbol{q}$
- Observer B: $\boldsymbol{q'}$