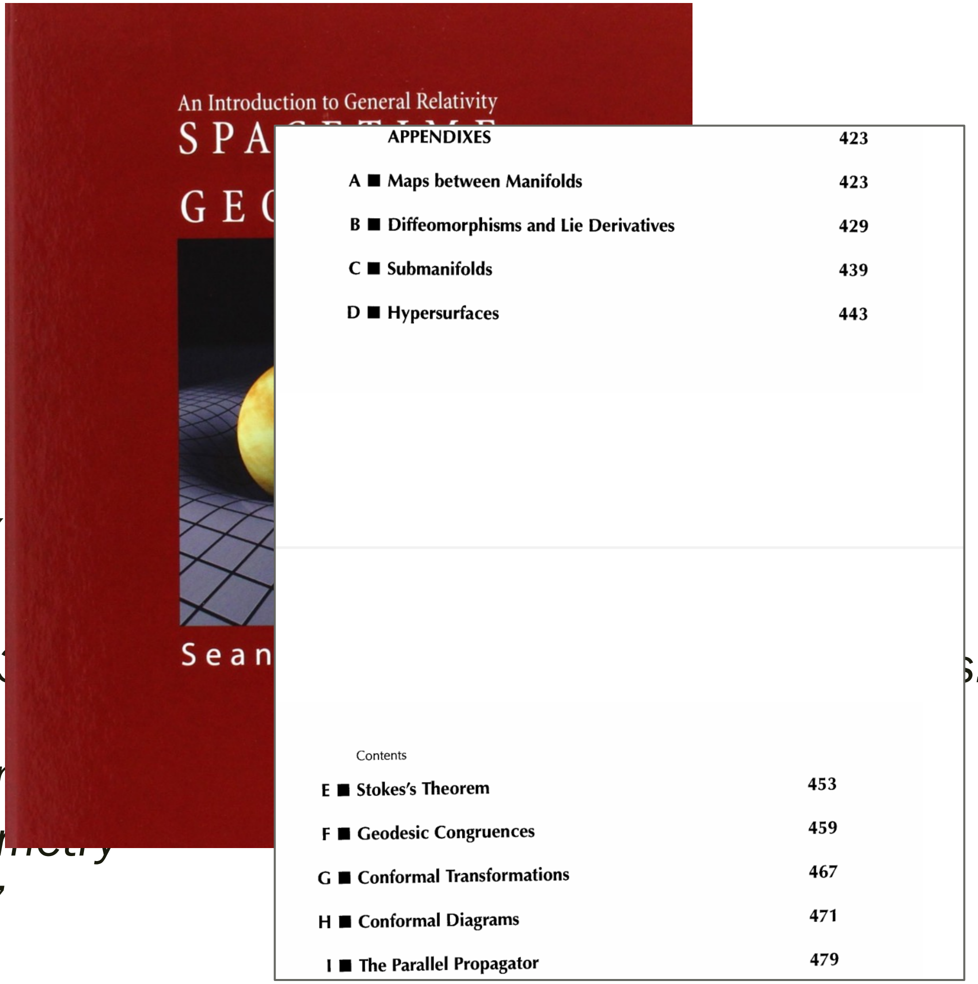
3+1 FORMALISM

2022년 수치상대론 중력파 여름학교 7.27, 7.28, 한국천문연구원 현영환

Outline and references

- Outline
  - *Initial value problem*
  - *Hypersurface*
  - *Gauss, Codazzi equation*
  - *3+1 decomposition of Einstein eq.*
  - *Initial data construction*
- Previous lecture (~2021 summer school
  - *3+1 형식론 (3+1 formalism) 노트 by*
- Reference:
  - *Baumgarte, T. W., & Shapiro, S. L. (200* *. In Physics Reports”*
  - *Gourgoulhon, E. (2012). “3+1 Formalis*
  - *Carroll, S. (2004). “Spacetime and Geo*
  - *Wald, R. M. (1984). “General Relativity”*
  - *....*

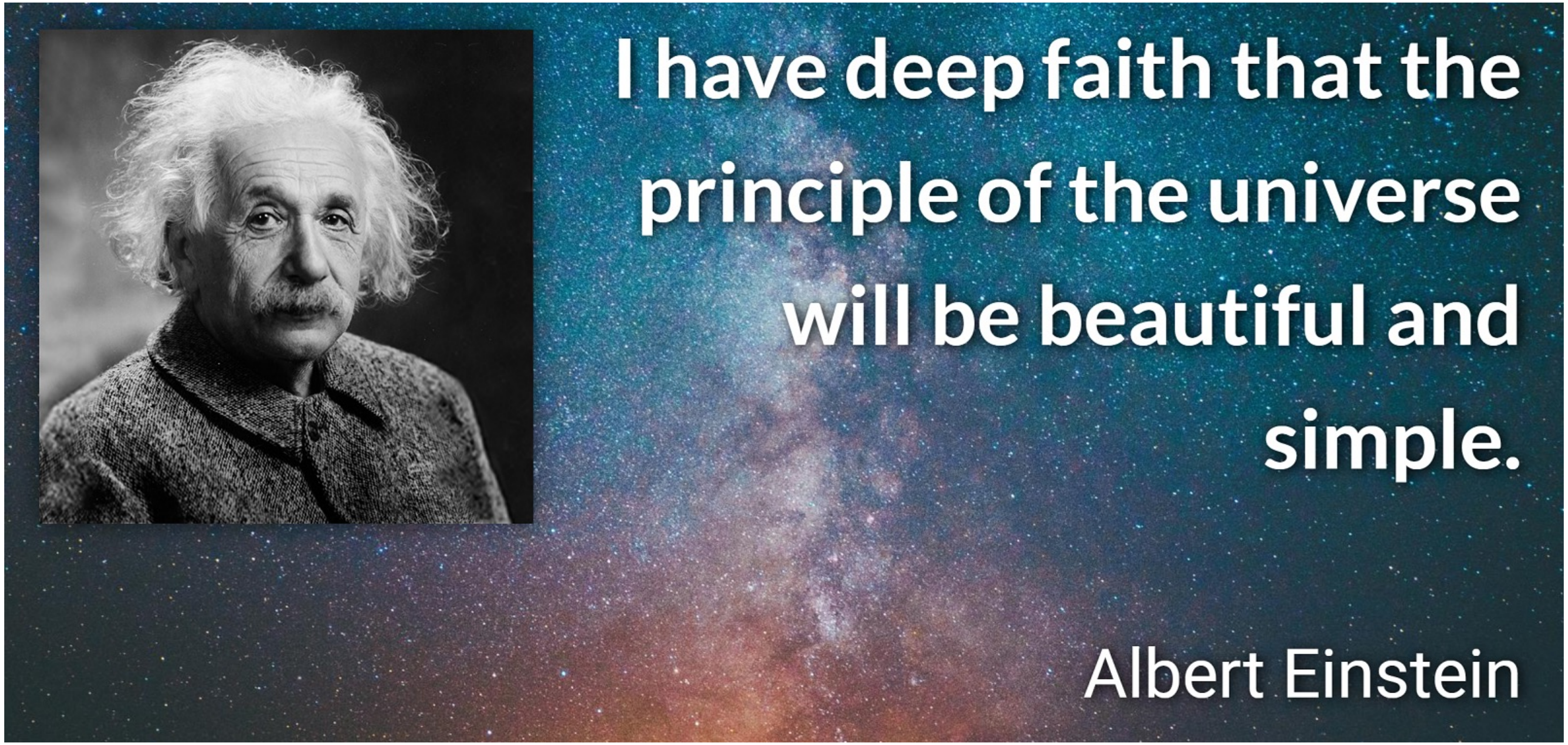


Contents

Why 3+1?

- 3+1=

Why 3+1? Let's see the final form….



Why 3+1? Einstein eq.

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho$$

$$(\bar{\Delta}_{\mathrm{L}}W)^j - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_i K = 8\pi G\psi^{10}p^j$$

$$\partial_t\gamma_{ij} = -2\alpha K_{ij}$$

$$\hookrightarrow 4\bar{\gamma}_{ij}\psi^3\partial_t\psi =$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$(\text{trace})\ \frac{1}{2}\gamma^{ij}\partial$$

$$\cdot\ \rho))$$

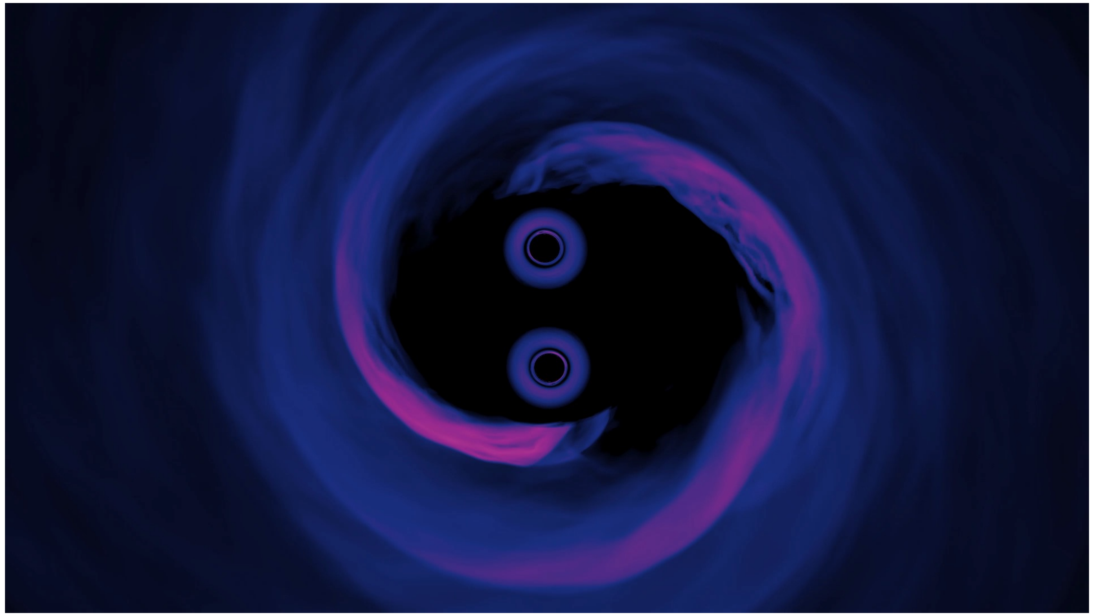
$$\partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_iD_j\alpha - \alpha 8\pi G(S_{ij}$$

$$+ \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k$$

$$(\text{trace})\ \partial_t K = -D^2\alpha + \alpha(K_{ij}K^{ij} + 4\pi(\rho + S)) + \beta^i D$$

Why 3+1?

- 3+1= 3 along 1



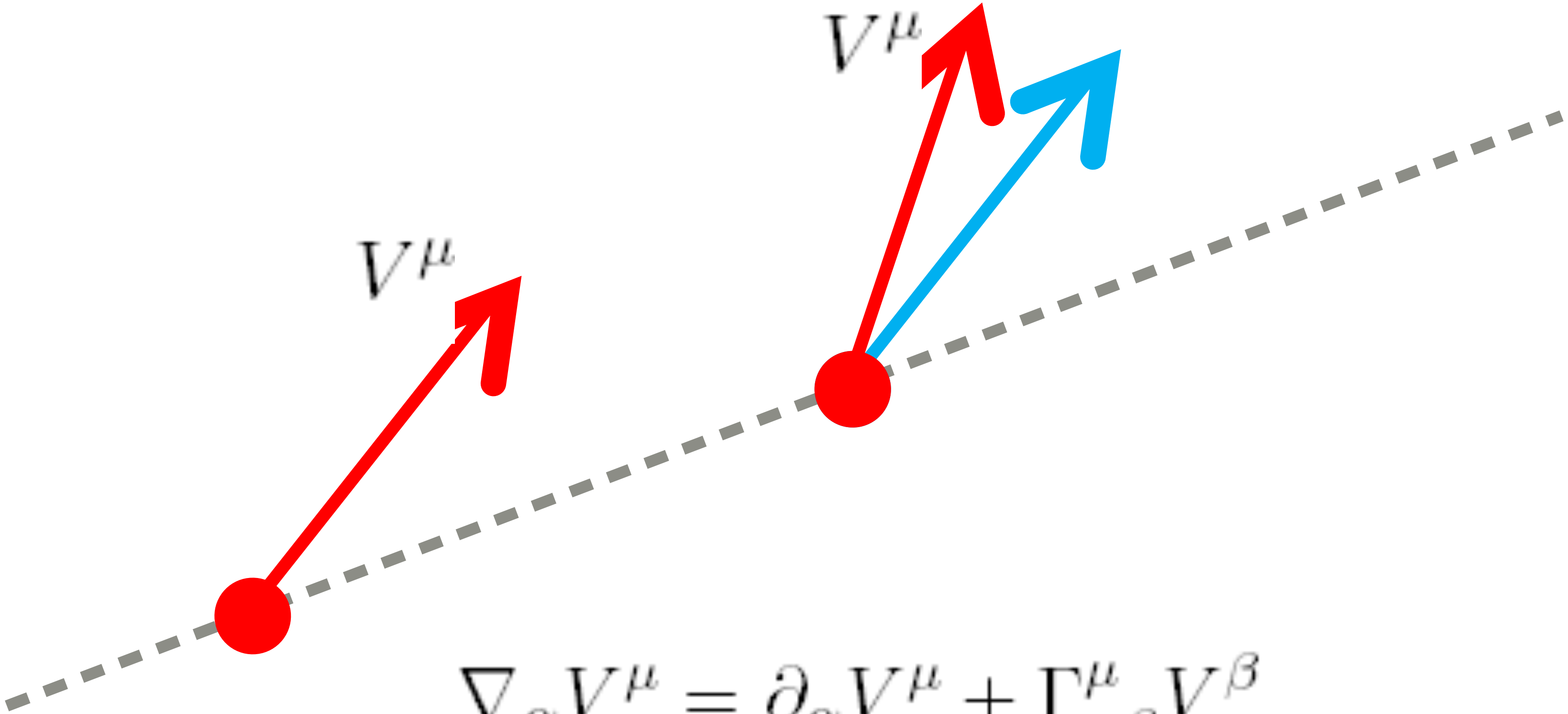
https://svs.gsfc.nasa.gov/13086

```
ADMBase::metric_type = "physical"

ADMBase::initial_data    = "twopunctures"
ADMBase::initial_lapse   = "twopunctures-averaged"
ADMBase::initial_shift   = "zero"
ADMBase::initial_dtlapse = "zero"
ADMBase::initial_dtshift = "zero"

# needed for AHFinderDirect
ADMBase::metric_timelevels = 3

TwoPunctures::par_b          =  1.168642873
TwoPunctures::par_m_plus     =  0.453
TwoPunctures::par_m_minus    =  0.453
TwoPunctures::par_P_plus [1] = +0.3331917498
TwoPunctures::par_P_minus[1] = -0.3331917498

#TODO# TwoPunctures::grid_setup_method = "evaluation"

TwoPunctures::TP_epsilon = 1.0e-2
TwoPunctures::TP_Tiny    = 1.0e-2

TwoPunctures::verbose = yes



ActiveThorns = "ML_BSSN ML_BSSN_Helper NewRad"

ADMBase::evolution_method         = "ML_BSSN"
ADMBase::lapse_evolution_method   = "ML_BSSN"
ADMBase::shift_evolution_method   = "ML_BSSN"
ADMBase::dtlapse_evolution_method = "ML_BSSN"
ADMBase::dtshift_evolution_method = "ML_BSSN"

ML_BSSN::harmonicN           = 1      # 1+log
ML_BSSN::harmonicF           = 2.0    # 1+log
ML_BSSN::ShiftGammaCoeff     = 0.75
ML_BSSN::BetaDriver          = 1.0
ML_BSSN::advectLapse         = 1
ML_BSSN::advectShift         = 1

ML_BSSN::MinimumLapse        = 1.0e-8

ML_BSSN::initial_boundary_condition = "extrapolate-gammas"
ML_BSSN::rhs_boundary_condition     = "NewRad"
Boundary::radpower                     = 2
```

Why 3+1?

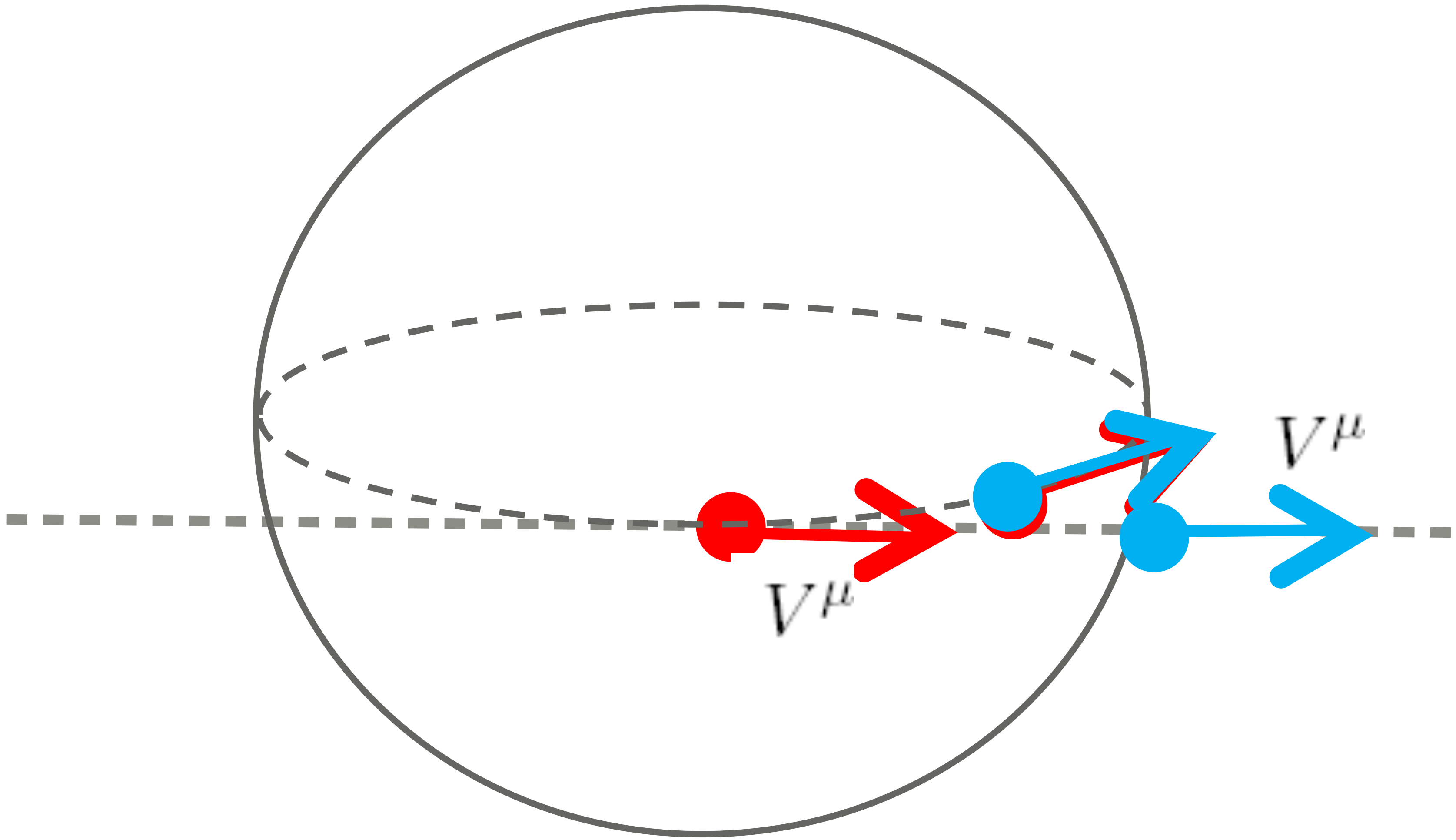
- 3+1= 3 along 1
- D.E., I.C. and solution…. So what?
  - *4D quantity ≠ 3D quantity*
  - *PDE (Cauchy problem)*
  - *causal structure of sol.*

1. 4D quantity ≠ 3D quantity (1)



$$\nabla_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma^\mu_{\alpha\beta} V^\beta$$

1. 4D quantity ≠ 3D quantity (2)



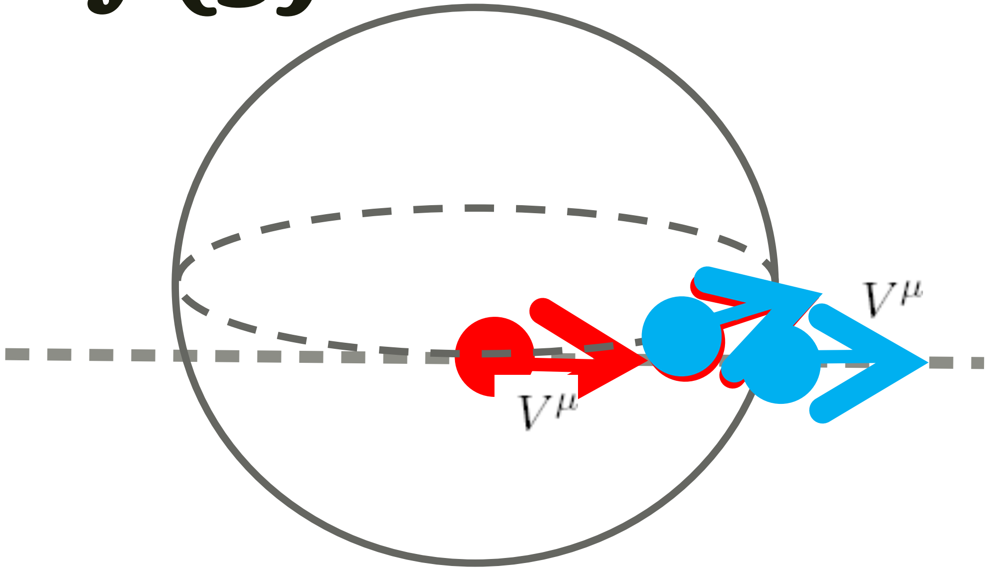
1. 4D quantity ≠ 3D quantity (3)

$$ds^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$\nabla_\phi(\boldsymbol{\partial}_\phi)^\mu = \nabla_\phi\delta^\mu_\phi = \partial_\phi\delta^\mu_\phi + \Gamma^\mu_{\phi\phi}$$

$$= -\frac{g_{\phi\phi,\mu}}{2g_{\mu\mu}} = -\frac{(r^2\sin^2\theta)_{,\mu}}{2g_{\mu\mu}}$$

$$= \begin{cases} \nabla_\phi(\boldsymbol{\partial}_\phi)^r = -\dfrac{(r^2\sin^2\theta)_{,r}}{2g_{rr}} = -\dfrac{2r\sin^2\theta}{2} \xrightarrow{\theta = \frac{\pi}{2}} = -r \\ \nabla_\phi(\boldsymbol{\partial}_\phi)^\theta = -\dfrac{(r^2\sin^2\theta)_{,\theta}}{2g_{\theta\theta}} = -\dfrac{2r^2\sin\theta\cos\theta}{2r^2} \xrightarrow{\theta = \frac{\pi}{2}} 0 \\ \nabla_\phi(\boldsymbol{\partial}_\phi)^\phi = -\dfrac{(r^2\sin^2\theta)_{,\phi}}{2g_{\phi\phi}} = 0 \end{cases}$$



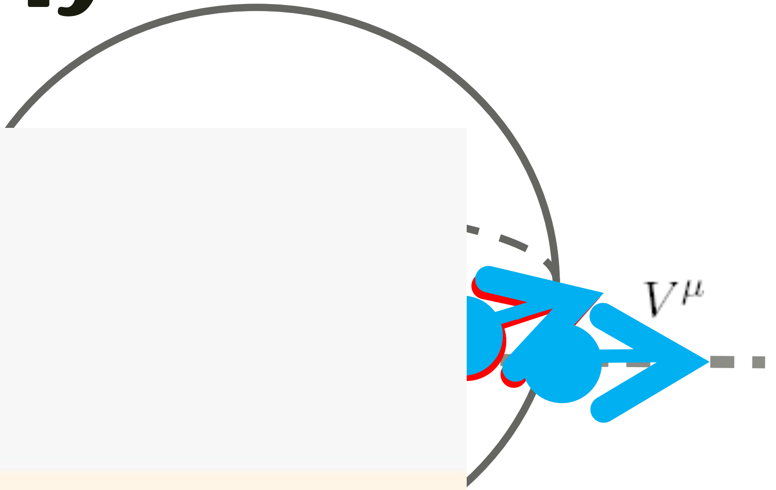
$$ds^2 = \gamma_{ij}\mathrm{d}\hat{x}^i\mathrm{d}\hat{x}^j = \hat{r}^2\mathrm{d}\hat{\theta}^2 + \hat{r}^2\sin^2\hat{\theta}\mathrm{d}\hat{\phi}^2, \quad \hat{x}^i = (\hat{\theta}, \hat{\phi})$$

$$\hat{\nabla}_{\hat{\phi}}(\boldsymbol{\partial}_{\hat{\phi}})^{\hat{\mu}} = \hat{\nabla}_{\hat{\phi}}\delta^{\hat{\mu}}_{\hat{\phi}} = \partial_{\hat{\phi}}\delta^{\hat{\mu}}_{\hat{\phi}} + \Gamma^{\hat{\mu}}_{\hat{\phi}\hat{\phi}}$$

$$= -\frac{\gamma_{\hat{\phi}\hat{\phi},\hat{\mu}}}{2\gamma_{\hat{\mu}\hat{\mu}}} = -\frac{(\hat{r}^2\sin^2\hat{\theta})_{,\hat{\mu}}}{2\gamma_{\hat{\mu}\hat{\mu}}}$$

$$= \begin{cases} \hat{\nabla}_{\hat{\phi}}(\boldsymbol{\partial}_{\hat{\phi}})^{\hat{\mu}=\hat{\theta}} = -\dfrac{(\hat{r}^2\sin^2\hat{\theta})_{,\hat{\theta}}}{2\gamma_{\hat{\theta}\hat{\theta}}} = -\dfrac{2\hat{r}^2\sin\hat{\theta}\cos\hat{\theta}}{2\hat{r}^2} \xrightarrow{\theta = \frac{\pi}{2}} 0 \\ \hat{\nabla}_{\hat{\phi}}(\boldsymbol{\partial}_{\hat{\phi}})^{\hat{\mu}=\hat{\phi}} = -\dfrac{(\hat{r}^2\sin^2\hat{\theta})_{,\hat{\phi}}}{2\gamma_{\hat{\phi}\hat{\phi}}} = 0 \end{cases}$$

1. 4D quantity ≠ 3D quantity (4)




```
Inverse[( gtt gtx gty gz
          gtx gxx gxy gxz
          gty gxy gyy gyz
          gtz gxz gyz gzz )] // MatrixForm // FullSimplify
역행렬                        행렬 형식       전체 간소화

Inverse[( gxx gxy gxz
          gyz gyy gyz
          gxz gyz gzz )] // MatrixForm // FullSimplify
역행렬                   행렬 형식       전체 간소화
```

xForm=

$$\begin{pmatrix}
\frac{gxz^2\,gyy-2\,gxy\,gxz\,gyz+gxy^2\,gzz+gxx\,(gyz^2-gyy\,gzz)}{gtt\,gxz^2\,gyy-2\,gtt\,gxy\,gxz\,gyz+gtt\,gxx\,gyz^2-gtz\,gxy^2\,gz+gtz\,gxx\,gyy\,gz-gtx\,(gxz\,gyy-gxy\,gyz)\,(gtz+gz)+gtt\,(gxy^2-gxx\,gyy)\,gzz+gty^2\,(-gxz^2+gxx\,gzz)+gty\,(gtz\,gxy\,gxz-gtz\,gxx\,gyz+2\,gtx\,gxz\,gyz+gxy\,gxz\,gz-gx} \\
\frac{gtz\,gxz\,gyy-gtz\,gxy\,gyz-gty\,gxz\,gyz+gtx\,gyz^2+gty\,gxy\,gzz-gtx\,gyy\,gzz}{-gtt\,gxz^2\,gyy+2\,gtt\,gxy\,gxz\,gyz-gtt\,gxx\,gyz^2+gtz\,gxy^2\,gz-gtz\,gxx\,gyy\,gz+gtx\,(gxz\,gyy-gxy\,gyz)\,(gtz+gz)-gtt\,gxy^2\,gzz+gtt\,gxx\,gyy\,gzz+gty^2\,(gxz^2-gxx\,gzz)+gty\,(-gtz\,gxy\,gxz+gtz\,gxx\,gyz-2\,gtx\,gxz\,gyz-gxy\,gxz\,g} \\
\frac{-gtz\,gxy\,gxz+gty\,gxz^2+gtz\,gxx\,gyz-gtx\,gxz\,gyz-gty\,gxx\,gzz+gtx\,gxy\,gzz}{-gtt\,gxz^2\,gyy+2\,gtt\,gxy\,gxz\,gyz-gtt\,gxx\,gyz^2+gtz\,gxy^2\,gz-gtz\,gxx\,gyy\,gz+gtx\,(gxz\,gyy-gxy\,gyz)\,(gtz+gz)-gtt\,gxy^2\,gzz+gtt\,gxx\,gyy\,gzz+gty^2\,(gxz^2-gxx\,gzz)+gty\,(-gtz\,gxy\,gxz+gtz\,gxx\,gyz-2\,gtx\,gxz\,gyz-gxy\,gxz\,g} \\
\frac{-gty\,gxy\,gxz+gtx\,gxz\,gyy+gtz\,(gxy^2-gxx\,gyy)+gty\,gxx\,gyz-gtx\,gxy\,gyz}{-gtt\,gxz^2\,gyy+2\,gtt\,gxy\,gxz\,gyz-gtt\,gxx\,gyz^2+gtz\,gxy^2\,gz-gtz\,gxx\,gyy\,gz+gtx\,(gxz\,gyy-gxy\,gyz)\,(gtz+gz)-gtt\,gxy^2\,gzz+gtt\,gxx\,gyy\,gzz+gty^2\,(gxz^2-gxx\,gzz)+gty\,(-gtz\,gxy\,gxz+gtz\,gxx\,gyz-2\,gtx\,gxz\,gyz-gxy\,gxz\,g}
\end{pmatrix}$$

xForm=

$$\begin{pmatrix}
\frac{-gyz^2+gyy\,gzz}{-gxz^2\,gyy-gxx\,gyz^2+gxz\,gyz\,(gxy+gyz)+gxx\,gyy\,gzz-gxy\,gyz\,gzz} & \frac{-gxz\,gyz+gxy\,gzz}{gxz^2\,gyy-gxz\,gyz\,(gxy+gyz)+gxy\,gyz\,gzz+gxx\,(gyz^2-gyy\,gzz)} & \frac{gxz\,gyy-gxy\,gyz}{gxz^2\,gyy-gxz\,gyz\,(gxy+gyz)+gxy\,gyz\,gzz+gxx\,(gyz^2-gyy\,gzz)} \\
\frac{gyz\,(gxz-gzz)}{-gxz^2\,gyy-gxx\,gyz^2+gxz\,gyz\,(gxy+gyz)+gxx\,gyy\,gzz-gxy\,gyz\,gzz} & \frac{gxz^2-gxx\,gzz}{gxz^2\,gyy-gxz\,gyz\,(gxy+gyz)+gxy\,gyz\,gzz+gxx\,(gyz^2-gyy\,gzz)} & \frac{(gxx-gxz)\,gyz}{gxz^2\,gyy-gxz\,gyz\,(gxy+gyz)+gxy\,gyz\,gzz+gxx\,(gyz^2-gyy\,gzz)} \\
\frac{gxz\,gyy-gyz^2}{gxz^2\,gyy-gxz\,gyz\,(gxy+gyz)+gxy\,gyz\,gzz+gxx\,(gyz^2-gyy\,gzz)} & \frac{gxy\,gxz-gxx\,gyz}{-gxz^2\,gyy-gxx\,gyz^2+gxz\,gyz\,(gxy+gyz)+gxx\,gyy\,gzz-gxy\,gyz\,gzz} & \frac{gxx\,gyy-gxy\,gyz}{-gxz^2\,gyy-gxx\,gyz^2+gxz\,gyz\,(gxy+gyz)+gxx\,gyy\,gzz-gxy\,gyz\,gzz}
\end{pmatrix}$$

2. PDE (1) GR

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu}$$

$$R \equiv R^{\mu\nu}{}_{\mu\nu}$$

$$\Gamma^{\rho}_{\mu\nu} \equiv \frac{1}{2} g^{\rho\sigma}\left(g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma}\right)$$

```
EinsteinCD[- ... g] & // Expand //
ToCanonical
```

확장

$-\frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} \ldots$, $\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \ldots \partial_\beta g_{\alpha\gamma} +$, $\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \ldots$, $\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \ldots$, $\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \ldots \partial_\nu \partial_\mu g_{\alpha\beta}$

2. PDE (2) ODE case

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = x \;\rightarrow\; f(x) = \frac{1}{2}x^2 + C \xrightarrow{f(0)=1} \frac{1}{2}x^2 + 1$$

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} = f(x) \;\rightarrow\; f(x) = Ae^{x} + Be^{-x} \xrightarrow[]{\substack{f(0)=2 \\ f'(0)=0}} e^{x} + e^{-x}$$

2. PDE (3) Cauchy problem

Formal statement [edit]

For a partial differential equation defined on $\mathbf{R}^{n+1}$ and a smooth manifold $S \subset \mathbf{R}^{n+1}$ of dimension n (S is called the Cauchy surface), the Cauchy problem consists of finding the unknown functions $u_1, \ldots, u_N$ of the differential equation with respect to the independent variables $t, x_1, \ldots, x_n$ that satisfies[2]

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = F_i \left(t, x_1, \ldots, x_n, u_1, \ldots, u_N, \ldots, \frac{\partial^k u_j}{\partial t^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \ldots \right)$$
$$\text{for } i, j = 1, 2, \ldots, N;\ k_0 + k_1 + \cdots + k_n = k \leq n_j;\ k_0 < n_j$$

subject to the condition, for some value $t = t_0$,

$$\frac{\partial^k u_i}{\partial t^k} = \phi_i^{(k)}(x_1, \ldots, x_n) \quad \text{for } k = 0, 1, 2, \ldots, n_i - 1$$

where $\phi_i^{(k)}(x_1, \ldots, x_n)$ are given functions defined on the surface S (collectively known as the **Cauchy data** of the problem). The derivative of order zero means that the function itself is specified.

Cauchy–Kowalevski theorem [edit]

The Cauchy–Kowalevski theorem states that *If all the functions F_i are analytic in some neighborhood of the point $(t^0, x_1^0, x_2^0, \ldots, \phi_{j,k_0,k_1,\ldots,k_n}^0, \ldots)$, and if all the functions $\phi_j^{(k)}$ are analytic in some neighborhood of the point $(x_1^0, x_2^0, \ldots, x_n^0)$, then the Cauchy problem has a unique analytic solution in some neighborhood of the point $(t^0, x_1^0, x_2^0, \ldots, x_n^0)$.*

[Wikipedia]

3. Initial value problem / well-posedness

- Initial value formulation:
  - *appropriate initial data* > *subsequent uniquely determined dynamical evolution*
- Appropriate initial data:
  - *small changes in initial data* > *small change in solution*
    > *predictable physics law*
  - *Any changes in initial data can not change solutions outside causal future.*

    > *"Initial value formulation" is well-posed.*

3. Initial value problem / well-posedness

- Initial value forr
  - *appropriate* *amical evolution*
- Appropriate init
  - *small changes in initial data* > *small change in solution*
    > *predictable physics law*
  - *Any changes in initial data can not change solutions outside causal future.*

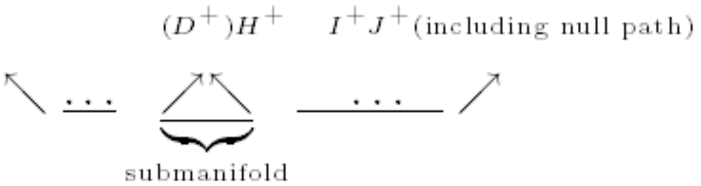
> *"Initial value formulation" is well-posed.*

Newtonian mechanics:

$$\frac{d^2 q_i}{dt^2} = F_i\left(q_1, \cdots, q_n; \frac{dq_1}{dt}, \cdots, \frac{dq_n}{dt}\right)$$

initial data: $q_{10}, \cdots, q_{n0}, \left(\frac{dq_1}{dt}\right)_0, \cdots, \left(\frac{dq_n}{dt}\right)_0$ at $t = t_0$

3. Causal structure

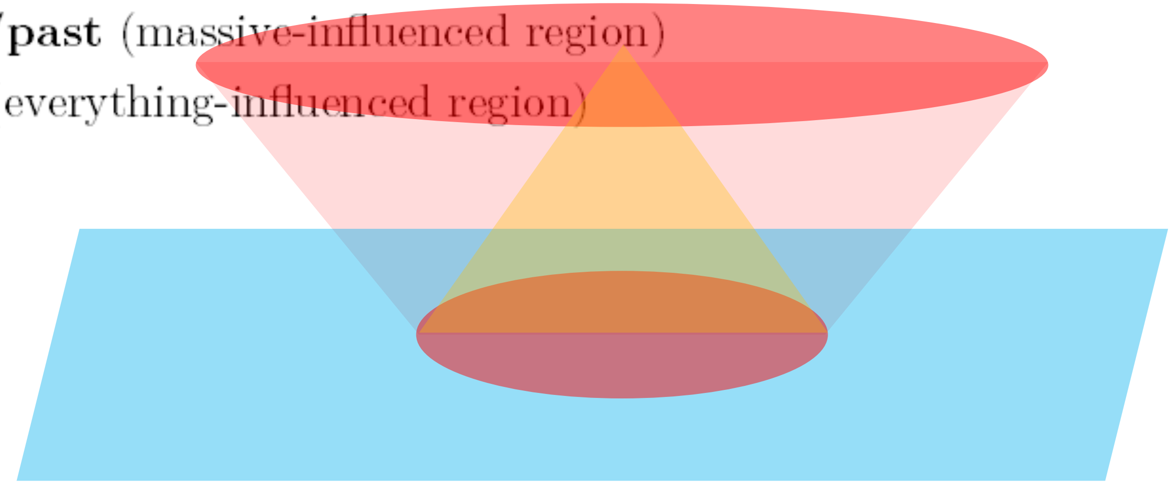


$D^{\pm}(S)$: future/past **domain** of dependence (determined region)

$H^{\pm}(S)$: future/past **Cauchy horizon** (determined region limit)

$I^{\pm}(S)$: chronological **future/past** (massive-influenced region)

$J^{\pm}(S)$: causal **future/past** (everything-influenced region)



3. Globally hyperbolic spacetime

- **Cauchy surface**

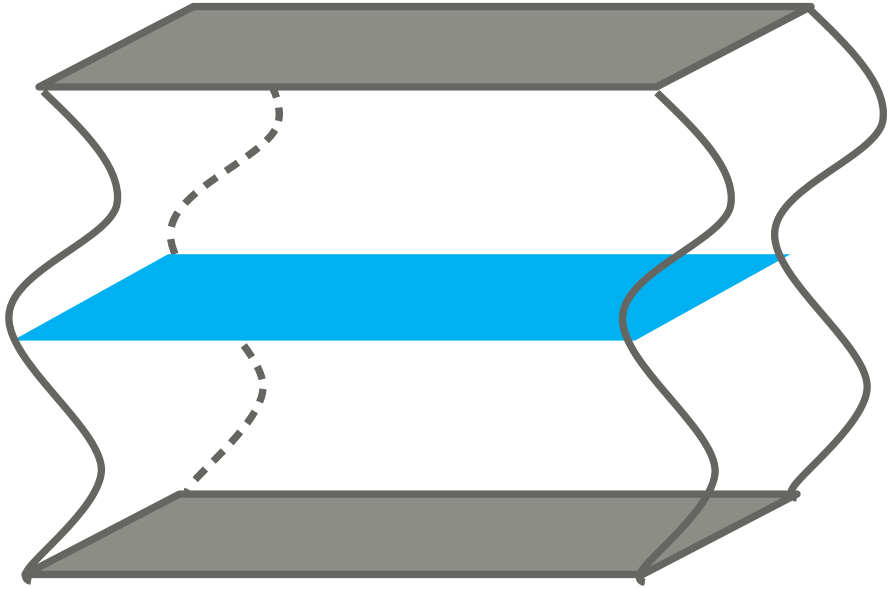
  Σ in $\mathcal{M}$ of one-time intersections with each causal curve

  $(\Sigma|_{\text{d.o.d of }\Sigma=\mathcal{M}})$

- **Globally hyperbolic spacetime**

  $(\mathcal{M}, g)$ which has Σ_{Cauchy}

  $\xrightarrow{\text{topology}} (\mathcal{M}, g)|_{\mathcal{M}=\Sigma\times\mathbb{R}}$



Initial value problem / well-posedness

$$g^{\mu\nu}(x;\phi_\alpha;\nabla_\sigma\phi_\alpha)\nabla_\mu\nabla_\nu\phi_\beta = F_\beta(x;\phi_\alpha;\nabla_\sigma\phi_\alpha)$$

THEOREM 10.1.3. *Let $(\phi_0)_1, \ldots, (\phi_0)_n$ be any solution of the quasilinear hyperbolic system (10.1.21) on a manifold M and let $(g_0)^{ab} = g^{ab}(x; (\phi_0)_j; \nabla_c(\phi_0)_j)$. Suppose $(M, (g_0)_{ab})$ is globally hyperbolic (or, alternatively, consider a globally hyperbolic region of this spacetime). Let Σ be a smooth spacelike Cauchy surface for $(M, (g_0)_{ab})$. Then, the initial value formulation of equation (10.1.21) is well posed on Σ in the following sense: For initial data on Σ sufficiently close to the initial data for $(\phi_0)_1, \ldots, (\phi_0)_n$, there exists an open neighborhood O of Σ such that equation (10.1.21) has a solution, $\phi_1, \ldots, \phi_n$, in O and $(O, g_{ab}(x; \phi_j; \nabla_c\phi_j))$ is globally hyperbolic. The solution is unique in O and propagates causally in the sense that if the initial data for $\phi'_1, \ldots, \phi'_n$ agree with that of $\phi_1, \ldots, \phi_n$ on a subset, S, of Σ, then the solutions agree on $O \cap D^+(S)$. Finally, the solutions depend continuously on the initial data in the sense described above for the Klein-Gordon field.*

[Wald (1984) p.251]

Initial value formulation in GR

THEOREM 10.2.2. *Let Σ be a three-dimensional C^∞ manifold, let h_{ab} be a smooth Riemannian metric on Σ, and let K_{ab} be a smooth symmetric tensor field on Σ. Suppose h_{ab} and K_{ab} satisfy the constraint equations (10.2.28) and (10.2.30). Then there exists a unique C^∞ spacetime, (M, g_{ab}), called the maximal Cauchy development of (Σ, h_{ab}, K_{ab}), satisfying the following four properties:* (i) *(M, g_{ab}) is a solution of Einstein's equation.* (ii) *(M, g_{ab}) is globally hyperbolic with Cauchy surface Σ.* (iii) *The induced metric and extrinsic curvature of Σ are, respectively, h_{ab} and K_{ab}.* (iv) *Every other spacetime satisfying* (i)–(iii) *can be mapped isometrically into a subset of (M, g_{ab}). Furthermore, (M, g_{ab}) satisfies the desired domain of dependence property in the following sense. Suppose (Σ, h_{ab}, K_{ab}) and $(\Sigma', h'_{ab}, K'_{ab})$ are initial data sets with maximal developments (M, g_{ab}) and (M', g'_{ab}). Suppose there is a diffeomorphism between $S \subset \Sigma$ and $S' \subset \Sigma'$ which carries (h_{ab}, K_{ab}) on S into (h'_{ab}, K'_{ab}) on S'. Then $D(S)$ in the spacetime (M, g_{ab}) is isometric to $D(S')$ in the spacetime (M', g'_{ab}). Finally, the solution g_{ab} on M depends continuously on the initial data (h_{ab}, K_{ab}) on Σ.* (A precise definition of the topologies on initial data and solutions which makes this map continuous is given in Hawking and Ellis 1973.)

[Wald (1984) p.264]

Why 3+1? Action as well.... (1)

$$\text{(point particle)}\ S = \int d\tau \ \rightarrow \ \frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\tau^2} + \Gamma^\beta_{\alpha\nu} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0$$

Why 3+1? Action as well.... (2)

$$S = \frac{1}{16\pi G}\int_{t_1}^{t_2}\left\{\int_{\Sigma_t} N(\widehat{R} - K^2 + K_{ij}K^{ij})\sqrt{\gamma}\, d^3x\right\}\mathrm{d}t$$
$$-\frac{1}{16\pi G}\int_{t_1}^{t_2}\left[\int_{\Sigma_t} 2(\widehat{\nabla}_i\widehat{\nabla}^i N)\sqrt{\gamma}\, d^3x\right]\mathrm{d}t + \frac{1}{16\pi G}\int_S 2\mathcal{K}\sqrt{\gamma'}\, d^3y$$

Why 3+1? Action as well.... (3)

$$H = -\frac{1}{16\pi G}\int_{\Sigma_t} d^3x\sqrt{\gamma}\,(NC_0 - 2\beta^i C_i)$$
$$+\frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\cdot 2[r_i(K^i{}_j\beta^j - K\beta^i) - N(\kappa-\kappa_0)]$$

where κ is an extrinsic curvature of S_t.

$$M_{\mathrm{ADM}} = -\frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\cdot 2(\kappa-\kappa_0)$$

Why 3+1? Action as well.... (4)

```
INFO (TwoPunctures): The two puncture masses are mp=0.48072940437916789 and mm=0
.48072940437915207
INFO (TwoPunctures): Puncture 1 ADM mass is 0.5
INFO (TwoPunctures): Puncture 2 ADM mass is 0.5
INFO (TwoPunctures): The total ADM mass is 1.15612
```

$$M_{\mathrm{ADM}} = -\frac{1}{16\pi G}\int_{S_t=\Sigma_t\cap S} d^2\theta\sqrt{\sigma}\cdot 2(\kappa-\kappa_0)$$

Now, Let's split Einstein eq.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

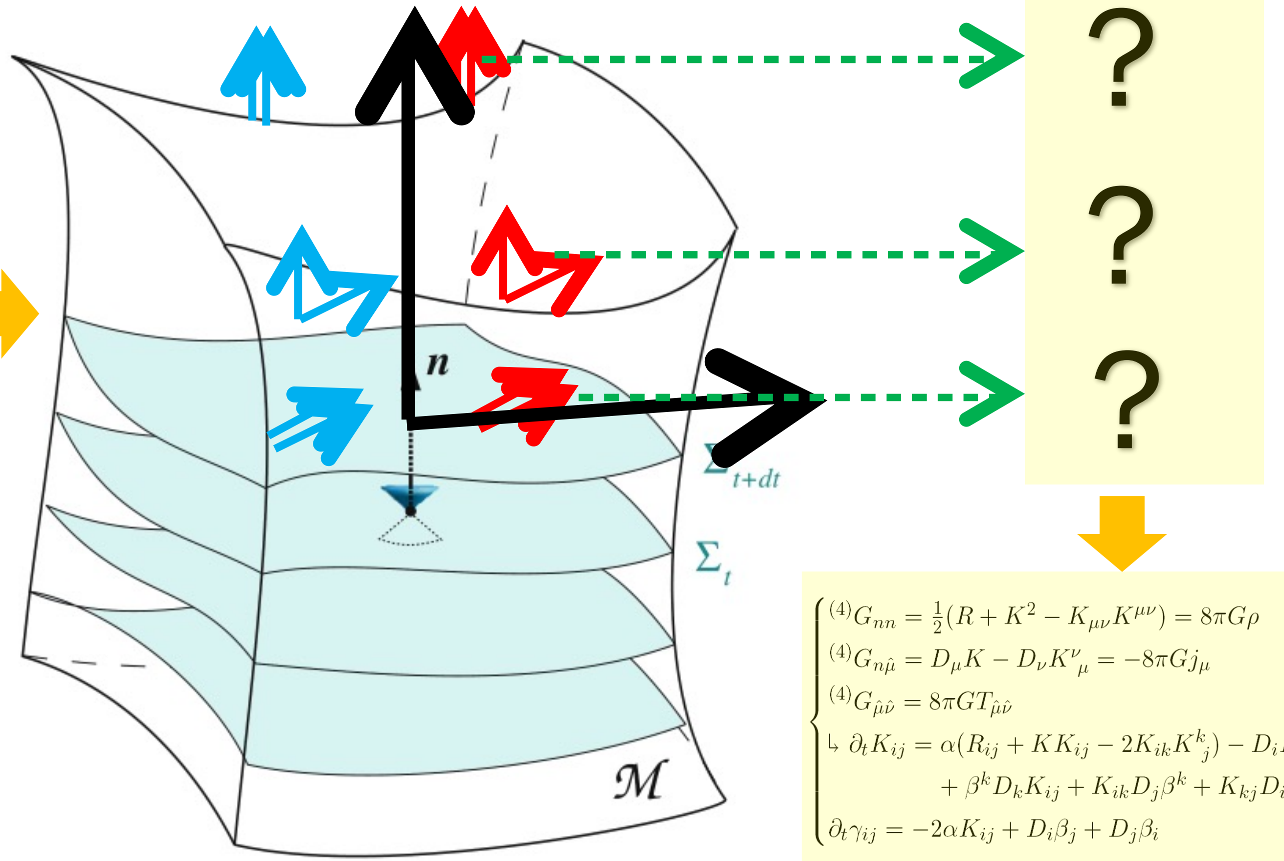
Our Goal

$$
\begin{cases}
{}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho \\
{}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu \\
{}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \\
\hookrightarrow \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_iD_j\alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \\
\qquad\qquad + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k \\
\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i
\end{cases}
$$

3+1 decomposition of Einstein ~~eq.~~ tensor

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Manifold/Hypersurface
- Foliation
- Normal vector
- 3+! metric
- Gauss-normal coordinate
- Projection
- Extrinsic curvature
- Intrinsic curvature
- Projection of Riemann tensor (Gauss..)



$$\begin{cases} {}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho \\ {}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu \\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \\ \Lsh\ \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_iD_j\alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)) \\ \qquad\qquad + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k \\ \partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

Manifold and submanifold/hypersurface

- Manifold
- Immersion/embedding

$$(\text{immersion}) \begin{cases} (\text{derivative one-to-one}) \\ (\text{points no need to be one-to-one} \to \text{self-interaction possible}) \end{cases}$$

$$(\text{embedding}) \begin{cases} (\text{immersion + topologically same (homeomorphic)}) \\ (\text{Locally } N \to M \text{ is homeomorphic, that is, locally immersion is embedding.}) \\ \quad \Lsh (\text{Local neighbourhood of a point } x \text{ on } N \\ \qquad \text{can not be mapped to a self-interacted image.}) \end{cases}$$

- Codimension/submanifold

$$(\text{immersion, } M \to N) \to [\ \dim(N) - \dim(M) : \text{codimension}\ ]$$

$$(\text{embedding, } M \to N) \ \to \ [\ M \text{ is a submanifold of } N\ ]$$

- hypersurface

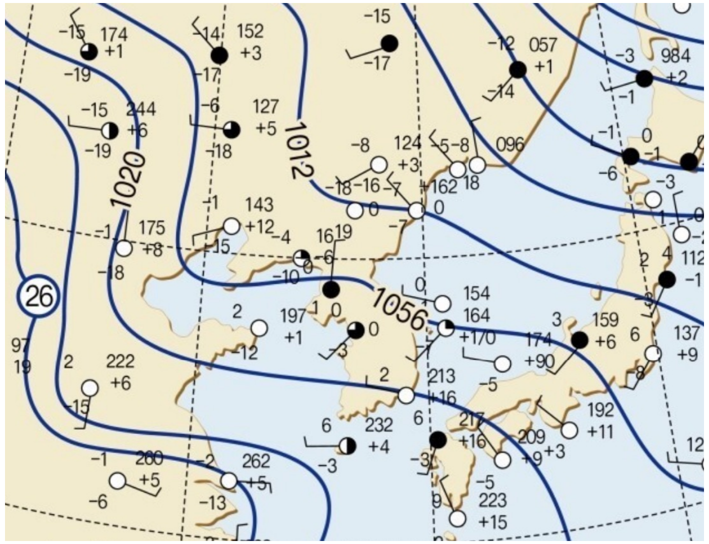
codimension-1 submanifold

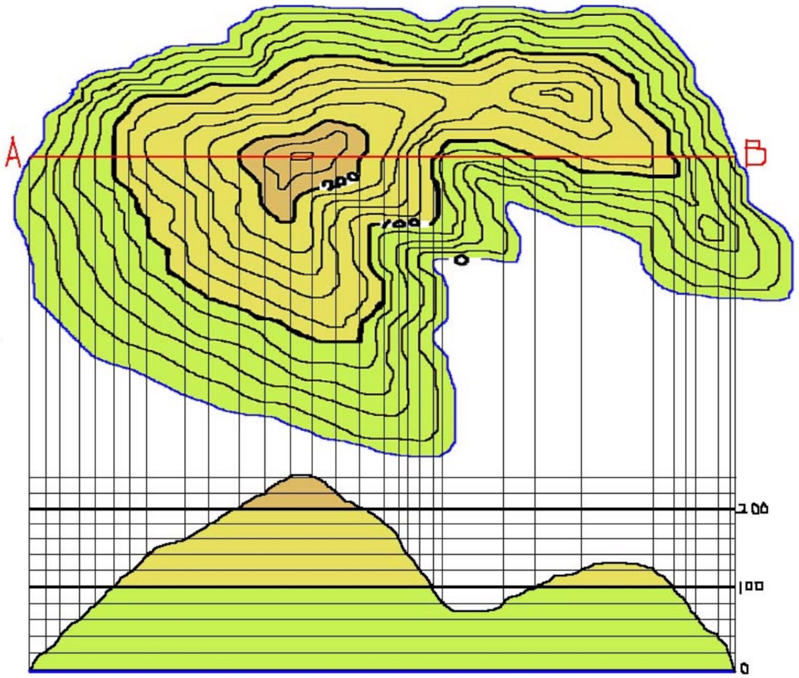
Submanifold and foliation

- Foliation? Slicing? How?

Submanifold and foliation

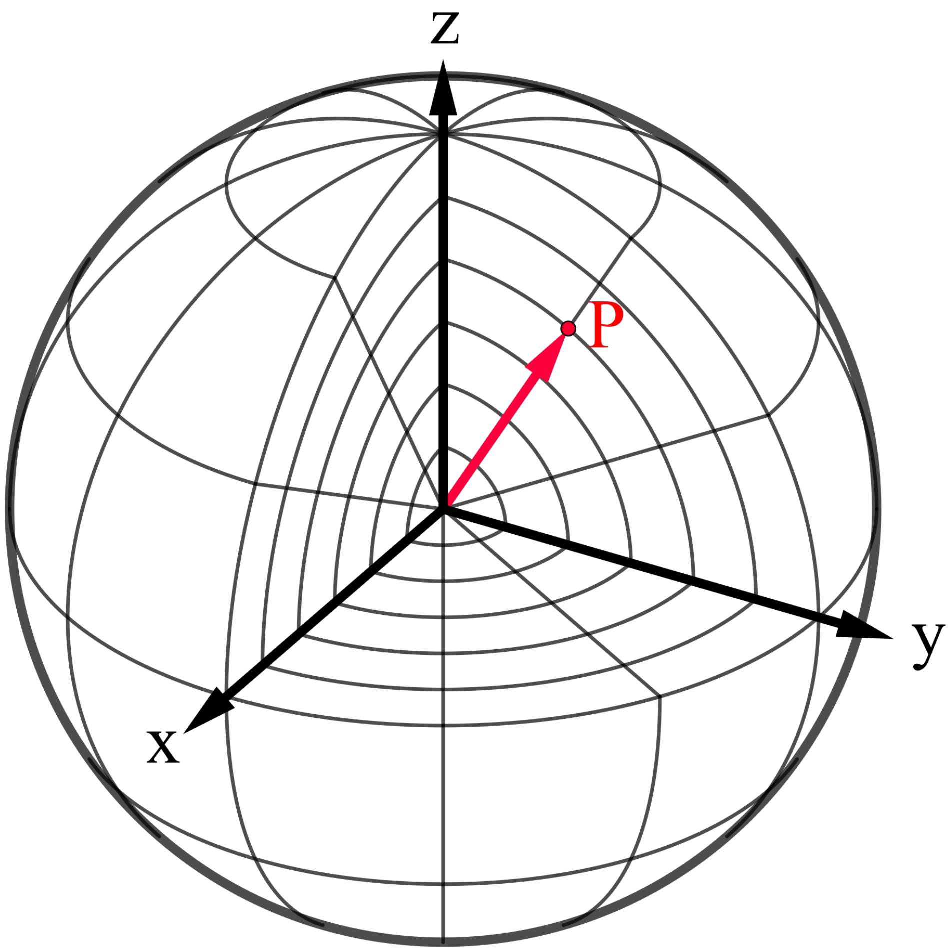
- Foliation? Slicing? How?





Submanifold and foliation

- Foliation? Slicing? How?
  For example, with r….



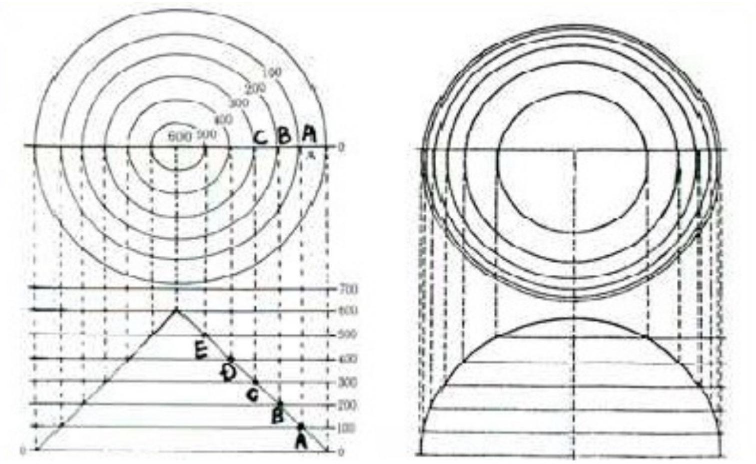
Submanifold and foliation

- Surface forming

Frobenius' theorem : $[V_{(a)}, V_{(b)}]^\mu = \alpha^c V^\mu_{(c)}$ (closed vector field) $\rightarrow$ integral submanifold

Frobenius' theorem : $\nabla_{[\mu} n^{(a)}_{\nu]} V^\mu W^\nu = 0$

for V^μ, W^μ such that $n^{(a)}_\mu V^\mu = 0,\ n^{(a)}_\mu W^\mu = 0$



- Non-degenerate scalar fields labels submanifolds. > foliation

$$\begin{cases} (f^i\text{: exterior coordinates}) \leftarrow (\text{foliation}) \leftarrow \begin{bmatrix} (M \rightarrow N)\text{'s Codimension number's} \\ \text{non-degenerated } f^i(x) \text{ labels submanifold.} \end{bmatrix} \\ (y^a\text{: coordinates on the submanifold } M) \end{cases}$$

$\Lsh$ [in a neighborhood of M, coordinates: $x^\mu = (f^i, y^a)$]

3+1 decomposition of Einstein ~~eq.~~ tensor

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- ~~Manifold/Hypersurface~~
- ~~Foliation~~
- Normal vector
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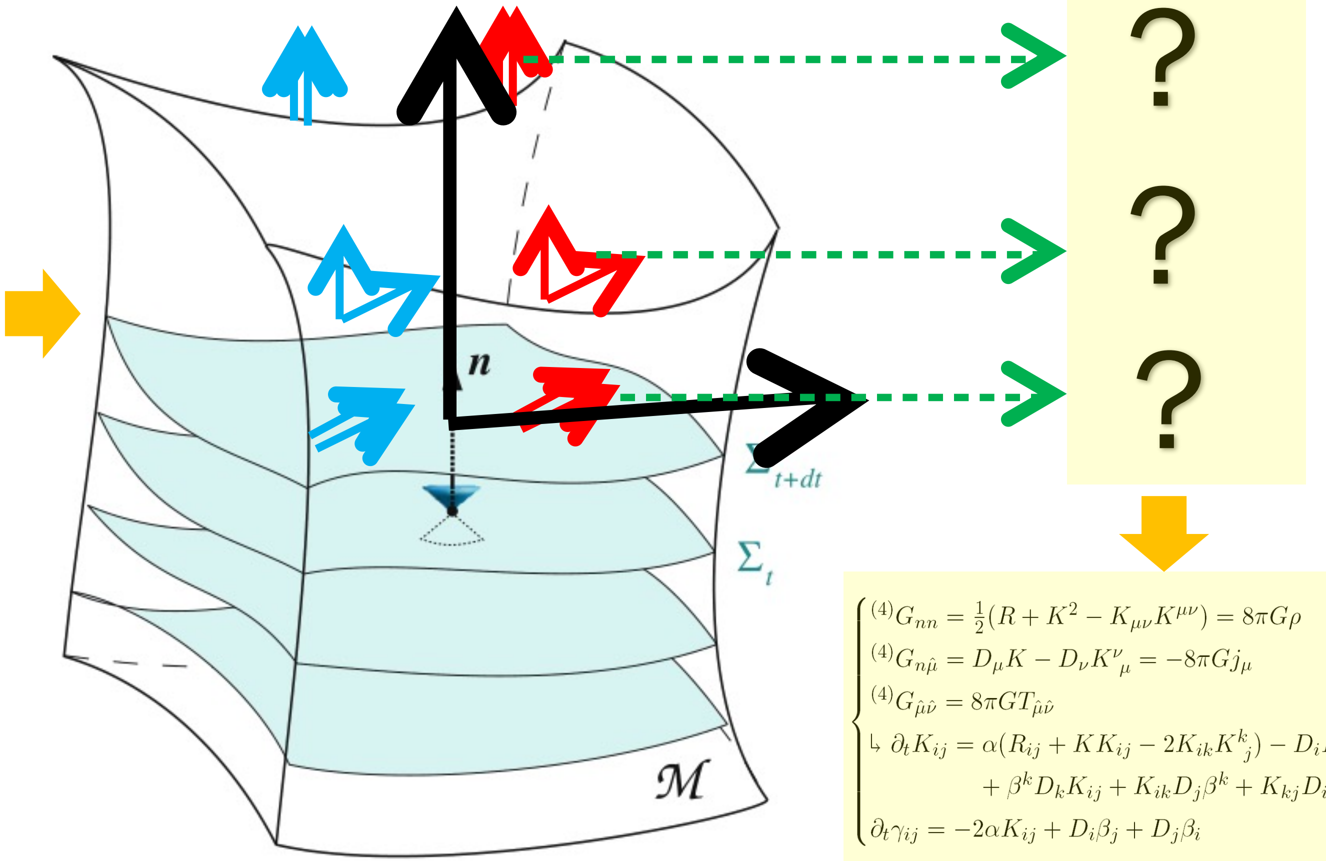
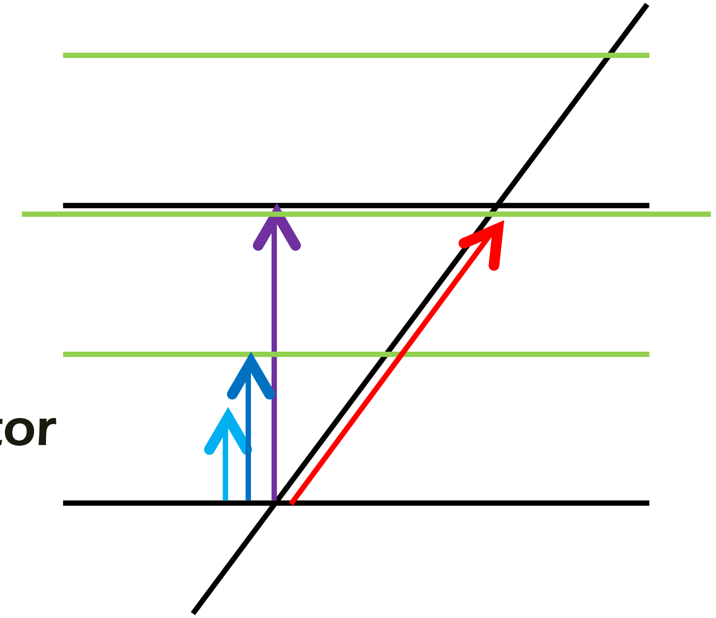


$$\begin{cases} {}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho \\ {}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu \\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \\ \Lsh\ \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_i D_j \alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \\ \qquad\qquad + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k \\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$


영화로운 덕후생활
영화로운 드라마 | 쌤은 대체 누구셈?
tvN
<이상한 나라의 수학자> vs. <명불허전>
자료 협조: SHOWBOX

Normal/tangent vector



- Scalar field > Submanifold Σ_t
- Gradient of Scalar field > normal vector (one-form basis)

$$\mathbf{d}t = \boldsymbol{\nabla} t \qquad \langle \boldsymbol{V} \text{ on } \Sigma_t,\ \boldsymbol{\nabla} t\rangle = 0$$

- A curve intersecting the hypersurface > tangent basis vector

$$\gamma(t) \text{ by } \Sigma_t :\ \boldsymbol{t} = \boldsymbol{\partial_t}$$

- Tangent basis and one-form

$$\gamma(t) \text{ by } \Sigma_t :\ \langle \boldsymbol{t}, \boldsymbol{\nabla} t\rangle = t^\alpha \nabla_\alpha t = t^\alpha \partial_\alpha t = \partial_t t = 1$$

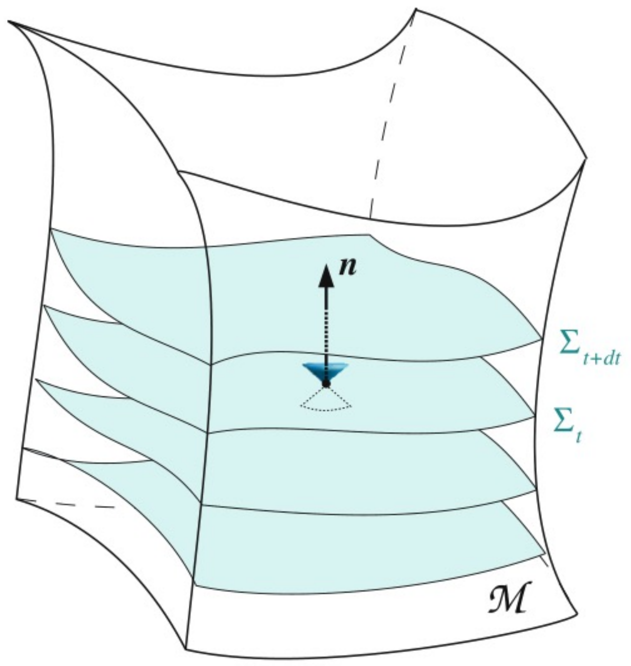
- A congruence of curves > coordinates > vector components

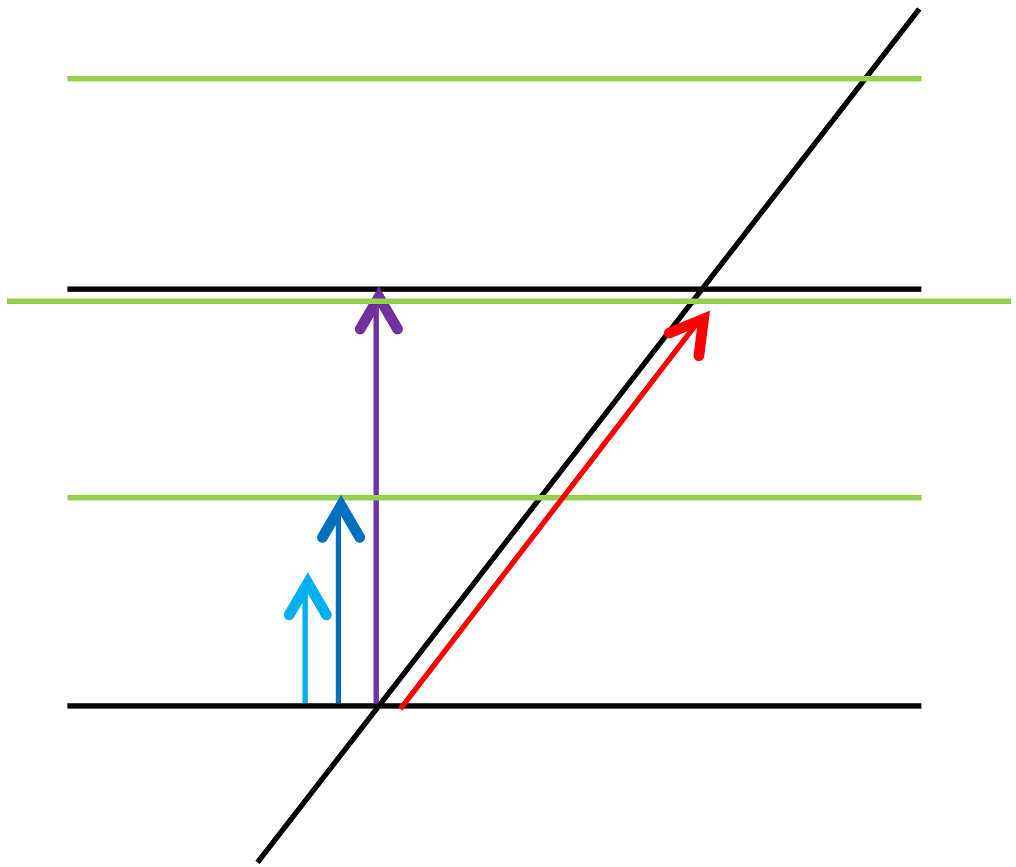
$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} \equiv (\partial_t)^\alpha \xrightarrow{\text{when } x^\alpha \equiv (t, y^a)} \delta_t{}^\alpha = (1,0,0,0)$$

$$(y_a)^\alpha = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t \equiv (\partial_{y^a})^\alpha \xrightarrow{\text{when } x^\alpha \equiv (t, y^a)} \delta_a{}^\alpha \xrightarrow{\alpha = 1} (0,1,0,0)$$

$$\nabla_\alpha t = \left(\frac{\partial t}{\partial x^\alpha}\right) \equiv (\mathrm{d}t)_\alpha \xrightarrow{\text{when } x^\alpha \equiv (t, y^a)} \delta^t{}_\alpha = (1,0,0,0)$$

> Not unit vectors





1. foliation with t
2. T=1 at T=2
3. dt is short
4. t long along grid line
5. dt.t=1
6. Two basis dt, t like V=Vx $\hat{x}$
7. X=0 along t axis
8. dx is not tangent
9. For intrinsic term, (mu to i)

Unit normal vector (normalization)

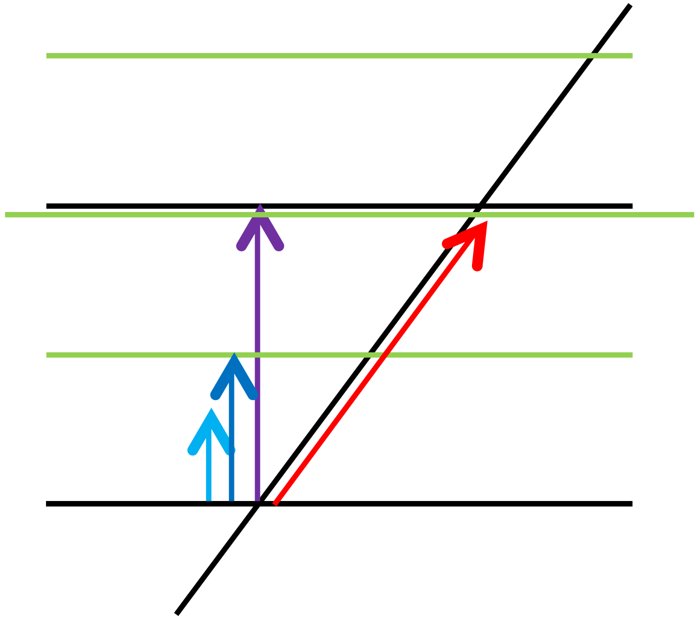
$$n_\mu = -N\partial_\mu t = -N\nabla_\mu t = -N(\mathrm{d}t)_\mu \quad \rightarrow \quad n_\mu (y_a)^\mu = 0$$

$$n^\mu = -N(\mathbf{d}t)^\mu = -N\nabla^\mu t = -N g^{\mu\nu}\nabla_\nu t = -N g^{\mu\nu}\delta^t_\nu = -N g^{\mu t}$$

↳ $\boxed{n^\mu = -N g^{\mu t} > 0}$

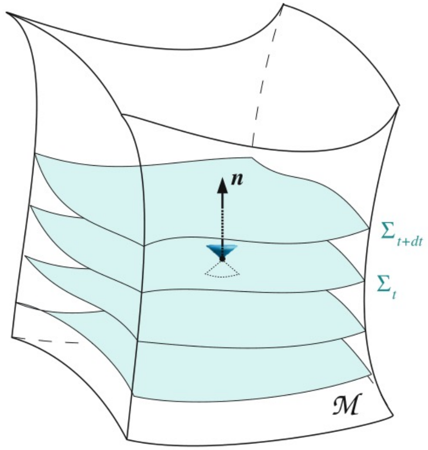
$$n_\mu = -N(\mathbf{d}t)_\mu = -N\nabla_\mu t = -N\delta^t_\nu = (-N, 0, 0, 0), \quad (\text{where } N > 0)$$

↳ $\boxed{n_\mu = -N\delta^t_\mu}$



$$n_\mu n^\mu = N^2 g^{\mu\nu}(\nabla_\mu t)(\nabla_\nu t) = N^2 g^{tt} = \pm 1 \equiv \sigma$$

$$\hookrightarrow \begin{cases} < 0 & \text{timelike normal vector} \rightarrow \text{spacelike hypersurface} \\ = 0 & \text{null normal vector} \rightarrow \text{null hypersurface} \\ > 0 & \text{spacelike normal vector} \rightarrow \text{timelike hypersurface} \end{cases}$$



Normal vector “n” to tangent basis “t”

$$t^\alpha = N' n^\alpha + N^a$$

$\circlearrowleft$ contracting with n_α

$$\underbrace{n_\alpha t^\alpha}_{=-N(\partial_\alpha t)t^\alpha=-N} = N' \underbrace{n^\alpha n_\alpha}_{=-1\ (\because\ \text{timelike})} + \cancel{N^a n_\alpha}$$

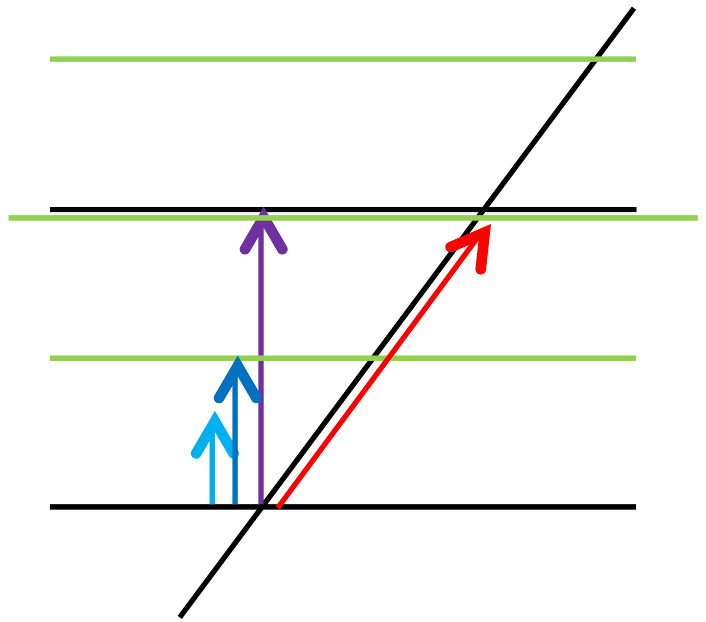
$$\Downarrow$$

$$N' = N$$

$$\therefore\ t^\alpha = \underbrace{N n^\alpha}_{\equiv m^\alpha} + N^a (y_a)^\alpha$$

$$\boxed{\boldsymbol{t} = \boldsymbol{m} + \boldsymbol{\beta}} \quad (\langle \mathbf{d}t, \boldsymbol{m} \rangle = \nabla_{\boldsymbol{m}} t = m^\mu \nabla_\mu t = 1)$$

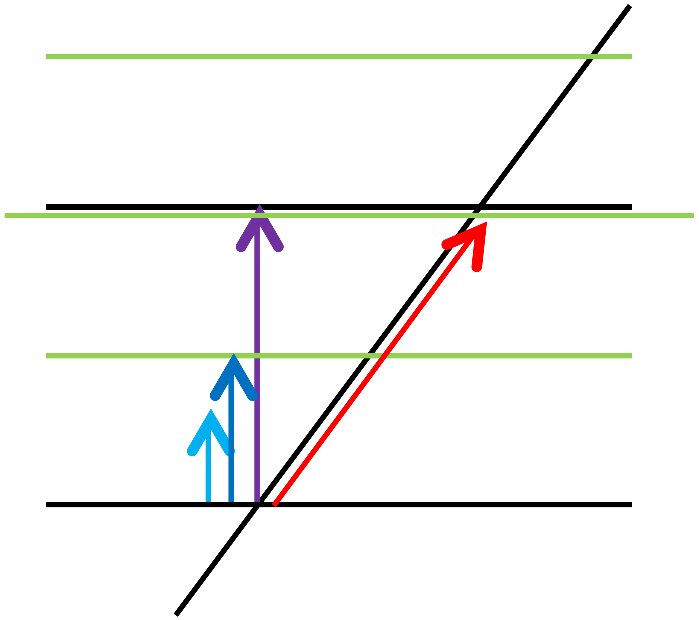
$$\begin{cases} N \equiv \alpha & \text{(lapse function)} \\ N^\alpha \equiv \beta^\alpha = (0, \vec{\beta}) & \text{(shift vector)} \\ m^\alpha = N n^\alpha = (1, -\vec{\beta}) & \text{(evolution vector)} \end{cases}$$



3+1 decomposition of metric (1)

$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu,\ g^{\mu\nu} = (\mathbf{d}x^\mu)\cdot(\mathbf{d}x^\nu)$$

$$\downarrow \begin{cases} \boldsymbol{t} = \boldsymbol{m} + \boldsymbol{\beta} & \\ N \equiv \alpha & \text{(lapse function)} \\ N^\alpha \equiv \beta^\alpha = (0, \vec{\beta}) & \text{(shift vector)} \\ m^\alpha = N n^\alpha = (1, -\vec{\beta}) & \text{(evolution vector)} \end{cases}$$



$$g_{00} = \boldsymbol{\partial}_t \cdot \boldsymbol{\partial}_t = -N^2 + \vec{\beta}\cdot\vec{\beta}$$

$$g_{0i} = \boldsymbol{\partial}_t \cdot \boldsymbol{\partial}_i = (\boldsymbol{m} + \boldsymbol{\beta}) \cdot \boldsymbol{\partial}_i = (\cancel{\boldsymbol{m}\cdot\boldsymbol{\partial}_i} + \boldsymbol{\beta}\cdot\boldsymbol{\partial}_i) = \beta_i$$

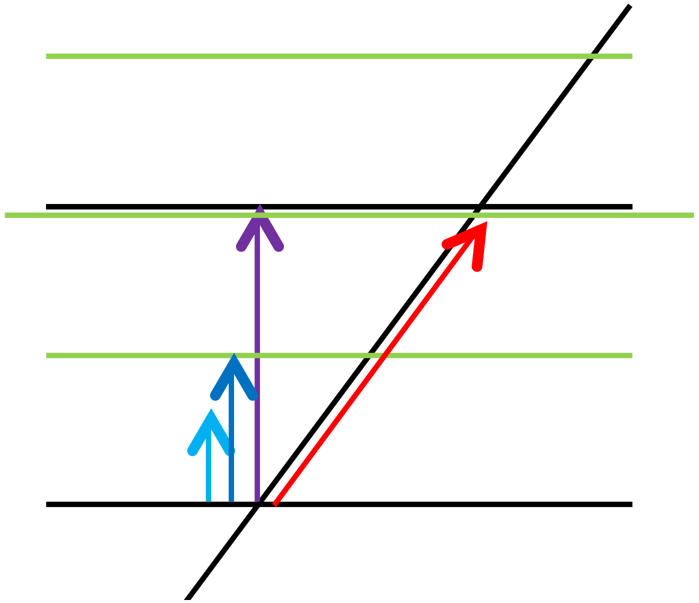
$$g_{ij} \equiv \gamma_{ij} \ \rightarrow \ \gamma_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j \equiv P_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

3+1 decomposition of metric (2)

$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu, \quad g^{\mu\nu} = (\mathbf{d}x^\mu)\cdot(\mathbf{d}x^\nu)$$

$$\downarrow \begin{cases} \boldsymbol{t} = \boldsymbol{m} + \boldsymbol{\beta} & \\ N \equiv \alpha & \text{(lapse function)} \\ N^\alpha \equiv \beta^\alpha = (0, \vec{\beta}) & \text{(shift vector)} \\ m^\alpha = N n^\alpha = (1, -\vec{\beta}) & \text{(evolution vector)} \end{cases}$$



$$\begin{aligned} ds^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -(N^2 - \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ &= -N^2\mathrm{d}t^2 + \gamma_{ij}\underbrace{(\mathrm{d}x^i + \beta^i\mathrm{d}t)}_{\mathrm{d}\hat{x}^i}\underbrace{(\mathrm{d}x^j + \beta^j\mathrm{d}t)}_{\mathrm{d}\hat{x}^j} \end{aligned}$$

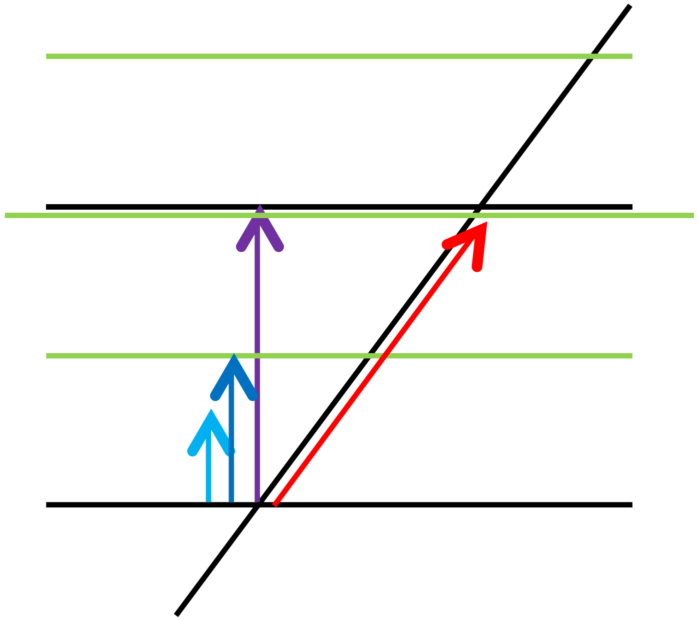
Note that $\mathrm{d}x^i$ is not on Σ_t when $\vec{\beta} \neq 0$, but $\mathrm{d}\hat{x}^i$ is on Σ_t.

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

3+1 decomposition of metric (3)

$$g_{\mu\nu} = \boldsymbol{\partial}_\mu \cdot \boldsymbol{\partial}_\nu, \quad g^{\mu\nu} = (\mathbf{d}x^\mu)\cdot(\mathbf{d}x^\nu)$$

$$\downarrow \begin{cases} \boldsymbol{t} = \boldsymbol{m} + \boldsymbol{\beta} & \\ N \equiv \alpha & \text{(lapse function)} \\ N^\alpha \equiv \beta^\alpha = (0, \vec{\beta}) & \text{(shift vector)} \\ m^\alpha = N n^\alpha = (1, -\vec{\beta}) & \text{(evolution vector)} \end{cases}$$



$$g^{00} = (\mathbf{d}t)\cdot(\mathbf{d}t) = \left(-\frac{1}{N}n^\mu\right)\left(-\frac{1}{N}n_\mu\right) = \sigma\frac{1}{N^2} = -\frac{1}{N^2}$$

$$g^{0\mu} = (\mathbf{d}t)\cdot(\mathbf{d}x^\mu) = \delta^\beta_\alpha\left(-\frac{1}{N}n^\alpha\right)\underbrace{(\mathbf{d}x^\mu)_\beta}_{=\delta^\mu_\beta} = -\frac{1}{N}n^\mu = \frac{1}{N^2}(-1, \vec{\beta})$$

$$g^{ij} = (\mathbf{d}x^i)\cdot(\mathbf{d}x^j) = g^{\mu\nu}(\mathbf{d}x^i)_\mu(\mathbf{d}x^j)_\nu = (\sigma n^\mu n^\nu + P^{\mu\nu})\underbrace{(\mathbf{d}x^i)_\mu}_{=\delta^i_\beta}\underbrace{(\mathbf{d}x^j)_\nu}_{=\delta^j_\nu}$$

$$= \sigma n^i n^j + \gamma^{ij} = -\frac{\beta^i\beta^j}{N^2} + \gamma^{ij}$$

$$g^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{N^2} & \dfrac{\beta^j}{N^2} \\ \dfrac{\beta^i}{N^2} & \gamma^{ij} - \dfrac{\beta^i\beta^j}{N^2} \end{pmatrix}$$

3+1 decomposition of metric (4)

$$\left(\begin{aligned}
&\text{Note that } g^{ij} \neq \gamma^{ij}, \\
&\qquad\qquad g^{ij} = g^{\mu i} g^{\nu j} g_{\mu\nu}, \\
&\qquad\qquad \gamma^{ij} = \gamma^{i\mu}\gamma^{j\nu}\gamma_{\mu\nu} = \gamma^{i\ell}\gamma^{jk}\gamma_{\ell k}, \\
&(1)\ g^{ik} g_{kj} = g^{i\mu} g_{\mu j} - g^{i0} g_{0j} = \delta^i{}_j - \frac{\beta^i}{N^2}\beta_j \\
&(2)\ g^{ik} g_{kj} = (\gamma^{ik} - \frac{\beta^i \beta^k}{N^2})\gamma_{kj}, \quad (\beta^i = \gamma^{ij}\beta_j)
\end{aligned}\right)$$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{\beta^j}{N^2} \\ \frac{\beta^i}{N^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{N^2} \end{pmatrix}$$

3+1 decomposition of metric (4)

$$
\begin{aligned}
ds^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\
&= -(N^2 - \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \\
&= -N^2\mathrm{d}t^2 + \gamma_{ij}\underbrace{(\mathrm{d}x^i + \beta^i\mathrm{d}t)}_{\mathrm{d}\hat{x}^i}\underbrace{(\mathrm{d}x^j + \beta^j\mathrm{d}t)}_{\mathrm{d}\hat{x}^j}
\end{aligned}
$$

$$
g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}
$$

Note that $\mathrm{d}x^i$ is not on Σ_t when $\vec{\beta} \neq 0$, but $\mathrm{d}\hat{x}^i$ is on Σ_t.

$$
g^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{N^2} & \dfrac{\beta^j}{N^2} \\ \dfrac{\beta^i}{N^2} & \gamma^{ij} - \dfrac{\beta^i\beta^j}{N^2} \end{pmatrix}
$$

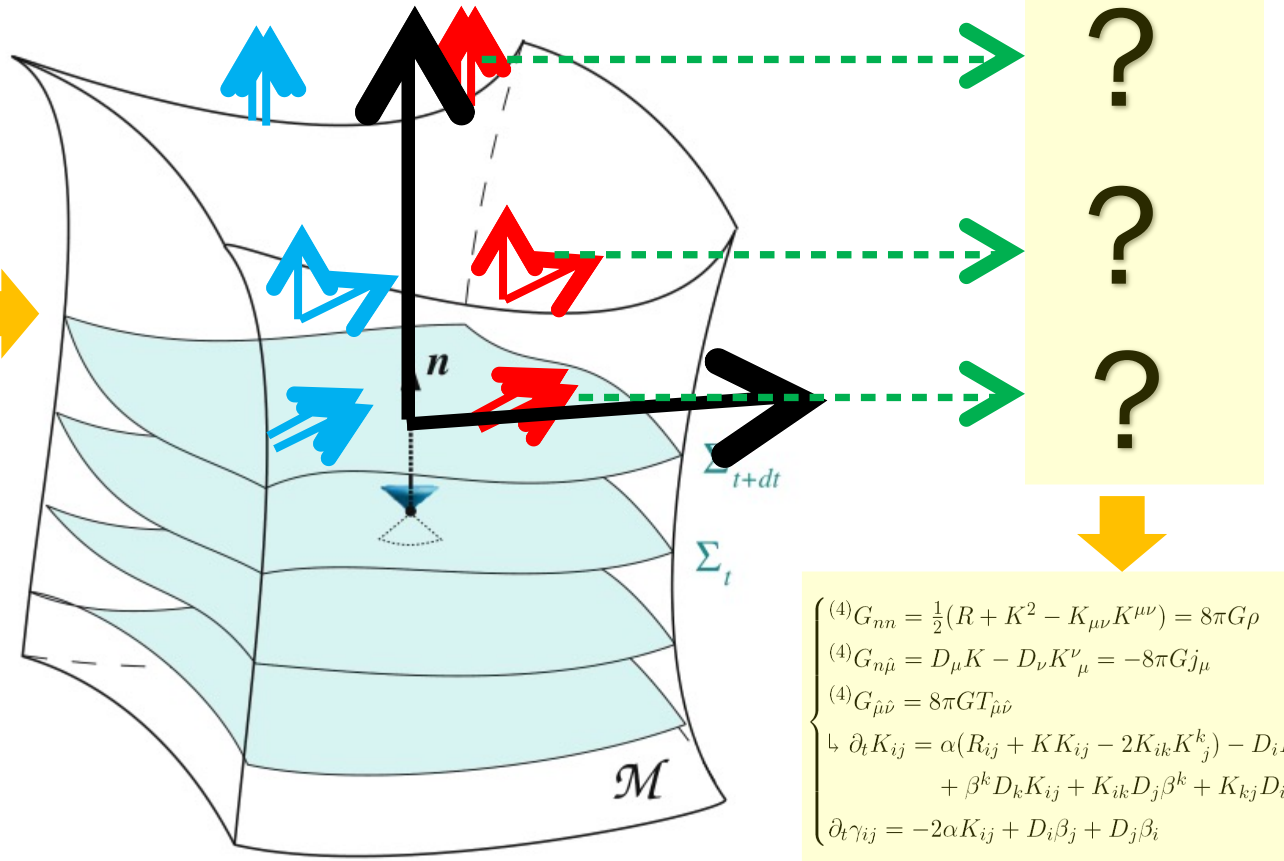
$$
g^{00} = -\frac{1}{N^2} = \frac{C_{00}}{\det(g_{\mu\nu})} = \frac{(-1)^{0+0}M_{00}}{\det(g_{\mu\nu})} = \frac{\det(\gamma_{ij})}{\det(g_{\mu\nu})}
$$

$$
\Rightarrow \boxed{\sqrt{-g} = N\sqrt{\gamma}}
$$

3+1 decomposition of Einstein ~~eq.~~ tensor

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- ~~Manifold/Hypersurface~~
- ~~Foliation~~
- ~~Normal vector~~
- ~~3+! metric~~
- Gauss-normal coordinate
- Projection
- Extrinsic curvature
- Intrinsic curvature
- Projection of Riemann tensor (Gauss..)



$$\begin{cases} {}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho \\ {}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu \\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \\ \hookrightarrow \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_iD_j\alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)) \\ \qquad\qquad + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k \\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

Gaussian-normal coordinates

The conditions for the construction of the Gaussian normal coordinates :

1. n-dimensional manifold $\mathcal{M}$, and its hypersurface Σ.
2. coordinates on Σ : $\{y_p^1, \cdots, y_p^{n-1}\}$
3. normal vector n^μ at p on Σ
4. geodesic curve parameterized by z, and the tangent vector $\partial_z = n^\mu$
5. the geodesic equation = 2nd order DE,
   the solution for the parameter is fixed by two initial condition.
6. normalization condition $(n_\mu n^\mu = \pm 1)$, and a value on Σ $(z(p) = 0)$
   $\rightarrow$ fixes z parameter on the geodesic curve.
7. With the geodesic congruence passes Σ, when the manifold is foliated with Σ's where the two conditions hold, we can have a unique coordinate system within the region where there is no intersection of geodesic curves.

> lapse? Shift?

Projection tensor (1)

$$\text{Gaussian normal coordinates}: \; ds^2 = \sigma \mathrm{d}z^2 + \gamma_{ij}\mathrm{d}y^i\mathrm{d}y^j$$

$$\hookrightarrow ds^2 = g_{\mu\nu}(\mathbf{d}x^\mu)\otimes(\mathbf{d}x^\nu)$$

$$= -\mathrm{d}t^2 + \gamma_{ij}\mathrm{d}y^i\mathrm{d}y^j \;\Leftrightarrow\; -(N^2 - \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i\mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

$$\circlearrowleft \; \mathbf{d}t = -N^{-1}\boldsymbol{n} \xrightarrow{N=1} -\boldsymbol{n} = -n_\mu(\mathbf{d}x^\mu) \qquad \hookrightarrow N = 1,\; \beta^i = 0$$

$$= -n_\mu n_\nu \mathrm{d}x^\mu\mathrm{d}x^\nu + \gamma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

$$= (-n_\mu n_\nu + \gamma_{\mu\nu})\mathrm{d}x^\mu\mathrm{d}x^\nu$$

$$\therefore \; \gamma_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu$$

$$= P_{\mu\nu} \quad (\text{projection tensor})$$

> Projection tensor = metric on hypersurface

Projection tensor (2)

- Definition: $\boxed{P_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu}$

- Projected vectors are tangent to the hypersurface.

$$(P_{\mu\nu}V^\mu)n^\nu = g_{\mu\nu}V^\mu n^\nu - \sigma n_\mu \overbrace{n_\nu V^\mu n^\nu}^{=\sigma V^\mu,\ \sigma^2=1}$$
$$= 0$$

- Act like the metric for tangent vectors

$$P_{\mu\nu}V^\mu W^\nu = g_{\mu\nu}V^\mu W^\nu - \cancelto{0}{\sigma n_\mu n_\nu V^\mu W^\nu}$$
$$= g_{\mu\nu}V^\mu W^\nu$$

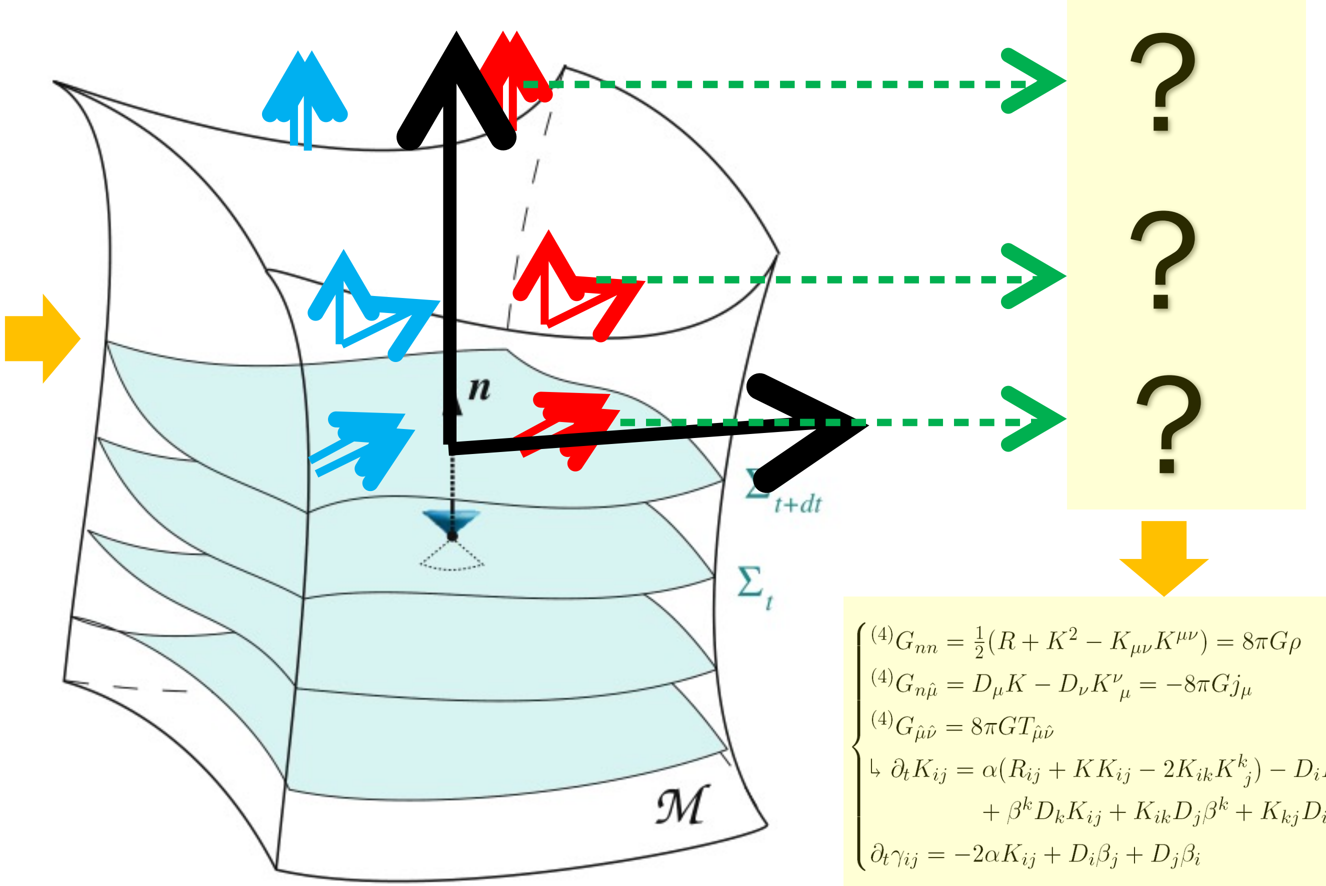
- Idempotent f(f(x)=f(x)

$$P^\mu{}_\lambda P^\lambda{}_\nu = \ldots = P^\mu{}_\nu$$

3+1 decomposition of Einstein ~~eq.~~ tensor

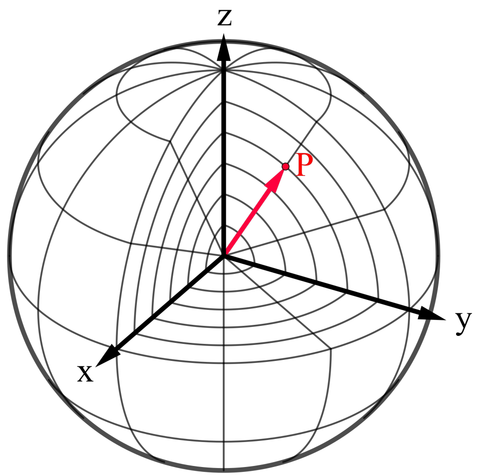
$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- ~~Manifold/Hypersurface~~
- ~~Foliation~~
- ~~Normal vector~~
- ~~3+! metric~~
- ~~Gauss-normal coordinate~~
- ~~Projection~~
- Extrinsic curvature
- Intrinsic curvature
- Projection of Riemann tensor (Gauss..)



$$\begin{cases} {}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho \\ {}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^{\nu}{}_{\mu} = -8\pi G j_\mu \\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \\ \Lsh\ \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^{k}{}_{j}) - D_i D_j \alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \\ \qquad + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k \\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$

Fundamental form of hypersurface



$$ds^2 = \gamma_{ij} \mathrm{d}\hat{x}^i \mathrm{d}\hat{x}^j = \hat{r}^2 \mathrm{d}\hat{\theta}^2 + \hat{r}^2 \sin^2\hat{\theta} \mathrm{d}\hat{\phi}^2, \quad \hat{x}^i = (\hat{\theta}, \hat{\phi})$$

- 1st fundamental form of the hypersurface:
  Projection tensor > project all tensors on to the hyper surface

$$\boxed{P_{\mu\nu}} = g_{\mu\nu} - \sigma n_\mu n_\nu$$

- 2nd fundamental form of the hypersurface:
  Change of projection tensor along the normal direction > bended hypersurface
  >> Extrinsic curvature

$$\boxed{K_{\mu\nu}} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu}$$
$$= \frac{1}{2} P^\alpha{}_\mu P^\beta{}_\nu \mathcal{L}_n g_{\alpha\beta}$$
$$= \nabla_\mu n_\nu - \sigma n_\mu a_\nu$$

Let's skip....

$$P^{\mu}_{\ \alpha} P^{\nu}_{\ \beta} \left(K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \right)$$

$$\Rightarrow K_{\alpha\beta} = \frac{1}{2} P^{\mu}_{\ \alpha} P^{\nu}_{\ \beta} \mathcal{L}_n (g_{\mu\nu} - \sigma n_\mu n_\nu)$$

$$= \frac{1}{2} P^{\mu}_{\ \alpha} P^{\nu}_{\ \beta} \mathcal{L}_n g_{\mu\nu} - \cancel{\frac{1}{2} P^{\mu}_{\ \alpha} P^{\nu}_{\ \beta} \mathcal{L}_n (\sigma n_\mu n_\nu)} \rightarrow = 0 \because P \cdot n = 0$$

$$\therefore \boxed{K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \rightarrow \frac{1}{2} P^{\alpha}_{\ \mu} P^{\beta}_{\ \nu} \mathcal{L}_n g_{\alpha\beta}}$$

$$K_{\mu\nu} = \frac{1}{2} P^{\alpha}_{\ \mu} P^{\beta}_{\ \nu} \underbrace{\mathcal{L}_n g_{\alpha\beta}}_{2\nabla_{(\alpha} n_{\beta)}}$$

$$= \frac{1}{2} P^{\alpha}_{\ \mu} P^{\beta}_{\ \nu} (2\nabla_\beta n_\alpha + \underbrace{\cancel{2\nabla_{[\alpha} n_{\beta]}}}_{\text{by the Frobenius theorem}})$$

$$= \frac{1}{2} P^{\alpha}_{\ \mu} P^{\beta}_{\ \nu} \cdot 2\nabla_\alpha n_\beta \quad \text{(by the symmetric property of } K_{\mu\nu})$$

$$= (\delta^\alpha_\mu - \sigma n^\alpha n_\mu)(\delta^\beta_\nu - \sigma n^\beta n_\nu) \nabla_\alpha n_\beta$$

$$= \nabla_\mu n_\nu - \cancel{\sigma n^\beta n_\nu \nabla_\mu n_\beta} - \sigma n^\alpha n_\mu \nabla_\alpha n_\nu + \cancel{n^\alpha n^\beta n_\mu n_\nu \nabla_\alpha n_\beta}$$

(where $\nabla_\mu (\underbrace{n_\nu n^\nu}_{\text{scalar}}) = \partial_\mu (\underbrace{n_\nu n^\nu}_{=\sigma}) = 0$

$(\nabla_\mu n_\nu) n^\nu = (\nabla_\mu n^\nu) n_\nu$ (by metric compatibility)

$\therefore n^\nu (\nabla_\mu n_\nu) = 0$)

$$\therefore \boxed{K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \rightarrow \nabla_\mu n_\nu - \sigma n_\mu a_\nu} \quad \text{, where } a^\mu = n^\nu \nabla_\nu n^\mu$$

Derivatives in Kμν

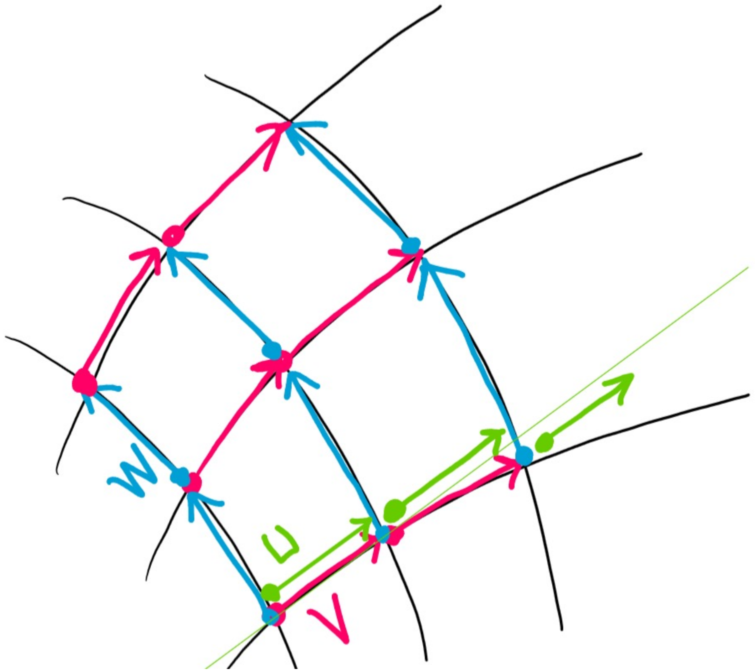
$$\boxed{K_{\mu\nu}} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu}$$
$$= \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta}$$
$$= \nabla_{\mu} n_{\nu} - \sigma n_{\mu} a_{\nu}$$

- Lie derivative:

$$\mathcal{L}_V U^{\mu}$$

- Covariant derivative:

$$\nabla_{\mu} V^{\nu}$$



$\mathcal{L}_V V =$

$\mathcal{L}_V W =$

$\mathcal{L}_W V =$

$\nabla_V V =$

$\nabla_U U =$

Lie derivative in Kμν

$$\boxed{K_{\mu\nu}} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2}P^{\alpha}{}_{\mu}P^{\beta}{}_{\nu}\mathcal{L}_n g_{\alpha\beta} = \nabla_{\mu}n_{\nu} - \sigma n_{\mu}a_{\nu}$$

- Lie derivative: change from a field

$$\mathcal{L}_V U^{\mu} = [V,U]^{\mu} = V^{\nu}\partial_{\nu}U^{\mu} - U^{\nu}\partial_{\nu}V^{\mu}$$

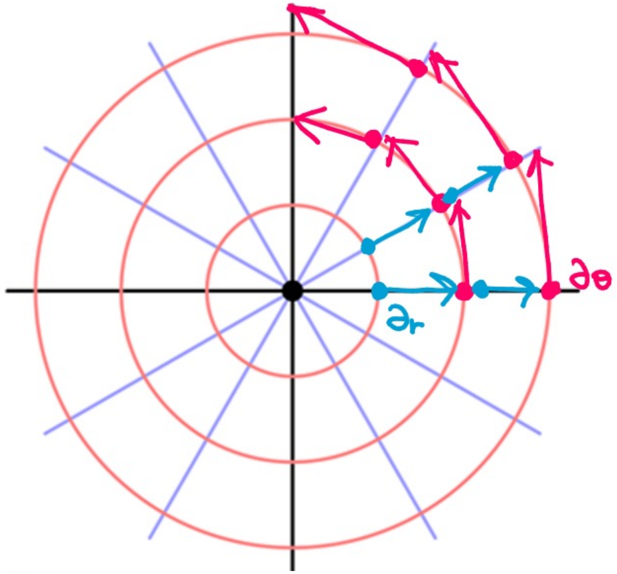
$$\mathcal{L}_V \omega_{\mu} = V^{\nu}\partial_{\nu}\omega_{\mu} + (\partial_{\mu}V^{\nu})\omega_{\nu}$$

↳ (basis vectors) $[\boldsymbol{X}, \boldsymbol{Y}] = 0$

↳ (for metric) $\mathcal{L}_{\boldsymbol{V}} g_{\mu\nu} = 2\nabla_{(\mu}V_{\nu)}$

$$\mathcal{L}_{\boldsymbol{\partial}_\theta}(\boldsymbol{\partial}_\theta)^i = [\boldsymbol{\partial}_\theta, \boldsymbol{\partial}_\theta]^i = 0$$

$$\mathcal{L}_{\boldsymbol{\partial}_r}(\boldsymbol{\partial}_\theta)^i = [\boldsymbol{\partial}_r, \boldsymbol{\partial}_\theta]^i = (\boldsymbol{\partial}_r)^j\partial_j(\boldsymbol{\partial}_\theta)^i = \partial_r\delta^i_\theta = 0$$



$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

- Extrinsic curvature ~ bending of hypersurface ~ area change along normal dir.

Covariant derivative in Kμν

$$\boxed{K_{\mu\nu}} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu}$$
$$= \frac{1}{2}P^{\alpha}_{\ \mu}P^{\beta}_{\ \nu}\mathcal{L}_n g_{\alpha\beta}$$
$$= \nabla_\mu n_\nu - \sigma n_\mu a_\nu$$

- Covariant derivative:
  - *change from a parallel transport (geodesic)*

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda$$

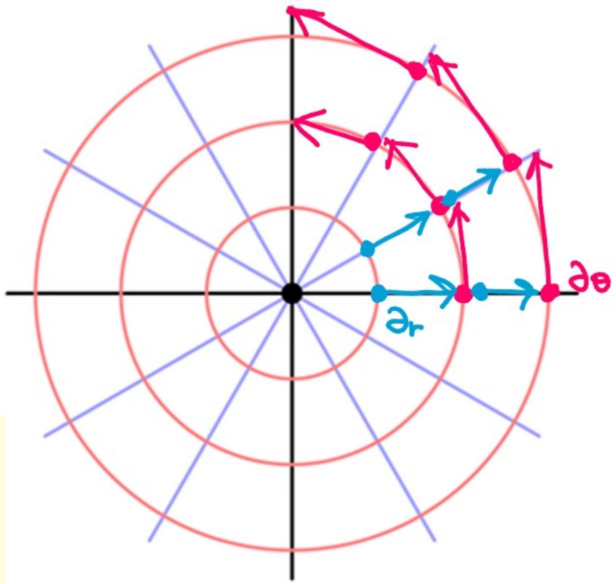
$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda$

↳ (parallel transport) $U^\mu \nabla_\mu V^\nu = \boldsymbol{\nabla_U} V^\nu = 0$

↳ (4-acceleration) $\boldsymbol{\nabla_U U} = \boldsymbol{a} \ \rightarrow 0$ (geodesic) $\left(U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)$

↳ (metric compatibility) $\nabla_\lambda g_{\mu\nu} = 0$

$$\boldsymbol{\nabla}_{\boldsymbol{\partial}_\theta}(\boldsymbol{\partial}_\theta)^i = (\boldsymbol{\partial}_\theta)^j \nabla_j (\boldsymbol{\partial}_\theta)^i = \nabla_\theta \delta^i_\theta = \Gamma^i_{\theta\theta} \neq 0 \text{ when } i = r$$
$$\boldsymbol{\nabla}_{\boldsymbol{\partial}_r}(\boldsymbol{\partial}_\theta)^i = \cdots$$



$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

- Extrinsic curvature ~ ∇ change of normal vector – non-geodesic change

Properties of Kμν

- Geometrical understanding:

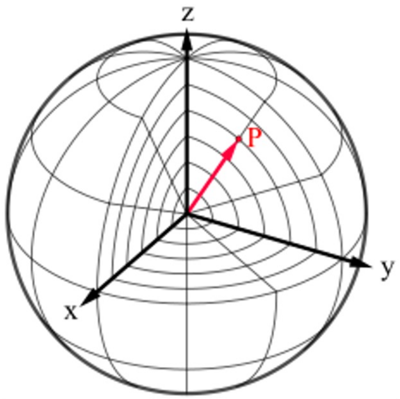
$$\boxed{K_{\mu\nu}} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu}$$
$$= \frac{1}{2} P^{\alpha}_{\ \mu} P^{\beta}_{\ \nu} \mathcal{L}_n g_{\alpha\beta}$$
$$= \nabla_\mu n_\nu - \sigma n_\mu a_\nu$$

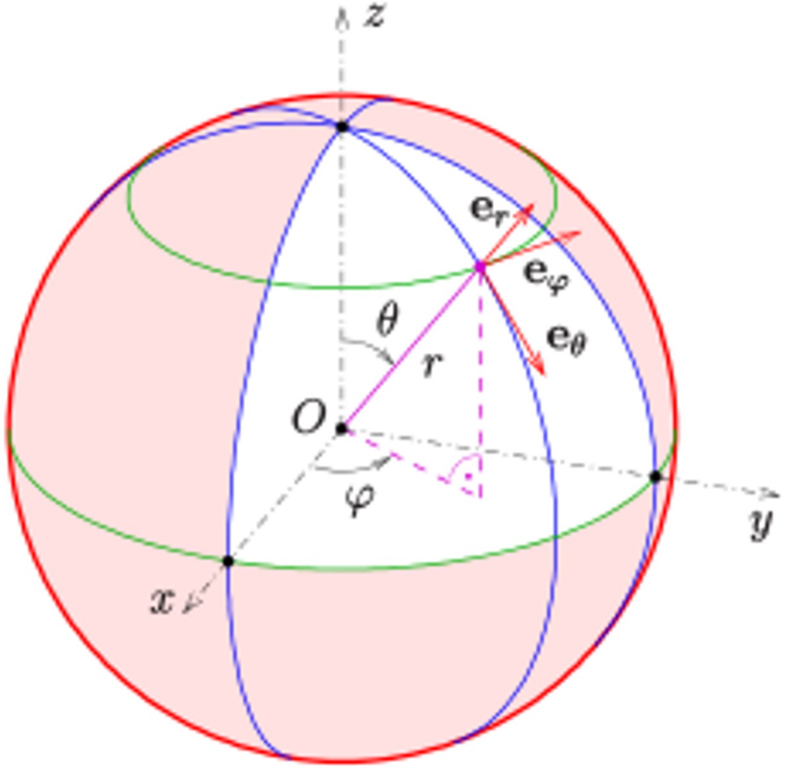
- Tangent to the hypersurface

- Symmetric tensor

- Dimensional analysis of K and R

Example of Kμν





(1) 3-dimensional manifold M

(2) foliation of M with $\Sigma_r' s$

(3) coordinates: $x^i = (r, \theta, \phi)$ (r along normal dir., (θ, ϕ) on Σ_r)

(4) metric: $ds^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + \underbrace{r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2}_{=\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j}$

(5) Christoffel symbol: $\Gamma^i_{jk} = \cdots$

(6) curvature: $M \to {}^{(3)}R = 0, \quad \Sigma_r \to {}^{(2)}R = \dfrac{2}{r^2}$

(7) normal vector: $(\mathbf{d}r)_i = \nabla_i r = \partial_i r = \delta^r_i = (1,0,0) \equiv r_i$

(8) extrinsic curvature: $K_{\theta\theta} = \begin{cases} = \dfrac{1}{2}\mathcal{L}_{\boldsymbol{r}}\gamma_{\theta\theta} = \dfrac{1}{2}(r^i \cancel{\nabla_i\gamma_{\theta\theta}} + 2(\nabla_\theta r^\mu)\gamma_{\mu\theta}) \\ \qquad = (\nabla_\theta r^\mu)\gamma_{\mu\theta} = \nabla_\theta r_\theta = \nabla_\theta \delta^r_\theta = -\Gamma^r_{\theta\theta} = -\dfrac{g_{\theta\theta,r}}{-2g_{rr}} = \dfrac{2r}{2} = r \\ = \nabla_\theta r_\theta - \sigma_r r_\theta \cancel{a_\theta} = \nabla_\theta r_\theta = r \end{cases}$

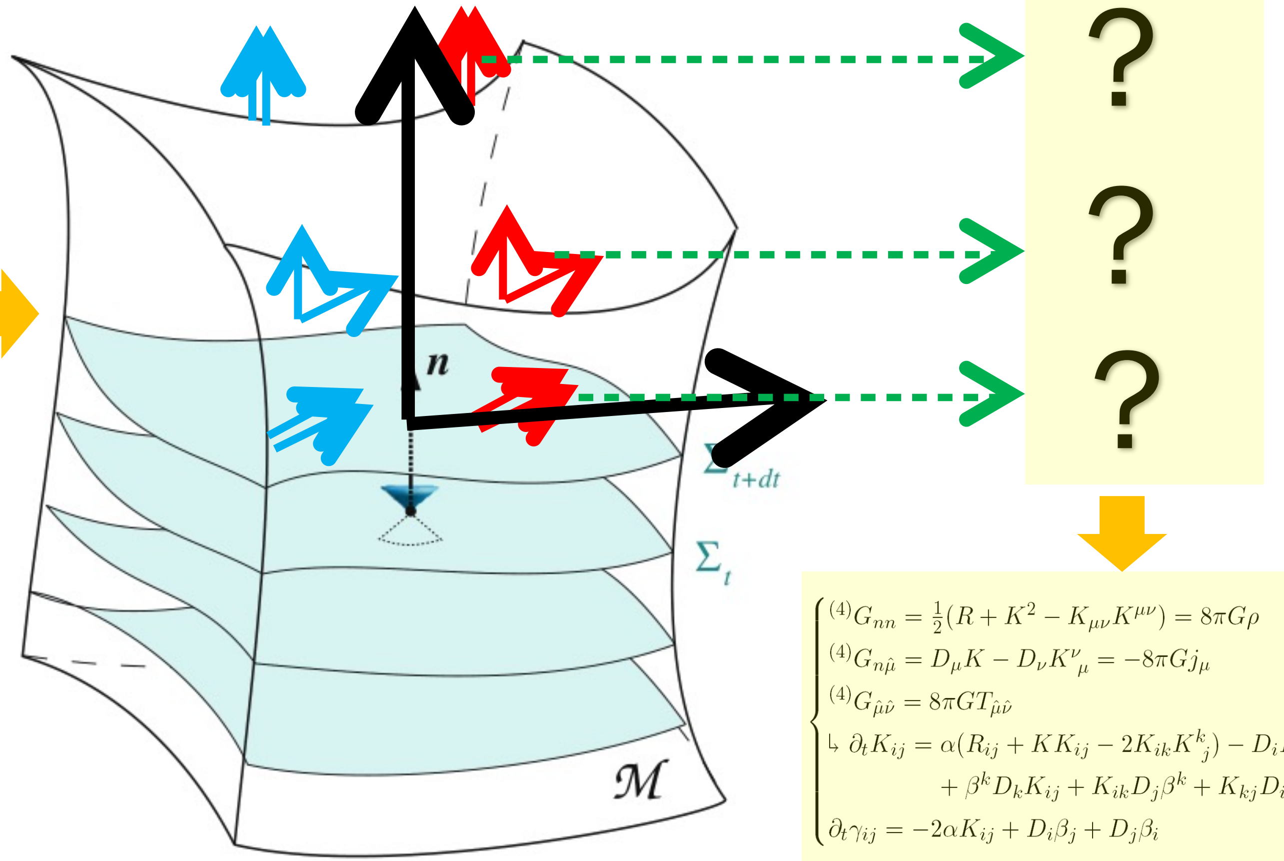
$$K_{\phi\phi} = -\Gamma^r_{\phi\phi} = -\frac{g_{\phi\phi,r}}{-2g_{rr}} = r\sin^2\theta$$

$$K = K^\theta{}_\theta + K^\phi{}_\phi = g^{\theta\theta}K_{\theta\theta} + g^{\phi\phi}K_{\phi\phi} = \frac{1}{r^2}\cdot r + \frac{1}{r^2\sin^2\theta}\cdot r\sin^2\theta = \frac{2}{r}$$

3+1 decomposition of Einstein ~~eq.~~ tensor

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

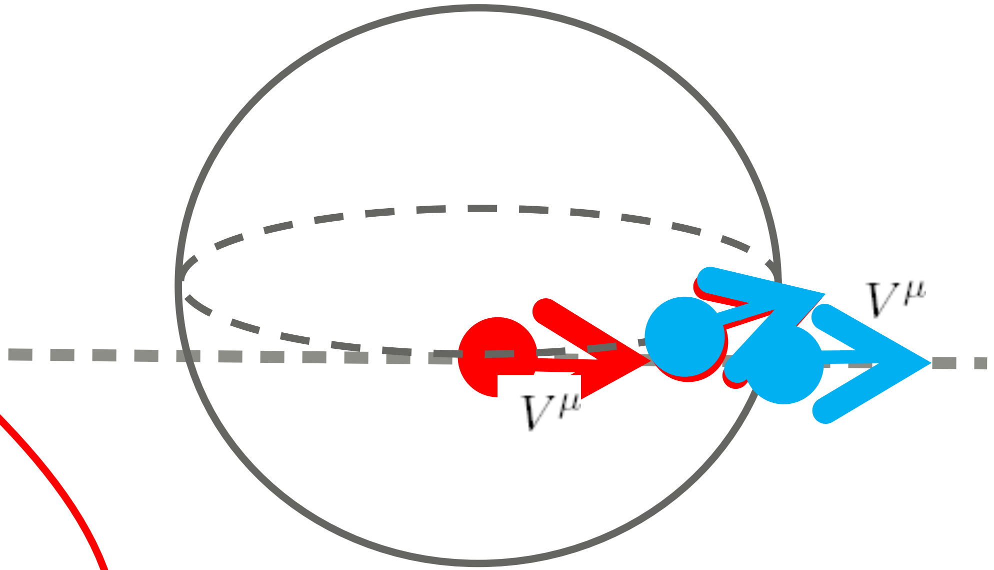
- ~~Manifold/Hypersurface~~
- ~~Foliation~~
- ~~Normal vector~~
- ~~3+! metric~~
- ~~Gauss-normal coordinate~~
- ~~Projection~~
- ~~Extrinsic curvature~~
- Intrinsic curvature
- Projection of Riemann tensor (Gauss..)



$$\begin{cases} {}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho \\ {}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu \\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \\ \hookrightarrow \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_iD_j\alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \\ \qquad\qquad + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k \\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

Intrinsic curvature (1)

$$M\,:\; g_{\mu\nu},\;\nabla_\mu[\Gamma(g)] \;\rightarrow\; [\nabla_\mu,\nabla_\nu]V^\lambda = R^\lambda{}_{\rho\mu\nu}V^\rho \;\rightarrow\; R$$
$$\Lsh\; \nabla_\mu g_{\nu\rho} = 0$$



$$\begin{aligned}
\circlearrowleft\; \nabla_\mu P_{\nu\rho} &= \nabla_\mu(g_{\nu\rho} + n_\nu n_\rho)\\
&= \nabla_\mu g_{\nu\rho} + (\nabla_\mu n_\nu)n_\rho + n_\nu(\nabla_\mu n_\rho)\\
&= K_{\mu\nu}n_\rho + n_\nu K_{\mu\rho} + n_\mu a_\nu n_\rho + n_\nu a_\mu n_\rho\\
&\neq 0
\end{aligned}$$

$$\circlearrowleft\; \hat{\nabla}_\sigma X^{\mu\cdots}_{\nu\cdots} = P^\alpha_\sigma P^\mu{}_\beta \cdots \nabla_\alpha X^{\beta\cdots}_{\gamma\cdots}$$
$$\hat{\nabla}_\mu\hat{\nabla}_\nu X^\rho = P_\mu{}^{\mu'} P_\nu{}^{\nu''} P_{\rho''}{}^{\rho'} \nabla_{\mu'}(P_{\rho'}{}^{\rho''} P_{\nu''}{}^{\nu'} \nabla_{\nu'} X^{\rho'})$$

$$\Sigma_t\,:\; \gamma_{\mu\nu},\;\hat{\nabla}_\mu[\Gamma(\gamma)] \;\rightarrow\; [\hat{\nabla}_\mu,\hat{\nabla}_\nu]V^\lambda = \hat{R}^\lambda{}_{\rho\mu\nu}V^\rho \;\rightarrow\; \hat{R}$$
$$\Lsh\; \hat{\nabla}_\mu \gamma_{\nu\rho} = 0$$

Intrinsic curvature (2. Gauss eq. (1))

$$\boxed{\widehat{\nabla}_\mu \widehat{\nabla}_\nu V^\rho} = P^\alpha{}_\mu P^\beta{}_\nu P^\rho{}_\gamma \nabla_\alpha \underbrace{(P^\delta{}_\beta P^\gamma{}_\lambda \nabla_\delta V^\lambda)}$$

$$= -\sigma n^\delta (\nabla_\alpha n_\beta) P^\gamma{}_\lambda \nabla_\delta V^\lambda - \sigma P^\delta{}_\beta (\nabla_\alpha n^\gamma) n_\lambda \nabla_\delta V^\lambda + P^\delta{}_\beta P^\gamma{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda$$

(using $\nabla_\mu P^\alpha{}_\beta = \nabla_\mu (\delta^\alpha{}_\beta - \sigma n^\alpha n_\beta) = -\sigma \nabla_\mu (n^\alpha n_\beta)$
and $P^\alpha{}_\beta n_\alpha = 0$)

$\rightarrow [\widehat{\nabla}_\mu, \widehat{\nabla}_\nu] V^\lambda = \widehat{R}^\lambda{}_{\rho\mu\nu} V^\rho \rightarrow \widehat{R}$

$\hookrightarrow \nabla_\mu g_{\nu\rho} = 0$

$$= P^\alpha{}_\mu P^\beta{}_\nu P^\rho{}_\gamma \left(-\sigma n^\delta (\nabla_\alpha n_\beta) P^\gamma{}_\lambda \nabla_\delta V^\lambda - \sigma (\nabla_\alpha n^\gamma) P^\delta{}_\beta \underbrace{n_\lambda \nabla_\delta V^\lambda}_{= -V^\lambda \nabla_\delta n_\lambda \ \because\ \nabla_\delta (n_\lambda V^\lambda) = 0} + P^\delta{}_\beta P^\gamma{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda\right)$$

$$= -\sigma \underbrace{P^\alpha{}_\mu P^\beta{}_\nu (\nabla_\alpha n_\beta)}_{K_{\mu\nu}} P^\rho{}_\gamma P^\gamma{}_\lambda n^\delta \nabla_\delta V^\lambda + \sigma \overbrace{P^\alpha{}_\mu P^\rho{}_\gamma (\nabla_\alpha n^\gamma)}^{K^\rho{}_\mu} \underbrace{P^\beta{}_\nu P^\delta{}_\beta (\nabla_\delta n_\lambda) V^\lambda}_{P^\delta{}_\nu (\nabla_\delta n_\lambda) \delta^\lambda{}_\kappa V^\kappa = P^\delta{}_\nu (\nabla_\delta n_\lambda) P^\lambda{}_\eta P^\eta{}_\kappa V^\kappa = K_{\nu\eta} P^\eta{}_\kappa V^\kappa = K_{\nu\lambda} V^\lambda}$$

$$+ P^\alpha{}_\mu \underbrace{P^\beta{}_\nu P^\delta{}_\beta}_{P^\delta{}_\nu} \underbrace{P^\rho{}_\gamma P^\gamma{}_\lambda}_{P^\rho{}_\lambda} \nabla_\alpha \nabla_\delta V^\lambda$$

$$= -\sigma K_{\mu\nu} P^\rho{}_\lambda n^\delta \nabla_\delta V^\lambda + \sigma K^\rho{}_\mu K_{\nu\lambda} V^\lambda + P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda$$

$$\boxed{\equiv \nabla_{\hat\mu} \nabla_{\hat\nu} V^{\hat\rho} - \sigma K_{\mu\nu} \nabla_n V^{\hat\rho} + \sigma K^\rho{}_\mu K_{\nu\lambda} V^\lambda}$$

(where we defined $\nabla_{\hat\mu} \nabla_{\hat\nu} V^{\hat\rho} \equiv P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda$)

Intrinsic curvature (3. Gauss eq. (2))

$$\widehat{\nabla}_\mu \widehat{\nabla}_\nu V^\rho = \nabla_{\widehat{\mu}} \nabla_{\widehat{\nu}} V^{\widehat{\rho}} - \sigma K_{\mu\nu} \nabla_n V^{\widehat{\rho}} + \sigma K^\rho{}_\mu K_{\nu\lambda} V^\lambda$$

$$(\text{where } \nabla_{\widehat{\mu}} \nabla_{\widehat{\nu}} V^{\widehat{\rho}} \equiv P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda)$$

$$\Rightarrow \; 2\widehat{\nabla}_{[\mu} \widehat{\nabla}_{\nu]} V^\rho \equiv \widehat{R}^\rho{}_{\sigma\mu\nu} V^\sigma$$

$$= 2(-\sigma \overset{0}{K_{[\mu\nu]}} P^\rho{}_\gamma n^\delta \nabla_\delta V^\lambda + \sigma K^\rho{}_{[\mu} K_{\nu]\lambda} V^\lambda + \underbrace{P^\alpha{}_{[\mu} P^\delta{}_{\nu]} P^\rho{}_\lambda \nabla_\alpha \nabla_\delta V^\lambda}_{})$$

$$= P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda \nabla_{[\alpha} \nabla_{\delta]} V^\lambda = \frac{1}{2} P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda R^\lambda{}_{\sigma\alpha\delta} V^\sigma$$

$$= 2(\sigma K^\rho{}_{[\mu} K_{\nu]\sigma} + \frac{1}{2} P^\alpha{}_\mu P^\delta{}_\nu P^\rho{}_\lambda R^\lambda{}_{\sigma\alpha\delta}) V^\sigma$$

$$\Rightarrow \; \widehat{R}^\rho{}_{\sigma\mu\nu} = P^\rho{}_\alpha P^\beta{}_\sigma P^\gamma{}_\mu P^\delta{}_\nu R^\alpha{}_{\beta\gamma\delta} + \sigma (K^\rho{}_\mu K_{\sigma\nu} - K^\rho{}_\nu K_{\sigma\mu})$$

$$\Rightarrow \; \boxed{\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma}} \quad \textbf{(Gauss eq.)}$$

(intrinsic)=(projected R)+(bending)

Intrinsic curvature (4. Gauss eq. (3))

Contracted Gauss' equation :

$$P^{\mu\rho}(\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2K_{\rho[\mu}K_{\nu]\sigma})$$

$$\rightarrow \quad \widehat{R}_{\nu\sigma} = \underbrace{P^{\mu\rho}R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}}}_{=P^{\mu\rho}P^{\alpha}{}_{\mu}P^{\beta}{}_{\nu}P^{\gamma}{}_{\rho}P^{\delta}{}_{\sigma}R_{\alpha\beta\gamma\delta}=P^{\alpha\gamma}P^{\beta}{}_{\nu}P^{\delta}{}_{\sigma}R_{\alpha\beta\gamma\delta}=(g^{\alpha\gamma}-\sigma n^{\alpha}n^{\gamma})P^{\beta}{}_{\nu}P^{\delta}{}_{\sigma}R_{\alpha\beta\gamma\delta}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma})$$

$$\rightarrow \quad \boxed{\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}}$$

Intrinsic curvature (4. Gauss eq. (4))

Scalar Gauss relation :

$$P^{\nu\sigma}(\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma})$$

$$\rightarrow \quad \widehat{R} = \underbrace{P^{\nu\sigma}P^{\alpha}_{\ \nu}P^{\beta}_{\ \sigma}}_{=P^{\alpha\beta}=g^{\alpha\beta}-\sigma n^{\alpha}n^{\beta}}(R_{\alpha\beta} - \sigma R_{n\alpha n\beta}) + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]}^{\ \nu}$$

$$= R - \sigma R_{nn} - \sigma R_{nn} + \sigma^2 \cancel{R_{nnnn}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]}^{\ \nu}$$

$$= R - \sigma 2R_{nn} + \sigma(K^2 - K^{\mu\nu}K_{\mu\nu})$$

$$\Rightarrow \boxed{\widehat{R} = R - \sigma 2R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})}$$

$$\Rightarrow \text{when } \sigma = -1,\ \widehat{R} = R + 2R_{nn} - (K^2 - K_{\mu\nu}K^{\mu\nu})$$

Intrinsic curvature (5. Codazzi eq.)

$$2\nabla_{[\widehat{\mu}}\nabla_{\widehat{\nu}]}n^{\widehat{\rho}} = R^{\widehat{\rho}}{}_{\lambda\widehat{\mu}\widehat{\nu}}n^{\lambda}$$

$$\hookrightarrow P_{\mu}{}^{\alpha}P_{\nu}{}^{\beta}P^{\rho}{}_{\gamma}\cdot 2\nabla_{[\alpha}\nabla_{\beta]}n^{\gamma} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}}$$

$$= 2P_{[\mu}{}^{\alpha}P_{\nu]}{}^{\beta}P^{\rho}{}_{\gamma}\nabla_{\alpha}\underbrace{\nabla_{\beta}n^{\gamma}}_{=K_{\beta}{}^{\gamma}+\sigma n_{\beta}a^{\gamma}}$$

$$= 2P_{[\mu}{}^{\alpha}P_{\nu]}{}^{\beta}P^{\rho}{}_{\gamma}(\nabla_{\alpha}K_{\beta}{}^{\gamma} + \sigma\underbrace{\nabla_{\alpha}n_{\beta}}_{=K_{\alpha\beta}+\sigma \cancel{n_{\alpha}}a_{\beta}}a^{\gamma} + \sigma\cancel{n_{\beta}}\nabla_{\alpha}a^{\gamma})$$

$$= 2P_{[\mu}{}^{\alpha}P_{\nu]}{}^{\beta}P^{\rho}{}_{\gamma}(\nabla_{\alpha}K_{\beta} + \sigma K_{\alpha\beta}a^{\gamma})$$

$$= 2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} + 2\sigma\cancel{K_{[\mu\nu]}}a^{\rho}$$

$$= 2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\rho}$$

$$\text{Codazzi's equation}: \boxed{2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}}}$$

$$\text{Contracted Codazzi's equation}: \boxed{2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}}}$$

Summary....

Scalar Gauss relation : $\boxed{\widehat{R} = R - \sigma 2R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})}$

$P^{\nu\sigma}$

Contracted Gauss' equation : $\boxed{\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}}$

$P^{\mu\rho}$

$\boxed{\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2K_{\rho[\mu}K_{\nu]\sigma}}$ **(Gauss eq.)**

$$2\widehat{\nabla}_{[\mu}\widehat{\nabla}_{\nu]}V^{\rho} \equiv \widehat{R}^{\rho}{}_{\sigma\mu\nu}V^{\sigma}$$

$$2\nabla_{[\widehat{\mu}}\nabla_{\widehat{\nu}]}n^{\widehat{\rho}} = R^{\widehat{\rho}}{}_{\lambda\widehat{\mu}\widehat{\nu}}n^{\lambda}$$

Codazzi's equation : $\boxed{2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}}}$

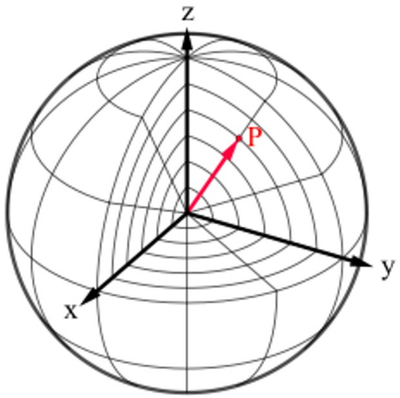
Contracted Codazzi's equation : $\boxed{2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}}}$

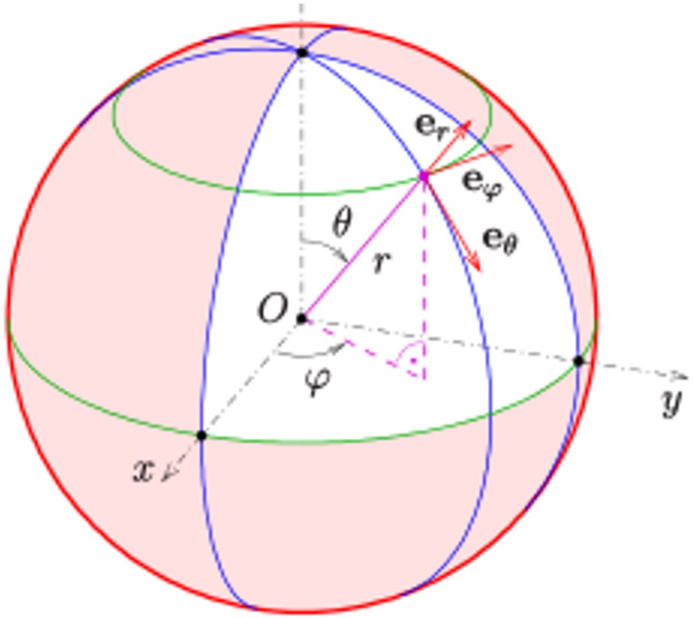
Example of intrinsic curvature

For a sphere of radius r in the flat space, we obtained,

$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

$$\widehat{R} = \frac{2}{r^2},\; K_{\theta\theta} = r,\; K_{\phi\phi} = r\sin^2\theta,\; K = \frac{2}{r}$$





The contracted Gauss equation in the flat space:

$$\widehat{R} = \cancel{R} - \sigma 2\cancel{R_{nn}} + \overbrace{\sigma}^{=1}(K^2 - K_{\mu\nu}K^{\mu\nu})$$

$$\widehat{R} = K^2 - K_{\mu\nu}K^{\mu\nu} = K^2 - K_{\theta\theta}K^{\theta\theta} - K_{\phi\phi}K^{\phi\phi}$$

$$\hookleftarrow\; K^{\theta\theta} = g^{\theta\theta}g^{\theta\theta}K_{\theta\theta},\; K^{\phi\phi} = g^{\phi\phi}g^{\phi\phi}K_{\phi\phi}$$

$$= K^2 - (g^{\theta\theta})^2(K_{\theta\theta})^2 - (g^{\phi\phi})^2(K_{\phi\phi})^2$$

$$= \left(\frac{2}{r}\right)^2 - \frac{1}{r^4}r^2 - \frac{1}{r^4\sin^4\theta}r^2\sin^4\theta$$

$$= \frac{4}{r^2} - \frac{1}{r^2} - \frac{1}{r^2}$$

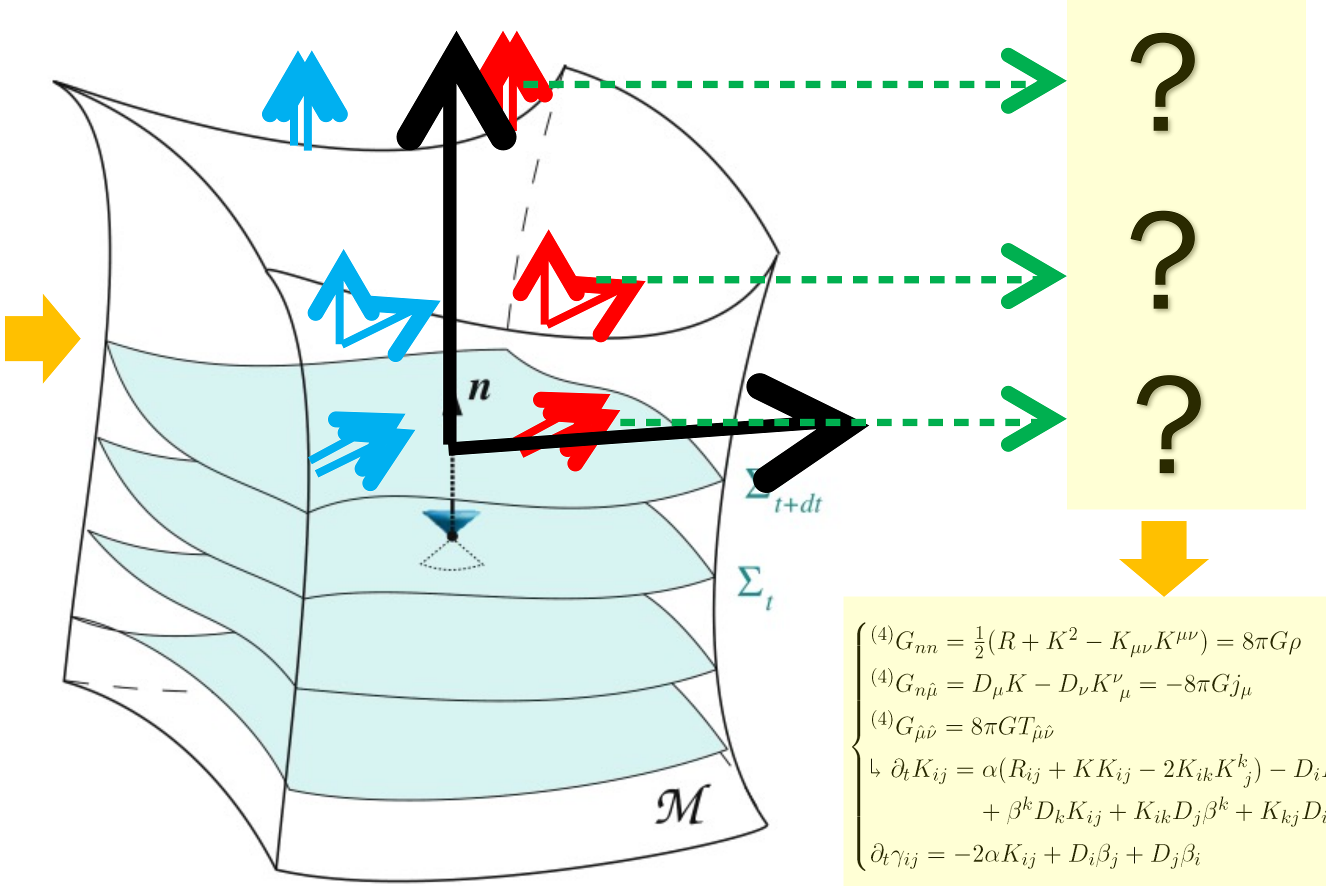
$$= \frac{2}{r^2}$$

Dimensional analysis of K and R

3+1 decomposition of Einstein ~~eq.~~ tensor

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

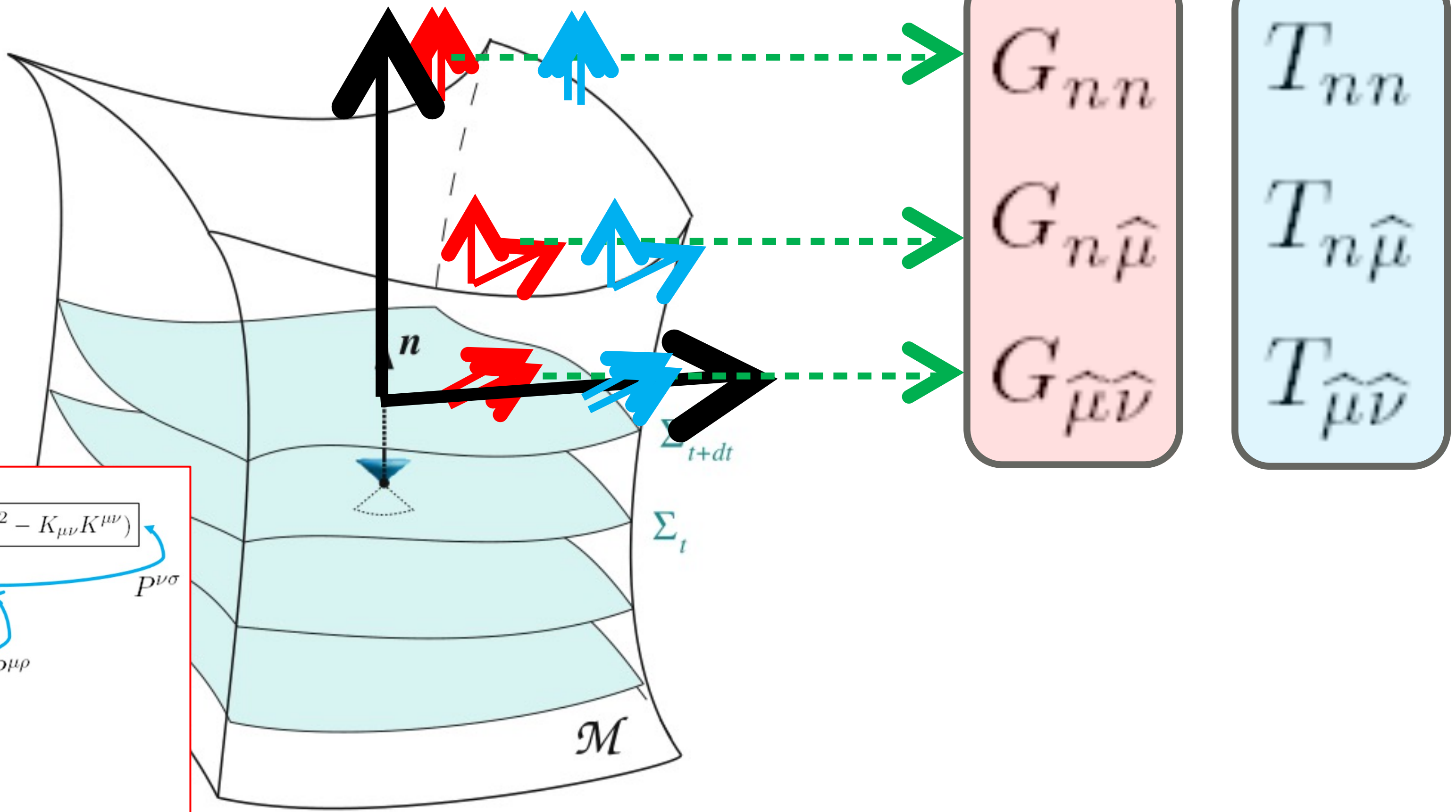
- ~~Manifold/Hypersurface~~
- ~~Foliation~~
- ~~Normal vector~~
- ~~3+! metric~~
- ~~Gauss normal coordinate~~
- ~~Projection~~
- ~~Extrinsic curvature~~
- ~~Intrinsic curvature~~
- ~~Projection of Riemann tensor (Gauss..)~~



$$\begin{cases} {}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho \\ {}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu \\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \\ \hookrightarrow \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_i D_j \alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \\ \qquad + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k \\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$

3+1 decomposition of Einstein eq. tensor

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$



[Gourgoulhon, 2021]

Summary....

Scalar Gauss relation : $\widehat{R} = R - \sigma 2R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})$

Contracted Gauss' equation : $\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}$ ($P^{\nu\sigma}$)

$\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2K_{\rho[\mu}K_{\nu]\sigma}$ **(Gauss eq.)** ($P^{\mu\rho}$)

$2\widehat{\nabla}_{[\mu}\widehat{\nabla}_{\nu]}V^{\rho} \equiv \widehat{R}^{\rho}{}_{\sigma\mu\nu}V^{\sigma}$

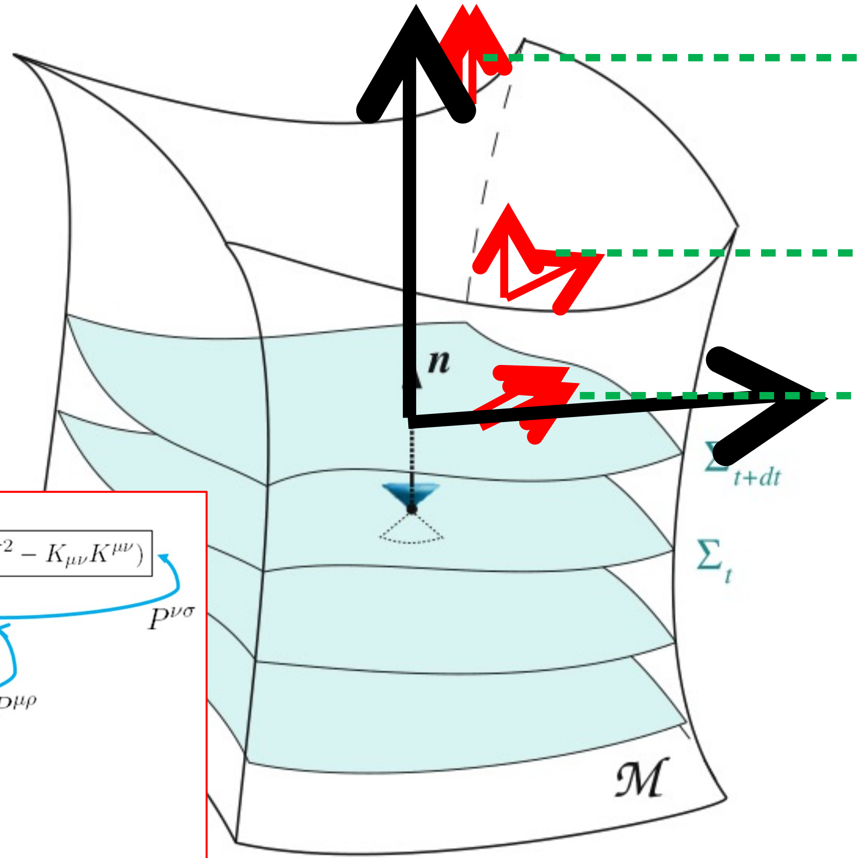
$2\nabla_{[\widehat{\mu}}\nabla_{\widehat{\nu}]}n^{\widehat{\rho}} = R^{\widehat{\rho}}{}_{\lambda\widehat{\mu}\widehat{\nu}}n^{\lambda}$

Codazzi's equation : $2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\rho} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}}$

Contracted Codazzi's equation : $2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}}$

3+1 decomposition of Einstein eq. tensor

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$



[Gourgoulhon, 2021]

$$G_{nn} = R_{nn} - \frac{1}{2} g_{nn} R$$

$$G_{n\hat{\mu}} = R_{n\hat{\mu}} - \frac{1}{2} g_{n\hat{\mu}} R$$

$$G_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2} g_{\hat{\mu}\hat{\nu}} R$$

Summary....

Scalar Gauss relation : $\widehat{R} = R - \sigma 2 R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})$

Contracted Gauss' equation : $\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma}$

$P^{\nu\sigma}$

$\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma}$ **(Gauss eq.)**

$P^{\mu\rho}$

$2\widehat{\nabla}_{[\mu}\widehat{\nabla}_{\nu]} V^{\rho} \equiv \widehat{R}^{\rho}{}_{\sigma\mu\nu} V^{\sigma}$

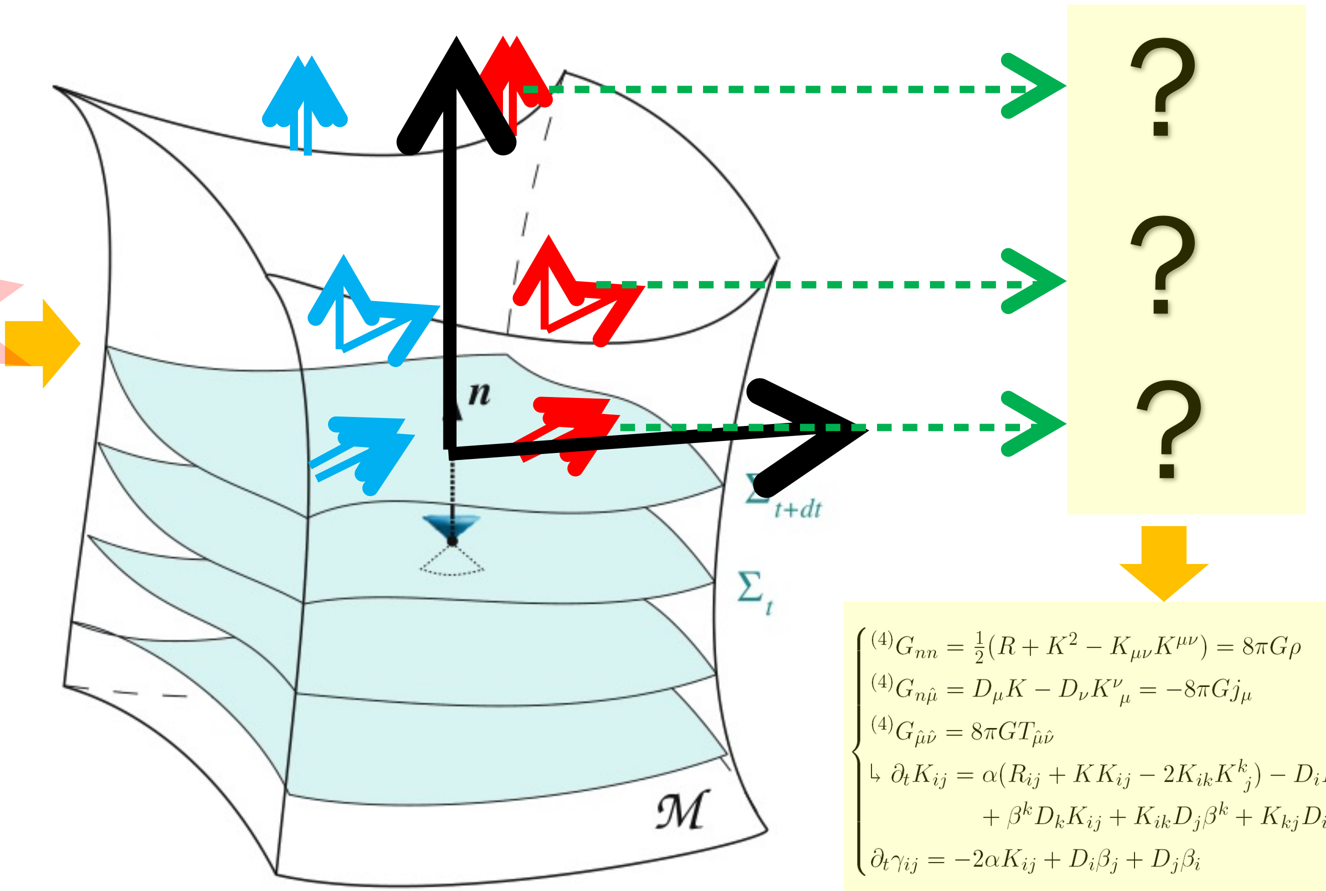
$2\nabla_{[\widehat{\mu}}\nabla_{\widehat{\nu}]} n^{\widehat{\rho}} = R^{\widehat{\rho}}{}_{\lambda\widehat{\mu}\widehat{\nu}} n^{\lambda}$

Codazzi's equation : $2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\rho} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}}$

Contracted Codazzi's equation : $2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}}$

3+1 decomposition of Einstein ~~eq.~~ tensor

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$



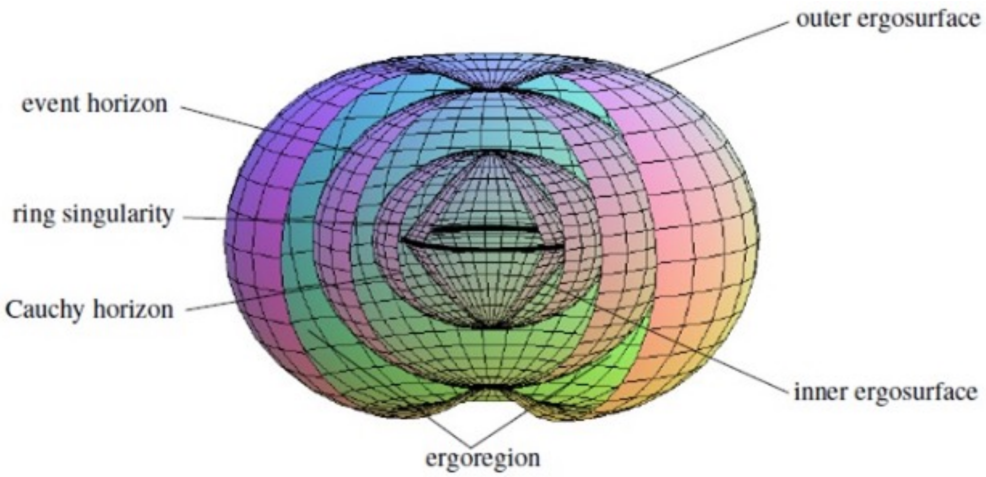
- ~~Manifold/Hypersurface~~
- ~~Foliation~~
- ~~Normal vector~~
- ~~3+! metric~~
- ~~Gauss-normal coordinate~~
- ~~Projection~~
- ~~Extrinsic curvature~~
- ~~Intrinsic curvature~~
- ~~Projection of Riemann tensor (Gauss..)~~

$$\begin{cases} {}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho \\ {}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu \\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \\ \hookrightarrow \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_iD_j\alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)) \\ \qquad\qquad + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k \\ \partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

3+1 decomposition of Tμν (1)

- Eulerian observer (observer moving along a normal vector)
  - *Fiducial observer*
  - *axisymmetric and stationary spacetimes,*
    > locally non-rotating observers
    > zero-angular-momentum observers (ZAMO)
  - *Observer's acceleration:*

$$\boldsymbol{a} = \nabla_{\boldsymbol{n}} \boldsymbol{n} = \hat{\boldsymbol{\nabla}} \ln N$$



Kerr black hole

"Schwarzschild and Kerr Solutions of Einstein's Field Equation", Christian Heinicke and Friedrich W. Heh

3+1 decomposition of Tμν (2)

$$\boldsymbol{u} = \boldsymbol{n},$$

$$\boldsymbol{a} = \nabla_{\boldsymbol{u}} \boldsymbol{u} = \nabla_{\boldsymbol{n}} \boldsymbol{n}$$

$$a_\mu = n^\nu \nabla_\nu n_\mu = \sigma n^\nu \nabla_\nu (N \nabla_\mu t) = \sigma n^\nu (\nabla_\nu N) \underbrace{(\nabla_\mu t)}_{=\sigma N^{-1} n_\mu} + \sigma N n^\nu \underbrace{(\nabla_\nu \nabla_\mu t)}_{=\nabla_\mu \nabla_\nu t = \sigma \nabla_\mu (N^{-1} n_\nu)}$$

$$= \frac{1}{N} n^\nu n_\mu \nabla_\nu N + N n^\nu \nabla_\mu \left(\frac{1}{N} n_\nu \right)$$

$$= \frac{1}{N} n^\nu n_\mu \nabla_\nu N + N \underbrace{n^\nu n_\nu}_{=\sigma} \underbrace{\nabla_\mu \left(\frac{1}{N} \right)}_{=-\frac{1}{N^2} \nabla_\mu N} + \underbrace{n^\nu \nabla_\mu n_\nu}_{=0 \because n_\mu n^\mu = \sigma}$$

$$= \frac{1}{N} (-\sigma) (-\sigma n^\nu n_\mu \nabla_\nu N + \underbrace{\nabla_\mu N}_{=\delta^\nu{}_\mu \nabla_\nu N})$$

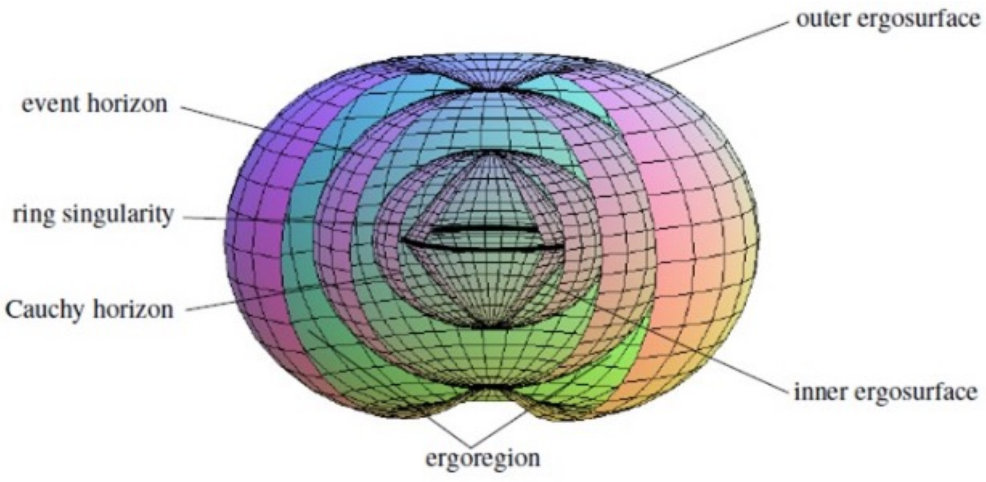
$$= \frac{1}{N} (-\sigma) (\delta^\nu{}_\mu - \sigma n^\nu n_\mu) \nabla_\nu N$$

$$= \frac{1}{N} (-\sigma) P^\nu{}_\mu \nabla_\nu N$$

$$= (-\sigma) \frac{1}{N} \widehat{\nabla}_\mu N = (-\sigma) \widehat{\nabla}_\mu \ln N$$

$$\rightarrow \; n_\mu a^\mu = 0$$

$$\hookrightarrow \; \boxed{\boldsymbol{a} = \nabla_{\boldsymbol{n}} \boldsymbol{n} = \widehat{\boldsymbol{\nabla}} \ln N}$$



ack hole

ınd Kerr Solutions of Einstein's Field Equation",
ke and Friedrich W. Heh

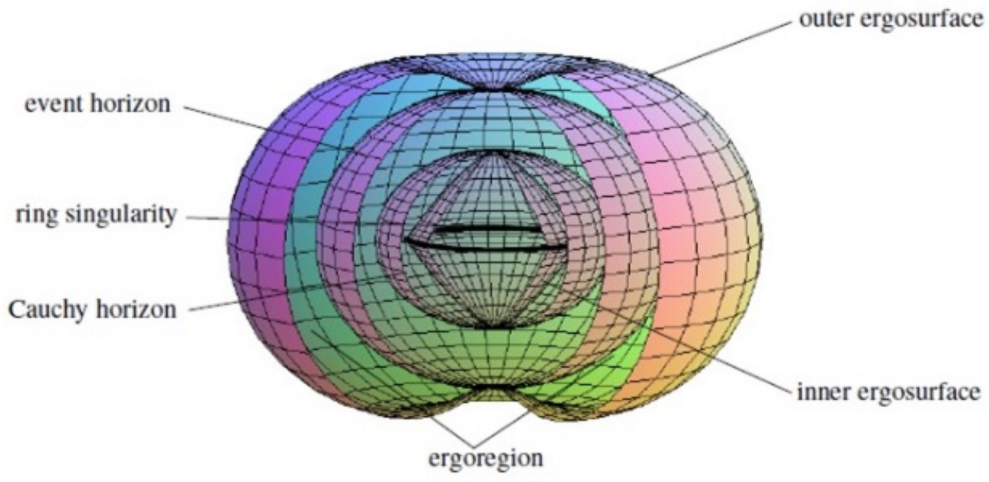
3+1 decomposition of Tμν (3)

- Eulerian observer (observer moving along a normal vector)
  - *Fiducial observer*
  - *axisymmetric and stationary spacetimes,*
    > locally non-rotating observers
    > zero-angular-momentum observers (ZAMO)
  - *Observer's acceleration:*

$$\boxed{\boldsymbol{a} = \nabla_{\boldsymbol{n}} \boldsymbol{n} = \widehat{\boldsymbol{\nabla}} \ln N}$$

  - *E,.. observed by Eulerian observer.*

$$\begin{cases} \text{energy density} & \rho_e = T_{nn} \\ \text{momentum density} & p_\alpha = -T_{n\widehat{\alpha}} \\ \text{stress tensor} & S_{\mu\nu} = T_{\widehat{\mu}\widehat{\nu}} \end{cases}$$



Kerr black hole
"Schwarzschild and Kerr Solutions of Einstein's Field Equation", Christian Heinicke and Friedrich W. Heh

3+1 decomposition of Tμν (4)

$$\begin{cases} \text{energy density} & \rho_e = T_{nn} \\ \text{momentum density} & p_\alpha = -T_{n\hat{\alpha}} \\ \text{stress tensor} & S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}} \end{cases}$$

- Eulerian observer (observer moving along a normal vector)

$$\mathbf{T} = \int \mathcal{N} \mathbf{p} \otimes \mathbf{p} \frac{d\mathcal{V}_p}{m} \left(\xrightarrow{\text{relativistic}} T^{\mu\nu} \equiv \int \mathcal{N} p^\mu p^\nu \frac{d\mathcal{V}_p}{\mathcal{E}} \right)$$

$$T^{\mu\nu} \equiv \frac{p^\mu \Delta x^\nu}{\Delta t \Delta x \Delta y \Delta z}$$

$$T^{00} = \frac{p^0 \Delta t}{\Delta t \Delta x \Delta y \Delta z} = \frac{E}{\Delta V} = \rho \text{ (energy density)}$$

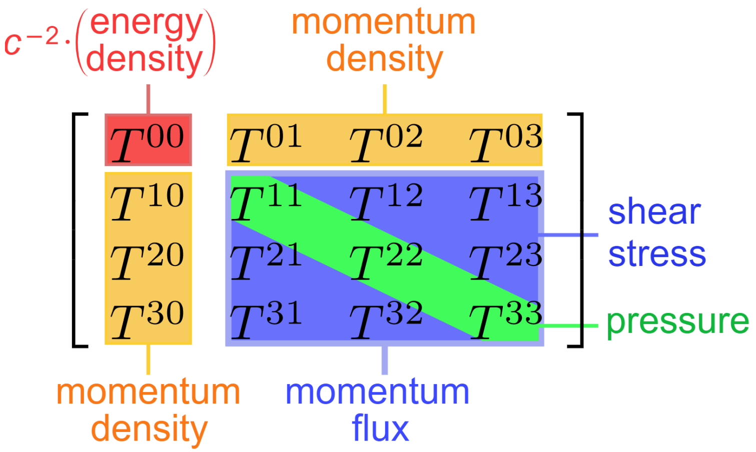
$$T^{0x} = \frac{p^0 \Delta x}{\Delta t \Delta x \Delta y \Delta z} = \frac{E}{\Delta t \Delta A_x} \text{ (energy flux)}$$

$$T^{x0} = \frac{p^x \Delta t}{\Delta t \Delta x \Delta y \Delta z} = \frac{p^x_{\text{tot}}}{\Delta V} = p^x \text{ (momentum density)}$$

$$T^{xx} = \frac{p^x \Delta x}{\Delta t \Delta x \Delta y \Delta z} = \frac{F^x}{\Delta A_x} = P^x \text{ (pressure=momentum flux)}$$

$$T^{xy} = \frac{p^x \Delta y}{\Delta t \Delta x \Delta y \Delta z} = \frac{F^x}{\Delta A_y} \text{ (shear stress=momentum flux)}$$

$$\hookrightarrow T^{\mu\nu} \begin{cases} \mu = 0 & \text{energy} \\ \mu = i & \text{momentum} \end{cases} \begin{cases} \nu = 0 & \text{density} \\ \nu = i & \text{flux} \end{cases}$$



3+1 decomposition of Tμν (5)

$$\begin{cases} \text{energy density} & \rho_e = T_{nn} \\ \text{momentum density} & p_\alpha = -T_{n\hat{\alpha}} \\ \text{stress tensor} & S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}} \end{cases}$$

- Eulerian observer (observer moving along a normal vector)

$$\mathbf{T} = \rho_0 \mathbf{v} \otimes \mathbf{v} + P\mathbf{g} \quad \text{(ideal fluid in Newtonian fluid mechanics)}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\boldsymbol{T} = \rho \boldsymbol{u} \otimes \boldsymbol{u} + P\boldsymbol{P} \quad \text{(ideal fluid in 4D spacetime)}$$

where $\mathbf{T} = T^{ij}\boldsymbol{\partial}_i\boldsymbol{\partial}_j$: stress tensor,

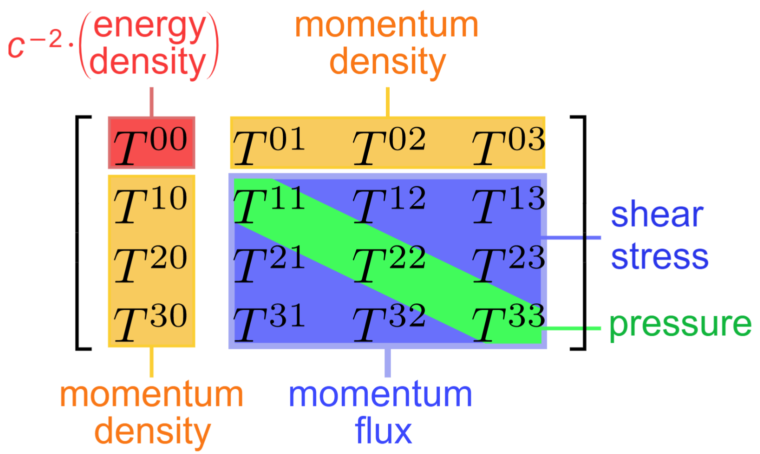
$\boldsymbol{T} = T^{\mu\nu}\boldsymbol{\partial}_\mu\boldsymbol{\partial}_\nu$: stress-energy (energy-momentum) tensor

ρ_0 : rest-mass density

$\boldsymbol{u} = u^\mu \boldsymbol{\partial}_\mu$: four-velocity

$\boldsymbol{P} = P^{\mu\nu}\boldsymbol{\partial}_\mu\boldsymbol{\partial}_\nu$: tensor projecting on a spacelike hypersurface

with $\boldsymbol{u} \otimes \boldsymbol{u}$, $\boldsymbol{P}$ we have $g^{\mu\nu} = -u^\mu u^\nu + P^{\mu\nu} \quad \cdots (P)$



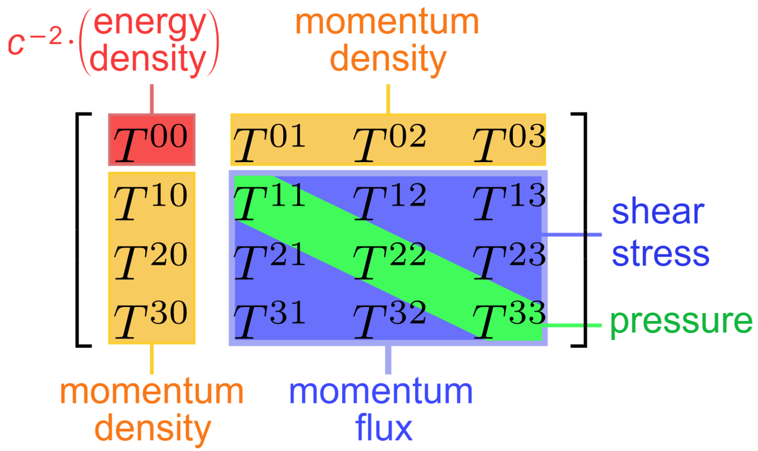
3+1 decomposition of Tμν (6)

$$\begin{cases} \text{energy density} & \rho_e = T_{nn} \\ \text{momentum density} & p_\alpha = -T_{n\hat{\alpha}} \\ \text{stress tensor} & S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}} \end{cases}$$

- Eulerian observer (observer moving along a normal vector)

$$\boldsymbol{T} = E\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{p} + \boldsymbol{p}\otimes\boldsymbol{n} + \boldsymbol{S}$$

$$\begin{aligned} T &= T^{\mu}{}_{\mu} \\ &= g^{\mu\nu}T_{\mu\nu} \\ &= (\sigma n^\mu n^\nu + P^{\mu\nu})T_{\mu\nu} \\ &= -n^\mu n^\nu T_{\mu\nu} + P^{\mu\nu}T_{\mu\nu} \\ &= -E + S \end{aligned}$$



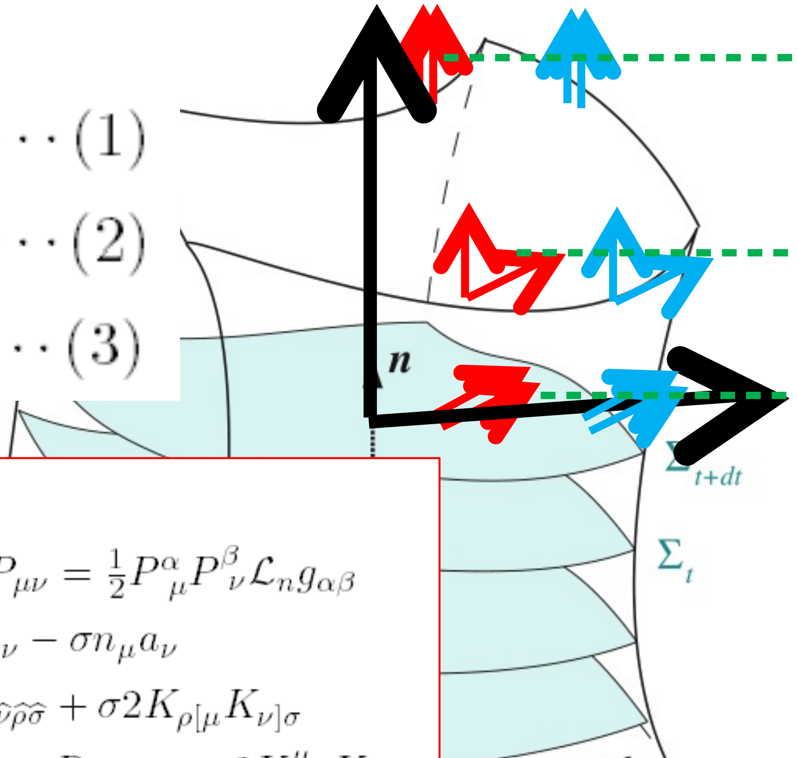
$$\begin{cases} (\text{energy density}) & \rho_e = T_{nn} \\ (\text{momentum density}) & p_\alpha = -T_{n\hat{\alpha}} \\ (\text{stress tensor}) & S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}} \end{cases}$$

3+1 decomposition of Einstein eq. (0)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\begin{cases} G_{nn} = 8\pi G T_{nn} \cdots (1) \\ G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \cdots (2) \\ G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \cdots (3) \end{cases}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$



G_{nn}	T_{nn}
$G_{n\hat{\mu}}$	$T_{n\hat{\mu}}$
$G_{\hat{\mu}\hat{\nu}}$	$T_{\hat{\mu}\hat{\nu}}$

Sum

Now we learned:

(extrinsic curvature)	$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta}$ $= \nabla_\mu n_\nu - \sigma n_\mu a_\nu$
(Gauss eq.)	$\hat{R}_{\mu\nu\rho\sigma} = R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma}$
(Contracted Gauss eq.)	$\hat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma R_{n\hat{\nu}n\hat{\sigma}} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma}$
(Gauss scalar eq.)	$\hat{R} = R - \sigma 2 R_{nn} + \sigma (K^2 - K_{\mu\nu} K^{\mu\nu})$
(Gauss-Codazzi eq.)	$2\hat{\nabla}_{[\mu} K_{\nu]}{}^{\rho} = R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}}$
(Contracted Codazzi eq.)	$2\hat{\nabla}_{[\mu} K_{\nu]}{}^{\mu} = R_{n\hat{\nu}}$

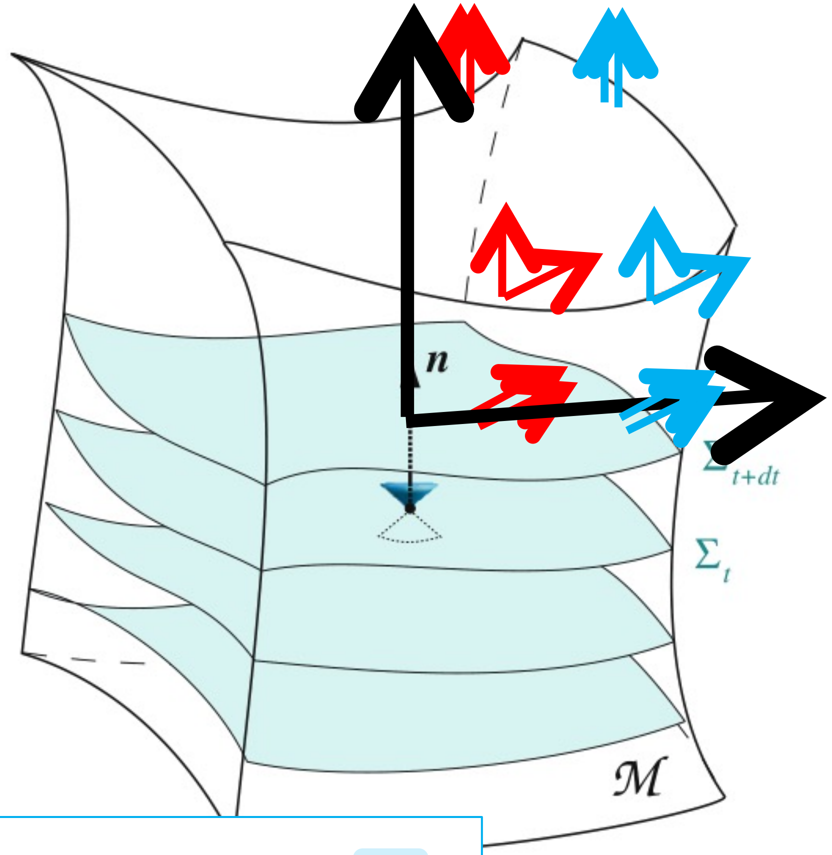
(energy density)	$\rho_e = T_{nn}$
(momentum density)	$p_\alpha = -T_{n\hat{\alpha}}$
(stress tensor)	$S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}}$

3+1 decomposition of Einstein eq. (1-1)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow G_{nn} = 8\pi G T_{nn} \cdots (1)$$

$$\Rightarrow R_{nn} - \frac{1}{2} g_{nn} R = 8\pi G T_{nn}$$



[Gourgoulhon, 2021]

Now we learned:

(extrinsic curvature) $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta} = \nabla_\mu n_\nu - \sigma n_\mu a_\nu$

(Gauss eq.) $\widehat{R}_{\mu\nu\rho\sigma} = R_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma}$

(Contracted Gauss eq.) $\widehat{R}_{\nu\sigma} = R_{\widehat{\nu}\widehat{\sigma}} - \sigma R_{n\widehat{\nu}n\widehat{\sigma}} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma}$

(Gauss scalar eq.) $\widehat{R} = R - \sigma 2 R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})$

(Gauss-Codazzi eq.) $2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\rho} = R^{\widehat{\rho}}{}_{n\widehat{\mu}\widehat{\nu}}$

(Contracted Codazzi eq.) $2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}}$

(energy density) $\rho_e = T_{nn}$

(momentum density) $p_\alpha = -T_{n\widehat{\alpha}}$

(stress tensor) $S_{\mu\nu} = T_{\widehat{\mu}\widehat{\nu}}$

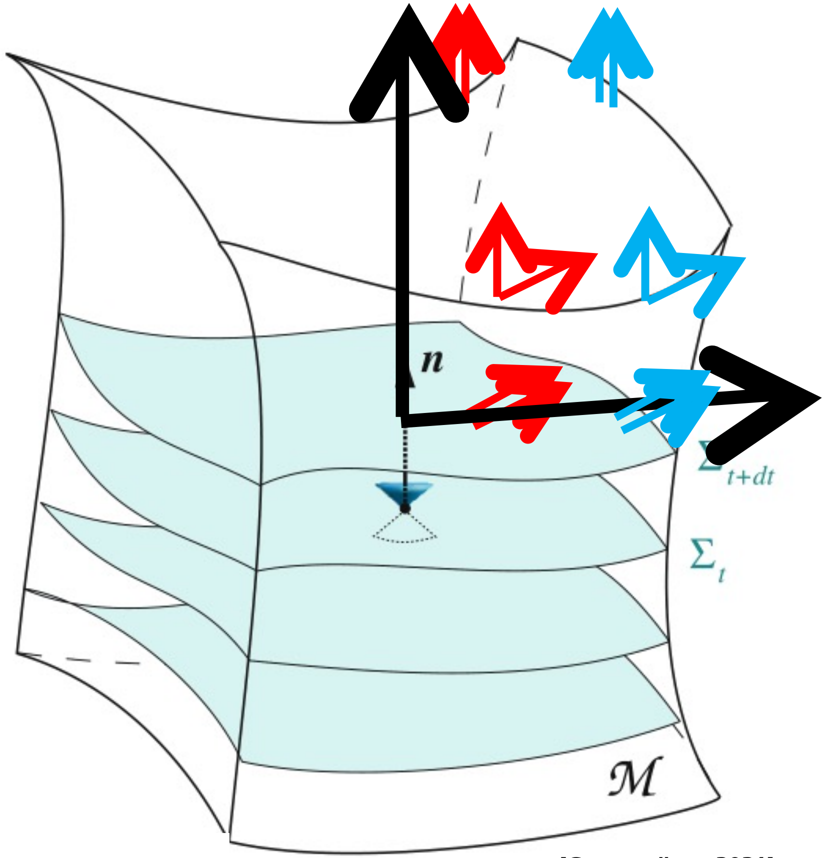
3+1 decomposition of Einstein eq. (1-2)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow G_{nn} = 8\pi G T_{nn} \cdots (1)$$

$$\Rightarrow \overbrace{R_{nn}}^{=\frac{1}{2}(\widehat{R} - R + K^2 - K_{ij}K^{ij})} - \frac{1}{2} R \underbrace{g_{nn}}_{=g_{\mu\nu}n^\mu n^\nu = n_\mu n^\mu = \sigma = -1} = 8\pi G \overbrace{T_{nn}}^{E}$$

$$\Rightarrow \boxed{\widehat{R} + K^2 - K_{ij}K^{ij} = 16\pi G E}$$



[Gourgoulhon, 2021]

$$\begin{cases} \text{(Gauss scalar eq.)} & \widehat{R} = R - \sigma 2 R_{nn} + \sigma (K^2 - K_{\mu\nu}K^{\mu\nu}) \\ \text{(energy density)} & \rho_e = T_{nn} \end{cases}$$

3+1 decomposition of Einstein eq. (2-1)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

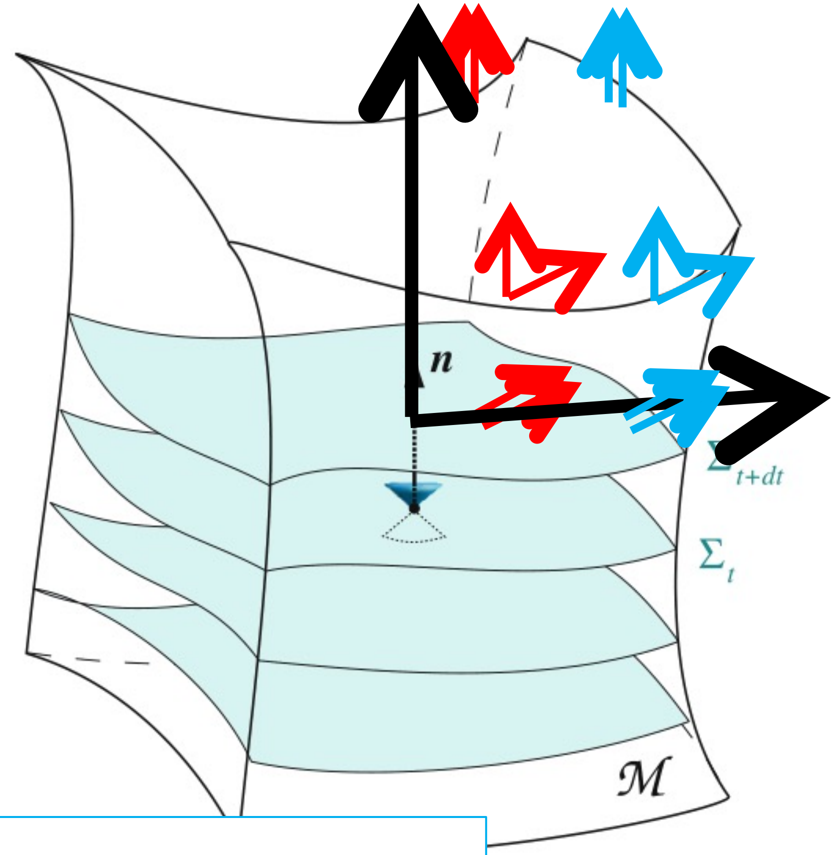
$$\hookrightarrow G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \cdots (2)$$

$$\Rightarrow R_{n\hat{\mu}} - \frac{1}{2} g_{n\hat{\mu}} R = 8\pi G T_{n\hat{\mu}}$$

Now we learned:

$$\begin{cases} \text{(extrinsic curvature)} & K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta} \\ & \quad = \nabla_\mu n_\nu - \sigma n_\mu a_\nu \\ \text{(Gauss eq.)} & \widehat{R}_{\mu\nu\rho\sigma} = R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma} \\ \text{(Contracted Gauss eq.)} & \widehat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma R_{n\hat{\nu}n\hat{\sigma}} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma} \\ \text{(Gauss scalar eq.)} & \widehat{R} = R - \sigma 2 R_{nn} + \sigma (K^2 - K_{\mu\nu} K^{\mu\nu}) \\ \text{(Gauss-Codazzi eq.)} & 2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\rho} = R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}} \\ \text{(Contracted Codazzi eq.)} & 2\widehat{\nabla}_{[\mu} K_{\nu]}{}^{\mu} = R_{n\hat{\nu}} \end{cases}$$

$$\begin{cases} \text{(energy density)} & \rho_e = T_{nn} \\ \text{(momentum density)} & p_\alpha = -T_{n\hat{\alpha}} \\ \text{(stress tensor)} & S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}} \end{cases}$$



[Gourgoulhon, 2021]

3+1 decomposition of Einstein eq. (2-2)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

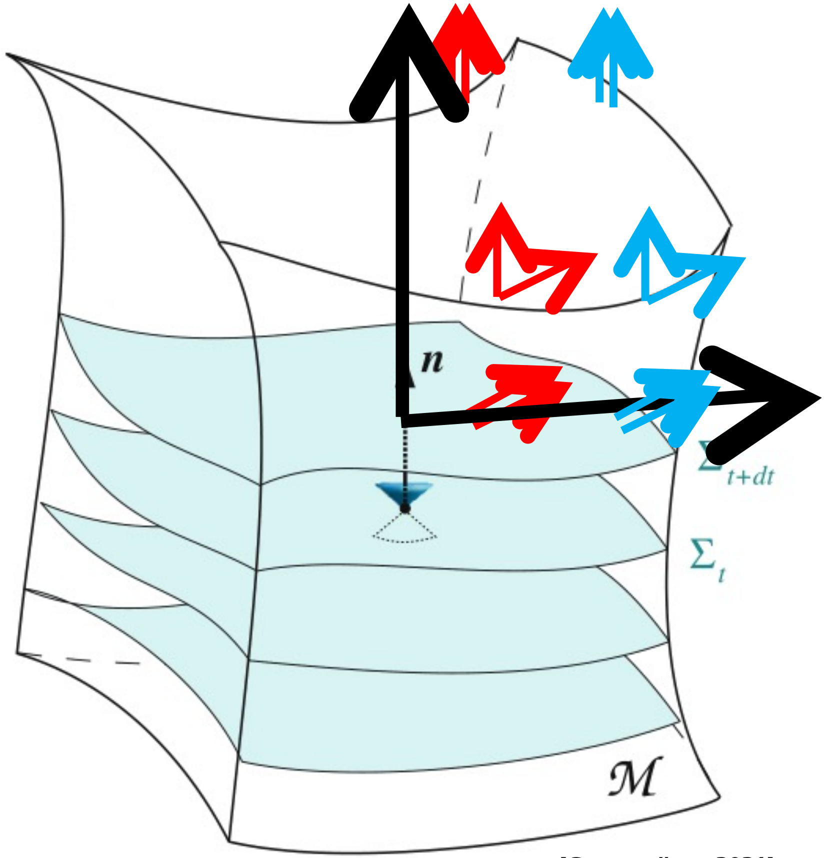
$$\hookrightarrow G_{n\widehat{\mu}} = 8\pi G T_{n\widehat{\mu}} \cdots (2)$$

$$\Rightarrow \overbrace{R_{n\widehat{\mu}}}^{2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu}} - \frac{1}{2} R \underbrace{g_{n\widehat{\mu}}}_{=g_{\alpha\beta}n^{\alpha}P^{\beta}{}_{\nu}=n_{\beta}P^{\beta}{}_{\mu}=0} = 8\pi G \underbrace{T_{n\widehat{\mu}}}_{=-p_{\mu}}$$

$$-2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = 8\pi G p_{\nu}$$

$$\boxed{\widehat{\nabla}_{\nu}K - \widehat{\nabla}_{\mu}K^{\mu}{}_{\nu} = 8\pi G p_{\nu}}$$

$$\begin{cases} \text{(Contracted Codazzi eq.)} & 2\widehat{\nabla}_{[\mu}K_{\nu]}{}^{\mu} = R_{n\widehat{\nu}} \\ \text{(momentum density)} & p_{\alpha} = -T_{n\widehat{\alpha}} \end{cases}$$



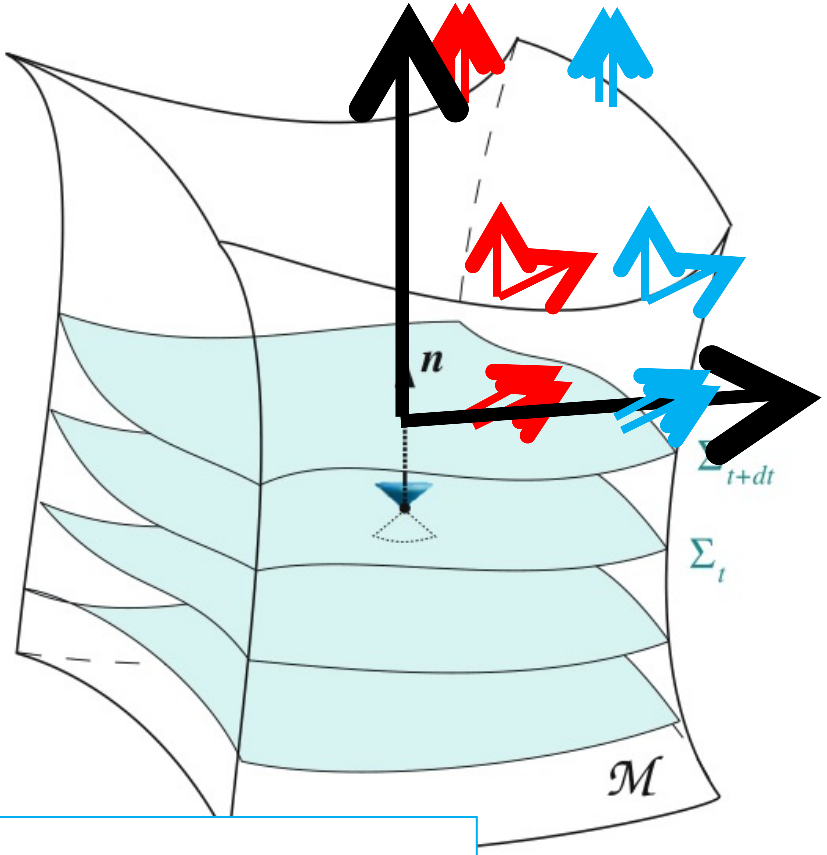
[Gourgoulhon, 2021]

3+1 decomposition of Einstein eq. (3-1)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow G_{\hat\mu\hat\nu} = 8\pi G T_{\hat\mu\hat\nu} \cdots (3)$$

$$\Rightarrow R_{\hat\mu\hat\nu} - \frac{1}{2} g_{\hat\mu\hat\nu} R = 8\pi G T_{\hat\mu\hat\nu}$$



[Gourgoulhon, 2021]

Now we learned:

(extrinsic curvature)	$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta}$
	$= \nabla_\mu n_\nu - \sigma n_\mu a_\nu$
(Gauss eq.)	$\hat R_{\mu\nu\rho\sigma} = R_{\hat\mu\hat\nu\hat\rho\hat\sigma} + \sigma 2 K_{\rho[\mu} K_{\nu]\sigma}$
(Contracted Gauss eq.)	$\hat R_{\nu\sigma} = R_{\hat\nu\hat\sigma} - \sigma R_{n\hat\nu n\hat\sigma} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma}$
(Gauss scalar eq.)	$\hat R = R - \sigma 2 R_{nn} + \sigma (K^2 - K_{\mu\nu} K^{\mu\nu})$
(Gauss-Codazzi eq.)	$2\hat\nabla_{[\mu} K_{\nu]}{}^{\rho} = R^{\hat\rho}{}_{n\hat\mu\hat\nu}$
(Contracted Codazzi eq.)	$2\hat\nabla_{[\mu} K_{\nu]}{}^{\mu} = R_{n\hat\nu}$

(energy density)	$\rho_e = T_{nn}$
(momentum density)	$p_\alpha = -T_{n\hat\alpha}$
(stress tensor)	$S_{\mu\nu} = T_{\hat\mu\hat\nu}$

3+1 decomposition of Einstein eq. (3-2)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\Lsh\ G_{\widehat{\mu}\widehat{\nu}} = 8\pi G T_{\widehat{\mu}\widehat{\nu}} \cdots (3)$$

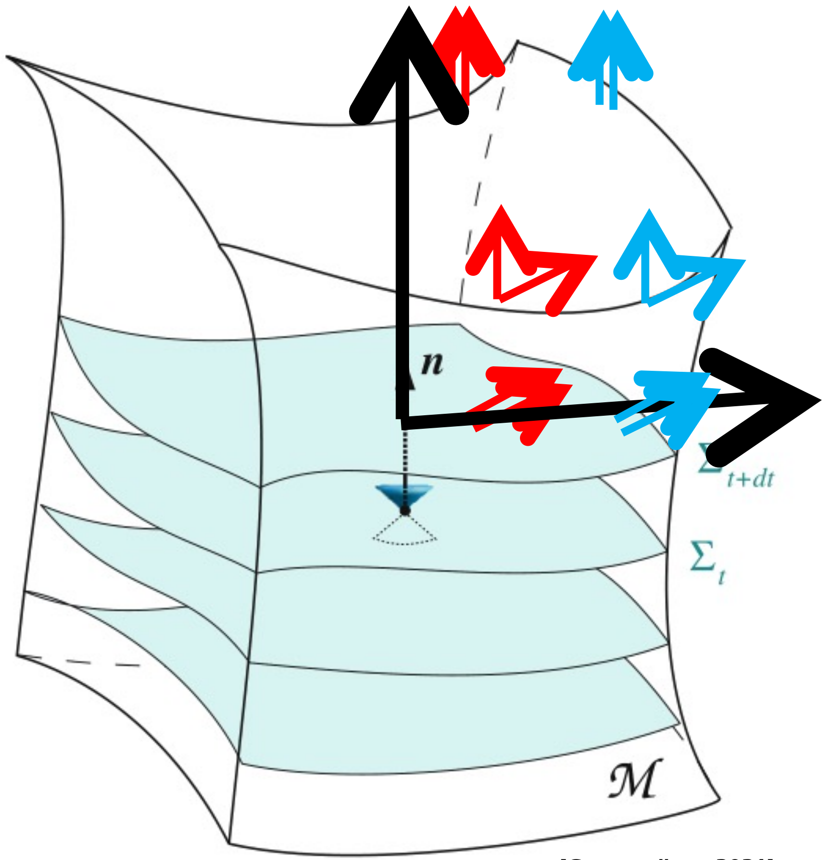
$$\Rightarrow \boxed{R_{\widehat{\mu}\widehat{\nu}} - \frac{1}{2} g_{\widehat{\mu}\widehat{\nu}} R = 8\pi G T_{\widehat{\mu}\widehat{\nu}}}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\Lsh\ G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} G^{\alpha}{}_{\alpha} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha})$$

$$\Lsh\ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \cancel{R} - \frac{1}{2} g_{\mu\nu} (R - \underbrace{\frac{1}{2} g^{\alpha}{}_{\alpha} R}_{=2R}) = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha})$$

$$\Rightarrow \boxed{R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha})}$$



[Gourgoulhon, 2021]

3+1 decomposition of Einstein eq. (3-3)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

↳ $G_{\hat\mu\hat\nu} = 8\pi G T_{\hat\mu\hat\nu} \cdots (3)$

$$R_{\hat\mu\hat\nu} = 8\pi G\left(T_{\hat\mu\hat\nu} - \frac{1}{2} g_{\hat\mu\hat\nu} T^{\alpha}{}_{\alpha}\right)$$

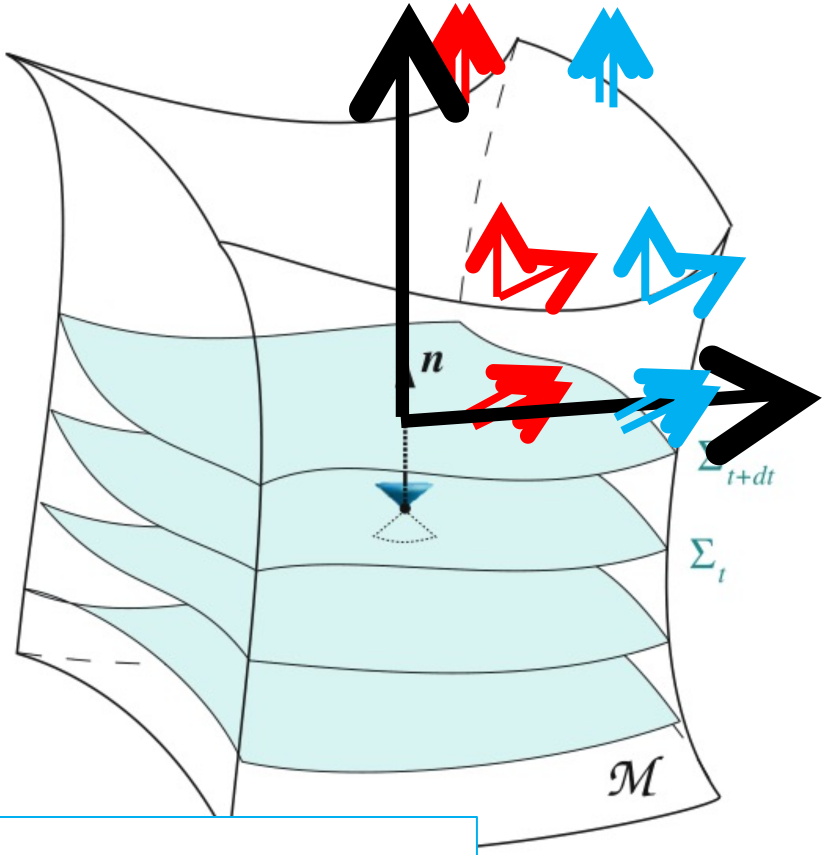
↻ $\hat R_{\nu\sigma} = R_{\hat\nu\hat\sigma} - \sigma R_{n\hat\nu n\hat\sigma} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}$

$$\hat R_{\mu\nu} - R_{n\hat\mu n\hat\nu} + 2K^{\alpha}{}_{[\alpha}K_{\mu]\nu} = 8\pi G\left[S_{\hat\mu\hat\nu} - \frac{1}{2} P_{\hat\mu\hat\nu}(S - E)\right]$$

Now we learned:

(extrinsic curvature)	$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta}$
	$= \nabla_\mu n_\nu - \sigma n_\mu a_\nu$
(Gauss eq.)	$\hat R_{\mu\nu\rho\sigma} = R_{\hat\mu\hat\nu\hat\rho\hat\sigma} + \sigma 2K_{\rho[\mu}K_{\nu]\sigma}$
(Contracted Gauss eq.)	$\hat R_{\nu\sigma} = R_{\hat\nu\hat\sigma} - \sigma R_{n\hat\nu n\hat\sigma} + \sigma 2K^{\mu}{}_{[\mu}K_{\nu]\sigma}$
(Gauss scalar eq.)	$\hat R = R - \sigma 2R_{nn} + \sigma(K^2 - K_{\mu\nu}K^{\mu\nu})$
(Gauss-Codazzi eq.)	$2\hat\nabla_{[\mu}K_{\nu]}{}^{\rho} = R^{\hat\rho}{}_{n\hat\mu\hat\nu}$
(Contracted Codazzi eq.)	$2\hat\nabla_{[\mu}K_{\nu]}{}^{\mu} = R_{n\hat\nu}$

(energy density)	$\rho_e = T_{nn}$
(momentum density)	$p_\alpha = -T_{n\hat\alpha}$
(stress tensor)	$S_{\mu\nu} = T_{\hat\mu\hat\nu}$



[Gourgoulhon, 2021]

3+1 decomposition of Einstein eq. (3-4)

$$R_{n\hat{\nu}n\hat{\sigma}} = R_{\hat{\nu}n\hat{\sigma}n} = P_{\nu\alpha} n^{\lambda} P_{\sigma}{}^{\beta} n^{\mu} R^{\alpha}{}_{\lambda\beta\mu}$$

$$= P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} R^{\alpha}{}_{\lambda\beta\mu} n^{\lambda}$$

$$= P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} [\nabla_{\beta}, \nabla_{\mu}] n^{\alpha}$$

$$= P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \nabla_{\beta} \underbrace{\nabla_{\mu} n^{\alpha}}_{= K_{\mu}{}^{\alpha} - n_{\mu} a^{\alpha}} - P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \underbrace{\nabla_{\mu} \nabla_{\beta} n^{\alpha}}_{= K_{\beta}{}^{\alpha} - n_{\beta} a^{\alpha}}$$

$$= P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \nabla_{\beta} K_{\mu}{}^{\alpha} - P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \nabla_{\beta} (n_{\mu} a^{\alpha}) - P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \nabla_{\mu} K_{\beta}{}^{\alpha} + P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \nabla_{\mu} (n_{\beta} a^{\alpha})$$

$$= \overbrace{P_{\nu\alpha} P_{\sigma}{}^{\beta} \overbrace{n^{\mu} \nabla_{\beta} K_{\mu}{}^{\alpha}}^{= -K_{\mu}{}^{\alpha} \nabla_{\beta} n^{\mu} = -K_{\mu}{}^{\alpha} (K_{\beta}{}^{\mu} - n_{\beta} a^{\mu})}}^{= -K_{\mu\nu} K_{\sigma}{}^{\mu}} - P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \nabla_{\beta} n_{\mu} a^{\alpha} - P_{\nu\alpha} P_{\sigma}{}^{\beta} \overbrace{n^{\mu} n_{\mu}}^{= -1} \nabla_{\beta} a^{\alpha}$$

$$- P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \nabla_{\mu} K_{\beta}{}^{\alpha} + P_{\nu\alpha} P_{\sigma}{}^{\beta} \underbrace{n^{\mu} \nabla_{\mu} n_{\beta}}_{= a_{\beta}} a^{\alpha} + P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} n_{\beta} \nabla_{\mu} a^{\alpha}$$

$$= -K_{\mu\nu} K_{\sigma}{}^{\mu} - P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \nabla_{\mu} K_{\beta}{}^{\alpha} + P_{\nu\alpha} P_{\sigma}{}^{\beta} \nabla_{\beta} a^{\alpha} + P_{\nu\alpha} P_{\sigma}{}^{\beta} a_{\beta} a^{\alpha}$$

$$= -K_{\mu\nu} K_{\sigma}{}^{\mu} - P_{\nu\alpha} P_{\sigma}{}^{\beta} n^{\mu} \nabla_{\mu} K_{\beta}{}^{\alpha} + \hat{\nabla}_{\sigma} a_{\nu} + a_{\sigma} a_{\nu}$$

3+1 decomposition of Einstein eq. (3-5)

$$\widehat{R}_{\mu\nu} - R_{n\widehat{\mu}n\widehat{\nu}} + 2K^{\alpha}{}_{[\alpha}K_{\mu]\nu} = 8\pi G[S_{\widehat{\mu}\widehat{\nu}} - \frac{1}{2}P_{\widehat{\mu}\widehat{\nu}}(S-E)] \quad \cdots (3)$$

$$\circlearrowleft \; R_{n\widehat{\mu}n\widehat{\nu}} = K_{\mu\alpha}K^{\alpha}{}_{\nu} - \frac{1}{N}\mathcal{L}_m K_{\mu\nu} + \frac{1}{N}\widehat{\nabla}_\mu\widehat{\nabla}_\nu N$$

$$R_{\widehat{\mu}\widehat{\nu}} = \widehat{R}_{\mu\nu} - K_{\mu\alpha}K^{\alpha}{}_{\nu} + \frac{1}{N}\mathcal{L}_m K_{\mu\nu} - \frac{1}{N}\widehat{\nabla}_\mu\widehat{\nabla}_\nu N + \underbrace{2K^{\alpha}{}_{[\alpha}K_{\mu]\nu}}_{=K^{\alpha}{}_{\alpha}K_{\mu\nu} - K^{\alpha}{}_{\mu}K_{\alpha\nu}}$$

$$= \widehat{R}_{\mu\nu} - 2K_{\mu\alpha}K^{\alpha}{}_{\nu} + \frac{1}{N}\mathcal{L}_m K_{\mu\nu} - \frac{1}{N}\widehat{\nabla}_\mu\widehat{\nabla}_\nu N + KK_{\mu\nu}$$

$$\circlearrowleft \; \mathcal{L}_m K_{\mu\nu} = \mathcal{L}_{(\partial_t - \beta)}K_{\mu\nu} = \partial_t K_{\mu\nu} + \beta^\alpha\partial_\alpha K_{\mu\nu} + \partial_\mu\beta^\alpha K_{\alpha\nu} + \partial_\nu\beta^\alpha K_{\mu\alpha}$$

$$= \widehat{R}_{\mu\nu} - 2K_{\mu\alpha}K^{\alpha}{}_{\nu} + \frac{1}{N}(\partial_t K_{\mu\nu} + \beta^\alpha\partial_\alpha K_{\mu\nu} + \partial_\mu\beta^\alpha K_{\alpha\nu} + \partial_\nu\beta^\alpha K_{\mu\alpha}) - \frac{1}{N}\widehat{\nabla}_\mu\widehat{\nabla}_\nu N + KK_{\mu\nu}$$

$$\Rightarrow \widehat{R}_{\mu\nu} - 2K_{\mu\alpha}K^{\alpha}{}_{\nu} + \frac{1}{N}(\partial_t K_{\mu\nu} + \beta^\alpha\partial_\alpha K_{\mu\nu} + \partial_\mu\beta^\alpha K_{\alpha\nu} + \partial_\nu\beta^\alpha K_{\mu\alpha}) - \frac{1}{N}\widehat{\nabla}_\mu\widehat{\nabla}_\nu N + KK_{\mu\nu}$$

$$= 8\pi G[S_{\widehat{\mu}\widehat{\nu}} - \frac{1}{2}P_{\widehat{\mu}\widehat{\nu}}(S-E)]$$

3+1 decomposition of Einstein eq. (3-6)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \cdots (3)$$

$$R_{\hat{\mu}\hat{\nu}} = 8\pi G (T_{\hat{\mu}\hat{\nu}} - \frac{1}{2} g_{\hat{\mu}\hat{\nu}} T^{\alpha}{}_{\alpha})$$

$$\hat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma R_{n\hat{\nu}n\hat{\sigma}} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma}$$

$$\hat{R}_{\mu\nu} - R_{n\hat{\mu}n\hat{\nu}} + 2K^{\alpha}{}_{[\alpha} K_{\mu]\nu} = 8\pi G [S_{\hat{\mu}\hat{\nu}} - \frac{1}{2} P_{\hat{\mu}\hat{\nu}} (S - E)]$$

$$R_{n\hat{\mu}n\hat{\nu}} = K_{\mu\alpha} K^{\alpha}{}_{\nu} - \frac{1}{N} \mathcal{L}_m K_{\mu\nu} + \frac{1}{N} \hat{\nabla}_\mu \hat{\nabla}_\nu N$$

Now

(ex $$\hat{R}_{\mu\nu} - 2K_{\mu\alpha} K^{\alpha}{}_{\nu} + \frac{1}{N} (\partial_t K_{\mu\nu} + \beta^\alpha \partial_\alpha K_{\mu\nu} + \partial_\mu \beta^\alpha K_{\alpha\nu} + \partial_\nu \beta^\alpha K_{\mu\alpha})$$

(G $$- \frac{1}{N} \hat{\nabla}_\mu \hat{\nabla}_\nu N + K K_{\mu\nu} = 8\pi G [S_{\hat{\mu}\hat{\nu}} - \frac{1}{2} P_{\hat{\mu}\hat{\nu}} (S - E)]$$

(Contracted Gauss eq.) $\hat{R}_{\nu\sigma} = R_{\hat{\nu}\hat{\sigma}} - \sigma R n\hat{\nu} n\hat{\sigma} + \sigma 2 K^{\mu}{}_{[\mu} K_{\nu]\sigma}$

(Gauss scalar eq.) $\hat{R} = R - \sigma 2 R_{nn} + \sigma (K^2 - K_{\mu\nu} K^{\mu\nu})$

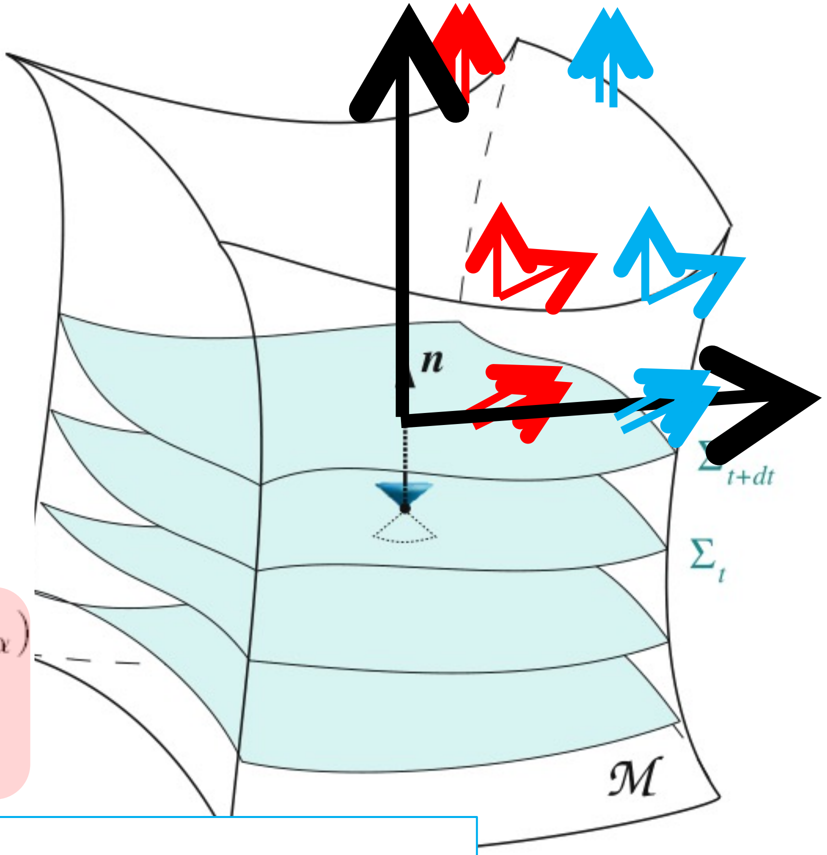
(Gauss-Codazzi eq.) $2\hat{\nabla}_{[\mu} K_{\nu]}{}^{\rho} = R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}}$

(Contracted Codazzi eq.) $2\hat{\nabla}_{[\mu} K_{\nu]}{}^{\mu} = R_{n\hat{\nu}}$

(energy density) $\rho_e = T_{nn}$

(momentum density) $p_\alpha = -T_{n\hat{\alpha}}$

(stress tensor) $S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}}$



[Gourgoulhon, 2021]

3+1 decomposition of Einstein eq. (4-1)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ G_{nn} = 8\pi G T_{nn} \ \rightarrow\ \hat{R} + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow\ \hat{\nabla}_\nu K - \hat{\nabla}_\mu K^\mu{}_\nu = 8\pi G p_\nu \\ (3)\ G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \ \rightarrow\ \hat{R}_{\mu\nu} - 2K_{\mu\alpha}K^\alpha{}_\nu \\ \qquad + \frac{1}{N}(\partial_t K_{\mu\nu} + \beta^\alpha \partial_\alpha K_{\mu\nu} + \partial_\mu \beta^\alpha K_{\alpha\nu} + \partial_\nu \beta^\alpha K_{\mu\alpha}) \\ \qquad - \frac{1}{N}\hat{\nabla}_\mu \hat{\nabla}_\nu N + K K_{\mu\nu} = 8\pi G[S_{\hat{\mu}\hat{\nu}} - \frac{1}{2}P_{\hat{\mu}\hat{\nu}}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \end{cases}$$

(Gauss scalar eq.)

(Gauss-Codazzi eq.) $2\hat{\nabla}_{[\mu}K_{\nu]}{}^\rho = R^{\hat{\rho}}{}_{n\hat{\mu}\hat{\nu}}$

(Contracted Codazzi eq.) $2\hat{\nabla}_{[\mu}K_{\nu]}{}^\mu = R_{n\hat{\nu}}$

(momentum density) $p_\alpha = -T_{n\hat{\alpha}}$

(stress tensor) $S_{\mu\nu} = T_{\hat{\mu}\hat{\nu}}$

3+1 decomposition of Einstein eq. (4-2)

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} P^{\alpha}{}_{\mu} P^{\beta}{}_{\nu} \mathcal{L}_n g_{\alpha\beta} = \nabla_\mu n_\nu - \sigma n_\mu a_\nu$$

$$\circlearrowleft \ \mathcal{L}_m P_{\mu\nu} = \underbrace{m^\alpha}_{=Nn^\alpha} \nabla_\alpha P_{\mu\nu} + \underbrace{\nabla_\mu m^\alpha}_{=\nabla_\mu N \cancel{n^\alpha} + N\nabla_\mu n^\alpha} P_{\alpha\nu} + \nabla_\nu m^\alpha P_{\mu\alpha}$$

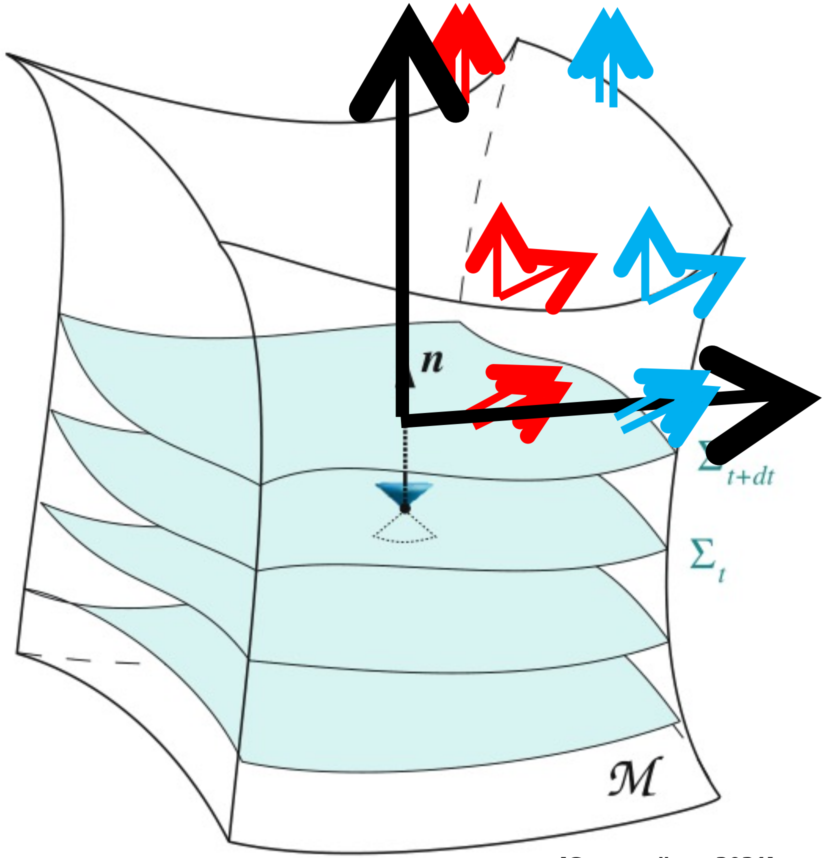
$$= N(n^\alpha \nabla_\alpha P_{\mu\nu} + \nabla_\mu n^\alpha P_{\alpha\nu} + \nabla_\nu n^\alpha P_{\mu\alpha}) = N\mathcal{L}_n P_{\mu\nu}$$

$$= \frac{1}{2N}\mathcal{L}_m P_{\mu\nu}$$

$$= \frac{1}{2N}\mathcal{L}_{\partial_t - \beta} P_{\mu\nu}$$

$$= \frac{1}{2N}(\partial_t P_{\mu\nu} - \cancel{\beta^\alpha \widehat{\nabla}_\alpha P_{\mu\nu}} - \overbrace{\widehat{\nabla}_\mu \beta^\alpha P_{\alpha\nu}}^{=\widehat{\nabla}_\mu \beta_\nu} - \overbrace{\widehat{\nabla}_\nu \beta^\alpha P_{\mu\alpha}}^{=\widehat{\nabla}_\nu \beta_\mu})$$

$$\Rightarrow \boxed{\partial_t P_{\mu\nu} = 2NK_{\mu\nu} + \widehat{\nabla}_\mu \beta_\nu + \widehat{\nabla}_\nu \beta_\mu}$$



[Gourgoulhon, 2021]

3+1 decomposition of Einstein eq. (4-3)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ G_{nn} = 8\pi G T_{nn} \ \rightarrow\ \widehat{R} + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ G_{n\widehat{\mu}} = 8\pi G T_{n\widehat{\mu}} \ \rightarrow\ \widehat{\nabla}_\nu K - \widehat{\nabla}_\mu K^\mu{}_\nu = 8\pi G p_\nu \\ (3)\ G_{\widehat{\mu}\widehat{\nu}} = 8\pi G T_{\widehat{\mu}\widehat{\nu}} \ \rightarrow\ \partial_t K_{\mu\nu} = N(-\widehat{R}_{\mu\nu} + 2K_{\mu\alpha}K^\alpha{}_\nu - KK_{\mu\nu}) \\ \qquad - (\beta^\alpha \partial_\alpha K_{\mu\nu} + \partial_\mu \beta^\alpha K_{\alpha\nu} + \partial_\nu \beta^\alpha K_{\mu\alpha}) \\ \qquad + \widehat{\nabla}_\mu \widehat{\nabla}_\nu N + 8\pi G N[S_{\widehat{\mu}\widehat{\nu}} - \frac{1}{2}P_{\widehat{\mu}\widehat{\nu}}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow\ \partial_t P_{\mu\nu} = 2NK_{\mu\nu} + \widehat{\nabla}_\mu \beta_\nu + \widehat{\nabla}_\nu \beta_\mu \end{cases}$$

>>> Intrinsic eq. with coordinates on the hypersurface

3+1 decomposition of Einstein eq. (4-4)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ G_{nn} = 8\pi G T_{nn} & \rightarrow\ \widehat{R} + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ G_{n\widehat{\mu}} = 8\pi G T_{n\widehat{\mu}} & \rightarrow\ \widehat{\nabla}_i K - \widehat{\nabla}_j K^j{}_i = 8\pi G p_i \\ (3)\ G_{\widehat{\mu}\widehat{\nu}} = 8\pi G T_{\widehat{\mu}\widehat{\nu}} & \rightarrow\ \partial_t K_{ij} = N(-\widehat{R}_{ij} + 2K_{ik}K^k{}_j - KK_{ij}) \\ & \qquad - (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ & \qquad + \widehat{\nabla}_i \widehat{\nabla}_j N + 8\pi G N [S_{ij} - \frac{1}{2} P_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} & \rightarrow\ \partial_t P_{ij} = 2NK_{ij} + \widehat{\nabla}_i \beta_j + \widehat{\nabla}_j \beta_i \end{cases}$$

>>> Intrinsic eq. with coordinates on the hypersurface

Now we change notation….;

$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ ^{(4)}G_{\widehat{\mu}\widehat{\nu}} = 8\pi G T_{\widehat{\mu}\widehat{\nu}} \ \rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\ \qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ \qquad - D_i D_j \alpha - 8\pi G\alpha[S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

$$N \rightarrow \alpha,\ K_{\mu\nu} \rightarrow -K_{\mu\nu},\ P_{ij} \rightarrow \gamma_{ij},\ \widehat{\nabla} \rightarrow D,\ R \rightarrow {}^{(4)}R,\ \widehat{R} \rightarrow R$$

From 3+1 Einstein eq. (1)

- Intrinsic eq.
- Hamiltonian constraint 1
- Momentum constraint 3
- Unknown K,P 12
- Degree to choose coordinates 4
- Dynamic d.o.f (12-4-4) = 4
- Gravitational field d.o.f 4/2 =2
- We can choose 8 independent variable for initial data and solve constraint eq's.

$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} & \rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} & \rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ ^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} & \rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\ & \qquad + (\beta^k \partial_k K_{ij} + \partial_i \beta^k K_{kj} + \partial_j \beta^k K_{ik}) \\ & \qquad - D_i D_j \alpha - 8\pi G \alpha [S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} & \rightarrow\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

Now…. Let's just see.

Conformal factor in the metric

from general spherical symmetric spacetime metric,

introducing a conformal factor ψ^4:

$$\boxed{ds^2 = -A(t,r)\mathrm{d}t^2 + 2B(t,r)\mathrm{d}t\mathrm{d}r + C(t,r)(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)}$$

$\hookrightarrow$ in terms of lapse and shift,

$$ds^2 = -(\alpha^2 - \vec{\beta}^2)\mathrm{d}t^2 + 2\beta_i\mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

$$\hookrightarrow \vec{\beta} = (\beta^r(t,r), 0, 0)$$

$$= -(\alpha^2 - \underbrace{\beta_r\beta^r}_{=g_{rr}(\beta^r)^2})\mathrm{d}t^2 + 2\underbrace{\beta_r}_{=g_{rr}\beta^r}\mathrm{d}t\mathrm{d}r + C(t,r)(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)$$

$$\hookrightarrow B = \beta^r \equiv \beta,\ C = g_{rr}$$

$$= -(\alpha^2 - C\beta^2)\mathrm{d}t^2 + 2C\beta\mathrm{d}t\mathrm{d}r + C(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)$$

$$\hookrightarrow C \equiv \psi^4$$

$$= -(\alpha^2 - \psi^4\beta^2)\mathrm{d}t^2 + 2\psi^4\beta\mathrm{d}t\mathrm{d}r + \psi^4\underbrace{(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)}_{=\bar{\gamma}_{ij}\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j}$$

$\therefore$ Any spherically symmetric metric is spatially conformally flat.

Conformal factor in the metric

$$ds^2 = -(\alpha^2 - \psi^4\beta^2)\mathrm{d}t^2 + 2\psi^4\beta\mathrm{d}t\mathrm{d}r + \psi^4\underbrace{(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)}_{=\bar{\gamma}_{ij}\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j}$$

$\therefore$ Any spherically symmetric metric is spatially conformally flat.

$$R = \psi^{-n}\bar{R} - 2n\psi^{-1-n}\bar{D}^2\psi + \underbrace{\frac{(4-n)n}{2}\psi^{-2-n}\bar{D}^i\psi\bar{D}_i\psi}_{\to 0 \text{ when } n=4}$$

(in flat case, since $\bar{\gamma}_{ij} = \eta_{ij}$ and $\bar{R}_{ij} = R = 0$,)

when $\gamma_{ij} = \psi^n\eta_{ij}$, that is , $n = 4$,

$$R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi$$

Conformal trace/traceless decomposition (1)

trace/traceless and conformal decomposition of K_{ij}:

(Note that we didn't fix $\bar{\gamma}_{ij} = \eta_{ij}$ yet.

We increased number of unknowns, γ_{ij} to $\psi, \bar{\gamma}_{ij}$ for convenience.)

$$K_{ij} \equiv A_{ij} + \frac{1}{3}\gamma_{ij}K \quad (A_{ij} : \text{traceless spatial extrinsic curvature})$$

$$A^{ij} \equiv \psi^{-10}\bar{A}^{ij}, \;\; A_{ij} = \psi^{-2}\bar{A}_{ij}$$

$$\therefore \;\; K_{ij} \rightarrow \psi^{-2}\bar{A}_{ij} + \frac{1}{3}\gamma_{ij}K$$

further decomposition of traceless $\bar{A}^{ij} = \bar{A}^{ij}_{\mathrm{T}}$

into transverse/longitudinal components:

$$\bar{A}^{ij} \equiv \bar{A}^{ij}_{\mathrm{TT}} + \bar{A}^{ij}_{\mathrm{L}} \quad \text{where } \bar{D}_j\bar{A}^{ij}_{\mathrm{TT}} = 0$$

$$\bar{A}^{ij}_{\mathrm{L}} = \bar{D}^iW^j + \bar{D}^jW^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_kW^k \equiv (\bar{L}W)^{ij}$$

$$\bar{D}_j\bar{A}^{ij} = (\bar{\Delta}_{\mathrm{L}}W)^i$$

Conformal trace/traceless decomposition (2)

$$^{(4)}G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\hookrightarrow \begin{cases} (1)\ ^{(4)}G_{nn} = 8\pi G T_{nn} \ \rightarrow\ R + K^2 - K_{ij}K^{ij} = 16\pi G E \\ (2)\ ^{(4)}G_{n\hat{\mu}} = 8\pi G T_{n\hat{\mu}} \ \rightarrow\ D_i K - D_j K^j{}_i = -8\pi G p_i \\ (3)\ ^{(4)}G_{\widehat{\mu\nu}} = 8\pi G T_{\widehat{\mu\nu}} \ \rightarrow\ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) \\ \qquad + (\beta^k\partial_k K_{ij} + \partial_i\beta^k K_{kj} + \partial_j\beta^k K_{ik}) \\ \qquad - D_iD_j\alpha - 8\pi G\alpha[S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)] \\ K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \ \rightarrow\ \partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{cases}$$

$$\begin{cases} G_{nn} : 8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho \\ G_{ni} : (\bar{\Delta}_{\mathrm{L}} W)^j - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_i K = 8\pi G\psi^{10}p^j \\ G_{ij} : \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_iD_j\alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \\ \qquad + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k \\ \quad (\text{trace})\ \partial_t K = -D^2\alpha + \alpha(K_{ij}K^{ij} + 4\pi(\rho + S)) + \beta^i D_i K \\ K_{ij} : \partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\ \quad \hookrightarrow 4\bar{\gamma}_{ij}\psi^3\partial_t\psi = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\ \quad (\text{trace})\ \frac{1}{2}\gamma^{ij}\partial_t\gamma_{ij} = \partial_t \ln\gamma^{1/2} = -\alpha K + D_i\beta^i \end{cases}$$

Ex. Schwarzschild metric

in the conformally flat, time-symmetric vacuum solution

whereby $(K_{ij} = -K^{ij} = 0,\ \rho = 0)$

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho$$

$$\Lsh\ \bar{D}^2\psi = \Delta^{\text{flat}}\psi = 0 \xrightarrow{\psi(\infty)=1} \psi = 1 + \frac{C}{r}$$

since the ADM mass is given by

$$M = -\frac{1}{2\pi G}\oint_\infty \bar{D}^i\left(1+\frac{C}{r}\right)\mathrm{d}^2S_i = -\frac{1}{2\pi G}\lim_{r\to\infty}(4\pi r^2 \times \left(-\frac{C}{r^2}\right) = \frac{2C}{G}$$

$$\therefore\ C = \frac{MG}{2}$$

we get $\psi = 1 + \dfrac{GM}{2r}$

$$ds^2 = -(\alpha^2 - \psi^4\beta^2)\mathrm{d}t^2 + 2\psi^4\beta\mathrm{d}t\mathrm{d}r + \psi^4\underbrace{(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)}_{=\bar{\gamma}_{ij}\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j}$$

$$ds^2 = -\left(\frac{1-\frac{m}{2r}}{2+\frac{m}{2r}}\right)^2\mathrm{d}t^2 + \left(1+\frac{m}{2r}\right)^4(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2)$$

$$\begin{cases} G_{nn}:\ 8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho \\ G_{ni}:\ (\bar{\Delta}_{\mathrm{L}}W)^j - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_iK = 8\pi G\psi^{10}p^j \\ G_{ij}:\ \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_iD_j\alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)) \\ \qquad\qquad + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k \\ \quad (\text{trace})\ \partial_t K = -D^2\alpha + \alpha(K_{ij}K^{ij} + 4\pi(\rho+S)) + \beta^i D_i K \\ K_{ij}\ :\ \partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\ \ \Lsh\ 4\bar{\gamma}_{ij}\psi^3\partial_t\psi = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\ \quad (\text{trace})\ \frac{1}{2}\gamma^{ij}\partial_t\gamma_{ij} = \partial_t \ln\gamma^{1/2} = -\alpha K + D_i\beta^i \end{cases}$$

Conformal trace/traceless decomposition (3)

$$G_{nn}:\ 8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho$$

$$G_{ni}:\ (\bar{\Delta}_L W)^j - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_i K = 8\pi G\psi^{10} p^j$$

G_{ij} : ∂…

(tra…

K_{ij} :

↳ $4\bar{\gamma}_{i}$…

(tra…

$$\begin{cases} G_{\mu\nu}(10\text{ eq.}) \to g_{\mu\nu}(10) \\ G_{\mu\nu}(10\text{ eq.}) + \dot{\gamma}_{ij}(6\text{ eq.}) \to g_{\mu\nu}(\overbrace{\alpha(1), \beta^i(3)}^{\text{related to remaining } \bar{\gamma}_{ij}, K \text{ but not appeared in constraint eq.}}, \psi(1), \bar{\gamma}_{ij}(3+2))(10) \\ \qquad + K_{\mu\nu}(K(1), \underbrace{A^{TT}_{ij}(2)}_{\text{GW dof.}}, A^{L}_{ij}(3))(6) \end{cases}$$

↳ by doubling dof with K_{ij}, γ_{ij},

$g_{\mu\nu}(10) \quad \to \quad$ constraint(4), gauge(4), GW dof(2)

$g_{\mu\nu}, K_{\mu\nu}(16) \quad \to \quad$ constraint(4), $\underbrace{\text{gauge(8), GW dof(4)}}_{\text{can be freely chosen}}$

Conformal trace/traceless decomposition (4)

G_{nn} :
G_{ni} :
G_{ij} :
(tr
K_{ij} :
↳ $4\bar{\gamma}$
(tr

background data and constraint equations:

$$\left\{ \begin{array}{l} g_{\mu\nu}(\overbrace{\alpha(1), \beta^i(3)}^{\text{related to remaining } \bar{\gamma}_{ij}, K \text{ but not appeared in constraint eq.}}, \psi(1), \boxed{\bar{\gamma}_{ij}(3+2)})(10) \\ +K_{\mu\nu}(\boxed{K(1)}, \boxed{\underbrace{A^{\mathrm{TT}}_{ij}(2)}_{\text{GW dof.}}}, A^{\mathrm{L}}_{ij}(3))(6) \end{array} \right. \rightarrow \left\{ \begin{array}{l} g_{\mu\nu}(\overbrace{\alpha(1), \beta^i(3)}^{\text{related to remaining } \bar{\gamma}_{ij}, K \text{ but not appeared in constraint eq.}}, \boxed{\psi(1)}, \bar{\gamma}_{ij}(3+2))(10) \\ +K_{\mu\nu}(K(1), \underbrace{A^{\mathrm{TT}}_{ij}(2)}_{\text{GW dof.}}, \boxed{A^{\mathrm{L}}_{ij}(3)})(6) \end{array} \right.$$

(chosen background data) will solve (the constraint equation)

for example, conformal flatness(CF), maximal slicing(MS), and no GW will give,

$$\left\{ \begin{array}{l} g_{\mu\nu}(\alpha(1), \beta^i(3), \psi(1), \boxed{\overbrace{\bar{\gamma}_{ij}}^{\equiv \eta_{ij}(\mathrm{CF})}(3+2)})(10) \\ +K_{\mu\nu}(\boxed{\underbrace{K(1)}_{\equiv 0(\mathrm{MS})}}, \boxed{\underbrace{A^{\mathrm{TT}}_{ij}(2)}_{\equiv 0\ (\mathrm{GW})}}, A^{\mathrm{L}}_{ij}(3))(6) \end{array} \right. \xrightarrow{\text{asymp'ly flat.}} \left\{ \begin{array}{l} g_{\mu\nu}(\alpha(1), \beta^i(3), \boxed{\overbrace{\psi(1)}^{=1+\frac{m}{2r}}}, \bar{\gamma}_{ij}(3+2))(10) \\ +K_{\mu\nu}(K(1), \underbrace{A^{\mathrm{TT}}_{ij}(2)}_{\text{GW dof.}}, \boxed{\underbrace{A^{\mathrm{L}}_{ij}(3)}_{=?}})(6) \end{array} \right.$$

Conformal thin-sandwich decomposition (1)

background data (γ_{ij}, K_{ij}) doesn't tell about evolution of anything,

$=$ difficult to use background data of a certain evolution

$$\left\{ \begin{array}{l} g_{\mu\nu}(\overbrace{\alpha(1), \beta^i(3)}^{\text{related to remaining } \bar{\gamma}_{ij}, K \text{ but not appeared in constraint eq.}}, \psi(1), \boxed{\bar{\gamma}_{ij}(3+2)})(10) \\ +K_{\mu\nu}(\boxed{K(1)}, \boxed{\underbrace{A^{\mathrm{TT}}_{ij}(2)}_{\text{GW dof.}}}, A^{\mathrm{L}}_{ij}(3))(6) \end{array} \right.$$

ς using $\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$

ς $K_{ij} \to \dfrac{1}{2\alpha}(-\partial_t\gamma_{ij} + D_i\beta_j + D_j\beta_i)$

$\boxed{\ ?\ }$ $\to$ We can use a background data of a specific $\dot{\gamma}_{ij}$.

Conformal thin-sandwich decomposition (2)

ς using $\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$

ς $K_{ij} \to \frac{1}{2\alpha}(-\partial_t\gamma_{ij} + D_i\beta_j + D_j\beta_i)$

$$\Lsh \begin{cases} 8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho \quad \to \quad \psi(1) \\ (\bar{\Delta}_{\mathrm{L}}\beta)^i - (\bar{L}\beta)^{ij}\bar{D}_j \ln(\alpha\psi^{-6}) = \alpha\psi^{-6}\bar{D}_j(\alpha^{-1}\psi^6\bar{u}^{ij}) + \frac{4}{3}\alpha\bar{D}^i K + 16\pi\alpha\psi^4 j^l \quad \to \beta^i(3) \\ \dot{K}_{ij} = \cdots \\ \dot{\gamma}_{ij} = \cdots \quad \to \begin{cases} \text{traceless} : \left[u^{ij} = \dot{\bar{\gamma}}_{ij} = -2\alpha A^{ij} + (L\beta)^{ij}\right] \to \left[\bar{A}^{ij} = \psi^6 \frac{(\bar{L}\beta)^{ij} - \bar{u}^{ij}}{2\alpha}\right] \\ \quad \Lsh \left[(\alpha, \beta^i, \psi, \bar{u}^{ij}) \to \bar{A}^{ij}\right] \\ \text{trace} : \left[\frac{1}{3}\gamma^{mn}\partial_t\gamma_{mn} = ft(\psi) = -\frac{2}{3}\alpha K + \frac{2}{3}D^k\beta_k\right] \\ \quad \Lsh \left[(\alpha, K, \psi) \to D^k\beta_k\right] \end{cases} \end{cases}$$

$$\Lsh \left[\text{choose } \bar{\gamma}_{ij},\ \dot{\bar{\gamma}}_{ij}(=\bar{u}_{ij})\right] \xrightarrow[\text{from } \rho,\, j^i \text{ eq.}]{\text{given } \alpha,\ K} (\psi,\ \beta^i) \to \left[\bar{A}^{ij} = \psi^6\frac{(\bar{L}\beta)^{ij} - \bar{u}^{ij}}{2\alpha}\right] \to (\gamma_{ij},\ K_{ij})$$

Conformal thin-sandwich decomposition (3)

[conformal transverse-traceless decomposition]

related to remaining $\bar{\gamma}_{ij}, K$

but not appeared in constraint eq.

$$\begin{cases} g_{\mu\nu}(\overbrace{\alpha(1), \beta^i(3)}, \psi(1), \boxed{\bar{\gamma}_{ij}(3+2)})(10) \\ +K_{\mu\nu}(\boxed{K(1)}, \boxed{\underbrace{A^{\rm TT}_{ij}(2)}_{\text{GW dof.}}}, \underbrace{A^{\rm L}_{ij}(3)}_{\text{constraint}})(6) \end{cases}$$

[background data (8) → constraint (4)]

→

[conformal thin-sandwich decomposition]

newly added in the constraint eq.

$$\begin{cases} g_{\mu\nu}(\overbrace{\boxed{\alpha(1)}, \beta^i(3)}, \psi(1), \boxed{\bar{\gamma}_{ij}(3+2)})(10) \\ +K_{\mu\nu}(\boxed{K(1)}, \underbrace{\underbrace{A^{\rm TT}_{ij}(2)}_{\text{GW dof.}}, A^{\rm L}_{ij}(3)})(6) \\ \to \boxed{(\dot{\gamma}_{ij} \to) u_{ij}(5),} \ \alpha,\ \beta^i \end{cases}$$

[background data (12) → constraint (4)]

$$\xrightarrow{K_{ij} \to \frac{1}{2\alpha}(-\partial_t \gamma_{ij} + D_i\beta_j + D_j\beta_i)}$$

Conformal thin-sandwich decomposition (4)

- *Assuming conformal flatness,*

$$\rho \;:\; \boxed{\Delta^{\text{flat}}\psi = -\frac{1}{8}\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} - 2\pi\psi^{5}G\rho} \quad \rightarrow (\text{Hamiltonian constraint for } \psi)$$

$$p^i \;:\; \boxed{(\Delta_{\text{L}}^{\text{flat}}\beta)^i = 2\bar{A}^{ij}\bar{D}_j(\alpha\psi^{-6}) + 16\pi\alpha\psi^{4}j^i} \quad \rightarrow (\text{constraint for } \beta^i\text{, minimal distortion})$$

$$\alpha \;:\; \boxed{\Delta^{\text{flat}}(\alpha\psi) = \alpha\psi\left(\frac{7}{8}\psi^{-8}\bar{A}_{ij}\bar{A}^{ij} + 2\pi G\psi^{4}(\rho + 2S)\right)} \quad \rightarrow (\text{maximal slicing for } \alpha)$$

$$\circlearrowleft\; D^2\alpha = \alpha(A_{ij}A^{ij} + 4\pi G(\rho + S)) \quad \cdots(\dot{K}\text{ eq., maximal slicing})$$

$$\circlearrowleft\; \Delta^{\text{flat}}\psi = -\tfrac{1}{8}\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} - 2\pi\psi^{5}G\rho \quad \cdots(\rho\text{ eq., Hamiltonian constraint})$$

$$\hookleftarrow\; \bar{D}^2(\alpha\psi) = \psi^{5}D^2\alpha + \alpha\bar{D}^2\psi \quad \cdots(\text{identity})$$

$$\dot{K}_{ij} = \cdots$$

$$\dot{\gamma}_{ij} \;:\; \text{traceless} \;:\; [u^{ij} = \dot{\tilde{\gamma}}_{ij} = -2\alpha A^{ij} + (L\beta)^{ij}] \rightarrow \boxed{\bar{A}^{ij} = \psi^{6}\frac{(\bar{L}\beta)^{ij} - \cancel{\bar{u}^{ij}}}{2\alpha}}$$

$$\Rsh\; [(\alpha, \beta^i, \psi, \bar{u}^{ij}) \rightarrow \bar{A}^{ij}]$$

$$\text{trace} \;:\; \boxed{\frac{1}{3}\gamma^{mn}\partial_t\gamma_{mn} = ft(\psi) = -\cancel{\frac{2}{3}\alpha K} + \frac{2}{3}D^k\beta_k}$$

$$\Rsh\; [(\alpha, K, \psi) \rightarrow D^k\beta_k]$$

Rewriting ADM evolution equation (1)

From Maxwell's equations

$$\begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} = 4\pi\rho \text{ (constraint eq. for } \boldsymbol{E}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B} = -\frac{1}{c}\partial_t \boldsymbol{B} \\ \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \epsilon_0 \mu_0 \partial_t \boldsymbol{E} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\partial_t \boldsymbol{E} \cdots (a) \end{cases}$$

introducing a potential, $B_i \equiv \epsilon_{ijk} D_j A_k$, $E_i \equiv -D_i \Phi - \partial_t A$

which leads $\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}$

Then, evolution eq. for $\boldsymbol{E}$

$$\begin{cases} (A_i) : \ \partial_t A_i = -E_i - D_i \Phi \\ (a) \ \ : \ \partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i \end{cases}$$

Rewriting ADM evolution equation (2)

(1) $\partial_t(A_i) \rightarrow (a)$,

$$\begin{cases} \nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} = 4\pi\rho \ (\text{constraint eq. for } \boldsymbol{E}) \\ (A_i):\ \partial_t A_i = -\boxed{E_i} - D_i\Phi \\ \qquad \swarrow \ \partial_t \ \hookleftarrow \\ (a)\ :\ \boxed{\partial_t E_i} = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i \end{cases}$$

$$\Lsh\ \underbrace{-\partial_t^2 A_i + D^2 A_j}_{\text{wave eq. form}} - \underbrace{(D_i D^j A_j + D_i \partial_t \Phi)}_{\text{mixed derivatives}\rightarrow 0 \text{ when we choose a Lo. gauge } \partial_t\Phi = -D^j A_j} = -4\pi J_i$$

↯ Lorenz gauge

$$\boxed{-\partial_t^2 A_i + D^2 A_j = -4\pi J_i}$$

This is the harmonic coordinates case in gravity,

But this approach may develop pathologies

which could prematurely end a numerical simulations

Rewriting ADM evolution equation (2)

(2) $\partial_t(a) \to (A_i)$,

$$\begin{cases} \boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0} = 4\pi\rho \ \text{(constraint eq. for } \boldsymbol{E}\text{)} \\ (A_i):\ \boxed{\partial_t A_i} = \boxed{-E_i - D_i\Phi} \\ (a)\ :\ \partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i \\ \qquad\qquad \circlearrowleft\ \partial_t \\ \partial_t^2 E_i = -D^j D_j \boxed{\partial_t A_i} + D_i D^j \boxed{\partial_t A_j} - 4\pi\partial_t J_i \end{cases}$$

$$\hookrightarrow\ \underbrace{\partial_t^2 E_i = -D^j D_j(-E_i}\ -D_i\Phi) + D_i D^j(-E_j - D_j\Phi) - 4\pi\partial_t J_i$$

$$\hookrightarrow\ \overbrace{D^j D_j E_i - \partial_t^2 E_i} = \cancel{-D^j D_j D_i\Phi} + D_i \underbrace{D^j E_j}_{=4\pi\rho\ \text{(from constraint)}} + \cancel{D_i D^j D_j\Phi} + 4\pi\partial_t J_i$$

$$\hookrightarrow\ \boxed{\underbrace{D^j D_j E_i - \partial_t^2 E_i}_{\text{gauge invariant}} = 4\pi D_i\rho + 4\pi\partial_t J_i}$$

$\hookrightarrow$ matter derivative could make a problem

Rewriting ADM evolution equation (2)

This is a gauge-independent way,

This method reveals some potential disadvantages

for simulations that involve matter.

In evolution calculations of neutron stars, for example,

the non-smoothness of the matter and fields on the stellar surface

or across shocks always pose numerical difficulties.

$$\hookrightarrow \boxed{\underbrace{D^j D_j E_i - \partial_t^2 E_i}_{\text{gauge invariant}} = 4\pi D_i \rho + 4\pi \partial_t J_i}$$

↳ matter derivative could make a problem

Rewriting ADM evolution equation (3)

(3) auxiliary variable method:

no mixed derivative term without gauge fixing,

no derivative on matter:

$$\begin{cases} \text{evolution:} & \begin{cases} \partial_t E_i = -D^j D_j A_i + D_i \Gamma - 4\pi J_i \\ \partial_t \Gamma = -D^2 \Phi - 4\pi\rho \quad \rightarrow \text{no derivative on matter} \end{cases} \\ \text{constraint:} & \begin{cases} D^i E_i = 4\pi\rho \\ \Gamma = D^j A_j \end{cases} \end{cases}$$

$$\partial_t \boxed{\Gamma} = \partial_t \boxed{D^i A_i} = D^i \partial_t A_i \longrightarrow \underbrace{D^i(-E_i - D_i\Phi)}_{=-4\pi\rho \text{ (constraint eq.)}} = -D^2\Phi - 4\pi\rho$$

BSSN formalism (1)

conformal transformation + trace/traceless decomposition

(a) $\gamma_{ij} = \gamma^{1/3}\bar{\gamma}_{ij} = \psi^4\bar{\gamma}_{ij},\ \bar{\gamma} \equiv 1$ (normalization)

$$\xrightarrow{\text{BSSN}} \boxed{\gamma_{ij} = \psi^4\bar{\gamma}_{ij} \equiv e^{4\phi}\bar{\gamma}_{ij}} \xrightarrow{\gamma_{ij}\gamma^{ij} = \bar{\gamma}_{ij}\bar{\gamma}^{ij}} \boxed{\gamma^{ij} = e^{-4\phi}\bar{\gamma}^{ij}}$$

(b) $K_{ij} = \underbrace{A_{ij}}_{\text{traceless}} + \underbrace{\frac{1}{3}\gamma_{ij}K}_{\text{trace}},\ A^{ij} \equiv \psi^{-10}\bar{A}^{ij},\ A_{ij} = \psi^{-2}\bar{A}_{ij}$

$$\xrightarrow{\text{BSSN}} \boxed{\tilde{A}_{ij} \equiv e^{-4\phi}A_{ij}}$$

$$\Rightarrow \boxed{\tilde{A}^{ij} == e^{4\phi}A^{ij} = e^{-6\phi}\bar{A}^{ij}}$$

$R_{ij} = \bar{R}_{ij} + R^{\phi}_{ij}$

when $\gamma_{ij} = \psi^4\eta_{ij}$,

$$\begin{aligned} R_{ij} &= \bar{R}_{ij} - 2(\bar{D}_i\bar{D}_j \ln\psi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m \ln\psi) \\ &\quad + 4((\bar{D}_i \ln\psi)(\bar{D}_j \ln\psi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l \ln\psi)(\bar{D}_m \ln\psi)) \\ &= \bar{R}_{ij} + R^{\phi}_{ij} \end{aligned}$$

BSSN formalism (2)

$$R^{\phi}_{ij} = -2(\bar{D}_i\bar{D}_j \ln\psi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m \ln\psi)$$
$$+ 4((\bar{D}_i \ln\psi)(\bar{D}_j \ln\psi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l \ln\psi)(\bar{D}_m \ln\psi))$$
$$\varsigma \quad \psi = e^{\phi}, \ (\ln\psi = \phi)$$
$$= -2(\bar{D}_i\bar{D}_j\phi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\phi)$$
$$+ 4[(\bar{D}_i\phi)(\bar{D}_j\phi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l\phi)(\bar{D}_m\phi)]$$

$$\bar{R}_{ij} = \underbrace{-\frac{1}{2}\bar{\gamma}^{lm}\bar{\gamma}_{ij,lm}}_{\text{only 2nd deri.}} + \underbrace{\bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k}_{\text{all other mixed derivatives are absorbed to } \bar{\Gamma}^k} + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}(2\bar{\Gamma}^k_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k_{im}\bar{\Gamma}_{klj})$$

where $\bar{\Gamma}^i \equiv \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} = -\bar{\gamma}^{ij}{}_{,j}$

$$\partial_t\bar{\Gamma} = -2\partial_j\alpha\tilde{A}^{ij} + 2\alpha(\bar{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi G\bar{\gamma}^{ij}p_j + 6\tilde{A}^{ij}\partial_j\phi)$$
$$+ \beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{ki}\beta^j{}_{,jk} + \bar{\gamma}^{kj}\beta^i{}_{,kj}$$

BSSN formalism (3)

[evolutions equations]

(1) $\partial_t \phi = -\frac{1}{6}\alpha K + \frac{1}{6}\partial_i \beta^i + \beta^i \partial_i \phi$

(2) $\partial_t K = -D^2\alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) + 4\pi\alpha(\rho + S) + \beta^i D_i K$

(3) $\partial_t \bar{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \beta^k \partial_k \bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j \beta^k + \bar{\gamma}_{kj}\partial_i \beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k \beta^k$

(4) $\partial_t \tilde{A}_{ij} = e^{-4\phi}[-(D_i D_j \alpha)^{\rm TF} + \alpha(\overbrace{R_{ij}^{\rm TF}}^{\text{includes mixed derivative} \rightarrow \text{absorbed to } \bar{\Gamma}} - 8\pi S_{ij}^{\rm TF})] + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j)$
$+ \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k \beta^k$

↳ until (4), there's no mixed derivatives except for R_{ij} term in (4).

to remove this by introducing an auxiliary field $\bar{\Gamma}^i \equiv \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} = -\bar{\gamma}^{ij}{}_{,j}$,

(4) $\rightarrow R_{ij} = \bar{R}_{ij} + R^{\phi}_{ij}$

$$= \underbrace{-\frac{1}{2}\bar{\gamma}^{lm}\bar{\gamma}_{ij,lm}}_{\text{only 2nd deri.}} + \underbrace{\bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k}_{\text{all other mixed derivatives are absorbed to } \bar{\Gamma}^k} + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}(2\bar{\Gamma}^k_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k_{im}\bar{\Gamma}_{klj})$$

$$-2(\bar{D}_i\bar{D}_j\phi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\phi) + 4[(\bar{D}_i\phi)(\bar{D}_j\phi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l\phi)(\bar{D}_m\phi)]$$

BSSN formalism (4)

evolution of Γ :

$$\partial_t \bar{\Gamma}^i = -2\partial_j \alpha \tilde{A}^{ij} + 2\alpha(\bar{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi G \bar{\gamma}^{ij} p_j + 6\tilde{A}^{ij}\partial_j \phi)$$

$$+ \beta^j \partial_j \bar{\Gamma}^i - \bar{\Gamma}^j \partial_j \beta^j + \frac{2}{3}\bar{\Gamma}^i \partial_j \beta^j + \frac{1}{3}\bar{\gamma}^{ki}\beta^j{}_{,jk} + \bar{\gamma}^{kj}\beta^i_{,kj}$$

[constraint equation]

(1) $^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho$

(2) $^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu$

(3) $\bar{\Gamma}^i \equiv \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} = -\bar{\gamma}^{ij}{}_{,j}$

Various slicing (1-1. Geodesic slicing)

- Geodesic slicing (Gaussian normal) $\boxed{\alpha \equiv 1,\ \beta^i \equiv 0}$

$$ds^2 = -(\underbrace{\alpha^2}_{1} - \cancel{\psi^4\beta^2})\mathrm{d}t^2 + \cancel{2\psi^4\beta\mathrm{d}t\mathrm{d}r} + \psi^4(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)$$

$$= -\mathrm{d}t^2 + \psi^4(\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega)$$

↳ flat Robertson-Walker metric in spherical coordinates

+ comoving homogeneous and isotropic perfect fluid ⇒ Friedmann eq.

Various slicing (1-2. Geodesic slicing)

- **Geodesic slicing** $\boxed{\alpha \equiv 1,\ \beta^i \equiv 0}$

$$\partial_t K = -D^2\alpha + \alpha(K_{ij}K^{ij} + 4\pi G(\rho + S)) + \beta^i D_i K$$

$$\hookleftarrow\ \alpha \equiv 1,\ \beta \equiv 0,\ \text{in the vacuum}$$

$$= K_{ij}K^{ij} = A_{ij}A^{ij} + \frac{1}{3}K^2 \ \rightarrow \text{non-negative} \rightarrow \text{monotonically increasing}$$

when $K = K_0,\ A_{ij} = 0$ at $t = 0$,

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{1}{3}K^2 \rightarrow 3\int \frac{1}{K^2}\mathrm{d}K = \int \mathrm{d}t = t + C \rightarrow \frac{-3}{t+C} = K \rightarrow C = \frac{-3}{K_0}$$

$$\therefore\ K = \frac{3K_0}{3 - K_0 t}$$

A coordinate singularity forms at $t = 3/K_0$.

When a small amplitude GW is located at the origin and disappeared,
in the beginning, observer will be attracted to the origin,
and even after GW is disappeared, it will form a coordinate singularity.

Various slicing (2. Maximal slicing)

- **Maximal slicing:** $\boxed{D^2\alpha = \alpha(K_{ij}K^{ij} + 4\pi(\rho + S)) \text{ \& time-sym. at } t = t_0}$

$$\Leftrightarrow \boxed{K \equiv 0,\ \partial_t K \equiv 0}$$

Since $K \equiv g^{ab}K_{ab} = -\nabla^a n_a = 0$, the normal vector doesn't converge.

$\boxed{\text{maximal slicing } (K = 0 = \partial_t K)} + \boxed{\text{flat vacuum spacetime } (\bar{R} = 0,\ \rho = 0 = S)}$

$$ds^2 = -\left(\frac{1 - m/(2r)}{1 + m/(2r)}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d^2\Omega)$$

$\therefore \boxed{\text{maximal slicing}} + \boxed{\text{flat time-symmetric vacuum}}$

$+ \boxed{\alpha\text{'s b.c.(not always } \alpha = 1) + (\beta^i = 0)}$

$\Rightarrow$ Schwarzschild metric in isotropic coordinate

Various slicing (3. Harmonic slicing)

- Harmonic slicing: $\boxed{{}^{(4)}\Gamma^0 \equiv 0}$

combining $\beta^j \equiv 0$,

$${}^{(4)}\Gamma^0 \equiv 0 \;\rightarrow\; (\partial_t - \beta^j\partial_j)\alpha = -\alpha^2 K$$

$$\hookleftarrow\; \beta^j \equiv 0$$

$$\partial_t \alpha = -\alpha^2 K \xrightarrow{\partial_t \ln \gamma^{1/2} = -\alpha K + D_i\beta^i} = \alpha \partial_t \ln \gamma^{1/2}$$

$$\Lsh\; \alpha = C(x^i)\gamma^{1/2}$$

$$\therefore\; ({}^{(4)}\Gamma^0 = 0 \;\&\; \beta^j \equiv 0) \;\Rightarrow\; (\hat{\alpha} = \gamma^{-1/2}\alpha = C(x^i) = \text{const.})$$

therefore, harmonic slicing & zero shift is identical to the constant densitized lapse.

Various slicing (4. etc....)

generalized harmonic slicing: $\boxed{\partial_t \alpha = -\alpha^2 f(\alpha) K, \ f(\alpha) > 0}$

(suggested by Bona et al. (1995)

because of the weaker singularity avoidance of harmonic slicing)

$$\begin{cases} f = 1, & \text{"harmonic slicing" (Ei eq. to non-linear wave eq., const. densitized lapse)} \\ f = 0, & \text{"geodesic slicing" (collapsing to singularity, gives Friedmann eq.)} \\ f \to \infty, & \text{"maximal slicing" (not-converging normal vector, singularity avoidance)} \\ f = 2/\alpha, & \text{"1 + log slicing"} \because \ \partial_t \alpha = -\alpha^2 \left(\frac{2}{\alpha}\right) K = -2\alpha K \xrightarrow{\partial_t \ln \gamma^{1/2} = -\alpha K + \cancel{D_i \beta^i}} = 2\partial_t \ln \gamma^{1/2} \\ & \qquad \hookrightarrow \alpha = C + \ln \gamma \equiv 1 + \ln \gamma \end{cases}$$

in 1+log slicing, $f \to \infty$ when $\alpha \to 0$, it behaves like the maximal slicing.

Now, ... And what?

- Minimal distortion (elimination purely coordinate-related fluctuation in γ_{ij})
- Locating horizon
- Constructing initial data

$$^{(4)}G_{nn} \to 8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5 G\rho \quad \text{(H. constraint)}$$

$$\hookrightarrow \Delta^{\text{flat}}\psi = 0 \text{ (Laplace eq.)} \xrightarrow[\psi|_{r\to\infty} \to 1]{\text{assuming asymptotic flatness}} \psi = 1 + \frac{m}{2r}$$

$$\hookrightarrow \text{linear eq.} \to \boxed{\psi = 1 + \sum_\alpha \frac{m_\alpha}{2r_{C_\alpha}}}$$

where $r_{C_\alpha} = |x^i - C^i_\alpha|$ and

C^i_α is the coordinate location of the αth black hole

- Bowen-York approach

initial data constructed using
- the conformal transverse-traceless decomposition
- maximal slicing ($K = 0$, not $K_{ij} = 0$)
- conformal flatness ($\bar{\gamma}_{ij} = \eta_{ij}$)

- Evolution

How about now?

$$\begin{cases} {}^{(4)}G_{nn} = \frac{1}{2}(R + K^2 - K_{\mu\nu}K^{\mu\nu}) = 8\pi G\rho \\ {}^{(4)}G_{n\hat{\mu}} = D_\mu K - D_\nu K^\nu{}_\mu = -8\pi G j_\mu \\ {}^{(4)}G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}} \\ \hookrightarrow \partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) - D_i D_j \alpha - \alpha 8\pi G(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \\ \qquad\qquad + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k \\ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{cases}$$

Thank You!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